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Jaffray's ideas on ambiguity

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Abstract This paper discusses Jean-Yves Jaffray's ideas on ambiguity and the views underlying his ideas. His models, developed 20 years ago, provide the most tractable separation of risk attitudes, ambiguity attitudes, and ambiguity beliefs available in the literature today.

Keywords Ambiguity · Total absence of information · Belief functions

JEL Classification D81 · D03

1 Introduction

Jean-Yves Jaffray passed away on February 26, 2009. This paper describes his ideas on ambiguity. It explains how a principle of total absence of information, together with a desire to entirely disentangle probabilistic information from ambiguous information, was at the basis of his works. Jaffray's adherence to the principle of total absence of information, initiated by Cohen and Jaffray (1980), serves to treat ambiguous information in a strictly objective manner. This extreme adherence to objectivity fits well with Jaffray's applied work in statistics and computer science.

Jaffray (1989a) is his key paper on ambiguity. It achieves a clean and complete separation of risk attitude, ambiguity attitude, ambiguous beliefs, and a degree of operationalizability that has not yet been achieved elsewhere. Jaffray's achievement is especially useful because of the increased interest in ambiguity today.

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2 Jaffray's style and my relationship with him

Jaffray was modest and sincere in propagating his ideas. He was not willing to play the academic game of overselling, and satisfying the opinions and desires of average referees with their own research agendas. The sacrifice of one's own ideas that is required to play the academic game was unacceptable to Jaffray. It led him to choose outlets that were open to his opinions even if those deviated from trends and fashions, and that allowed him to express his ideas exactly as he wanted. Given the depth of his ideas, his clear visions, and his concise and to-the-point writing with no overselling involved, reading his papers closely and using his ideas are highly rewarding. The quality of his papers ensures their long-term survival.

I first met Jaffray in the FUR (Foundations of Utility and Risk) Conference in Venice 1984, where he presented fascinating work on the foundations of statistics, later published as Jaffray and Said (1994). In our first conversation, I told him that I was interested in the foundations of statistics, both the book (Savage 1954) and the topic, and this was the beginning of long and intense interactions. FUR 1984 was my second conference, at the beginning of my career, and Jaffray with his brilliancy became a guide for me then, young and lucky researcher that I was.

At least once per year I would visit Jaffray and give a seminar for him, Alain Chateauneuf, Michèle Cohen, and their fascinating group in Paris. I would then read his papers, primarily on ambiguity, and spend an afternoon discussing with him, learning much from his visions and motivations. We would also interact much at conferences and other occasions. For every model I knew, Jaffray's judgment was my first criterion. Our different judgments on the rationality of Bayesianism in the Ellsberg paradox, on consequentialism, and on some other topics led to intense discussions, with me continuously learning from his ideas. Our interactions became less intense after 1995, when Jean-Yves started working on Bayesian networks. This topic is outside my expertise. In those days, I also reached the age where human beings become less open to new visions and directions.

3 Basic concepts of decision under uncertainty

Savage (1954) formulated axioms for decision making that imply expected utility maximization for decision under uncertainty. In decision under uncertainty, *acts* map a *state space* $\mathcal{X}$ to an *outcome space*. The outcome space can, for instance, be $\mathbb{R}$, with outcomes designating money amounts. We will need no symbol for the outcome space in what follows. The state space models the uncertainty of the decision maker. One state is true, the other states are not true, and the decision maker does not know for sure which state is the true one. An act yields the outcome corresponding to the true state. As the decision maker has uncertainty about which state is true, he has uncertainty about the outcome that results from an act and, hence, has to make his decisions under uncertainty. The decision maker has preferences between acts. An *event* can consist of one or several states of nature. In the latter case, it is the event that any of its states contained is the true one.

Under expected utility, a *utility function* from the outcome set to the reals describes the subjective value of the outcomes, and a *probability measure* on the state space describes the uncertainty of the decision maker. Acts are evaluated by their *expected utility*: their probability-weighted average utility. The question arises how to apply Savage's model if no objective probabilities of the states are known, say because we lack the statistical data to determine these. Since Savage's axioms imply that probabilities *have* to be used, we then, for lack of better, have to use our best guesses of probabilities based on whatever information these can be based on. We then call the probabilities *subjective*. They can be different for different persons in the same way as utilities can be. For people who, like the author of this paper, believe in the rationality of Savage's axioms, there is no escape from using subjective probabilities, even if one wants to minimize their use.

4 Jaffray's philosophy of total absence of information

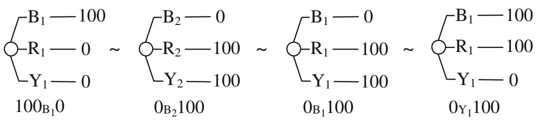
Jaffray did not believe in the rationality of Savage's axioms. For Jaffray, there was an escape from subjective probabilities. More than that, throughout his work, one of the driving principles was to completely and entirely avoid subjectivity in the modelling of uncertain information. The basis of his way of doing this is described in this section, and two applications are presented in the following sections. As throughout this paper, I will not formalize the analysis, with natural conditions such as continuity and weak monotonicity[1] implicitly assumed, and domains not exactly specified.

Imagine the following variation of the Ellsberg paradox. One unknown (ambiguous) urn A_1 contains 90 balls that are blue (B_1), red (R_1), or yellow (Y_1), in unknown proportions. A second urn A_2 has the same number of balls, but its proportions of the colors B_2, R_2, and Y_2 can be different than in the first. From each urn a ball will be drawn randomly and its color will be inspected. R_2 denotes the event that the ball drawn from the second urn is red; other events are denoted similarly. Jaffray always sought to obtain a complete disentanglement of probabilizable uncertainty for which we have objective probabilities given, and uncertainty for which we have no such probabilities. The latter I refer to as total absence of information. For the two urns we assume this total absence of information.

Consider the act $100_{B_1}0$ (on the extreme left in Fig. 1; circles indicate event nodes), yielding 100 (with euros as the unit) if the ball from A_1 is blue and nothing otherwise. The act $0_{B_2}100$ (second to the left in Fig. 1) yields 0 (nothing) if the color from A_2 is blue and 100 otherwise. Which act can be preferred? A preference for $0_{B_2}100$ may seem to be natural because it gives the prize for two colors and not for one, and two colors in A_2 may be more likely (in some sense) than one color in A_1. Yet such a preference uses more subjective beliefs about uncertain information than Jaffray would want to accept. Maybe there is some principle of insufficient reason underlying this subjective reasoning, treating the three colors as equal in some sense in each urn and

[1] If a first act yields at least as good an outcome for each state of nature as a second act does, then the first act is *weakly* preferred to the second. Requiring strict preference between the acts if some outcomes are strictly preferred is less self-evident because it requires nonnullness of the event of strictly better outcomes. Nonnullness is a subtle concept under nonexpected utility.

Fig. 1 Indifferences under total absence of information



then taking two as more than one. Such an assumption of insufficient reason comprises more subjectivity in beliefs (for instance, through a subjective choice of the state space and its partition) than Jaffray was willing to accept.

Jaffray accepted no strict preference between the two acts. Then one can assume no preference at all, and work with incompleteness and possibly intransitivity (Cohen and Jaffray 1980). In most other papers, Jaffray assumed indifference (denoted $\sim$), maintaining completeness according to the decision principle that one has to choose if one has to choose. The second and third indifferences in Fig. 1 are natural. Transitivity then implies the indifference

$$100_{B_1}0 \sim 0_{Y_1}100.$$

The two acts have the same maximal and minimal outcomes, but the second act yields at least as much as the first for each event and it yields strictly more under event R_1. Yet it is indifferent. While this finding does not violate weak monotonicity, it does violate stricter versions of monotonicity (Milnor 1954, p. 55). Jaffray was willing to pay this price to escape from subjective beliefs. The violation can be justified by a kind of context-dependent nullness of event R_1.

We generalize the logic of total absence of information in the above example to general acts. First consider a partition $\{E_1, \ldots, E_n\}$ of the state space and only acts measurable with respect to this partition. Jaffray assumed that we have no objective reason to treat the events in the partition considered differently from one another, implying a symmetric evaluation of acts: permuting outcomes over states does not affect preference. Consider a general act $(E_1 : x_1, E_2 : x_2, \ldots, E_n : x_n)$ with, say, $x_1 \geq \cdots \geq x_n$. By weak monotonicity, the act, weakly dominated by $(E_1 : x_1, E_2 : x_1, \ldots, E_{n-1} : x_1, E_n : x_n)$ and weakly dominating $(E_1 : x_1, E_2 : x_n, \ldots, E_n : x_n)$, is between the latter two acts in preference. However, the latter two are indifferent by the same logic as in the three-color urns above (consider the three events E_1, $E_2 \cup \cdots \cup E_{n-1}$, and E_n). Thus, any general act is equivalent to the best and also to the worst possible act with the same maximal and minimal outcomes for a given partition. All acts with the same maximal and minimal outcomes must be equivalent. This can be demonstrated for more and more refined partitions, making all acts with the same maximal and minimal outcomes indifferent. Hence, each act can be characterized by the pair (m, M) of their minimal and maximal outcomes. General infinite-valued acts are characterized by their infimum outcome m and their supremum outcome M, by limiting arguments.

Because of weak monotonicity, the preference value of any act will be between its infimum m and its supremum M, so that it will be an $\alpha/(1-\alpha)$ mixture of the values of m and M. Here, in general, α will depend on m and M, and $0 \leq \alpha = \alpha(m, M) \leq 1$. The $\alpha(m, M)$ weights are subjective and vary from individual to individual. Their subjectivity does not entail a violation of Jaffray's objectivity requirement for modelling beliefs because $\alpha(m, M)$ is not to be interpreted as a belief component. It is a component of decision attitude, which is allowed to be subjective just as utility is allowed to be subjective.

The above modelling of total absence of information is not new (Luce and Raiffa 1957 §13.4; see also Milnor 1954). It is a local α-Hurwicz evaluation as Jaffray (1989a) called it, where local reflects dependence of α on the outcomes. What is new is that Jaffray recognized the role of this philosophy of total absence of information in many situations and developed many sound models based on it. He always sought to completely and strictly separate probabilizable uncertainty from ambiguous uncertainty, where the latter is subject to the principles of total absence of information. Any uncertain information is disentangled into two such components.

5 Jaffray's foundation of belief functions

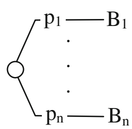
Belief functions, introduced by Dempster (1967) and Shafer (1976), were popular in artificial intelligence in the 1980s and 1990s. They became less so when Bayesian networks became popular. Nevertheless, belief functions have a number of attractive properties and continue to be of interest. Recently, there has been a renewed interest in them (Denoeux 2008). I will develop their definition in the following text.

Assume Savage's (1954) usual state space $\mathcal{X}$, with one state being true but it being unknown which one. Assume that a message will be received of the form $B \subset \mathcal{X}$, meaning that the true state will be contained in B. In Bayesian approaches, probabilities will then be updated using Bayes' formula if the appropriate ceteris paribus assumptions are satisfied. Under the philosophy of total absence of information, things are different. As a result of the new information, minimal and maximal outcomes are now to be taken over B, rather than over the whole space $\mathcal{X}$ as it was prior to the receipt of information.

Assume next, as a variation, that we will receive a message, but we are not sure which one. It will be one of $B_1, \ldots, B_n$. Assume that the uncertainty about the message to be received is completely probabilized, through probabilities $p_1, \ldots, p_n$ (Fig. 2). That is, we receive a random message as Dempster (1967) called it.

In this model, there is a strict separation between the probabilized information in the first stage and the total absence of information in the second stage. The assumption that the probabilized uncertainty is in the first stage and the ambiguous uncertainty in the second is a restrictive assumption applicable only in special cases, but nevertheless is of wide interest. Many practical situations can be modelled this way (Dempster 1967; Shafer 1976). This two-stage model can be compared with the version of the Anscombe and Aumann (1963) model that is popular in the analysis of nonexpected utility today, where the first stage comprises the ambiguous uncertainty and the second

Fig. 2 A random message



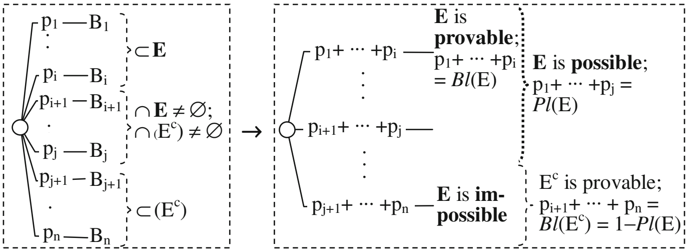


Fig. 3 Random messages and belief functions

stage comprises probabilized uncertainty.[2] So then probabilities appear in the second stage and not in the first. Both models are restrictive but yet of considerable interest. Sometimes, the probabilized stage is an artificially or experimentally added stage serving to clarify the processing of ambiguity. It can serve this purpose when taken as first stage and also when taken as second stage.

In Jaffray's model (Fig. 2), assume that we consider an event E that is relevant for our decisions (i.e., it has an impact for the outcomes that can result from acts) and want to infer what we can about its probability $P(E)$. We have:

Observation 5.1 $\sum_{B_j \subset E} p_j \leq P(E) \leq \sum_{B_i \cap E \neq \emptyset} p_i.$

Proof If we are sure to receive a message B and if B is contained in E (it implies E), then given this information E is sure to happen and has probability 1 of obtaining (E then is provable). If a random message B_1 with probability p_1 is contained in E, then we know that E happens with a probability of at least p_1. If then, further, a random message B_2 with probability p_2 is contained in E too, then E's probability must be at least $p_1 + p_2$. From this reasoning, the lower bound follows (see Fig. 3). It is the probability of E being provable.

If random messages B_n and B_{n-1} do not intersect E, then E will surely not happen under these messages (E^c is provable; or, E is impossible), and the probability of E not happening is at least $p_n + p_{n-1}$. Then E's probability cannot exceed $1 - (p_n + p_{n-1})$. From this reasoning the upper bound follows. □

The lower bound in Observation 5.1 can be interpreted as the probability of E being provable and the upper bound as the probability of E being possible. The function that assigns to each event E its lower bound is the *belief function*, *denoted* Bl(E) and the upper bound is the *plausibility function*, denoted Pl(E). As can readily be verified from the complement E^c (Fig. 3), $\text{Pl}(E) + \text{Bl}(E^c) = 1$ for all events E. Bl is therefore called the dual of Pl. Pl can be derived from Bl, so that the belief function comprises all the information about the plausibility function. These facts were well known, and I had known them, and belief functions in general, long before June 1988.

My real understanding of belief functions came only in a conversation with Jaffray in a hotel lounge in Budapest on June 7, 1988, around 7:00 PM. We discussed the phi-

[2] In the original model in Anscombe and Aumann (1963), there was a probabilized stage both before and after the uncertain events.

losophy of total absence of information and belief functions. At that moment I came to understand that, under this philosophy, belief functions are the only way possible to describe the uncertain information. Using a statistical term in a different context, they provide a sufficient statistic for the informational status. I will not fully formalize the following reasoning, which underlies Jaffray and Wakker (1993), the more so as a behavioral foundation will come in the following section. Ghirardato (2001) is another paper inspired by interactions with Jaffray and by Jaffray's views on belief functions.

Observation 5.2 *Under the principle of total absence of information, belief functions capture all relevant information from random messages.*

Proof Under total absence of information, the only thing we can say about the likelihood of an event after having received a random message is whether E is sure to happen, sure not to happen, or neither of the two (possible). It is a kind of three-valued logic.

For the next step, bear in mind that (1/4:100, 1/4:100, 1/2:0) and (1/2:100, 1/2:0) denote the same lottery for money, yielding 100 with probability 1/2 and 0 with probability 1/2, for decision under risk. Similarly, it should not matter how we obtain a total probability $\text{Bl}(E)$ of the "outcome" of E being provable, and a total probability $\text{Pl}(E)$ of the "outcome" of E being possible. That is, in Fig. 3, it does not matter if we collapse the upper i branches into one branch, the next j branches into a second branch, and the remaining branches into a third. In Fig. 3, we can move from the first dashed rectangle to the second. Bl and Pl thus comprise all relevant information. As Pl can be derived from Bl, Bl contains all relevant information. □

Induction with respect to the number of states in an event easily shows that, if we are given a belief function, then we can uniquely recover the probabilities p_j (Dempster 1967). They are called the *Möbius inverse* of the belief function. Möbius inverses can be defined for general (nonadditive) set functions, and they are nonnegative if and only if the set function is a belief function. Jaffray wrote several papers on Möbius inverses (Chateauneuf and Jaffray 1989; Philippe et al. 1999).

6 Jaffray's behavioral foundation of belief functions and ambiguity

Jaffray (1989a) bridged the gap between the literature on belief functions and the literature on decision theory by providing a behavioral foundation for belief functions. Assume an act f and a random message as in the preceding section. Denote by $m_j \leq M_j$ the infimum and supremum outcomes of f over B_j. By the logic of total absence of information conditional on each B_j of the preceding section, all relevant information of the act f is contained in Fig. 4b. That is, using the philosophy of total absence of information, we have reduced decision under ambiguity to decision under risk with probability distributions over ordered pairs of numbers! The latter is studied extensively in, for instance, Keeney and Raiffa (1976).

Jaffray assumed expected utility (EU) for given probabilities and, in particular, assumed the von Neumann-Morgenstern independence axiom there. A key step in Jaffray's preference foundation, which will lead to his decision model, is that this

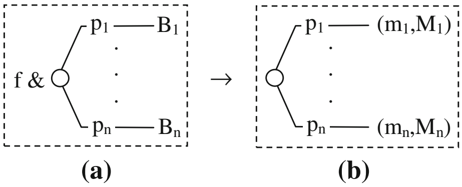


Fig. 4 A sufficient representation of acts

independence condition, and hence the implied EU, can be imposed on the probability distributions over pairs of outcomes just established in a subtle manner. I explain the point informally. A formal and complete explanation is in Jaffray (1991).

Assume that E is an event with known probability λ, and consider Fig. 5, explained in detail later. We assume the preference $f \succeq g$ between two acts; c denotes a third act. Dispreferred acts are indicated by bars in Fig. 5.

I first give three interpretations of the mixture notation $\lambda f + (1 - \lambda)c$ in the right part (d) of Fig. 5. The three interpretations will be mutually consistent. First, if we take acts as probability distributions over pairs of outcomes as just explained (Fig. 4), then the mix is the usual probabilistic mix. This is the easiest way to understand what follows. For completeness, I give two other interpretations. For the second one, we can equate acts with the belief functions that they generate over outcomes (again, justified by Jaffray 1991). Belief functions can be mixed. For example, for $0 \leq \lambda \leq 1$, $\lambda \text{Bl} + (1 - \lambda)\text{Bl}'$ denotes the belief function assigning $\lambda \text{Bl}(T) + (1 - \lambda)\text{Bl}'(T)$ to each set T. Thus, $\lambda f + (1 - \lambda)c$ can be taken as the mix of the belief functions over outcomes generated by f and c. Third, we can equate f and c with the Möbius inverses of the belief functions generated over outcomes (those Möbius inverses are probability distributions over subsets of the outcome space), and then take the mixture as probabilistic mixtures of those. Jaffray (1991) showed that the interpretations are mutually consistent. For the following reasoning, Jaffray (1991) adopted the second interpretation above, but I prefer the first. For each interpretation, once EU is accepted for risk, all implications in Fig. 5 are as reasonable as they are for decision under risk.

Many dynamic decision principles have been invoked in the literature, often with subtle hidden assumptions slipping in implicitly, and often with the term Dutch book (mis)used (Machina 1989). Jaffray was one of the greatest specialists on this topic. His reasoning in Fig. 5 has been very carefully designed and is fully appropriate. The essence is that we use a kind of separability of event E only, because event E is not ambiguous. Nowhere in the reasoning is any kind of separability or its dynamic counterparts assumed for ambiguous events; it always concerns objective probabilities

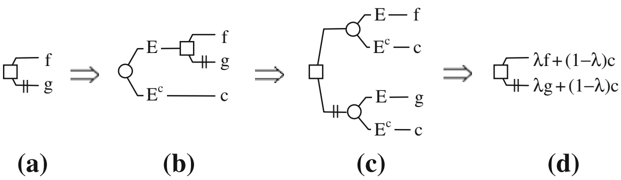


Fig. 5 Mixture independence for belief functions

appearing in the first stage. Jaffray emphasized this point in personal communication when working on Jaffray and Wakker (1993).

Before discussing the case more, I turn to the model characterized by the independence condition just presented:

$$f \mapsto \sum_{j=1}^{n} p_j v(m_j, M_j). \tag{6.1}$$

Here v is much the same as a von Neumann-Morgenstern utility.

I remember well when I first became acquainted with Jaffray's technique of using linear mixing for belief functions, in the autumn of 1988, when reading a working paper on it and not being in his company. All my perceived ideas were turned on their head, and for a considerable time I could not grasp the model conceptually. How can we be doing risk and expected utility if we should be doing ambiguity and nonexpected utility? This just cannot be possible? It reminds me of a joke about a Dutch farmer who has spent all his life breeding cows and now, visiting a zoo, sees a giraffe for the first time. He stands motionless, continuously staring at the animal. After having stood there for some hours, he turns around and walks away, nodding his head, and says: "Such animals don't exist." This is what I kept going through the first day.

Whereas it is easy to write Eq. 6.1 after the fact, the recognition of a linear structure in something as nonlinear as belief functions and, more importantly, the recognition of the preferential relevance of this linear structure, entail a remarkable advance. It took me a whole day, 22 years ago, before I could see the path opened up here by Jaffray, with this linearity not just a mathematical construct but also preferentially sensible. I hope that these ideas are less difficult to grasp (but not less fascinating) to the reader.

What we have obtained in Eq. 6.1 is Anscombe and Aumann (1963) model as it is used in the nonexpected utility literature today, but turned upside down. The probabilities precede the ambiguous events and do not come after. As argued before, this reversed version of the model can be adopted just as well. There is in fact a pro of this reversed approach, Jaffray's approach, over the use that is popular today. In the latter, in the first stage ambiguous uncertainty is resolved and then, in a second stage, afterward, probabilized uncertainty (risk) is resolved. The evaluation of the latter, contingent on an ambiguous event. In the first stage, is done independently of what happens outside the first-stage ambiguous event. This entails a weak form of separability of the ambiguous first-stage event that I find unreasonable for nonexpected utility evaluations of ambiguous events This problem is avoided in Jaffray's reversed approach, with any separability-like condition only imposed on events with known probabilities. Let me rephrase the point. In the second stage, we condition on first-stage events. For being conditioned on, events with known probabilities are better suited than events with unknown probabilities. Hence, Jaffray's two-stage model is more convincing than the Anscombe–Aumann model as commonly used today. This point provides another signal of Jaffray's depth.

If we focus on the special case where $m_i = M_i$ for all i, then there is no ambiguity and we are simply dealing with risk. With $u(x) = v(x, x)$, u captures the risk attitude of the decision maker. By monotonicity, it is natural that $u(m) \leq v(m, M) \leq u(M)$, so that we can define $0 \leq \alpha(m, M) \leq 1$ such that

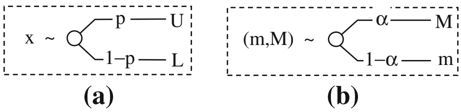


Fig. 6 Elicitation of risk and ambiguity attitudes in Jaffray's model

$$v(m, M) = \alpha(m, M)u(M) + (1 - \alpha(m, M))u(m). \tag{6.2}$$

Here, $\alpha(m, M)$ is a local anti-index of ambiguity aversion. The model of Eqs. 6.1 and 6.2 is tractable. The risk and ambiguity attitudes of the decision maker can easily be elicited from observed decisions. For example, assume that there are an upper outcome U and a lower outcome L, with all other outcomes in between. We may normalize $u(U) = 1$ and $u(L) = 0$. For each outcome x we can measure $u(x) = p$ from an equivalence in Fig. 6a. We can measure $\alpha(m, M) = \alpha$ from an indifference in Fig. 6b.

7 Discussion of Jaffray's model and extensions

Jaffray's model is highly remarkable. We have a complete and perfect separation of:

- the risk attitude, captured by u;
- the ambiguous beliefs, captured by the random messages;
- the ambiguity attitude, captured by $\alpha(m, M)$.

Remember that the pair (m, M) reflects an ambiguous situation of receiving something between m and M, but with total absence of information otherwise. The subjective components of the model, concerning risk and ambiguity attitudes, can readily be elicited from preferences (Fig. 6). For N outcomes, less than N^2 measurements have to be made, reflecting only a polynomial complexity. Other models of ambiguity popular today, such as Choquet expected utility (Gilboa 1987; Schmeidler 1989), multiple priors (Gilboa and Schmeidler 1989), and also prospect theory (Tversky and Kahneman 1992: they incorporated uncertainty and ambiguity), are of exponential complexity and have not yet obtained separations of ambiguous beliefs and ambiguity attitudes as clearly as Jaffray's model did from the beginning.

Jaffray (1989b) developed generalizations of his model to measures of belief more general than belief functions (then the p_js of the Möbius inverse can be negative and have no simple interpretation as probability); Jaffray (1992) considered updating; and Jaffray (1994, §3.4) used his linearity technique to characterize the α maxmin model for objectively given sets of priors.[3] Jaffray and Philippe (1997) extended Jaffray's model to the case of subjective sets of priors, using objective sets of priors for calibration purposes.

After working on Bayesian networks from the mid-1990s, a topic not covered in this paper, Jaffray returned to his decision model in Jaffray and Jeleva (2010), a valuable contribution to this issue of this journal. This paper, again, obtains a strict separation

[3] Ghirardato et al. (2004) and Eichberger et al. (2009) discussed the case of endogenous sets of priors.

between probabilized uncertainty and total absence of information. However, the separation is not in different stages. The model is one-stage and the probabilizable and ambiguous uncertainty concern two disjoint events that partition the universal event. It thus again fits into Jaffray's philosophy.

Gajdos et al. (2008) proposed a separation of ambiguity beliefs and attitudes for the multiple priors model. Their work was inspired by Jaffray (1989a), the more so as three of the authors were located in Paris, the place where Jaffray spent all his professional life, and influenced a generation of researchers. Gajdos (2008) emphasized this intellectual heritage. Ghirardato et al. (2004) initiated an alternative separation of ambiguity beliefs and ambiguity attitudes for the multiple priors models, but more work remains to be done (Eichberger et al. 2009).

Given the increased interest in ambiguity today and the remarkable achievements of Jaffray's (1989a) model for ambiguity, his ideas continue to deserve our attention.

References

Anscombe, F. J., & Aumann, R. J. (1963). A definition of subjective probability. *Annals of Mathematical Statistics, 34*, 199–205.

Chateauneuf, A., & Jaffray, J.-Y. (1989). Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences, 17*, 263–283.

Cohen, M., & Jaffray, J.-Y. (1980). Rational behavior under complete ignorance. *Econometrica, 48*, 1281–1299.

Dempster, A. P. (1967). Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics, 38*, 325–339.

Denoeux, T. (2008). Conjunctive and disjunctive combination of belief functions induced by non distinct bodies of evidence. *Artificial Intelligence, 172*, 234–264.

Eichberger, J., Grant, S., Kelsey, D., & Koshevoy, G. A. (2009). The alpha-meu model: A comment, mimeo.

Gajdos, T. (2008). Zoom, *La Lettre du CES*, no. 3, University of Paris I, 1–2 December 2008.

Gajdos, T., Hayashi, T., Tallon, J.-M., & Vergnaud, J.-C. (2008). Attitude towards imprecise information. *Journal of Economic Theory, 140*, 27–65.

Ghirardato, P. (2001). Coping with ignorance: unforeseen contingencies and non-additive uncertainty. *Economic Theory, 17*, 247–276.

Ghirardato, P., Maccheroni, F., & Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory, 118*, 133–173.

Gilboa, I. (1987). Expected utility with purely subjective non-additive probabilities. *Journal of Mathematical Economics, 16*, 65–88.

Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics, 18*, 141–153.

Jaffray, J.-Y. (1989a). Linear utility theory for belief functions. *Operations Research Letters, 8*, 107–112.

Jaffray, J.-Y. (1989b). Généralisation du Critère de l'Utilité Espérée aux Choix dans l'Incertain Régulier. *RAIRO-RO/Operations Research, 23*, 237–267.

Jaffray, J.-Y. (1991). Linear utility theory and belief functions: A discussion. In A. Chikan (Ed.), *Progress in decision, utility and risk theory*. Dordrecht: Kluwer.

Jaffray, J.-Y. (1992). Bayesian updating and belief functions. *IEEE Transactions on Systems, Man, and Cybernetics, 22*, 1144–1152.

Jaffray, J.-Y. (1994). Dynamic decision making with belief functions. In R. R. Yager, M. Fedrizzi, & J. Kacprzyk (Eds.), *Advances in the Dempster–Shafer theory of evidence* (pp. 331–352). New York: Wiley.

Jaffray, J.-Y., & Meglena, J. (2010). How to deal with partially analyzed acts? *Theory and Decision* (this issue).

Jaffray, J.-Y., & Philippe, F. (1997). On the existence of subjective upper and lower probabilities. *Mathematics of Operations Research, 22*, 165–185.

Jaffray, J.-Y., & Said, T. (1994). Optimal hypothesis testing with a vague prior. In S. Rios (Ed.), *Decision theory and decision analysis: Trends and challenges* (pp. 207–223). Dordrecht, The Netherlands: Kluwer.

Jaffray, J.-Y., & Wakker, P. P. (1993). Decision making with belief functions: Compatibility and incompatibility with the sure-thing principle. *Journal of Risk and Uncertainty, 7*, 255–271.

Keeney, R. L., & Raiffa, H. (1976). *Decisions with multiple objectives*. New York: Wiley. (Cambridge, UK: Cambridge University Press, 2nd edn, 1993).

Luce, R. D., & Raiffa, H. (1957). *Games and decisions*. New York: Wiley.

Machina, M. J. (1989). Dynamic consistency and non-expected utility models of choice under uncertainty. *Journal of Economic Literature, 27*, 1622–1688.

Milnor, J. (1954). Games against nature. In R. M. Thrall, C. H. Coombs, & R. L. Davis (Eds.), *Decision processes* (pp. 49–59). New York: Wiley.

Philippe, F., Debs, G., & Jaffray, J.-Y. (1999). Decision making with monotone lower probabilities of infinite order. *Mathematics of Operations Research, 24*, 767–784.

Savage, L. J. (1954). *The foundations of statistics*. New York: Wiley. (New York: Dover Publications, 2nd edn, 1972)

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica, 57*, 571–587.

Shafer, G. (1976). *A mathematical theory of evidence*. Princeton, NJ: Princeton University Press.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty, 5*, 297–323.