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THE REPETITIONS APPROACH TO CHARACTERIZE CARDINAL UTILITY

ABSTRACT. Building on previous work of A. Camacho, we give necessary and sufficient conditions for the existence of a cardinal utility function to represent, through summation, a preference relation on sequences of alternatives.

1. INTRODUCTION

Till recently, there were mainly three ways to derive cardinal utility. One is the approach, using strength of preference as a primitive. A second approach uses lotteries. Thirdly there is the approach where alternatives have several coordinates, and the utility function is a sum of coordinate functions.

Recently Camacho came with a new approach, the repetitions approach. For a careful exposition of this approach, a comparison to other approaches, and an explanation of its intuitive virtues, the reader is referred to Camacho [1–4]. The purpose of this paper is to use the ideas of Camacho to give a set of necessary and sufficient conditions, alternative to his set, and to give some supplement to his work. Where Camacho works with finite sequences, we use infinite sequences with tails α_0 ("zero"); in Section 3 we shall show that our set-up is in fact equivalent to Camacho's. We only work with these infinite sequences for their convenience in our present mathematical work.

We assume we have a nonempty set $\mathscr{A}$ of alternatives, with one special element α^0, the "receive nothing" alternative. By $\mathscr{X} \subset \mathscr{A}^{\mathbb{N}}$ we denote the set of those infinite sequences $x = (x_j)_{j \in \mathbb{N}}$, for which

$$N_x := \sup (\{0\} \cup \{j \in \mathbb{N} : x_j \neq \alpha^0\})$$

is finite, so x has a "tail", constant α^0. Furthermore we assume a binary relation $\succcurlyeq$ on $\mathscr{X}$, called *preference relation*, present. Usual notations are $x \preccurlyeq y$ for $y \succcurlyeq x$, $x \succ y$ for $x \succcurlyeq y$ & not $y \succcurlyeq x$, $x \prec y$ for $y \succ x$, and

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$x \approx y$ for $x \succcurlyeq y$ & $y \succcurlyeq x$. $\succcurlyeq$ is a *weak order* if it is transitive and complete ($x \succcurlyeq y$ or $y \succcurlyeq x$, for all $x, y \in X$).

Our purpose is to find a function

$$u : \mathscr{A} \to \mathbb{R} \text{ s.t. } x \succcurlyeq y \text{ iff } \Sigma_{j=1}^{\infty} [u(x_j) - u(y_j)] \geqslant 0.$$

For such a function to exist, $\succcurlyeq$ must certainly satisfy the following four axioms, as can be checked straightforwardly and is not elaborated here.

AXIOM 1. $\succcurlyeq$ is a weak order.

AXIOM 2 (The *Permutation Axiom*). For all $x, y \in \mathscr{X}$, $N \in \mathbb{N}$, permutations π on $\{1, \ldots, N\}$, s.t. $x_j = y_{\pi(j)}$ for all $j \leqslant N$, $x_j = y_j$ for all $j > N$; we have $x \approx y$.

(A reordering of alternatives does not change desirability).

AXIOM 3 (The *Independence Axiom*). For all $x, y, x', y' \in \mathscr{X}$, $i \in \mathbb{N}$, s.t. $x_i = y_i$, $x_i' = y_i'$, $x_j = x_j'$ and $y_j = y_j'$ for all $j \neq i$, we have $x \succcurlyeq y \Leftrightarrow x' \succcurlyeq y'$.

(The preference between x and y is independent of coordinates i at which x and y are equal.)

AXIOM 4 (The *Archimedean Axiom*). For all $x, y, v, w \in \mathscr{X}$ with $x \succ y$, $v \succ w$, there exists $M \in \mathbb{N}$ s.t. $p \succcurlyeq q$ where $p_{kN_x+j} = x_j$ for all $0 \leqslant k \leqslant M - 1$, $1 \leqslant j \leqslant N_x$, $p_{MN_x+j} = w_j$ for all $1 \leqslant j \leqslant N_w$, and $p_n = \alpha^0$ for all $n > MN_x + N_w$; and where $q_{lN_y+i} = y_i$ for all $0 \leqslant l \leqslant M - 1$, $1 \leqslant i \leqslant N_y$, $q_{MN_y+i} = v_i$ for all $1 \leqslant i \leqslant N_v$, and $q_m = \alpha^0$ for all $m > MN_y + N_v$.

(The difference between v and w can be compensated by a sufficient number of differences between x and y.)

Constructions such as that of p above will more often be carried out in the sequel. One can imagine the "untailed" part of p to consist of M replicas of the "untailed" part of x, followed by one replica of the "untailed" part of w. Axiom 4 has not been used by Camacho, but he indicated it more or less in Section 2.1, page 364, (d), in [3].

Our main result:

THEOREM 1.1. The following two assertions are equivalent:

1.1. (i) "There exists u: $\mathscr{A} \to \mathbb{R}$ s.t.
$x \succcurlyeq y \Leftrightarrow \Sigma_{j=1}^{\infty} [u(x_j) - u(y_j)] \geqslant 0$."

1.1. (ii) "$\succcurlyeq$ satisfies axioms 1 to 4."

Furthermore, if (i) holds, then u can be replaced by $\bar{u}$: $\mathscr{A} \to \mathbb{R}$ if and only if real τ and positive σ exist s.t. $\bar{u} = \tau + \sigma u$.

The implication (i) $\Rightarrow$ (ii) is straightforward. In the next section we assume (ii), and derive (i), and the "Furthermore . . ." statement.

2. PROOF OF THEOREM 1.1

Assume Axioms 1 to 4 are satisfied. We define an equivalence relation E on $\mathscr{X}$ by xEy if $\|\{j | x_j = \beta\}\| = \|\{j | y_j = \beta\}\|$ for all $\beta \neq \alpha^0$ in $\mathscr{A}$. By $[x]$ we denote the equivalence class $\{y \in \mathscr{X} | yEx\}$, and $[\mathscr{X}] := \{[x] | x \in \mathscr{X}\}$. By the permutation Axiom we have $E \subseteq \approx$. We may write $[x] = \Sigma_{j=1}^{n} n_j[\alpha_j]$, with $n, n_j \in \mathbb{N}$ for all j, $\{x_i: i \in \mathbb{N}\} = \{\alpha_j: 1 \leqslant j \leqslant n\} \cup \{\alpha^0\}$; and $n_j = \|\{i: x_i = \alpha_j\}\|$ if $\alpha_j \neq \alpha^0$ for all j, and $\alpha_j \neq \alpha_k$ if $j \neq k$. We define $[x] + [y]$, and $n[x]$ for $n \in \mathbb{N} \cup \{0\}$ in the usual way. The operation + on $[\mathscr{X}]$ is associative and commutative, has neutral element $[(\alpha^0, \alpha^0, \ldots)]$.

We define the binary relation $\succcurlyeq'$ on $[\mathscr{X}]$ by $[x] \succcurlyeq' [y]$ if there exist $v \in [x]$, $w \in [y]$, s.t. $v \succcurlyeq w$. By Axioms 1 and 2 this is iff $v \succcurlyeq w$ for all $v \in [x]$, $w \in [y]$. So we have $x \succcurlyeq y \Leftrightarrow [x] \succcurlyeq' [y]$. The notations $\preccurlyeq'$, $\succ'$, $\prec'$, $\approx'$ are as usual. We have, for all $x, y, v, w \in \mathscr{X}$:

LEMMA 1′. $\succcurlyeq'$ is a weak order.

LEMMA 2′. $x \succcurlyeq y \Leftrightarrow [x] \succcurlyeq' [y]$.

LEMMA 3′. (*Additivity*). $[x] \succcurlyeq' [y] \Leftrightarrow [x] + [v] \succcurlyeq' [y] + [v]$.

Proof. $(v_1, \ldots, v_{N_v}, x_1, \ldots, x_{N_x}, \alpha^0, \ldots) \in [x] + [v]$; $(v_1, \ldots, v_{N_v}, y_1, \ldots, y_{N_y}, \alpha^0, \ldots) \in [y] + [v]$; $(\alpha^0, \ldots, \alpha^0, x_1, \ldots, x_{N_x}, \alpha^0, \ldots)$ (first N_v coordinates α^0) $\in [x]$; $(\alpha^0, \ldots, \alpha^0, y_1, \ldots, y_{N_y}, \alpha^0, \ldots)$ (first N_v coordinates α^0) $\in [y]$. Now apply independence of $\succcurlyeq$, N_v times; then Lemma 2′.

LEMMA 4′. (*Archimedean Axiom* for $\succcurlyeq'$). If $[x] \succ' [y]$, $[v] \succ' [w]$, then $M \in \mathbb{N}$ exists s.t. $M[x] + [w] \succcurlyeq' M[y] + [v]$.

Proof. Define p, q as in Axiom 4. Then $p \in M[x] + [w]$, $q \in M[y] + [v]$. Apply Axiom 4, and Lemma 2′. ■

These four lemmas enable us to apply Theorem 3.2.1.1 of Krantz *et al.* [5]. In this we do not need commutativity of $+$.

THEOREM 2.1. For any binary relation $\succsim'$ on $[\mathscr{X}]$ the following two assertions are equivalent.

2.1. (i) "There exists $\phi\colon [\mathscr{X}] \to \mathbb{R}$ s.t. $[x] \succsim' [y] \Leftrightarrow \phi([x]) \geqslant \phi([y])$ and s.t. $\phi([x] + [y]) = \phi([x]) + \phi([y])$, for all $x, y \in \mathscr{X}$."

2.1. (ii) "$\succsim'$ is a weak order that satisfies additivity and the Archimedean Axiom."

Furthermore, another function $\bar{\phi}$ satisfies (i) if and only if positive σ exists s.t. $\bar{\phi} = \sigma\phi$.

Proof. By Theorem 3.2.1.1 of Krantz *et al.* [5]. Note for this that, if $[w] \succcurlyeq' [v]$ and $[x] \succ' [y]$, then by repeated application of Lemmas 1′ and 3′, $[x] + [w] \succcurlyeq' [y] + [w] \succcurlyeq' [y] + [v]$. So still the result of Lemma 4′ holds, with $M = 1$. ■

LEMMA 5. Let $\succsim$ be a binary relation on $\mathscr{X}$, $\succsim'$ one on $[\mathscr{X}]$, s.t. $x \succsim y \Leftrightarrow [x] \succsim' [y]$. Then Assertion 2.1. (i) implies Assertion 1.1. (i) with $\succsim$ for $\succcurlyeq$, by the definition $u(\alpha) := \phi([\alpha, \alpha^0, \ldots])$. And then Assertion 1.1. (i) with $\succsim$ for $\succcurlyeq$ implies Assertion 2.1. (i) by the definition

$$\phi(\Sigma_{j=1}^{n} n_j[\alpha_j]) = \Sigma_{j=1}^{n} n_j[u(\alpha_j) - u(\alpha^0)].$$

Proof. Let Assertion 2.1. (i) be satisfied. Define u as above. Then

$$x \succsim y \Leftrightarrow [x] \succsim' [y] \Leftrightarrow (N = \max\{N_x, N_y\})\ \Sigma_{j=1}^{N} 1[x_j] \succsim' \succsim' \Sigma_{j=1}^{N} 1[y_j] \Leftrightarrow \Sigma_{j=1}^{N} \phi[(x_j, \alpha^0, \ldots)] \geqslant \Sigma_{j=1}^{N} \phi[(y_j, \alpha^0, \ldots)] \Leftrightarrow \Leftrightarrow \Sigma_{j=1}^{\infty} [u(x_j) - u(y_j)] \geqslant 0\text{: Assertion 1.1. (i).}$$

Next let Assertion 1.1. (i) be satisfied, with $\succsim$ for $\succcurlyeq$. Define ϕ as above. Then $\phi([x] + [y]) = \phi([x]) + \phi([y])$ for all x, y. And

$$\phi(\Sigma_{j=1}^{n} n_j[\alpha_j]) \geqslant \phi(\Sigma_{i=1}^{m} m_i[\beta_i]) \Leftrightarrow \Sigma_{j=1}^{n} n_j[u(\alpha_j) - u(\alpha^0)] \geqslant \geqslant \Sigma_{i=1}^{m} m_i[u(\beta_i) - u(\alpha^0)] \Leftrightarrow \{\text{let } x_{(\Sigma_{k=1}^{j-1} n_k)+n_j'} = \alpha_j \text{ for all } 1 \leqslant j \leqslant n,\ 1 \leqslant n_j' \leqslant n_j,\ x_a = \alpha^0 \text{ for all } a > \Sigma_{k=1}^{n} n_k;\ y \text{ analogously from } m, i, \beta_i \text{ i.s.o. } n, j, \alpha_j\}\ \Sigma_{j=1}^{\infty} [u(x_j) - - u(y_j)] \geqslant 0 \Leftrightarrow x \succsim y \Leftrightarrow [x] \succsim' [y] \Leftrightarrow \Sigma_{j=1}^{n} n_j[\alpha_j] \succsim' \succsim' \Sigma_{i=1}^{m} m_i[\beta_i]\text{: Assertion 2.1. (i).} \quad \blacksquare$$

Now we can complete the proof of Theorem 1.1. Assertion (ii) there implies Assertion 2.1. (ii), as we saw by Lemmas 1′ to 4′. Thus it implies Assertion 2.1. (i). Lemma 5 now gives Assertion 1.1. (i). That function u, satisfying 1.1. (i), can be replaced by any $\bar{u} = \tau + \sigma u$ for real τ and positive σ, is straightforward. Conversely, suppose u in 1.1. (i) can be replaced by $\bar{u}$. Then derive ϕ from u as in Lemma 5, and analogously $\bar{\phi}$ from $\bar{u}$. By Theorem 2.1 we get that $\bar{\phi} = \sigma\phi$ for a positive real σ. This can only be if $\bar{u} = \tau + \sigma u$ with $\tau = \bar{u}(\alpha^0) - \sigma u(\alpha^0)$.

3. EQUIVALENCE OF OUR SET-UP WITH CAMACHO'S

First we formulate some consequences of Axioms 1 to 4, which directly follow from Theorem 1.1.

DEFINITION. We say $\succcurlyeq$ satisfies the *Repetitions Axiom* if $[x \succcurlyeq y \Leftrightarrow x' \succcurlyeq y']$ for all $x, y, x', y' \in \mathscr{X}$ for which $n, m \in \mathbb{N}$ exist s.t. $x \succcurlyeq y$, $m \geqslant N_x$, $m \geqslant N_y$, $x'_{km+j} = x_j$ and $y'_{km+j} = y_j$ for all $1 \leqslant j \leqslant m$, $0 \leqslant k \leqslant n - 1$, $x'_i = y'_i = \alpha^0$ for all $i > nm$. .

COROLLARY 1. Axioms 1 to 4 imply the Repetitions Axiom.

Proof. $x \succcurlyeq y \Leftrightarrow \Sigma_{j=1}^{\infty} [u(x_j) - u(y_j)] \geqslant 0 \Leftrightarrow n\, \Sigma_{j=1}^{\infty} [u(x_j) - u(y_j)] \geqslant 0 \Leftrightarrow \Sigma_{j=1}^{\infty} [u(x'_j) - u(y'_j)] \geqslant 0 \Leftrightarrow x' \succcurlyeq y'$. ■

DEFINITION. We say $\succcurlyeq$ satisfies the *Rate of Substitution Axiom* if for all $\alpha, \beta, \gamma, \delta \in \mathscr{A}$ with $(\alpha, \alpha^0, \ldots) \succcurlyeq (\beta, \alpha^0, \ldots)$ and $(\gamma, \alpha^0, \ldots) \succ (\delta, \alpha^0, \ldots)$, there exists $R(\alpha, \beta, \gamma, \delta) \in \mathbb{R}$ s.t. for all $x, y \in \mathscr{X}$, $n, m \in \mathbb{N} \cup \{0\}$, $A, B \subset \mathbb{N}$ with $A \cap B = \phi$, $\|A\| = n > 0$, $\|B\| = m$, $x_j = y_j$ for all $j \notin A \cup B$, $x_j = \alpha$ and $y_j = \beta$ for all $j \in A$, $x_j = \delta$ and $y_j = \gamma$ for all $j \in B$, we have $x \succ y$ iff $m/n < R(\alpha, \beta, \gamma, \delta)$, $x \approx y$ iff $m/n = R(\alpha, \beta, \gamma, \delta)$, and $x \prec y$ iff $m/n > R(\alpha, \beta, \gamma, \delta)$.

COROLLARY 2. Axioms 1 to 4 imply the Rate of Substitution Axiom.

Proof. Let, for $\alpha, \beta, \gamma, \delta$ as above, $R(\alpha, \beta, \gamma, \delta) = [u(\alpha) - u(\beta)]/[u(\gamma) - u(\delta)]$. ■

In Camacho [2–4], also a nonempty set $\mathscr{A}$ is the point of departure, but the set $\mathscr{X}^f := \bigcup_{n=1}^{\infty} \mathscr{A}^n$ of all *finite* sequences of alternatives is considered. If $x \in \mathscr{A}^n$, we say x has *length n*. There is assumed to be a binary (preference) relation $\succcurlyeq''$ present on $\mathscr{X}^f$, such that the restriction of $\succcurlyeq''$ to $\mathscr{A}^n$ is a weak order for every $n \in \mathbb{N}$, and such that $x \succcurlyeq'' y$ for no x, y of different length. From Camacho's set-up we can come to our approach as follows. Take an arbitrary element of $\mathscr{A}$, denote it as α^0. Assign to every $(x_1, \ldots, x_n) \in \mathscr{X}^f$ the element $(x_1, \ldots, x_n, \alpha^0, \ldots)$ of $\mathscr{X}$. And write $x \succcurlyeq y$ if $(x_1, \ldots, x_n) \succcurlyeq'' (y_1, \ldots, y_n)$, with $n = \max\{N_x, N_y\}$, for all $x, y \in \mathscr{X}$.

Conversely, from our set-up we can come to Camacho's set-up by defining $x \succcurlyeq'' y$ whenever

$$(x_1, \ldots, x_n, \alpha^0, \ldots) \succcurlyeq (y_1, \ldots, y_n, \alpha^0, \ldots),$$

for all $x, y \in \mathscr{A}^n$; for all $n \in \mathbb{N}$.

Now the combination of our weak order and independence Axioms for $\succcurlyeq$ is equivalent to the combination of Camacho's weak order and independence Axioms for $\succcurlyeq''$. For brevity we do not elaborate that here. Also, in the presence of weak orderness and independence, the permutation, repetition, and rate of substitution Axioms in our set-up are equivalent to those in Camacho's set-up. The key role in all this is played by the independence Axioms.

4. AN EXAMPLE

In this section we give an example to illustrate the necessity of the Archimedean Axiom. Also this should illustrate where we deviate from Camacho's set-up. Let $\mathscr{A} = \{\alpha^0, \beta, \gamma, \delta, \varepsilon\}$, and let $f\colon \mathscr{A} \to \mathbb{R}$ be s.t. $f(\alpha^0) = 0, f(\beta) = 1, f(\gamma) = \sqrt{2}, f(\delta) = \sqrt{3}, f(\varepsilon) = 1 + \sqrt{2} + \sqrt{3}$. For $x, y \in \mathscr{X}$, we have $x \succ y$ if $\Sigma_{j=1}^{\infty} [f(x_j) - f(y_j)] > 0$ or $\Sigma_{j=1}^{\infty} [f(x_j) - f(y_j)] = 0$ and $\| \{j\colon x_j = \varepsilon\} \| > \| \{j\colon y_j = \varepsilon\} \|$. We have $x \approx y$ if $\Sigma_{j=1}^{\infty} [f(x_j) - f(y_j)] = 0$ and $\| \{j\colon x_j = \varepsilon\} \| = \| \{j\colon y_j = \varepsilon\} \|$, which can be seen to occur only if xEy, with E as in Section 2. Of course $x \succcurlyeq y$ if $x \succ y$ or $x \approx y$. It can be seen that $\succcurlyeq$ is a weak order, it satisfies the permutation and independence Axioms. But it does *not* satisfy the Archimedean Axiom. To see this, take $x = (\varepsilon, \alpha^0, \ldots)$, $y = (\beta, \gamma, \delta, \alpha^0, \ldots)$, $v = (\beta, \alpha^0, \ldots)$, $w = (\alpha^0, \alpha^0, \ldots)$. Then $x \succ y$, $v \succ w$, but for all $M \in \mathbb{N}$, and p, q as in Axiom 4 we have $p \prec q$ since

$$\Sigma_{j=1}^{\infty} [f(p_j) - f(q_j)] = M\, \Sigma_{j=1}^{\infty} [f(x_j) - f(y_j)] + f(w) - f(v) < 0.$$

So Axiom 4 is violated. It can easily be seen that $\succcurlyeq$ satisfies the Repetitions Axiom, this in fact is implied by Axioms 1 to 3. We finally show that $\succcurlyeq$ satisfies the Rate of Substitution Axiom. To every four μ, ν, σ, τ in $\mathscr{A}$ s.t. $(\mu, \alpha^0, \ldots) \succcurlyeq (\nu, \alpha^0, \ldots)$ and $(\sigma, \alpha^0, \ldots) \succ (\tau, \alpha^0, \ldots)$, we assign

$$R(\mu, \nu, \sigma, \tau) := [f(\mu) - f(\nu)]/[f(\sigma) - f(\tau)].$$

Let then $x, y \in \mathscr{X}$, $n, m \in \mathbb{N} \cup \{0\}$, $A, B \subset \mathbb{N}$ with $\|A\| = n > 0$, $\|B\| = m$, $x_j = y_j$ for all $j \notin A \cup B$, $x_j = \mu$ and $y_j = \nu$ for all $j \in A$, $x_j = \tau$ and $y_j = \sigma$ for all $j \in B$. If now $m/n < R(\mu, \nu, \sigma, \tau)$, then

$$\Sigma_{j=1}^{\infty} [f(x_j) - f(y_j)] = n[f(\mu) - f(\nu)] - m[f(\sigma) - f(\tau)] > 0,$$

so $x \succ y$. If $m/n > R(\mu, \nu, \sigma, \tau)$, then analogously $x \prec y$.

Remains the case $m/n = R(\mu, \nu, \sigma, \tau)$.

Apparently then $[f(\mu) - f(\nu)]/[f(\sigma) - f(\tau)]$ is rational. There are only a few possibilities for this: either $\mu = \nu$, or $\mu = \sigma$ and $\nu = \tau$. In either case $x \approx y$ follows.

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