The Brownian motion in finance: an epistemological puzzle

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Introduction

While in medicine, comparison of the data supplied by a clinical syndrome with the data supplied by the biological system is used to arrive at the most accurate diagnosis, the same cannot be said of financial economics: the accumulation of statistical results that contradict the Brownian hypothesis used in risk modelling, combined with serious empirical problems in the practical implementation of the Black-Scholes-Merton model, the benchmark theory of mathematical finance founded on the Brownian hypothesis, has failed to change the Brownian representation, which has endured for more than fifty years despite the extent of its invalidation by experience. Without any statistical foundations, one mathematical representation (Brownian motion) has become the established approach, acting in the minds of practitioners as a "prenotion" in the sense the word is used by Durkheim (1894), i.e. a "schematic, summary representation" which has produced a kind of spontaneous epistemology. The question arises of the persistence of this mathematical (Brownian) representation, which has been the basis for every financial risk modelling approach: how can its long life be explained? How was this spontaneous epistemology formed, and why did it prove to be so persistent?

To address this question and offer an answer, I will test the various dynamics of scientific knowledge used with reference to financial modelling. All these dynamics are specific ways of describing the relationship between knowledge of a phenomenon (here, the representation of a stock market dynamic) and the phenomenon itself (here, stock price fluctuations). Observing that it is impossible for the positivist approach to solve the financial puzzle, I turn to the three principal postpositivistic dynamics, developed by Kuhn, Lakatos and Quine. I shall try to make to speak these representations of science for financial research, in order to reflect on the dominance of the Brownian representation in finance. We shall see that none of the epistemologies examined can explain why the Brownian representation continues to be used in mathematical finance research. I shall then propose an alternative hypothesis, concerning a significant pervading mental model that has irrigated both academics and practitioners in the financial sector: the "principle of continuity" introduced into economics by Alfred Marshall in 1890. I consider that this principle of continuity has become a "persistent idea" in the form of a viral approach to representations, and I give this persistent idea the metaphorical name of the "Brownian virus". Then, to explain the spread of this Brownian virus through the financial sector, the contamination of financial practices by this mental representation founded on the principle of continuity, I introduce the concept of the "financial *Logos*", a discourse that structures practices and organisations, calculations, prudential regulations and accounting standards, leading to a general financialisation of society from the 1980s onwards.

The article is organized as follows. The first part reviews the main characteristics of the Brownian representation, stressing two of the fundamental properties of Brownian motion: the continuity of trajectories, and the normality of the marginal distributions. I then describe what is known as "non-Brownian syndrome", the general invalidation of both properties. This leads on to introduction of the epistemological puzzle of the Brownian representation's persistence despite the non-Brownian syndrome. The second part of the article considers the inability of the positivist epistemological approach to solve the financial puzzle. The third part reviews the options offered by the approaches of Kuhn, Lakatos and Quine. The fourth part presents the hypothesis concerning the mental model of the principle of continuity.

1. The Brownian representation of price fluctuations

Asset prices move as time passes: prices are dynamic. Descriptions of price changes generally use a stochastic analytical framework. I call the mathematical equation that describes the time-path of market dynamics, the evolution of cumulative returns over a given time period, the *standard model* asset price dynamic. The standard model makes it possible to produce statements and "risk numbers" about the probability distribution governing future prices. The foundation of this standard model is a specific stochastic process, Brownian motion and the Brownian representation generally. A "Brownian return" is a return whose description of the movement can be represented by Brownian motion.

1.1. The Brownian framework

Stochastic processes are an important component in contemporary financial modelling of the market dynamic of asset prices. Let us fix some notations. In what follows, the price of any security is denoted at time t by $S(t)$. The "simple return" on this security corresponds to real monetary gains or losses. The gain (or loss) is simply the difference $S(t) - S(0)$, and the "natural" arithmetic return is given by the basic formula $(S(t) - S(0)) / S(0)$. Academics and practitioners (traders, risk managers, etc.) are generally interested in the continuous rate of return between time 0 and time t. This quantity is denoted $X(t) = \ln S(t) - \ln S(0)$, where "ln" is the natural logarithm. The move from the natural return to the logarithmic return represents the logarithmic convention used in return calculations in financial modelling. Financial computations are usually performed using $X(t)$. In the financial modelling literature, the quantity $X(t)$ is conceived as a *stochastic process* which describes the return dynamic.

1.1.1 The Brownian motion in theory: one-dimensional market risk

I now present the standard model in a single picture. The most commonly used continuous-time stochastic process in finance is Brownian motion, one of the best known Lévy processes, which are càdlàg[1] stochastic processes with stationary independent increments[2]. The Brownian representation of the cumulative return dynamic $X(t)$ has been based on Brownian motion since the seminal works of Louis Bachelier (Bachelier, 1900) and Maury Osborne (Osborne, 1959). In this specific representation, the return dynamic of any financial asset at time t is given by an equation associating a return "trend" and the "risk" of the given asset:

$$X(t) = m\, t + \sigma W(t) \qquad (1)$$

In this equation, m is the mean parameter which gives the trend growth (proportional to time t) of the cumulative return and σ is the standard deviation parameter (square root of variance, diffusion coefficient of Brownian motion), termed "volatility" in the markets, designed to capture the *scale* of

[1] French: "continue à droite, limite à gauche", i.e. right continuous with left limits.

[2] Lévy processes, labelled after the French mathematician Paul Lévy, are continuous-time stochastic processes with independent and identically distributed (IID) increments. With the exception of Brownian motion with drift, they consist entirely of jumps. See for example Bertoin (1996) and Sato (1999).

the distribution of possible returns around the mean (± 5% or ± 50%), the degree of uncertainty regarding future returns. As a result, with the standard[3] Wiener process *W* (*t*), "risk" is considered as a random excursion around the trend, and the shape and trajectory are given by the Wiener process, scaled by the value of the standard deviation. To put it differently, in the standard model, the volatility of a financial asset is considered a proxy for risk. This is a *one-dimensional assessment* of risk: the scale of risk. The *morphology* of risk (the shape of the risk profile), which is supposed to capture the uncertainty of future returns, is described by the standard Wiener process.

Moving from asset returns to asset prices, the equation for the price dynamic is:

$$S(t) = \exp(X(t)) \tag{2}$$

Equation (2) defines an exponential (or geometric) Brownian motion. This equation became the standard model of market dynamics in 1965 when it was recommended by Paul Samuelson.

Because the financial quantities described by the standard model are the mean (average trend of return fluctuations) and the volatility, we get a representation of the mean-variance map associated with Brownian dynamics. I call this mean-variance map the standard model representation of the risk-return analysis. This map contains all the first-generation elementary information about the market dynamics of returns. Hence, the mathematics of the standard model of cumulative returns is clearly linked to the Brownian representation.

One main feature of the Brownian representation is its combination of the two characteristics of path-continuity in the return dynamics and normality in return distributions. The continuity property reflects a liquid market, where there are many buyers and sellers for a given security. The normality property reflects the risk of a given security: returns are more frequently close to the mean return than far from the mean return. I now turn to these characteristics.

1.1.2 Brownian motion in financial practice: normality and path-continuity of returns

Path-continuity and normality of returns form the common core of the portfolio theory, option pricing theory, and fundamental asset pricing methods. Portfolio theory is considered a central factor in the making of finance into a scientific discipline (MacKenzie, 2006). It emerged between 1950 and 1965, that is, between the validation of the random walk model and the introduction of the efficient market hypothesis in asset management practices. Portfolio theory was first developed by Harry Markowitz in 1952, then refined by James Tobin in 1958, and achieved its classic twofold formulation from William Sharpe in 1963 with respect to the linear probability model and in 1964 with respect to the equilibrium model.

This latest theoretical development had a deep impact on the portfolio management profession: it gave portfolio managers an incentive to aim for maximum diversification of assets and extensive indexation on so-called "benchmarks": market indices created to consist of securities representing some aspect of the total market (such as the S&P 500). This widespread indexation has been both the norm and the limitation of the asset management industry, in theory and in practice too. It is closely linked to the Brownian representation of return dynamics because of Quetelet's shadow about

[3] "Standard" means $W_0 = 1$, the increments are independent and for $0 < s < t$, the increment $W(t) - W(s)$ is normally distributed with mean zero and variance t-s

averages (Walter, 1996). I labelled this influence "Quetelet's influence" on asset management methodology (Walter, 2005). In introducing the idea of the optimal mean-variance portfolio, Markowitz and Sharpe used and thus validated the Bachelier-Osborne representation. To solve the problem of optimizing portfolios, it was necessary to hypothesize a stochastic process for the time series of securities' returns on the market, since calculation of the variance-covariance matrix requires a probability characterization of all the comovements by these securities, in the form of a probability vector concerning the whole market: a joint distribution of returns for all securities. This distribution was a joint multinormal distribution. Assuming multinormality in securities' price changes made it possible for the calculations to be made, and for Markowitz MV-optimal portfolios to be constructed. Therefore, the foundations for the quantitative approach to investment management come from the academic research on the random nature of stock market prices and the Brownian representation of price variations (Walter, 1996).

A few years later came the birth and development of the Black-Scholes-Merton (1973) option pricing theory. This consolidated the key role of Brownian motion in the financial industry. The development of option pricing tools became so important in finance in the 1970's and 1980's, with intensive use of second-order diffusion processes, i.e. a marginal distribution with finite variance, that it was apparently impossible to question the use of Brownian motion in finance.

1.1.3 Brownian motion in the regulatory framework: time scaling of risk

Assumptions of continuity and normality also underlie almost all prudential regulation worldwide. They are key elements of regulatory frameworks (Basel III, Solvency II).

These regulations require banks and insurance companies to hold sufficient capital to buffer their risks. Brownian motion entails time scaling of distributions – and consequently time scaling of risk – in the sense that one given horizon (e.g., t) of a return distribution is scaled to another (e.g., $t \times a$). This means that the distribution of $X(t \times a)$ is the same as the distribution of $X(t) \times \sqrt{a}$. This is called the *scaling property* of Brownian motion or the *square-root-of-time rule* of scaling. This scaling property leads to scaling of volatility (risk) in the sense that the volatility at scale t multiplied by the length of duration a is equal to the volatility at scale t multiplied by the square-root-of-time duration, i.e. $\sigma(t) \times \sqrt{a}$. The square-root-of-time rule is widely used in the Basel III and Solvency II regulations which promote the calculation and implementation of a probabilistic measure of market risk called `Value-at-Risk' (Jorion, 2000) (hereafter VaR). The minimum capital requirement is an estimated quantile of a return distribution (10-day 95% VaR metrics). The 10-day VaR is obtained simply by applying time scaling of risk using the square-root-of-time rule: VaR (10-day) = VaR (1-day)$\times \sqrt{10}$. The square-root-of-time rule underlying the regulatory requirements for calculating minimum capital is a very narrow subset of the time-scaling rule of risk, and is directly attributable to the assumption that return dynamics can be modelled by the Brownian representation of price dynamics.

This scaling property of the Brownian representation also promotes the widespread practice of calculating annualized volatility from weekly volatility: based on the weekly percentage change, the annualized volatility is equal to the weekly volatility (the standard deviation of the data) multiplied by $\sqrt{52}$. This shows how far the Brownian representation has penetrated the calculation routines of financial practitioners, even if they do not know its theoretical foundations.

1.2. Non-Brownian syndrome

There are many discrepancies between theory and empirical financial phenomena, and I have labelled this the “non-Brownian syndrome” (Walter, 2013, p. 262). A large number of practical difficulties have been encountered in financial applications of Brownian movement in the tools of practical finance. Also, a large number of “stylized empirical facts” have emerged from statistical analysis of price variations in financial markets (for recent reviews see for example Cont, 2001; Sewell, 2011). The term “stylized fact" is used in finance to refer to empirical findings that are pervasive across time and consistent across markets (e.g. heavy tails, intermittency, volatility clustering etc.). These stylized facts invalidate many of the test implications resulting from the Brownian representation of return dynamics. The “view from outside” on finance (Ippoliti, 2017a) strongly supports general properties and representation of stock markets behaviour that cannot be explained with reference to Brownian representation of price dynamics.

1.2.1 Problems arising from experience: practical difficulties

Difficulties were quick to appear back in the 1960s, when the Brownian representation was first used by financial professionals (Press, 1967; Barnea and Downes, 1973). Practical problems arose when attempts were made to apply this representation directly to portfolio and trading position management tools, and for the pricing of options (Jarrow and Rosenfeld 1984; Ball and Torous, 1985; Jorion, 1988). Nearly all the professions of market finance and portfolio management were affected by these operational difficulties, asking for new methodologies, for example a “Non-Gaussian Merton-Black-Scholes theory” (Boyarchenko and Levendorskii, 2002) or a “jump-diffusion model for option pricing” (Kou, 2002). On the options market for example, one of the main problems was the level of volatility for options that were far from their strike price, a situation known as the “volatility smile” because the implied volatility curve looks like a smiling mouth, whereas the Black-Scholes-Merton model predicts that the implied volatility curve is flat (Eberlein, Keller and Prause, 1998). The volatility smile is in fact more like a “volatility smirk”: the reverse skew pattern of volatility is a stylized fact that has often been documented in the statistical literature. By analogy with Maxwell’s demon, I would liken it to the market demon grinning at the operators’ powerlessness to reliably quantify sudden moves in listing prices. The diabolical grin of volatility reminds market actors that their representation system (the way they see financial uncertainty) is faulty.

Portfolio management also encountered innumerable practical difficulties when Brownian-based models were put into operation must be pointed out: these difficulties resulted from the non-validation of the underlying premises: first, the market indices are not optimal as assumed in the Sharpe-Markowitz approach; and second, the multi-normality of stock market variations is not confirmed, resulting in constant instability in the variance-covariance matrix, and making it difficult to assess the quality of the supposedly optimal portfolio. Finally, the expected return is not easy to estimate, and a change in estimation leads to a modification of the optimal portfolio. One year after the 2008 crisis, a leading global player in the asset management industry acknowledged that “the primary reason for this underestimation of risk *lies in the conventional approach to applying mean-variance theory*, which was pioneered by Harry Markowitz in 1952” (Sheik and Qiao, 2009, our italics). The impact of this underestimation of risk was huge: “we believe that conventionally derived portfolios carry a higher level of downside risk than many investors believe, or current portfolio modelling techniques can identify”.

I now turn to the stylized facts which contradict the normality and path-continuity of returns. The Brownian representation of stock market fluctuations faces a twofold problem. Two assumptions must be confirmed: the normality of the empirical distributions, and the continuity of the cumulative time-path of returns. In fact, these two assumptions of normality and continuity, these two *hypotheses*, are not corroborated by the data: the morphology of stock market fluctuations – the structure of stock market uncertainty – shows neither marginal normality, nor a continuous trajectory. Hence, for convenience, I am going to group "violations" of the Brownian representation into two groups: non-normality and discontinuity. I shall now present them separately. As I have addressed this issue in a previous work to which I refer the reader for further details (Walter, 2019), I simply summarize the main points of the debate here.

1.2.2 Problems with normality: the leptokurtic phenomenon

Non-normality was the point on which the Brownian representation was first attacked in academic finance research (Osborne, 1959; Alexander, 1961; Mandelbrot, 1963, 1967; Teichmoller, 1971; Officer, 1972; Fielitz and Smith, 1972; Press, 1972; Praetz, 1972; Barnea and Downes, 1973; Brenner, 1974; Blattberg and Gonedes, 1974; Hsu *et al.*, 1974; Hagerman, 1978). From a technical standpoint, increments in Brownian motion (and thus periodic returns) follow a normal distribution: empirical distributions are expected to be normal. The violation of normal distribution takes the form of a precise stylized fact: on nearly all the chronological market series examined, at almost all scales of resolution, for almost all analysis periods, we see that the kurtosis of the empirical distribution obtained is significantly higher than its theoretical value under a normal distribution in calendar time (Ané and Geman, 2000). I have labelled this characteristic the "leptokurtic phenomenon" (Walter, 2002). The leptokurtic phenomenon has been the cornerstone of all the theoretical debates in the academic finance community regarding which stochastic process best models the stock market dynamic, splitting the modelling consensus into several mutually irreducible schools of thought (Walter, 2017).

The normal distribution is another expression of the idea that a price "jump" or large stock market move is impossible. Such events, due to their magnitude, would lie in the tails of the distribution, and make them "fat": fat tails reflect an excessive probability of "extreme" observations in a distribution. The probability of these tail events in a normal distribution is assumed to be too low for them to have a realistic chance of occurring on the markets. These events, known as "rare", have no real existence in the probabilistic framework of Brownian motion, and yet they are very important to fully grasp the form of financial uncertainty, because of the extreme value problem they highlight (Walter, 2017).

If the empirical distribution of stock price returns could really be modelled by a normal distribution, then it would be possible to demonstrate that the period elapsing between two major financial crises would be ten thousand years on average. The series of crises since the 1987 crash clearly shows that this is not the case. The implied consequence of this observation is simple: models that use a normal distribution for marginal returns cannot accommodate unforeseeable financial developments. The Brownian representation of market uncertainty cannot be used to model market fluctuations and stock price dynamics (Madan *et al.*, 1998; Prause, 1999).

1.2.3 Problems with path continuity: jumpy markets

From a time-path standpoint, the path-continuity property indicates that Brownian motion, although it visually presents trajectories that can rise or fall, sometimes rapidly, over short periods, is always continuous. But this kind of continuity is not confirmed by the facts. First, it is clearly contradicted by large jumps caused by sudden price movements. Continuity suggests there should be no crashes, no breaks in trading, and no liquidity crises. In reality, prices move almost instantaneously from one quoted price to another, which may be very different from the first price, *especially* during liquidity crises or market crashes. Limit down or limit up phenomena are then observed: when demand (or supply) outstrips supply (or demand) for equities, the price-setter can no longer find an equilibrium price, and trading is suspended, either downwards (limit down), or upwards (limit up). When trading can resume, there is a significant price difference from the previous listing. Stock market price fluctuation modelling using exponential Brownian motion cannot incorporate such interruptions.

Second, continuity is also contradicted by small jumps in price. It is possible to object that the presence of small jumps in asset prices is not necessarily incompatible with a continuous representation: quoted prices simply have to be considered as prices recorded along a continuous trajectory. It all depends on the size of the jumps in relation to the Brownian motion diffusion coefficient. If the stock market dynamic could really be characterized by a continuous (Brownian) representation, then the observed discontinuities (small jumps) would remain within a theoretical range of fluctuations defined by volatility. In reality, this is not the case, as the empirically observed jumps exceed the calculated volatility: price drops and sudden changes, even small, invalidate the assumption of continuous representation for market price variations (Aït-Sahalia and Jacod, 2009, 2010). The observed price variations cannot be modelled by a Brownian representation of stock market uncertainty. It is, so to speak, too regular, too "smooth": the Brownian representation of stock market fluctuations gives a smooth structure to stock market uncertainty, with no jolts or sudden movements. As the basic structure of any financial market is discrete, finance is now widely understood to experience jumps at the microstructure level, even if only on a very small scale (Carr *et al.*, 2002).

1.3. The puzzle: prevailing ignorance of non-Brownian syndrome

Despite the large number of practical difficulties and stylized facts, the Brownian representation is stubbornly persistent, and has withstood the contradictions of experience as if it were immune to doubt. The Brownian representation cannot accommodate discontinuity, and the continuity assumption has been firmly maintained even in the face of contradictory evidence. Historically and epistemologically, this is an intriguing phenomenon. While statisticians were finding increasing evidence backing the diagnosis of non-Brownian dynamics in market movements, finance academics appear to have deliberately ignored that evidence, opting for the standard model of reducing return processes to Brownian motion, and even going as far as modifying the data to fit the theory (De Bruin and Walter, 2017). Even despite a large number of criticisms coming from physicists (Bouchaud and Potters, 1997, 2001; Mantegna and Stanley, 2000; Sornette, 2003), the Brownian representation remained dominant. Consequently, the Brownian representation of asset price dynamics has become part and parcel of finance curricula across the globe, the point of reference of most top financial economics journals, and the dominant view in the financial industry itself.

The question arises of the tenacity of this Brownian representation: how can its long life be explained? The models based on the Brownian representation do not stand up to scientific testing, but they are still being used. Why? What reasons can there be for clinging to such a mistaken belief? I call this situation an "epistemological puzzle", which I now try to answer by drawing on the main methodologies of the philosophy of science.

2. Entering the puzzle: the positivist epistemology

We now turn to the philosophy of science, starting by examining the positivist approach. Positivist epistemology assumes that only "facts" derived from the scientific method can make legitimate knowledge claims. In the positivist view, science must stick to what can be observed and measured ("data"), and the researcher is assumed to be separate from the topic of research, and not to affect its outcomes. This implies that financial theory has no influence on financial markets: searching for the most accurate representation of price dynamics can be achieved by sticking to purely financial data.

2.1. Hempel's schema

Let us start our very brief discussion of philosophy of science with Hempel's schema. How can the logical validity and explanatory power of the Brownian representation be assessed in a positivist framework? For a formal expression of this question, I follow the traditional positivist approach, as presented for example in Carl Hempel's *Philosophy of Natural Science* (Hempel, 1966). Following Hempel, I will call any statement being tested a hypothesis, whether it describes some particular fact or event, expresses a general law or is some other, more complex, proposition. Hypothesis H will be tested by reference to test implication I. "I" is a statement describing the "observable occurrences to be expected" (Hempel, 1966, p. 7). The method is summarized in the following schema.

If hypothesis H is true, then so is I
But (as the evidence shows), I is not true

H is not true

This form of argument is labelled the *modus tollens* in logic. The first line of this schema represents the world of theory, the second line the empirical world of the phenomenon studied (in this case, asset price movements), and the third line the result of the scientific approach. The observation/theory distinction is central in this representation of knowledge. In this positivist framework, objectivity rests on a clear separation of testable statements about financial data from theoretical representations like Brownian-based theories of price dynamics.

Hypothesis H: Brownian representation
In finance, market dynamics can be modelled using Brownian representation

Hypothesis H is compared with the phenomenon it seeks to explain, here the stock market dynamics observed on available sets of market data, for example the "tick-by-tick" data which are the most detailed record of a market's trading information. If hypothesis H here is "stock return dynamics can be modelled using Brownian representation", the test implications of H could be: I1: "the increments (i.e. the successive returns) are stationary and independent"; I2: "the empirical distributions of returns are normal" etc. From these data, "facts" are extracted, for example the implied volatility of stocks. A "practical" proposition P can then be derived from H, for example the shape of the volatility curve. In this example, proposition P could be "with Brownian dynamics, the Black-Scholes-Merton model predicts that the implied volatility curve is flat". In other words, Hempel's positivist approach can be summed up as follows: put hypothesis H to the test by comparing I or P with the "data".

2.2. The positivist methodology's failure to explain the dominant model

In the traditional view of the philosophy of science, scientific knowledge is knowledge that can be proved to be accurate. The traditional conception of the proof of truth is a verificationist epistemology. The classical rational variant of verificationism accepts the existence of extralogical proofs (intellectual intuition), while the classical empiricist variant finds some proof in the simple experience of "facts" that support the theory ("hard facts"). Whether rationalists or empiricists, the verificationist epistemologists agree that just a single "fact" that contradicts a general theory is enough to disprove it. This dissymmetry between proof and disproof is the hallmark of the classical approach to validation of scientific knowledge, which has been the dominant tradition in rational thought. It was clearly identified as such and commented on by the main authors of this period (see for example the commentary on Pascal by Richard Popkin in *Scepticism, Theology and the Scientific Revolution in the Seventeenth Century*, 1968). It required invention of a new kind of reasoning to move from the "facts" to the theories: induction, a problem laid out in the classic expositions found in *An enquiry concerning human understanding* (1748) and *A treatise of human nature* (1739-1740) by David Hume (1711-1776). The aim is to extract a principle of order from an analysis of phenomena. As a result, the verificationist approach to knowledge requires an inductivist epistemology of which Rudolf Carnap is the modern representative. With a verificationist epistemology, the aim appears to be to prove that the Brownian representation of stock market dynamics is the "true" representation of market fluctuations. But the "facts" do not seem to provide a sound foundation for the Brownian representation. Empirical findings in financial time series exhibit non-stationarity, non-linearity in the mean and the variance, non-trivial scaling properties, long-range dependence on autocorrelation, etc. Consequently, the dissymmetry between proof and disproof that is characteristic of the verificationist epistemology has no effect in the case of finance. "Facts" invalidate the Brownian representation - but nothing happens in response.

Karl Popper applied a radical reversal to this traditional representation and substituted an epistemology based on *falsification*, arguing that it was impossible to prove a theory was true (confirm it); it was only possible to prove it was untrue (falsify it). This turned the classical viewpoint on its head and showed that only the falsity of theories could be proved, never their truth. The epistemology moved from a verificationist to a falsificationist approach. In the Popperian schema, the dynamic of knowledge is as follows. If proposition P is not corroborated by experience, for instance if "the volatility smile (or smirk)" is observed, then H is rejected. A hypothesis that survives the falsification test is *corroborated* but not proven. As a result of this, the falsificationist approach to knowledge requires a deductivist epistemology.

As Popper shows in *The logic of scientific discovery* (1959), these two methods (verification and thus an inductive model, falsification and thus a deductive model) are incompatible. They are two radically different ways of understanding the scientific method. And yet from the standpoint of the dynamic of knowledge, the Popperian conception of progress in science follows the traditional conception, and the contrast between verificationism and falsificationism covers a more fundamental agreement on the nature of science: continuous, constant rectification of errors. In Popper's view, scientific knowledge progresses continuously by trial and error, and rational reason selects hypotheses by eliminating theories as they are invalidated. Bold hypotheses are responded to by robust refutations, and this alternating movement enables strict refinement of knowledge. Finally, bold hypotheses require a form of audacity and risk-taking: hypothesis rejection is a possible result of comparison with proposition P. The boldness demanded by Popper in the search for knowledge confers a “heroic” dimension on the scientific undertaking (in the sense that courage is required). In the case of finance, the bold hypothesis is the use of the Brownian representation for price movements. Robust falsifications have appeared, documented by the stylized facts, but there has been no rectification of errors. Even the Pareto tail, the power law of empirical distributions, was not enough to discard the normal distribution assumption. The “tale of fat tails” was considered irrelevant. There has been nothing like a “heroic dimension” in the scientific research on finance. Something else was at work.

To summarize this brief examination of the philosophy of science, if the Hempel-Popper schema were relevant in finance, highlighting non-normal distributions (or non-independent increments or non-trivial scaling etc.) would be enough to refute (falsify) the Brownian representation. There are in fact many examples of non-corroboration of the Brownian dynamic, and no verifiable implication survived the falsification test. But that has not led the academic community to change the representation. We can only conclude that the Hempel-Popper schema cannot account for the Brownian representation’s persistence in finance. Neither the verificationist epistemology nor its opposite, the falsificationist epistemology, can explain its enduring acceptance. This means we must look elsewhere for the reason why this model is still used, turning to some approach other than the dynamic of knowledge, an approach that also includes the social dimension of knowledge. We have to consider the postpositivist view of science. We must consider that theories, background, knowledge and values can influence what is “observed”, and think of “financial data” as more closely related to “convention” than “pure facts”.

3. Exploring the puzzle: Kuhn, Lakatos, Quine

Following this observation that the longstanding persistence of the Brownian representation cannot be explained using the resources of traditional positivist philosophy, I now turn to postpositvist conceptual schemas, with a particular focus on ways of understanding the science promoted by Thomas Kuhn, Imre Lakatos and Willard van Orman Quine.

3.1. The Brownian representation: a Kuhnian paradigm?

Thomas Kuhn attacked the enchanted view of science, a unified image of science as a world that advances by trial and error in a continuous, rational manner, where knowledge is cumulative, a clear distinction is possible between observation statements (empirical basis) and theory statements, and the difference between a scientific theory and an unscientific belief is firmly-established (Popper’s criterion of demarcation). Kuhn sees science as neither inductive nor cumulative. He reasons in terms of paradigms. Kuhn argues that if a proposition deriving from theory is not corroborated by experience, that does not lead to doubts about the theory. When a theory is changed, that is not because it has been invalidated by experience, but because the paradigm disappears and is replaced by a new paradigm. This is a sociological position: the “true” theory is the one found in the social consensus.

This position has often been interpreted as relativistic or even irrational, with the rational continuity of Popper's progress in scientific knowledge giving way to the irrational discontinuity of Kuhn's paradigm shifts. As a result, Kuhn has been compared to certain epistemologists of a radical critique of science, such as Paul Feyerabend and certain sociologists of science in the English-speaking world's field of "science studies". Yet Kuhn's epistemology is not relativistic. Kuhn draws attention to disruptions and discontinuities, and suggests that the driver of change is to be found in the confrontation within science between orthodoxy and heresy.

3.1.1 Kuhn's schema

Let us consider this in more detail. Kuhn's epistemology is founded on a set of concepts that characterize science based on the concept of the cycle, whose effect is to historicize science by inserting it into a repeatable process that always follows a pre-established schema. Instead of the logical ahistorical process of theory-hypothesis-falsification-new hypothesis, Kuhn proposes the schema of normal science-crisis-revolution-new normal science. In other words, the ahistorical science of the logical positivist epistemology is replaced by the history of science and its vicissitudes and disruptions. History plays the role of a countermodel to Viennese logical positivism.

The principal Kuhnian concepts are: normal science, crisis and the scientific revolution, the paradigm, incommensurability between paradigms and the worldview carried by paradigms. What interests us here is the concept of the paradigm, and whether it can explain the persistence of the Brownian representation. Kuhn argues that the two essential functions of a paradigm are the normative function (defining judgement criteria) and the cognitive function (defining methods, types of tests, tools and applications). Paradigms are "universally recognized scientific achievements that for a time provide model problems and solutions to a community of practitioners" (Kuhn, 1970, p. viii). More specifically, "a paradigm is what the members of a scientific community share, *and*, conversely, a scientific community [here, the community of mathematicians of finance and 'quant' researchers] consists of men who share a paradigm" (Kuhn, 1970, p. 176). This means that "a paradigm governs, in the first instance, not a subject matter but rather a group of practitioners" (Kuhn, 1970, p. 180). In the most general view, a paradigm is a conception of the universe that imprints on researchers and practitioners what the object and direction of a large amount of their research and work should be. Finally, a paradigm lasts until it reaches the point of intellectual exhaustion. In this respect Kuhn proposes an internalist view of scientific change.

In his 1969 postscript, Kuhn calls a paradigm a disciplinary matrix that defines a language and rules: "disciplinary because it refers to the common possession of the practitioners of a particular discipline" (Kuhn, 1970, p. 182). A scientific revolution is thus a change of language and therefore, although Kuhn had not read Carnap's *The Logical Structure of the World* (1928) when he was writing *The structure of scientific revolutions*, Carnap appears to be a direct forerunner of Kuhn even though Carnap considers that not every shift in language necessarily contains the revolutionary dimension of a paradigm shift.

This characteristic (the idea that experience depends on theories and language) is the source of the practical effectiveness of paradigms, since they define mainstream research activity (normal science) by scientists, without seeking to challenge the validity of the paradigm (the paradigm is not put to the test in mainstream research). "Normal" science defines scientific research marked by a research community consensus on the foundations of the explanation of the world, a consensus that protects it from controversy. Financial science is thus “normal” in a dual sense: not only through the assumed normality of stock movement distributions, but also in the ways academic finance research is conducted between two paradigms in the Kuhnian schema.

Kuhn considers that there are no “givens”, only “hard-won” information. A paradigm leads to a voluntary limitation of research questions, self-censorship both intellectual and social, produced by the social construction of the facts to be analysed. This reflects a stream of thought from French epistemology, including Bachelard and Meyerson. In *The New Scientific Spirit* (1934/1984), Gaston Bachelard showed how every experiment is influenced by a deposited theory: ``when one has fully comprehended (…) that experimentation is always dependent on some prior intellectual construct, then it is obvious why one should look to the abstract for proof of coherence in the concrete'' (Bachelard, 1934, p. 41). And Meyerson, in *Identity and reality* (1908) agrees with Duhem’s idea about “the close dependence of experiments upon scientific theories” (Meyerson, 1908, p. 391-392; cf. Duhem, 1906, p. 300).

The concept of the paradigm confers a relative, or constructed, nature on what is called an “objective fact” in the empiricist epistemology: the “facts” of a paradigm do not exist in the framework of the previous paradigm. When there is a change of paradigm, the “facts” change and so do the “experiments” designed to detect the “facts” and test the theory. In the new paradigm, “the givens themselves change”. The article by Granger and Orr (1972) in which the tails of an empirically observed return distribution are truncated in order to statistically validate a normal morphology of empirical uncertainty provide a Kuhnian validation, in the context of finance research, of the way the “givens” depend on the paradigms.

The dependence of the “given” on the theory raises questions concerning the demarcation line between observation statements (empirical basis) and theory statements. These questions are systematized in Quine (see below), putting Kuhn at a radical remove from Carnap and the neopositivist epistemology. As Carnap said, “Quine does not acknowledge the distinction which I emphasize above, because according to him there are no sharp boundary lines between logical and factual truth, between questions of meaning and questions of fact” (Carnap, 1947).

3.1.2 Is Kuhn’s schema relevant for solving the puzzle in financial modelling?

The appearance in mainstream research activity of what are called “anomalies” compared to expectations brings about a crisis period for the paradigm. In a crisis period, there are three possible outcomes: a last-minute resolution of the problem by normal science (normal shared knowledge), arriving just in time to save the threatened paradigm; resistance by the problem, which is then handed on to the next generation (the question of distribution tails, for example, whose importance was noted as early as the 1960s, was evaded until the market crash of 1987); or a shaky adaptation of the former paradigm, followed by an abundance of models all competing for the monopoly on scientific authority, and a scholastic drift in the specialist literature.

I have proposed (Walter, 1994, p. 278) that the exponential of Brownian motion should be considered as a paradigm as defined by Kuhn, and that the boom in competing models of the post-1987 crash years should be analysed as the sign of a paradigm crisis in line with Kuhn’s conjecture, which should lead to a replacement. More specifically, my proposal was as follows. Adopting the Kuhnian schema, the financial modelling situation can be analysed in the following way. In the representation of stock market movements, the basic model is the exponential of Brownian motion that constitutes the paradigm for the academic community in finance. An increasing number of anomalies have been noted when testing this paradigm, leading to the emergence of other models, which are all competing for the representation of stock movements, such as the ARCH family (Engle, 1982), chaos models (Hsieh, 1991), Lévy processes (Eberlein and Keller, 1995), and stable processes (Mandelbrot, 1963). None of these models has really succeeded in gaining the foreground, as each one can explain some aspect of the stock market phenomenon and no statistical test exists to choose between the different representations. As a result, each one can claim scientific authority, but none of them is determinant. This abundance of competing models is leading researchers to ask fundamental questions about their probabilistic choices. This is how the history of scientific work on the morphology of empirical financial uncertainty could be written under a Kuhnian schema.

The Kuhnian schema thus provides a useful clue to understanding one of the reasons for the dominance of the Brownian representation: while many competing representations have long been in existence, no replacement paradigm has yet been established.

3.2. The Brownian representation: a Lakatosian research programme?

The concept of the research programme was introduced by Imre Lakatos in his benchmark publication *History of science and its rational reconstructions* (1970) to describe the operation of science, in response to the debate between Kuhn and Popper about the way scientific knowledge advances. Lakatos defines research programmes in the following way: "The basic unit of appraisal must be not an isolated theory or conjunction of theories but rather a '*research programme*', with a conventionally accepted (and thus by provisional decision 'irrefutable') '*hardcore*' and with a '*positive heuristic*' which define problems, outlines the construction of a belt of auxiliary hypotheses, foresees anomalies and turns them victoriously into examples, all according to a preconceived plan" (Lakatos, 1970, 99).

3.2.1 Lakatos' schema

A research programme is characterized by a principal hypothesis, a "hard core" of theory that is never attacked, and auxiliary hypotheses (auxiliary statements, or adjustments/amendments A) which are useful to specify the principal theoretical hypothesis in testable form, but can be abandoned if violated by experience (once again raising the problem of the "empirical basis"). Changing the auxiliary hypothesis saves the hard core by displacing the problem (problem shift). A research programme can be progressing or stagnating: it is progressing "as long as its theoretical growth anticipates its empirical growth, that is, as long as it keeps predicting novel facts with some success"; it is stagnating "if its theoretical growth lags behind its empirical growth, that is, as long as it gives only post hoc explanations" (Lakatos, 1970, 100).

Now I use the representation put forward by Hilary Putnam (1975) which generalises that of Hempel (1966). The schema is

Theory T
Auxiliary statements A
___
Prediction P: True or false?

The first line of this schema represents the statement of theory, the second line the auxiliary statements and the third line a derived prediction. The theory is confronted with the phenomenon it seeks to explain, the stock market dynamics. From this phenomenon, "facts" are extracted: successive empirical returns. Next, a proposition (or "statement") P derived from this hypothesis is tested. The prediction P here could be "with Brownian dynamics, the implied volatility curve is flat". In other words, the positivist approach is: put theory T to the test by comparing P to the "facts". There may be adjustments A ("auxiliary statements") to the theory T, stating the conditions in which the theory must be applied and tested. In this case, proposition P is derived from the combination of H and A: P = T+A.

Theory T: the market dynamics follows a Brownian motion with drift
Prediction P: with Brownian dynamics, the implied volatility curve is flat

This schema can be subsumed into two general frameworks which are often believed to oppose "facts" and theories. This divide echoes other dualisms familiar to philosophy: the couples of nature/culture, questions of fact/questions of meaning and factual truth/logical truth. In general, this general schema represents the coexistence of two types of statement: theory statements and observation statements. Observation statements are also called the "empirical basis" of science, here the financial data of price quotes. The observation/theory distinction is central in this representation of knowledge. This schema is illustrated in figure 1 below.

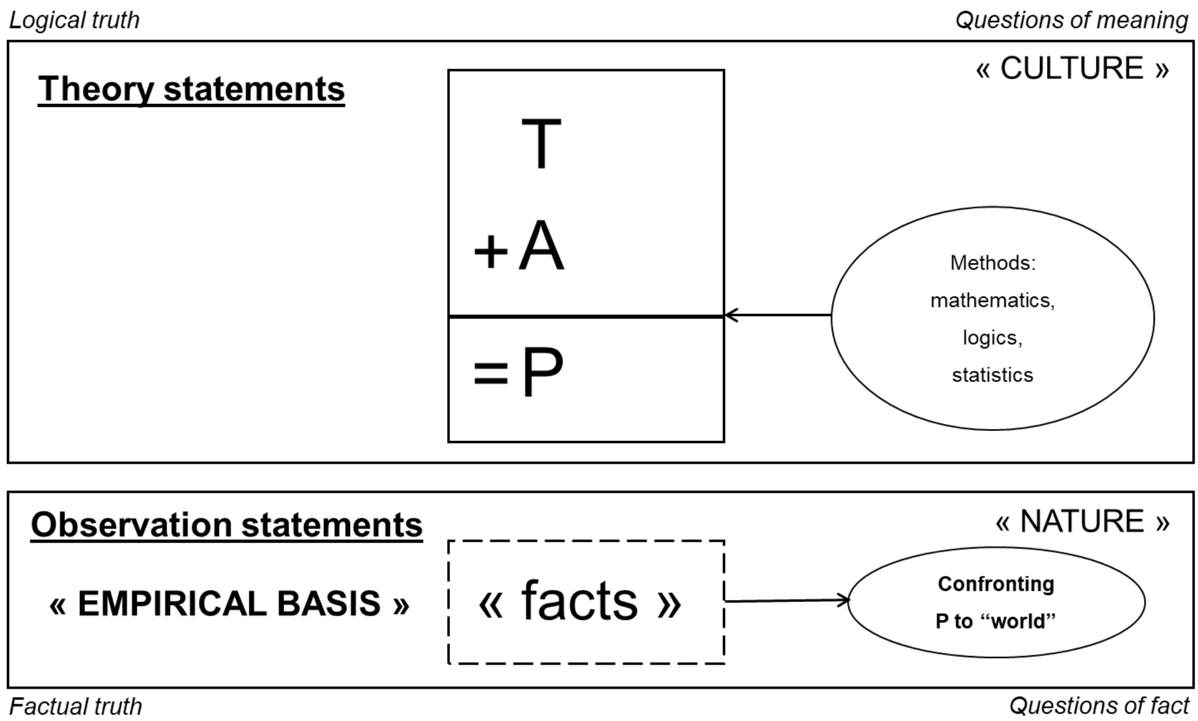

Figure 1. A simple way of representing the epistemological grid useful for finance

The question of the "empirical basis", its constitution and the measure of the fundamental magnitudes that characterise the disciplinary field under examination (Putnam, 1975, p. 319) is an important one. An empirical basis is never totally independent of the theoretical objective that is being pursued. As Ippoliti (2017b, 180) has recently pointed out, "raw data" do not exist and "data are not a form of zero-degree kind of knowledge". For describing the various stages of construction of an empirical basis to have market data that can be used for a statistical test of the Brownian representation, Walter (2013, Ch. 1) presents how data emerge from models, showing how this representation (the “model”) interacts with the formation of the basis itself (the “data”).

When confronted with an experiment whose results fail to corroborate the theory (an invalidating experiment), scientists do not in fact abandon the hard core of the theory, but seek a workaround that can provisionally save theory T. When T is not corroborated, the academic community finds a way of working with T despite a prediction P, either by using A, or by using new mathematical or statistical methods. It is possible to question auxiliary hypotheses and technical methods, but never the hard core of theory, T – until the day there are too many anomalies. This situation can last a long time. In the end, T falls down under the weight of “repairs”, but it takes time for this to happen.

In the gradual realisation of the need to change the principal hypothesis, Lakatos’ theory echoes Duhem, who argues that a principal hypothesis is continued by convention until the point is reached when “the worm-eaten columns of a building tottering in every part can no longer bear the building itself” (Duhem 1906, p. 217). Lakatos’ position is thus a position of refutationist epistemology that applies only to auxiliary hypotheses. A protective belt saves the hard core from any questioning of the principal theory due to invalidation by experience. Two possible directions can be identified in the mainstream scientific work: one is “allowed” (considered acceptable: positive heuristics) and the other is “forbidden” (considered unacceptable: negative heuristics).

With the schema of figure 1, it is possible to compare the traditional view, the Popperian view and the Kuhnian view of science (figure 2 below): with the traditional view, T is validated because P is true. With the Popperian view, T is rejected because P is false. With the Kuhn view, T is not rejected even if P is false.

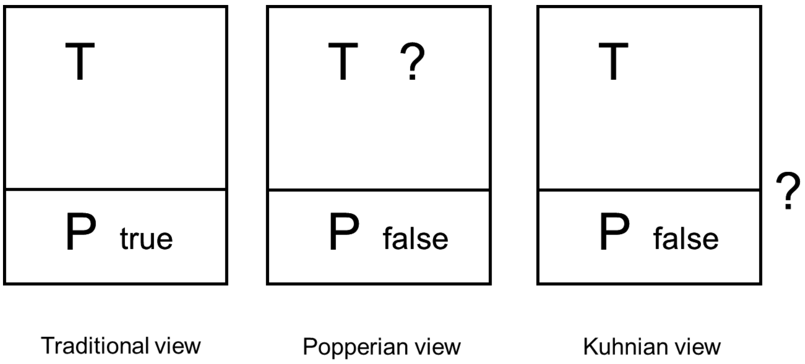

Figure 2. A simple way of comparing epistemological views (traditional, Popper, Kuhn)

3.2.2 Is Lakatos’ schema relevant for solving the puzzle in financial modelling?

The operative schema in mainstream scientific practice, as described by Lakatos, in this respect resembles Kuhn’s paradigm. But the difference between the Kuhnian approach and the Lakatosian approach lies in the understanding of a change in the principal hypothesis (paradigm for Kuhn, hard core for Lakatos). Kuhn – a reader of Alexandre Koyré – sees a paradigm as relating to a metaphysical view of the universe which it expresses in scientific terms, and any change in paradigm is a sort of metaphysical cataclysm: he sees it as the “revolutionary” dimension of periods of change, relating more to irrational conversions than rational reasoning; while in the view of Lakatos, the change in the hard core takes place more gradually, using the resources of rational thinking under Popper’s schema.

Adopting Lakatos' epistemology, the financial modelling situation examined earlier through the Kuhnian schema takes the following turn. The Brownian representation is the untouchable hard core of theory. Successive demonstrations of operational difficulties with its application in professional practice, or statistical divergences from this representation, are addressed one by one by adding auxiliary hypotheses or making amendments to the principal model. For example, in tests of the Black-Scholes-Merton model, what is challenged and refuted is the calibration of the model, rather than the model itself (Rainelli, 2003, p. 99). Lakatos' negative heuristic is in full operation here: the *modus tollens* must not be a pretext for rejecting the Brownian representation. The contradictions noted between the theoretical predictions of the hard core, and the financial phenomenon observed, lead to construction of a certain number of auxiliary hypotheses that form the protective belt around the hard core. ARCH processes, stochastic volatility models, and volatility smile (or smirk) modelling techniques are all ways of solving the problem caused by an uncorroborated theory, without changing the hard core.

The Brownian representation and its consequences delimit the hard core, i.e. the set of principles and hypotheses that are held to despite Kuhnian anomalies. The decision made by American academics in the 1970s was to make the Brownian representation irrefutable for all research programmes concerning financial modelling: this is the negative heuristic. The positive heuristic can then come into effect: an ad hoc anomaly modelling strategy must be devised to improve formulae based on the Brownian representation, such that the ad hoc research programme components that form the protective belt can become refutable. The ensuing debates then focus solely on these improvements, and research will advance under a schema that then echoes Kuhn's "banal" scientific activity.

In the light of Lakatos' work, some explanations can be found for the puzzling question of why an uncorroborated model and its probabilistic foundation (the Brownian representation) continue to be used. Although no competing paradigm can be put forward to replace the Brownian paradigm (the Kuhnian schema), interpretation of the auxiliary models associated with the research programme on stock price dynamics and the components of the protective belt around this representation that uphold the hard core of this programme, channelling refutability to ad hoc amendments (such as ARCH modelling or volatility surfaces) opens the door to an explanation of why this model is still used, and its ramifications and complexifications over the next following decades. The Brownian research programme is stubbornly resilient, and its defenders are making the fullest use of all the options offered by the protective belt. The falsificationist viewpoint alone cannot explain the continuing use of the Brownian representation. Lakatos' epistemology thus adds some interesting reflections on the peak of the Brownian representation in the 1970s and 1980s.

In the two epistemologies of Kuhn and Lakatos, the empirical basis issue remained unanswered. In both approaches the focus is on the reasons why an academic community changes its mind or does not change its mind, but the relationship between the theory and the empirical basis that refutes statements is not examined. The demarcation between theory statements and observation statements is still operative, although as seen with Kuhn, the "given" is "hard-won". This demarcation is what is radically questioned by Quine, who considers it impossible to distinguish between what derives from experience and what derives from language. We now turn to this understanding of the dynamic of knowledge which enhances previous knowledge and will connect epistemological analyses of the reasons for continuing to use the Brownian representation with approaches that come from science studies.

3.3. The Brownian representation: a Quine convention?

Up to this point, the two kinds of statement – theory and observation statements – were considered as distinct: the observation/theory distinction remained decisive for assessing the progress of scientific knowledge and for legitimizing the models. However, we have already started to see (for instance in Bachelard and in Kuhn) that the empirical basis is never totally independent of the theoretical aim, and that it can be constructed in accordance with a particular theory: what, then, does it mean to expect theories to correspond to facts? If it is impossible to consider facts as independent of conceptualization, the idea of "theories fitting the facts" loses clarity; and it seems to vanish completely when there is porosity between the empirical basis and the theory being tested. As Ippoliti (2017, 193) has emphasised, "in the specific case of finance, data turns out to be not only the result of cutting out 'reality', but also a way of shaping 'reality' by cutting it out".

In a famous article published in *The Philosophical Review* in 1951 entitled "Two Dogmas of Empiricism", Quine challenged the fundaments of Viennese logical empiricism by arguing that facts have no existence of their own, independently of the propositions stating them: propositions and facts increasingly look like two sides of the same coin - "facts" and "propositions" are inseparable. The very principle of the crucial experience is questioned, because the possibility of falsification disappears with this inseparability.

3.3.1 Quine's schema

One important consequence of this conception of the relationship between facts and propositions is the idea of underdetermination of theories by facts: reference to "facts" is not enough to decide between two competing theories. Therefore,

> "the total field [of science] is so underdetermined by its boundary conditions, experience, that there is much latitude of choice as to what statements to reevaluate in the light of any single contrary experience" (Quine, 1961, p. 42).

One or more invalidating experiences will not necessarily lead to rejection of a theory, because:

> "the edge of the system must be kept square with experience; the rest, with all its elaborate myths or fictions, has as its objectives the simplicity of laws" (Quine 1961, p. 45).

It is therefore the whole theory, not only one of its propositions, which is put to the test of experience:

> "our statements about the external world face the tribunal of experience not individually but only as a corporate body'' (Quine 1961, p. 41).

Therefore, it is systematically, and never individually, that the elements of a theory face the test of experience. In a footnote to page 41 of his article "Two dogmas of empiricism", Quine remarks that Duhem has defended this doctrine in *La théorie physique*. *Son objet et sa structure* (1906, p. 303-328). On this point, Quine agrees with Duhem:

> "To seek to separate each of the hypotheses of theoretical physics from the other assumptions on which this science rests in order to subject it in isolation to observational test is to pursue a chimera; for the realization and interpretation of no matter what experiment in physics imply adherence to a whole set of theoretical propositions" (Duhem 1906, p. 199-200).

That is the reason why this thesis is called the Duhem-Quine thesis. It springs from a holistic conception of confirmation, and one of its effects is that "any statement can be held true come what may, if we make drastic enough adjustments elsewhere in the system" (Quine 1961, p. 43). Analysing Newton's method, Duhem said that "experimental verifications are not the base of theory but its crown" (Duhem 1906, p. 204).

3.3.2 Is Quine's schema relevant for solving the puzzle in financial modelling?

The consequence of this holism is a form of conservatism which, given the nature of the relationship between theory and experience, is inherent to the functioning of science. Scientists are willing to stick with a theory as long as it is convenient. Theory thus becomes a convention as described by Poincaré:

> "The geometrical axioms are therefore neither synthetic a priori intuitions nor experimental facts. They are *conventions*. Our choice among all possible conventions is *guided* by experimental facts; but it remains *free*, and it is only limited by the necessity of avoiding every contradiction" (Poincaré, 1902, p. 66).

This is the reason why:

> "In the course of its development, *a physical theory is free to choose any path it pleases provided that it avoids any logical contradiction; in particular, it is free not to take account of experimental facts.* (Duhem, 1906, p. 206)

The Duhem-Quine thesis can account for the persistence of the Black-Scholes-Merton model and its combination with the Brownian representation of price movements: the internal logic of theoretical statements of mathematical finance prevails over empirical verification of their validity. "The edge of the system must be kept square with experience; the rest […] has as its objectives the simplicity of laws" (Quine, 1961, p. 45): here, the marginal adjustments of Brownian modelling could represent the edge of the system, while all the rest is supposed to remain constant. The Brownian representation cannot therefore be refuted. As previously pointed out in this article, Duhem believed that change would only take place on the day "the worm-eaten columns of a building tottering in every part" finally collapsed (Duhem, 1906, p. 217). Once again, using the schema of figure1, a simple representation of Quine view is possible, given in figure 3 below.

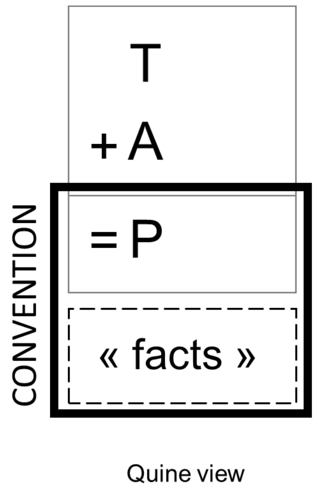

Figure 3. A simple way of representing Quine view for finance

Quine's criticism showed the limits of a purely epistemological analysis of the dynamics of knowledge. At the same time, the sociological implications of Kuhn's "paradigm" became increasingly relevant to the approach promoted by the rising field of social studies of science. Launched by the "Strong Programme" headed by Barry Barnes, David Bloor and Harry Collins, this sociological approach to scientific knowledge was systematically applied to finance in the 2000s (Callon, Licoppe, Muniesa, 2003; MacKenzie, 2006). This approach has many merits, in particular directing attention to external factors, especially dealing with "the social conditions and effects of science, and with the social structures and processes of scientific activity" (Ben-David and Sullivan, 1975). One of its hallmarks is to leave away from internal cognitive considerations. I, instead, would like to stress an explanation of the Brownian representation which remains "epistemological", in the sense that while it takes into account the "sociological" impact of this representation on the financial professions, it focuses on the fact that it relates to a "mental model", working as an implicit cognitive bias, in a sort of spontaneous epistemology.

4. A new proposal: the Brownian representation as the consequence of a mental model

I will now propose an alternative explanation for the persistence of the Brownian representation in mathematical modelling of financial risk, adopting the "view from inside" suggested by Ippoliti (2017b). I argue that a shared mental model has underlain the intellectual choices and mathematical representations of risk – or at least, has not curbed them.

A mental model is an explanation of a person's thought process about how something works in the real world (source: Wikipedia). For example, a runner participating in a mountain race on steep paths mentally represents to himself the characteristics of a risk (falling stones, tripping over a precipice, dangerous parts of the path, etc.) and uses that representation to anticipate the outcome of an action (running faster, slowing down, etc.). Mental models shape actors' normative issues (see for example Mantzavinos, 2001). Hence, if "rules, laws, institutions, regulators, the behaviour and the psychology of traders and investors are the key elements to the understanding of finance, and stock markets in particular" (Ippoliti, 2017b, 121), what I propose here is to consider that both finance professionals (bankers, traders, investors, regulators, "quants" etc.) and research professionals (academics in financial mathematics) have been influenced (shaped?) by a specific mental model, which is the *principle of continuity*. This principle has become so entrenched in our ways of thinking about risk and risk prevention that it has given rise to a fully-fledged, contagious mental model that I will now describe.

4.1. The principle of continuity

The principle of continuity is a principle from natural philosophy postulating that in nature, things change gradually rather than suddenly.

4.1.1 Expression of the principle of continuity

The most compact expression is found in the famous Latin saying *Natura non facit saltus* (meaning "nature does not make jumps"), which we owe to Leibniz (1646-1716). The principle of continuity was the source of calculus (differential and integral) as performed by Leibniz, then Newton (1643-1727). Note the original ambiguity of this principle, which can be understood as mathematical or metaphysical. The same principle provided the foundations for the ideas of the Swedish naturalist Carl von Linné (1707-1778) on classification of species, and later Charles Darwin (1809-1882) on the theory of evolution (1859). It was then taken up by Alfred Marshall (1842-1924), who quoted it in his *Principles of Economics* (1890). Marshall's aim in doing so was to show that calculus was the fundamental mathematical instrument to develop economic science: "If the book has any special character of its own, that may perhaps be said to lie in the prominence which it gives to […] the *Principle of Continuity*." As Norbert Wiener observed, "just as primitive peoples adopt the Western modes of denationationalized clothing and of parliamentarism out of a vague feeling that these magic rites and vestments will at once put them abreast of modern culture and technique, so the economists have developed the habit of dressing up their rather imprecise ideas *in the language of the infinitesimal calculus*" (Wiener, 1966, p. 90, emphasis added). Hence, to answer the question asked by Chen (2017, 17) "what's wrong with economic math", I would say: the principle of continuity.

4.1.1 The principle of continuity in financial modelling

This principle subsequently trickled down into all of neoclassical economic thought, which was the source of contemporary finance. The financial theory mathematically modelled since 1952 came in the wake of this principle of continuity, and one of its biggest successes was providing a way to value derivatives using the formulas developed by Fisher Black, Myron Scholes and Robert Merton in 1973, and later the fundamental valuation theorem for financial assets derived from the work of Harrison, Kreps and Pliska between 1979 and 1981. The principle of continuity was the mental model that governed researchers' intuition in the mathematical writing of financial risks, in their research work, and then in their teaching of finance.

The principle of continuity thus became the cornerstone of a representation of the probable in finance, which contained methods of reasoning for professional practices derived from financial mathematics based on the same principle. With a mental representation built on continuity, financial risk disappears as if by magic since if things change gradually and steadily, their development is always predictable and safeguards can be found in techniques of financial derivatives, which are all based on the principle of continuity. As MacKenzie and Spears (2014, p. 401) put it: "it is the strategy of Black-Scholes modelling writ large: find a perfect hedge, a *continuously-adjusted portfolio* of more basic securities that will have the same payoff as the derivative, whatever happens to the price of the underlying asset" (emphasis added); which means that the continuity assumption is at the core of a large number of financial techniques, like the central tenet of portfolio replication[4].

This principle was still predominant in the 1990s despite the emerging evidence of extreme values in the tails of empirical distributions. At the end of the 20th century, many financial techniques such as portfolio insurance or the calculation of capital requirements in the insurance industry still assumed that (financial) nature does not make jumps, and therefore promoted continuity. Despite many empirical difficulties encountered in attempts to apply this principle in practice, and despite academic warnings from outside neoclassical finance, the principle of continuity remained vastly more popular than its discontinuous competitors. Yet, as early as 1966, Wiener pointed out that "here some recent work of Mandelbrot is much to the point. He has shown the intimate way in which the commodity market is both theoretically and practically subject to random fluctuations arriving from the very contemplation of its own irregularities is something much wilder and much deeper than has been supposed, and that *the usual continuous approximations to the dynamics of the market must be applied with much more caution* than has usually been the case, or not at all" (Wiener, 1966, 92, emphasis added). Mandelbrot's entire work is based on the description and modelling of discontinuities, an objective for which he forged the notion of "fractal", a true mathematical devil for the financial theory based on the principle of continuity (Mandelbrot, 1997).

In the 20th century, both physics and genetics abrogated the principle of continuity. Quantum mechanics postulated discrete energy levels, and genetics took discontinuities into account. But economics – including modern financial theory – stood apart from this intellectual revolution. As Chen (2017, 41) noticed, neoclassical economics constructed a mathematical utopian of the market, in which the math selection was due to a philosophical preference. I complete this view in finance by hypothesizing that this philosophical preference was the principle of continuity.

4.2. A persistent idea

I will now suggest that the mental model of the principle of continuity has become a "persistent idea", a representation that finds its way into our minds at an infra-rational level and is thus protected from invalidation by experience and able to withstand events. I have metaphorically labelled this persistent idea the "Brownian virus", for as noted just above, in the 20th century the principle of continuity was disproved in the physical sciences (by the existence of discrete levels of energy in mechanics) and then in genetics, and consideration of discontinuity gradually became part of the new paradigm that formed in opposition to the old understanding of nature. But economic science remained aloof from these re-evaluations, and the construction of modern finance ignored it. Passive index-tracking asset management and portfolio insurance techniques are visible signs of the dominance of the principle of continuity in professional financial practices, and in work on mathematical modelling in finance.

If evidence were needed of the dominance of this principle, it can be found in a *Financial Times* interview published on 16 March 2008 with Alan Greenspan, president of the US Federal Reserve from 1987 to 2006, in which Greenspan declared: "We will never be able to anticipate all discontinuities in financial markets." The word "discontinuities" clearly indicates that in Greenspan's view, financial nature does not make jumps, and the "natural" dynamic of the financial markets is

[4] A replicating portfolio for a given asset is a portfolio of assets with the same financial properties (e. g. cash flows).

continuous. In other words, the mental model underlying this declaration is the principle of continuity.

To explain how the Brownian virus spread through the financial sector, and how financial practices were contaminated by this mental representation founded on the principle of continuity, I introduced the concept of the "financial *Logos*" (Walter, 2016). This term was coined to describe the discourse that structures practices and organisations, calculations, prudential regulations and accounting standards, leading to the generalized financialisation of society from the 1980s onwards. This *Logos* is both a guiding principle for professional practices and a discourse that justifies those practices. This discourse exists in three forms: written (such as formal rules for investment or spreading risks), oral (such as the discourse on what makes good financial management by a pension fund or investment bank) and technical (such as methods to calculate risks and equity). In short, the financial *Logos* is a structuring discourse that is incorporated into the financial management of banks, insurance companies, asset management companies, and oversight and control practices for financial activities. This structuring discourse particularly concerns representations of risk, a specific culture of risk overseen by the epistemological authorities of financial standardization, i.e. shared mandatory knowledge built on the principle of continuity. It creates a form of "epistemic culture" in the Knorr-Cetina (1999)'s sense, a way of determining how we know what we know, and defines a financial "quantification convention" in the Chiapello-Walter (2016)'s sense, a way of sharing assumptions about market dynamics.

Conclusion

My aim in this paper was to bring a major puzzle in the history of financial modelling to the attention of historians and philosophers of science. Historically and epistemologically, the persistence of the Brownian representation is an intriguing phenomenon: in around 1965, at a time when statisticians were finding increasing evidence backing the diagnosis of non-Gaussian distributions in market dynamics, finance academics deliberately chose to ignore that evidence and opted for a standard model that reduces return processes to Brownian motion. This paper aims to make a contribution to solving this puzzle in the philosophy of science literature. After noting the limitations of the usual explanations drawn from the philosophy of science to account for the persistence of the Brownian representation, I conclude with a proposition that will be examined in detail in a later work: this mathematical representation is to be considered as the sign of a deeper mental model whose cognitive structure is founded on the principle of continuity.

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