The Geometry of Conventionality

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Abstract

Hans Reichenbach famously argued that the geometry of spacetime is conventional in relativity theory, in the sense that one can freely choose the spacetime metric so long as one is willing to postulate a "universal force field". Here we make precise a sense in which the field Reichenbach defines fails to be a "force". We then argue that there is an interesting and perhaps tenable sense in which geometry is conventional in classical spacetimes. We conclude with a no-go result showing that the variety of conventionalism available in classical spacetimes does not extend to relativistic spacetimes.

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Reichenbach (1958) famously argued that spacetime geometry in relativity theory is conventional, in the following precise sense. Suppose that the geometry of spacetime is given by a model of general relativity, (M, g_{ab}).[1] Reichenbach claimed that one could equally well represent spacetime by any other (conformally equivalent) model,[2] $(M, \tilde{g}_{ab})$, so long as one

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[1] A *relativistic spacetime* is an ordered pair (M, g_{ab}), where M is a smooth, connected, paracompact, Hausdorff 4-manifold and g_{ab} is a smooth Lorentzian metric. Relativistic spacetimes are models of general relativity. For background, including details of the "abstract index" notation used here, see Malament (2012) or Wald (1984).

[2] Two metrics g_{ab} and $\tilde{g}_{ab}$ are said to be *conformally equivalent* if there is some non-vanishing scalar field Ω such that $g_{ab} = \Omega^2 \tilde{g}_{ab}$. Two spacetime metric are conformally equivalent just in case they agree on causal structure, i.e., they agree with regard to which vectors at a point are timelike or null. Reichenbach did not insist on conformal equivalence when he originally stated his conventionality thesis, but, as Malament (1986) argues, given the he argued elsewhere that the causal structure of spacetime was non-conventional, to make his views consistent it seems one needs to insist that metric structure is conventional only up to a conformal transformation.

was willing to postulate a "universal force field" F_{ab}, defined by $g_{ab} = \tilde{g}_{ab} - F_{ab}$. Various commentators have had the intuition that this universal force field is "funny"i.e., that it is not a "force field" in any standard sense.[3] We will begin by presenting a concrete example that, we believe, undermines the interpretation of as a "force field" at all. We will next show that in classical spacetimes, there is a robust sense in which arbitrary choices of spacetime geometry can be off-set by postulating a universal force field, albeit with a rather different trade-off equation from the one Reichenbach proposed. Indeed, the force field one needs to postulate in that context is not so funny after all: its behavior is strikingly similar to standard force fields, such as the electromagnetic field. Turning back to relativity theory, we will prove a no-go result to the effect that the trade-off equation we describe for classical spacetimes does not have a relativistic analog. The upshot is that there is an interesting and perhaps tenable sense in which geometry is conventional in classical spacetimes, but a the relativistic setting Reichenbach's position seems much less appealing.[4]

Consider the following example. Let (M, η_{ab}) be Minkowski spacetime and let ∇ be the Levi-Civita derivative operator compatible with η_{ab}.[5] Choose a coordinate system t, x, y, z such that $\eta_{ab} = \nabla_a t \nabla_b t - \nabla_a x \nabla_b x - \nabla_a y \nabla_b y - \nabla_a z \nabla_b z$. Now consider a second spacetime $(M, \tilde{g}_{ab})$, where $\tilde{g}_{ab} = \Omega^2 \eta_{ab}$ for $\Omega(t, x, y, z) = x^2 + 1/2$, and let $\tilde{\nabla}$ be the Levi-Civita derivative operator compatible with $\tilde{g}_{ab}$. Then $\tilde{\xi}^a = \Omega^{-1} \left(\frac{\partial}{\partial t}\right)^a$ is a smooth timelike vector field on M with unit length relative to $\tilde{g}_{ab}$. Let γ be the maximal integral curve of $\tilde{\xi}^a$ through the point $(0, 1/\sqrt{2}, 0, 0)$. One can confirm that the acceleration of this curve, relative to $\tilde{\nabla}$, is given by $\tilde{\xi}^n \tilde{\nabla}_n \tilde{\xi}^a = 2\sqrt{2} \left(\frac{\partial}{\partial t}\right)^a$ for all points on $\gamma[I]$. Meanwhile, γ is a geodesic

[3] We get the term "funny force" from Malament (1986), though it may predate him. Other classic discussions of Reichebach's conventionality thesis, including various expressions of skepticism, can be found in Sklar (1977), Glymour (1977), and Norton (1994).

[4] Of course, there are many reasons why one might be skeptical about claims concerning the conventionality of geometry, aside from the character of the force law. (See Sklar (1977) for a detailed discussion.) Our point here is to clarify just how a conventionality thesis would go if one were serious about postulating a universal force field in any recognizable sense.

[5] Minkowski spacetime is a relativistic spacetime (M, η_{ab}) where M is $\mathbb{R}^4$ and (M, η_{ab}) is flat and geodesically complete.

(up to reparameterization) of ∇, the Levi-Civita derivative operator compatible with g_{ab}. According to Reichenbach, it would seem to be a matter of convention whether (1) $\gamma[I]$ is the worldline of a free massive point particle in (M, η_{ab}) or (2) $\gamma[I]$ is the worldline of a massive point particle in $(M, \tilde{g}_{ab})$, accelerating due to the universal force field $F_{ab} = \tilde{g}_{ab} - \eta_{ab}$. But now observe: along $\gamma[I]$, the conformal factor Ω is equal to 1which means that along $\gamma[I]$, $g_{ab} = \tilde{g}_{ab}$ and thus $F_{ab} = \mathbf{0}$. And so, if one adopts option (2) above, one is committed to the view that the universal force field F_{ab} can accelerate particles even where the field vanishes. It follows that the field F_{ab} cannot be proportional to the acceleration of a body.

This example shows that F_{ab} is an unusual force field, indeedso unusual that it hardly deserves the name force at all. This is especially striking because there is a sense, which we will presently describe, in which classical spacetimes do support a kind of Reichenbachian conventionality about geometry, though the construction is quite different from what Reichenbach describes. To motivate our construction, we will begin by considering (an analog of) Reichenbach's trade-off equation in a classical spacetimes. Suppose the geometry of spacetime is given by a classical spacetime (M, t_a, h^{ab}, ∇).[6] Direct analogy with Reichenbach's trade-off equation would have us consider classical metrics $\tilde{t}_a$ and $\tilde{h}^{ab}$ and universal force fields F^a and G_{ab} satisfying $t_a = \tilde{t}_a + F_a$ and $h^{ab} = \tilde{h}^{ab} + G^{ab}$. We might want to assume that G^{ab} must be symmetric, since $\tilde{h}^{ab}$ is assumed to be a classical spatial metric. And as in the relativistic case, we might insist that these new metrics preserve causal structure—which here would mean that the compatibility condition $\tilde{t}_a \tilde{h}^{ab} = \mathbf{0}$ must be met, and that simultaneity relations between points must be preserved by the transformation, which means that $t_a \tilde{h}^{ab} = \mathbf{0}$ and $\tilde{t}_a h^{ab} = \mathbf{0}$. Together, these imply that $G^{ab} F_b = \mathbf{0}$.

[6] A classical spacetime is an ordered quadruple (M, t_a, h^{ab}, ∇), where M is a smooth, connected, paracompact, Hausdorff 4-manifold, t_a and h^{ab} are smooth fields satisfying $t_a h^{ab} = \mathbf{0}$ that can be interpreted as a "temporal metric" and a "spatial metric", respectively, and ∇ is a smooth derivative operator satisfying the compatibility conditions $\nabla_a t_b = \mathbf{0}$ and $\nabla_a h^{ab} = \mathbf{0}$. Classical spacetimes are models of Newtonian gravitation and geometrized Newtonian gravitation (sometimes, Newton-Cartan theory). For more on classical spacetimes, see Malament (2012).

Given these trade-off equations, Reichenbachian conventionalism about classical spacetime geometry might go something like this: the metrics (t_a, h^{ab}) are merely conventional since we could always use $(\tilde{t}_a, \tilde{h}^{ab})$ instead, so long as we also postulate universal forces F_a and G^{ab}. One could perhaps investigate this proposal to see how changes in the classical metrics affect the associated families of compatible derivative operators, or even just to understand what the degrees of freedom are. But there is an immediate sense in which this proposal is ill-formed. The issue is that the metrical structure of a classical spacetime does not have a close relationship to the acceleration of curves or to the motion of bodies. Acceleration is determined relative to a choice of derivative operator, ∇, and in general there are infinitely many derivative operators compatible with any pair of classical metrics. All of these give rise to different standards of acceleration. And so it is not clear that the fields F_a and G^{ab} bear any relation to the acceleration of a body. As in the relativistic example given above, this counts against interpreting them as force fields at all.

These considerations suggest that Reichenbach's force field does not do any better in Newtonian gravitation than it does in general relativity. But it also points in the direction of a different route to conventionalism about classical spacetime geometry. The proposal above failed because acceleration is determined relative to a choice of derivative operator, not classical metrics. Could it be that the choice of derivative operator in a classical spacetime is a matter of convention, so long as the choice is appropriately offset by some sort of universal force field? We claim that the answer is "yes". And, perhaps surprisingly, the universal force field is not all that funny.

Proposition 1. *Fix a classical spacetime (M, t_a, h^{ab}, ∇), and consider an arbitrary derivative operator on M, $\tilde{\nabla}$, which we assume to be compatible with t_a and h^{ab}. Then there exists a unique anti-symmetric field F_{ab} such that given any timelike curve γ with unit tangent vector field ξ^a, $\xi^n \nabla_n \xi^a = \mathbf{0}$ if and only if $\xi^n \tilde{\nabla}_n \xi^a = F^a{}_n \xi^n$, where $F^a{}_n \xi^n = h^{am} F_{mn} \xi^n$.*

Proof. If such a field exists, then it is necessarily unique, since the defining relation determines its action on all vectors (because the space of vectors at a point is spanned by

the timelike vectors). So it suffices to prove existence. Since $\tilde{\nabla}$ is compatible with t_a and h^{ab}, it follows from Prop. 4.1.3 of Malament (2012) that the $C^a{}_{bc}$ field relating it to ∇ must be of the form $C^a{}_{bc} = 2h^{an}t_{(b}\kappa_{c)n}$, for some anti-symmetric field κ_{ab}.[7] Pick some timelike geodesic γ of ∇, and suppose that ξ^a is its unit tangent vector field. Then the acceleration relative to $\tilde{\nabla}$ is given by $\xi^n\tilde{\nabla}_n\xi^a = \xi^n\nabla_n\xi^a - C^a{}_{nm}\xi^n\xi^m = -2h^{ar}t_{(n}\kappa_{m)r}\xi^n\xi^m = -2h^{ar}\kappa_{mr}\xi^m$. So we can take $F_{ab} = 2\kappa_{ab}$ and we have existence. □

This proposition means that one is free to choose any derivative operator one likes (compatible with the fixed classical metrics) and, by postulating a universal force field F_{ab}, one can recover all of the allowed trajectories of either a model of standard Newtonian gravitation or a model of geometrized Newtonian gravitation. Thus, since the derivative operator determines both the collection of geodesics—i.e., non-accelerating curvesand the curvature of spacetime, there is a Reichenbachian sense in which both acceleration and curvature are conventional in classical spacetimes. Most importantly, the field F_{ab} makes good geometrical sense as a force field. Like the Faraday tensor, which represents the electromagnetic field, the field defined in Prop. 1 is an anti-symmetric, rank 2 tensor field; moreover, this field is related to the acceleration of a body in precisely the same way that the Faraday tensor is. So F_{ab} as defined in Prop. 1 is not a "funny" force at all.

It is interesting to note that from this perspective, geometrized Newtonian gravitation and standard Newtonian gravitation are just special cases of a much more general phenomenon. Specifically, one can always choose the derivative operator associated with a classical spacetime in such a way that the curvature satisfies the geometrized Poisson equation and the allowed trajectories of bodies are geodesics (yielding geometrized Newtonian gravitation), or one can choose the derivative operator so that the curvature vanishesand

[7]The notation of $C^a{}_{bc}$ fields used here is explained in Malament (2012, Ch. 1.7) and Wald (1984, Ch. 3). Briefly, fix a derivative operator ∇ on a smooth manifold M. Then any other derivative operator $\tilde{\nabla}$ can be written as $\tilde{\nabla} = (\nabla, C^a{}_{bc})$, where $C^a{}_{bc}$ is a smooth, symmetric (in the lower indices) tensor field that allows one to express the action of $\tilde{\nabla}$ on an arbitrary tensor field in terms of the action of ∇ on that field.

when one makes this second choice, if other background geometrical constraints are met, the force field takes on the particularly simple form $F_{ab} = 2\nabla_{[a}\varphi t_{b]}$, for some scalar field φ that satisfies Poisson's equation (yielding standard Newtonian gravitation). These are non-trivial facts, but they arguably indicate that some choices of derivative operator are more convenient to work with than others (because the associated F_{ab} fields take simple forms), and not necessarily that these choices are canonical.[8]

Now let us return to the original question, concerning conventionality about geometry in relativity theory. We have now seen that in classical spacetimes, there is a trade-off between choice of derivative operator and a not-so-funny universal force field that does yield a kind of Reichenbachian conventionality. Does a similar result hold in relativity? The analogous proposal would go as follows. Fix a relativistic spacetime (M, g_{ab}), and let ∇ be the Levi-Civita derivative operator associated with g_{ab}. Now consider another torsion-free derivative operator $\tilde{\nabla}$.[9] We know that $\tilde{\nabla}$ cannot be compatible with g_{ab}, but we can insist that causal structure is preserved, and so we can require that there be some metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$ such that $\tilde{\nabla}$ is compatible with $\tilde{g}_{ab}$. The question we want to ask is this. Is there some (anti-symmetric, rank 2) tensor field F_{ab} such that, given a curve γ, γ is a geodesic (up to reparameterization) relative to g_{ab} just in case its acceleration relative to $\tilde{\nabla}$ is given by $F^a{}_n\tilde{\xi}^n$, where $\tilde{\xi}^a$ is the tangent field to γ with unit length relative to $\tilde{g}_{ab}$? The answer is "no", as can be seen from the following proposition.

[8]There is certainly more to say here regarding what, if anything, makes the classes of derivative operators associated with standard Newtonian gravitation and geometrized Newtonian gravitation "special", in light of Prop. 1. Several arguments in the literature might be taken to apply. For instance, though he does not show anything as general as Prop. 1, Glymour (1977) has observed that one can think of the gravitational force in Newtonian gravitation as a Reichenbachian universal force. He goes on to resist conventionalism by arguing that geometrized Newtonian gravitation is better confirmed, since it is empirically equivalent to Newtonian gravitation (with the funny force), but postulates strictly less. For an alternative perspective on the relationship between Newtonian gravitation and geometrized Newtonian gravitation, see Weatherall (2013). A second argument for why geometrized Newtonian gravitation should be preferred to standard Newtonian gravitationone that can likely be extended to the present contexthas recently been offered by Knox (2013). But we will not address this question further in the present paper.

[9]An interesting question that we do not address here is whether the torsion of the derivative operator can be seen as conventional in a Reichenbachian sense.

Proposition 2. *Let (M, g_{ab}) be a relativistic spacetime, let $\tilde{g}_{ab} = \Omega^2 g_{ab}$ be a metric conformally equivalent to g_{ab}, and let ∇ and $\tilde{\nabla}$ be the Levi-Civita derivative operators compatible with g_{ab} and $\tilde{g}_{ab}$, respectively. Suppose Ω is non-constant.*[10] *Then there is no tensor field F_{ab} such that an arbitrary curve γ is a geodesic relative to ∇ if and only if its acceleration relative to $\tilde{\nabla}$ is given by $F^a{}_n \tilde{\xi}^n$, where $\tilde{\xi}^n$ is the tangent field to γ with unit length relative to $\tilde{g}_{ab}$.*

Proof. Since g_{ab} and $\tilde{g}_{ab}$ are conformally equivalent, their associated derivative operators are related by $\tilde{\nabla} = (\nabla, C^a{}_{bc})$, where $C^a{}_{bc} = -\frac{1}{2}\Omega^{-1}\left(\delta^a{}_b \nabla_c \Omega^2 + \delta^a{}_c \nabla_b \Omega^2 - g_{bc} g^{ar} \nabla_r \Omega^2\right)$. Moreover, given any smooth timelike curve γ, if ξ^a is the tangent field to γ with unit length relative to g_{ab}, then $\tilde{\xi}^a = \Omega^{-1}\xi^a$ is the tangent field to γ with unit length relative to $\tilde{g}_{ab}$. A brief calculation reveals that if γ is a geodesic relative to ∇, then the acceleration of γ relative to $\tilde{\nabla}$ is given by $\tilde{\xi}^n \tilde{\nabla}_n \tilde{\xi}^a = \tilde{\xi}^n \nabla_n \tilde{\xi}^a - C^a{}_{nm} \tilde{\xi}^n \tilde{\xi}^m = \Omega^{-3}\left(\xi^a \xi^n \nabla_n \Omega - g^{ar} \nabla_r \Omega\right)$. Now suppose that a tensor field F_{ab} as described in the proposition existed. It would have to satisfy $\Omega^{-1} \tilde{g}^{an} F_{nm} \xi^m = \Omega^{-3}\left(\xi^a \xi^n \nabla_n \Omega - g^{ar} \nabla_r \Omega\right)$ for every unit (relative to g_{ab}) vector field ξ^a tangent to a geodesic (relative to ∇). Note in particular that F_{ab} must be well-defined as a tensor at each point, and so this relation must hold for *all* unit timelike vectors at any point p, since any vector at a point can be extended to be the tangent field of a geodesic passing through that point. Pick a point p where $\nabla_a \Omega$ is non-vanishing (which must exist, since we assume Ω is non-constant), and consider an arbitrary pair of distinct, co-oriented unit (relative to g_{ab}) timelike vectors at that point, μ^a and η^a. Note that there always exists some number α such that $\zeta^a = \alpha(\mu^a + \eta^a)$ is also a unit timelike vector. Then it follows that,

$$\tilde{g}^{an} F_{nm} \zeta^m = \frac{1}{\Omega^2}\left(\zeta^a \zeta^n \nabla_n \Omega - g^{ar} \nabla_r \Omega\right) = \frac{1}{\Omega^2}\left(\alpha^2\left(\mu^a \mu^n + \mu^a \eta^n + \eta^a \mu^n + \eta^a \eta^n\right) \nabla_n \Omega - g^{ar} \nabla_r \Omega\right).$$

[10] If Ω were constant, then the force field $F_{ab} = \mathbf{0}$ would meet the requirements of the proposition. But metrics related by a constant conformal factor are usually taken to be physically equivalent, since they differ only by an overall choice of units.

But since F_{ab} is a linear map, we also have

$$\tilde{g}^{an}F_{nm}\zeta^m = \alpha\tilde{g}^{an}F_{nm}\mu^m + \tilde{g}^{an}F_{nm}\eta^m = \frac{\alpha}{\Omega^2}\left(\mu^a\mu^n\nabla_n\Omega - g^{ar}\nabla_r\Omega\right) + \frac{\alpha}{\Omega^2}\left(\eta^a\eta^n\nabla_n\Omega - g^{ar}\nabla_r\Omega\right).$$

These two expressions must be equal, which, with some rearrangement of terms, implies that

$$(2\alpha - 1)g^{ar}\nabla_r\Omega = \alpha\left[(1-\alpha)(\mu^a\mu^n + \eta^a\eta^n) - 2\alpha\eta^{(a}\mu^{n)}\right]\nabla_n\Omega.$$

But this expression yields a contradiction, since the left hand side is a vector with fixed orientation, independent of the choice of μ^a and η^a, whereas the orientation of the right hand side will vary with μ^a and η^a, which were arbitrary. Thus F_{ab} cannot be a tensor at p. □

So it would seem that we do not have the same freedom to choose between derivative operators in general relativity that we have in classical spacetimes—at least not if we want the "universal force field" to take the form of familiar force fields. Of course, one might object that the force law could well be more complicated. And indeed, given a relativistic spacetime (M, g_{ab}), a conformally equivalent metric $\tilde{g}_{ab}$, and their respective derivative operators, ∇ and $\tilde{\nabla}$, there is always *some* tensor field such that we can get a "funny force" trade-off. Specifically, a curve γ will be a geodesic relative to ∇ just in case its acceleration relative to $\tilde{\nabla}$ is given by $\tilde{\xi}^n\tilde{\nabla}_n\tilde{\xi}^a = G^a{}_{nm}\tilde{\xi}^n\tilde{\xi}^m$, where $\tilde{\xi}^a$ is the unit (relative to $\tilde{g}_{ab}$) vector field tangent to γ, and $G^a{}_{bc} = -(\Omega^{-1}\delta^a{}_b\nabla_c\Omega + C^a{}_{bc})$, with $C^a{}_{bc}$ the field relating $\tilde{\nabla}$ to ∇. That the field $G^a{}_{bc}$ exists should be no surprise—it merely reflects the fact that the action of one derivative operator can always be expressed in terms of any other derivative operator and a rank three tensor. This $G^a{}_{bc}$ field presents a more compelling force law than Reichenbach's own universal force field, since $G^a{}_{bc}$ will always be proportional (in a generalized sense) to the acceleration of a body, just as one should expect of a force field. In particular, it will vanish precisely when the acceleration of the body does, which as we have seen is not the

case for Reichenbach's force field. So if one wants to hold on to a Reichenbachian position regarding the conventionality of geometry in general relativity, one can, though one should prefer a trade-off equation involving $G^a{}_{bc}$ to Reichenbach's own proposal. Still, a "force field" represented by a rank three tensor cannot be expected to behave like familiar forces, and so although this proposal may meet a bare threshold of coherence, it is still unappealing insofar as it requires one to postulate a novel force field that likely has strange properties.

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