How can a line segment with extension be composed of extensionless points? From Aristotle to Borel, and Beyond*

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Abstract

We provide a new interpretation of Zeno's Paradox of Measure that begins by giving a substantive account, drawn from Aristotle's text, of the fact that points lack magnitude. The main elements of this account are 1) the Axiom of Archimedes which states that there are no infinitesimal magnitudes, and 2) the principle that all assignments of magnitude, or lack thereof, must be grounded in the magnitude of line segments, the primary objects to which the notion of linear magnitude applies. Armed with this account, we are ineluctably driven to introduce a highly constructive notion of (outer) measure based exclusively on the total magnitude of potentially infinite collections of line segments. The Paradox of Measure then consists in the proof that every finite or potentially infinite collection of points lacks magnitude with respect to this notion of measure. We observe that the Paradox of Measure, thus understood, troubled analysts into the 1880's, despite their knowledge that the linear continuum is uncountable. The Paradox was ultimately resolved by Borel in his thesis of 1893, as a corollary to his celebrated result that every countable open cover of a closed line segment has a finite sub-cover, a result he later called the "First Fundamental Theorem of Measure Theory." This achievement of Borel has not been sufficiently appreciated. We conclude with a metamathematical analysis of the resolution of the paradox made possible by recent results in reverse mathematics.

1 The Paradox Interpreted

The question posed in our title, or rather the quandary it represents, has been referred to as "Zeno's Paradox of Measure."[1] Its source is the following passage from Aristotle's *On Generation and Corruption.*

> For to suppose that a body (i.e. a magnitude) is divisible through and through, and that this division is possible, involves a difficulty [...] What will remain [after the division]? A magnitude? No, that is impossible, since then there will be something not divided, whereas *ex hypothesi* the body was divisible through and through. But if it be admitted that neither a body nor a magnitude will remain, and yet division is to take place, the constituents of the body will either be points (i.e. without magnitude) or absolutely nothing. If its constituents are nothings, then it might both come-to-be out of nothings and exist as a composite of nothings [...] But if it consists of points, a similar absurdity will result: it will not possess any magnitude.[2]

In this section we present a new interpretation of Zeno's Paradox of Measure. The main novelty of our interpretation derives from the fact that we provide a substantive gloss on the claim that points lack magnitude, a gloss that is implicit in Aristotle's text, and is rooted in the mathematical practice of the ancient world. On this basis, we are driven ineluctably to an interpretation of the Paradox that reveals a cogent argument, entirely accessible

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[1] See [Skyrms, 1983]. We will adopt this label throughout the paper. It has also been called "Zeno's metrical paradox" [Grünbaum, 1952], "Zeno's metrical paradox of extension" [Friedman, 2012], and the "Paradox of Infinite Divisibility" [Huggett, 2019].

[2] Aristotle, *On Generation and Corruption*, 316a15-30, trans. Joachim in [Barnes, 1984].

> Ἔχει γὰρ ἀπορίαν, εἴ τις θήσει σῶμά τι εἶναι καὶ μέγεθος πάντη διαιρετόν, καὶ τοῦτο δυνατόν. [...] Τί οὖν ἔσται λοιπόν; μέγεθος οὐ γὰρ οἷόν τε· ἔσται γάρ τι οὐ διῃρημένον, ἢν δὲ πάντη διαιρετόν. Ἀλλὰ μὴν εἰ μηδὲν ἔσται σῶμα μηδὲ μέγεθος, διαίρεσις δ' ἔσται, ἢ ἐκ στιγμῶν ἔσται, καὶ ἀμεγέθη ἐξ ὧν σύγκειται, ἢ οὐδὲν παντάπασιν, ὥστε κἂν γίνοιτο ἐκ μηδενὸς κἂν εἴη συγκείμενον [...] Ὁμοίως δὲ κἂν ᾖ ἐκ στιγμῶν, οὐκ ἔσται ποσόν. (Greek text from [Rashed, 2005].)

The argument comes in the context of Aristotle's refutation of Democritus' theory of atomic magnitudes. For a detailed treatment of Aristotle's overall strategy in this chapter, see [Sedley, 2004]. Although Aristotle does not explicitly attribute this paradox to Zeno, Simplicius, who had access to Zeno's work, does attribute it to him in his commentary *On Aristotle's Physics*, 139.20-24 ([Simplicius, 2011]).

within the framework of ancient mathematics, that every potential infinity of points lacks magnitude. We will see that this very argument was still considered paradoxical by mathematicians of the late-nineteenth century – mathematicians who were well-acquainted with the fact that the linear continuum is uncountable.

1.1 Points lack magnitude

We begin by addressing a critical element of the paradox: "But if it be admitted that neither a body nor a magnitude will remain, and yet division is to take place, the constituents of the body will either be points (i.e. without magnitude) or absolutely nothing." We do not pursue the alternative that "absolutely nothing" remains, though recent work suggests that this horn of the dilemma may admit of fruitful mathematical treatment.[3] Rather, we focus on the alternative that what remain are points, and in particular, on the parenthetical observation that points are "without magnitude." Our first task is to provide a suitable understanding of the sense in which points are without magnitude. This will lead us immediately to one of the fundamental contributions of ancient mathematics to the understanding of the continuum, and simultaneously take us some way into the nexus of concepts that underlie our interpretation of the Paradox.

Without loss of generality, we will focus on linear magnitudes, so for us, the paradigmatic body is a line segment, and its magnitude is its length.[4] Why is it that points lack magnitude? The key to answering this question is implicit in the passage quoted above: "What will remain [after the division]? A magnitude? No, that is impossible, since then there will be something not divided, whereas *ex hypothesi* the body was divisible through and through." We understand the significance of this rhetorical interchange as follows. First, it is apparent that

1 (***Covering Principle***): *the magnitude of a point p is no greater than the magnitude of any line segment* l *on which it lies,*

and that

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[3] See [Lando and Scott, 2019].

[4] Thus, we restrict our project to interpreting, and resolving, the paradox understood as a *mathematical* antinomy.

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2 *every point p lies on some line segment* l *of unit length.*[5]

Next, we understand the divisibility of a line segment l "through and through," to imply that

3 *(**Iterated Bisection Principle**): for every positive integer n,* l *may be successively bisected up to n-times.*

It follows at once that

4 *if a point p lies on a line segment* l *of unit length, then for every positive integer n, p lies on a sub-segment* J *of* l *of magnitude* $1/2^n$.

It now follows from 1, 2, and 4 that

5 *for every point p and every positive integer n, the magnitude of p is at most* $1/2^n$.

There remains one further crucial step in order to conclude that p lacks magnitude. Namely, an axiom originally articulated by Eudoxus in the fourth century B.C.E.[6] but dubbed the 'Axiom of Archimedes' by Otto Stolz in the nineteenth century.[7]

6 *(**Axiom of Archimedes**): If M is a (non-zero) magnitude, then for some positive integer n,* $M > 1/n$.

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[5]When we say a point p lies on a line segment l, we require that p is not an endpoint of l, in other words, p is interior to l.

[6]There is no direct evidence from Eudoxus for this attribution, but there is little disagreement over the inductive evidence in its favor. Archimedes explicitly relies on Eudoxus for the 'method of exhaustion' in *On the Sphere and Cylinder*. On which topic, see Archimedes' précis to Dositheus: "For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus..." ([Heath, 2002], p. 2). Also, given that the method of exhaustion is essentially an alternative formulation of *Elements* V, Definition 4, and a crucial component of *Elements* X, Proposition 1, there can be no doubt that Archimedes is not the originator. Moreover, since Eudoxus spent time at Plato's Academy, it would not be implausible to suppose that his ideas reached Aristotle through the teachings of the Academy. Wilbur Knorr, for example, suggests that Eudoxus was active in Plato's Academy at roughly the same time as Aristotle ([Knorr, 1990], p. 318).

[7]See [Stolz, 1883]. Although his title for this axiom is something of a misnomer, we retain it.

This axiom is among the great treasures bequeathed to us by the mathematicians of ancient Greece. It is a fundamental property of the linear continuum: there are no infinitesimal magnitudes! We may now conclude that a point has no (positive) magnitude as follows. It is evident that

7 *for every positive integer n, $1/2^n$ is less than $1/n$.*

Hence, by 5, 6, and 7,

8 *for every point p, p has no (positive) magnitude, or, as we will say henceforth, p lacks magnitude.*

It is important to observe that at this point we have already made a crucial departure from many contemporary commentators on the Paradox of Measure:[8] *we insist on construing the assertion that points lack magnitude in terms of assertions about the magnitude of those geometric objects, namely line segments, to which the notion of magnitude first and foremost applies.* Surely, the ancients recognized that a point can have no greater magnitude than a line segment upon which it lies, and this recognition, together with the Iterated Bisection Principle and the Axiom of Archimedes, implies that a point has no positive magnitude.[9] Throughout our analysis and resolu-

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[8]Indeed, all commentators of whom we are aware.

[9]See, for example, *Physics* 215b12-20, trans. Hardie and Gaye in [Barnes, 1984]. Henceforth, and unless otherwise noted, translations of Aristotle's *Physics* are from Hardie and Gaye, occasionally modified for terminological consistency: "Now there is no ratio in which the void is exceeded by body, as there is no ratio of 0 to a number. For if 4 exceeds 3 by 1, and 2 by more than 1, and 1 by still more than it exceeds 2, still there is no ratio by which it exceeds 0; for that which exceeds must be divisible into the excess and that which is exceeded, so that 4 will be what it exceeds 0 by and 0. For this reason, too, a line does not exceed a point—unless it is composed of points."

> τὸ δὲ κενὸν οὐδένα ἔχει λόγον ᾧ ὑπερέχεται ὑπὸ τοῦ σώματος, ὥσπερ οὐδὲ τὸ μηδὲν πρὸς ἀριθμόν. εἰ γὰρ τὰ τέτταρα τῶν τριῶν ὑπερέχει ἑνί, πλείονι δὲ τοῖν δυοῖν, καὶ ἔτι πλείονι τοῦ ἑνὸς ἢ τοῖν δυοῖν, τοῦ δὲ μηδενὸς οὐκέτι ἔχει λόγον ᾧ ὑπερέχει· ἀνάγκη γὰρ τὸ ὑπερέχον διαιρεῖσθαι εἴς τε τὴν ὑπεροχὴν καὶ τὸ ὑπερεχόμενον, ὥστε ἔσται τὰ τέτταρα ὅσῳ τε ὑπερέχει καὶ οὐδέν. διὸ οὐδὲ γραμμὴ στιγμῆς ὑπερέχει, εἰ μὴ σύγκειται ἐκ στιγμῶν. (Greek text from [Ross, 1936].)

Aristotle's reason for claiming that there is no ratio between any whole number (ἀριθμός) and 0 (μηδέν), and by analogy that there is no ratio between a line (γραμμή) and a point (στιγμή), is a decisive invocation of the Archimedean proscription of infinitesimal elements. There is no n-th multiple of μηδέν such that μηδέν exhausts any ἀριθμός m. This

tion of the Paradox, we insist on what we call the **Primacy of Segment-Magnitude** as we introduce concepts that extend the applicability of the notion of magnitude to geometric objects other than line segments: these concepts are defined directly in terms of the lengths of (non-degenerate) line segments, and the conclusions we draw that various geometric objects lack positive magnitude relative to these concepts are derived by using the Iterated Bisection Principle and the Axiom of Archimedes in just the way we've applied these to conclude that points lack positive magnitude. In particular, *at no point do we argue that a geometric object lacks positive magnitude directly on account of the fact that it is an agglomeration of geometric objects that lack magnitude - we never engage in the exercise of summing zeroes.*[10] In the foregoing argument, the Covering Principle embodies our commitment to the Primacy of Segment-Magnitude. This approach allows us to retain close contact with the mathematical framework of the ancient world almost to the very resolution of the Paradox, indeed, as close to the resolution as the analysts of the late nineteenth century were able to approach, until Borel actually resolved it in 1893.[11]

1.2 What the Paradox Demands

The Paradox demands that we conclude that collections of points lack magnitude, based on the fact that points lack magnitude. Since (non-degenerate) line segments possess (positive) magnitude, this will allow us to conclude that line segments are not (composed of) collections of points. Just as we were able to give a cogent argument that points lack magnitude, based on the Iterated Bisection Principle and the Axiom of Archimedes, and under

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arithmetical application of the method of exhaustion pointedly recalls the iterated διαίρεσις on which the Paradox of Measure turns. Heath rightly connects Aristotle's reasoning in this passage to Definition IV in Book V of Euclid's *Elements* (*cf.* footnote 8 above), which is the Axiom of Archimedes in another guise ([Heath, 1970], pp. 116-117). *Cf.* [Heath, 1981], pp. 384-5 and *Elements* Book X, Proposition I. For additional evidence, see footnote 16 below.

[10]That is, we refrain from appealing to additivity properties of measures, or the failure thereof, in our interpretation of the Paradox or its resolution.

[11]Borel's resolution of the Paradox does not flow from his development of measure theory, but from his Finiteness Theorem, that predated this development and served as a *prolegomenon* thereto, as we discuss at length in Section 3.4. The Finiteness Theorem first appeared in his doctoral thesis, *Sur quelques points de la théorie des fonctions*, submitted in 1893, defended in 1894, and published a year later as [Borel, 1895].

the requirement that the notion of magnitude itself apply directly only to non-degenerate line segments, we will next extend that argument first to finite collections, and then to potential infinities, of points. In order to do so, we will generalize the Covering Principle which was the basis for our argument that individual points lack magnitude. To this end, it will be useful to introduce some terminology.

We say a collection of line segments Ξ *covers* a collection of points X just in case for every point p in X there is a line segment S in Ξ on which p lies. The *total magnitude* of a collection of line segments Ξ is the sum of the lengths of the line segments contained in Ξ. Our generalization of the Covering Principle may now be stated as follows.[12]

9 (*Covering Principle*): *The magnitude of a collection of points X is no greater than the total magnitude of any collection of line segments Ξ that covers X.*

1.2.1 Finite collections of points lack magnitude

The next proposition provides an important, if simple, application of the Covering Principle. We present the proof, because the summation of "finite geometric series" it involves is a useful prelude to our argument for Theorem 1.

Proposition 1 *If X is a finite collection of points, then X lacks magnitude.*

Proof: Let $X = \{p_1, \ldots, p_k\}$ be a finite collection of points. As discussed above, it follows from the Iterated Bisection Principle that for each $1 \leq j \leq k$, and each positive integer n, there is a line segment S_j such that p_j lies on S_j and the length of S_j is less than $1/2^{n+j}$.[13] Let Ξ be the finite collection of line segments $\{S_1, \ldots, S_k\}$. It is apparent by construction that Ξ covers X. Moreover, for each positive integer n, the total length of Ξ is the sum

$$1/2^{n+1} + \ldots + 1/2^{n+k}.$$

But,

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[12]We retain the name "Covering Principle" for this generalization of the earlier Principle from points to point-sets.

[13]Indeed, given p_j, we could construct S_j with midpoint p_j and length $1/2^{n+j}$.

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$$1/2^{n+1} + \ldots + 1/2^{n+k} = 1/2^n - 1/2^{n+k} < 1/2^n < 1/n,$$

a fact well-known to the ancients. Therefore, the Covering Principle together with the Axiom of Archimedes and 10 allow us to conclude that X lacks magnitude. ∎

1.2.2 Potential infinities of points lack magnitude

In order to extend Proposition 1 to potential infinities of points, we will need to explicate the notion of potential infinity. We may think of a potential infinity as a process of construction, any stage in the execution of which may be succeeded by another stage. For example, the collection of even positive integers can be understood as a potential infinity insofar as it may be generated by the process which begins with 2 and at each succeeding stage, adds 2 to the number realized at the stage preceding it. This description reduces the notion of potential infinity to two primitives: first the notion of a process of construction, and second the notion of its stage-wise execution.[14] The central point is the following: *a potentially infinite collection is not thought of as a "completed infinite totality," indeed, it is hardly to be thought of as a collection at all; it is rather given by a process, a finitely describable effective mode of generation, that can be applied at an ever finite, but inexhaustible, or limitless sequence of stages, to generate new instances.* Of course, this description of a potentially infinite collection lacks the precision required of a modern mathematical definition. This is hardly surprising, since its intent is to characterize an intuitive notion current in ancient times. Certainly, the wealth of ancient mathematical practice accords very well with this characterization. Whenever one comes across an infinite sequence of points in the context of an ancient geometrical argument, it is given by an explicit construction; moreover, the intent of the construction is to generate finite initial segments of any given length – the construction is limitless (ἄπειρον).[15] We

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[14]From a contemporary perspective, it is virtually irresistible to identify a process with an algorithm, and its stage-wise execution with computations via this algorithm on numerals representing successive positive integers, thereby identifying the notion of potentially infinite set with the notion of computably enumerable set. For our purposes, we need not make any such identification.

[15]We offer one explication of the notion of potential infinity that we regard as particularly fruitful in understanding the Paradox of Measure. We are of course aware that this notion has received extensive treatment in the literature. See, for example, Lear's authoritative

now discuss in detail a paradigmatic geometric example of potential infinity invoked earlier, the points and line segments generated by iterated bisection of a line segment of unit length. This example will serve both to illustrate the concept of potential infinity, and to provide the basis for substantive developments in later sections.

The Binary Ruler   Recall the Iterated Bisection Principle: given a line segment I, for every positive integer n, I may be successively bisected up to n-times. The process of iterating bisection is the paradigm of potential infinity in the realm of geometry. Let I be a line segment of unit length, with endpoints labelled 0 and 1. The process of iterated bisection may be applied to it without limit. After a finite number of applications, say n, it yields what we call the n-th *Binary Ruler*, B_n, a partition of the segment I into 2^n subsegments, each of length $1/2^n$, with endpoints labeled $k/2^n, (k+1)/2^n$, for $0 \leq k \leq 2^n$. Since the process is without limit, the partition it generates may at any stage be further refined to one consisting of twice as many subsegments of I.

If we like, we may think of the process of iterated bisection as a means of generating either a sequence of points or a sequence of intervals. In the case of points, we begin by listing 0 and 1, the endpoints of I, at the first and second positions of our sequence, and then continue with the points labeled $1/2$, $1/4$, and $3/4$. At each stage in the execution of this process, we enumerate from left to right[16] those members of B_n that have yet to be enumerated at earlier stages. We call this process of listing points the *bisection point process* and write b_n for the point listed at the n-th position in the execution of this process. Thus, b_1, b_2, b_3, b_4 and b_5 are the points on I labeled $0, 1, 1/2, 1/4$ and $3/4$. *The bisection point process is itself a potential infinity of points - it is a process for constructing a limitless sequence of distinct points, that is, each stage in the execution of the process yields a finite sequence of points, all of which are labeled on a single binary ruler B_n, but every such stage may be succeeded by further stages that label points yet to be labeled on B_n.* A deeper understanding of our treatment of potential infinities will emerge from our argument that the bisection point process lacks magnitude.

account in [Lear, 1980].

[16]We imagine 0 to the left of 1 along the unit segment I.

The bisection point process lacks magnitude In light of the Covering Priniciple, in order to establish that the bisection point process lacks magnitude, we must show that

11 *for every n there is a collection of intervals Ξ_n such that Ξ_n covers the bisection point process, and the total length of Ξ_n is less than $1/n$.*

Of course, we must understand the requisite collection of intervals Ξ_n as potentially infinite in exactly the sense in which we regard the bisection point process as potentially infinite. For a fixed number n, we must describe a process Ξ_n so as to meet the requisite condition 11. Moreover, we will need to explicate the meaning of the phrase "the total length of Ξ_n is less than $1/n$" in application to a potential infinity of line segments Ξ_n.

   The process Ξ_n operates as follows. At the j-th stage of its execution, the process Ξ_n generates a line segment of length $1/2^{n+j}$ with midpoint b_j, the point labeled at the j-th stage in the execution of the bisection point process. It is apparent that the potential infinity of line segments Ξ_n covers the binary point process. Moreover, by 10,

12 *for every stage k, the total length of the line segments generated by the process Ξ_n through stage k is equal to $1/2^n - 1/2^{n+k}$, and is thus strictly less than $1/2^n$, which is strictly less than $1/n$ for every n.*

Now 12 is exactly what is required to conclude that the total length of the potential infinity Ξ_n is less than $1/n$. Why? Because our understanding of potential infinities is based on facts about the finite stages of their execution. Thus, in order to assert that the total length of a potential infinity Ξ of line segments is strictly less than some value a, it is both necessary and sufficient to establish that for some value b strictly less than a, and for every stage k, the sum of the lengths of the line segments generated by the execution of Ξ through stage k is less than b. Thus, it follows at once from 12 that for each n, the total length of Ξ_n is strictly less than $1/n$. In other words, for every n, there is a potential infinity of line segments that covers the bisection point process and has total length less than $1/n$. As before, we may conclude by the Axiom of Archimedes that the bisection point process lacks magnitude.[17]

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[17]The argument here is implicit in Aristotle's observation that the sum of a geometric series is finite which can be found in *Physics*, 206b4-12: "In a way the infinite by addition is the same thing as the infinite by division. In a finite magnitude, the infinite by addition comes about in a way inverse to that of the other. For just as we see division going on

It is evident that our argument can be applied to any potential infinity of points - there is nothing special about the bisection point process in this regard. Given any process for generating successive points, we may construct line segments Ξ_n in the same way as above that cover these points and have total length less than $1/2^n$. We may thus conclude

Theorem 1 *Every potential infinity of points lacks magnitude.*

1.3   The Paradox

Our interpretation places Theorem 1 at the heart of the Paradox of Measure. Its proof, which is entirely intelligible within the framework of ancient mathematics, shows that if a line segment is composed of points, then it is *not* potentially infinite, since every line segment possesses magnitude,[18] while every potential infinity of points lacks magnitude. Insofar as the ancients would have found it difficult to conceive of a collection of points as neither finite nor potentially infinite, so far would they have rejected the claim that a line segment is constituted out of points.

Even apart from scruples about actual completed infinities of points, Theorem 1 might create considerable puzzlement via another route, again one easily traversed by mathematicians of the ancient world. In this instance,

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ad infinitum, so we see addition being made in the same proportion to what is already marked off. For if we take a determinate part of a finite magnitude and add another part determined by the same ratio (not taking in the same amount of the original whole), we shall not traverse the given magnitude. But if we increase the ratio of the part, so as always to take in the same amount, we shall traverse the magnitude; for every finite magnitude is exhausted by means of any determinate quantity however small."

> τὸ δὲ κατὰ πρόσθεσιν τὸ αὐτό ἐστί πως καὶ τὸ κατὰ διαίρεσιν· ἐν γὰρ τῷ πεπερασμένῳ κατὰ πρόσθεσιν γίγνεται ἀντεστραμμένως· ᾗ γὰρ διαιρούμενον ὁρᾶται εἰς ἄπειρον, ταύτῃ προστιθέμενον φανεῖται πρὸς τὸ ὡρισμένον. ἐν γὰρ τῷ πεπερασμένῳ μεγέθει ἂν λαβών τις ὡρισμένον προσλαμβάνῃ τῷ αὐτῷ λόγῳ, μὴ τὸ αὐτό τι τοῦ ὅλου μέγεθος περιλαμβάνων, οὐ διέξεισι τὸ πεπερασμένον· ἐὰν δ' οὕτως αὔξῃ τὸν λόγον ὥστε ἀεί τι τὸ αὐτὸ περιλαμβάνειν μέγεθος, διέξεισι, διὰ τὸ πᾶν πεπερασμένον ἀναιρεῖσθαι ὁτῳοῦν ὡρισμένῳ.

The connection between this passage and the Axiom of Archimedes has been noted by commentators. See [Heath, 1921], pp. 342-3; [Ross, 1936], p. 556; [Hussey, 1983], p. 84. Indeed, this passage lends considerable support to our approach, insofar as it emphasizes the role of the Axiom of Archimedes in explicating the fact that points lack magnitude.

[18]One may view this principle as yet another manifestation of the Axiom of Archimedes.

the focus is an important property of the bisection point process itself, which appears to conflict with the fact that it lacks magnitude.

Theorem 2 *Let I be a line segment of unit length, and let J be a line segment whose endpoints lie on I. Then, for some n, some point marked on the Binary Ruler B_n lies on the line segment J.*

Proof: Let J be a line segment whose endpoints a and b lie on the unit segment I, with a to the left of b. By the Axiom of Archimedes, we may choose n such that the length of J is greater than $1/n$. Let j be the greatest number such that the point on the Binary Ruler B_n marked $j/2^n$ is to the left of a, or coincides with a; such a j exists by the Axiom of Archimedes. Since $1/2^n < 1/n$, it follows at once that the point on the Binary Ruler B_n marked $(j+1)/2^n$ lies on the line segment J. ∎

Theorem 2 implies that the bisection point process is *dense in the unit line segment*; that is, for any two points on the unit line segment, there is a point generated by the bisection point process that lies between them. The ancients certainly understood that this was the case. Imagine how puzzling it must have seemed that, nonetheless, this potential infinity of points lacks magnitude - puzzling to the point of paradoxical. How could a line segment with magnitude be composed of points, if it can be shown that a potential infinity of points dense in such a segment lacks magnitude!? Even those with little concern for potential versus completed infinities might well find such a result troubling.[19]

2   The Paradox Reformulated

To this point we have provided a novel interpretation of the Paradox of Measure. It consists in a cogent mathematical proof, using concepts and techniques readily intelligible to students of mathematics in Plato's Academy, that every potential infinity of points, including such that are dense in a line segment of unit length, lack magnitude. Our argument involves no mysterious summation of infinities of zeroes. Quite the contrary, it provides a finitary understanding of the claim that points lack magnitude, an understanding that forces itself upon us as we reflect on Aristotle's text in the context of the mathematics of his day, in particular, the Axiom of Archimedes. We

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[19] As did analysts of the 1880's. See Section 3.3.

then generalize this understanding to collections of points in a way that exploits only finitary properties of potential infinities of non-degenerate line segments. The Covering Principle is the crucial ingredient: "the magnitude of a collection of points X is no greater than the total magnitude of any collection of line segments Ξ that cover X." Insofar as the intuitive notion of magnitude, in the one dimensional case, applies exclusively to non-degenerate line segments, it is the Covering Principle that allows us to attach significance to claims concerning the lack of magnitude of collections of points from which line segments may, or may not, be composed. As we proceed toward the resolution of the Paradox, it will be useful to recast this role of the Covering Principle via the definition of an alternative notion of magnitude that applies to collections of points.

2.1 Ancient-Measure

As we have emphasized, the intuitive notion of magnitude, in the case of one dimension, applies primarily, if not exclusively, to line segments. This is why we have insisted on presenting an explicit gloss on the assertion that points lack magnitude which is couched entirely in terms of the application of the notion of magnitude to (non-degenerate) line segments. The Covering Principle permits us to extend this gloss to collections of points, and indeed allows us to infer that every potential infinity of points lacks magnitude. But if we wish to allow for the possibility that line segments are composed of points, with a view toward resolving the Paradox, we need to provide a gloss on the assertion that a collection of points has (strictly positive) magnitude.

In this section, we introduce a notion of *ancient-measure* that allows us to attach a magnitude to arbitrary collections of points.[20] It is an ancient variant[21] of a notion first considered by Axel Harnack.[22] The definition of this notion will again be couched entirely in terms that apply the notion of magnitude to (non-degenerate) line segments. If Ξ is a potential infinity of line segments, and c is the length (that is, magnitude) of some line segment,

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[20]Of course, we are only interested in its application to finite and potentially infinite collections of points, and to line segments, insofar as they might legitimately be regarded as collections of points. The notion of an arbitrary collection of points only developed through the work of nineteenth-century mathematicians, and is thus not germane to our interpretation of the Paradox of Measure, nor even to its resolution.

[21]It is "ancient" for deploying potentially infinite, rather than countably infinite, covers.

[22]See [Harnack, 1885]. We introduce Harnack's notion below in Definition 5.

we say that the total length of Ξ exceeds c if and only if for some n the sum of the lengths of the line segments generated by Ξ through stage n exceeds c.[23]

Definition 1 *Let X be a collection of points, and let c be a magnitude. We say X has* ancient-measure c *if and only if*

1. *for every potential infinity of line segments Ξ, if Ξ covers X, then the total length of Ξ exceeds c, and*

2. *for every $n > 0$ there is a potential infinity of line segments Ξ such that Ξ covers X and the total length of Ξ is less than $c + 1/n$.*

First, note that, by Definition 1, a collection of points X has ancient-measure 0 if and only if X lacks magnitude in the sense heretofore explicated through the use of the Covering Principle as elaborated in the application given by 11. That is,

13 *X lacks magnitude if and only for every $n > 0$, there is a potential infinity of line segments Ξ such that Ξ covers X and the total length of Ξ is less than $1/n$.*

For convenience of future reference, we restate Theorem 1 in terms of the notion of measure as follows.

Theorem 3 *The ancient-measure of every potential infinity of points is zero.*

2.2   The Paradox Formulated in Terms of Ancient-Measure

We may now reformulate our interpretation of the Paradox as the following argument that a line segment is not composed of a collection of points.

1. The magnitude of a collection of points is its ancient-measure.

2. Every collection of points is either finite or potentially infinite.

3. The ancient-measure of every finite or potentially infinite collection of points is zero.

4. The magnitude of a line segment is its length, and is thus non-zero.

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[23]Note that we admit the case that $c = 0$, the magnitude of a "degenerate" line segment.

5. A line segment is not composed of a collection of points.

Claim 1 is the central thesis of our interpretation. We review the main elements in our argument for this claim. First, when we extend the notion of magnitude from line segments, which are intuitively its only object, to collections of points, we must define this extension in terms of the magnitude of line segments. How else? Moreover, our definition must assign magnitude zero to collections consisting of a single point, since it is clear from our text that this is required for a correct interpretation of the Paradox. Moreover, the extended notion should assign magnitude 0 to those collections of points familiar to mathematicians of the ancient world, such as the bisection point process, else it is obscure why the argument would be a source of puzzlement at all. Finally, the mathematical and logical resources deployed in the definition should be accessible within the framework of ancient mathematics.[24] All these desiderata are satisfied by the highly constructive notion of ancient-measure we have defined.[25] We take claim 2 as an expression of the fundamentally constructive approach of the mathematicians and philosophers of the ancient world to the notion of infinity. Claim 3 is Theorem 3, a result whose highly constructive proof is easily accessible within the framework of ancient mathematics. Claim 4 is the fundamental principle that the magnitude of a line segment is its length; the fact that this length is strictly greater that $1/n$ for some n, and is thus non-zero, is a corollary to the Axiom of Archimedes. The conflict between the intuitive notion of magnitude applied to line segments and the constructed notion of magnitude of collections of points expressed in Claim 5 is the essence of the Paradox as we interpret

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[24]We would like to thank Jeremy Avigad for pointing out that there is a significant gap between the mathematical resources necessary to articulate the definition of ancient-measure in the case that $c = 0$ and $c > 0$. In particular, the requirement expressed in Definition (1.1) is trivial to verify in the case that $c = 0$. Thus, in our argument for Claim 3, we only needed to provide a substantive verification of the requirement expressed in Definition (1.2). On the other hand, it is obscure that the requirement expressed in Definition (1.1), in the case $c > 0$, could have been grasped by the ancients, insofar as its verification in a particular case would involve refuting the existence of a potentially infinite cover of total length less than some strictly positive c. It is exactly such verification, in application to potentially infinite covers of a non-degenerate line segment, that would be required to resolve the Paradox as we understand it. Thus, from our point of view, this represents a fundamental conceptual obstacle to the resolution of the Paradox in antiquity.

[25]The forthcoming resolution of the Paradox will provide further, indeed compelling, justification for identifying the magnitude of a collection of points with its ancient-measure.

it.[26]

3 The Paradox Resolved

Borel resolved the Paradox of Measure in his doctoral dissertation, submitted in 1893, defended in 1894, and published in 1895.[27] Borel's significant contribution to the history of ideas in this respect has so far gone unnoticed. In order to appreciate the depth and novelty of his fundamental contribution, we need to consider some developments in mathematical analysis in the 1880's that provide the immediate context for his work. We begin with the notion of *content*.

3.1 Content

The concept of content was first defined by Otto Stolz in 1884 and was discovered independently by other analysts shortly thereafter, among them Cantor.[28] We will adopt the canonical definition given by Axel Harnack in 1885.[29]

Definition 2 *Let X be a collection of points, and let c be a magnitude. We say X has* content *c if and only if*

1. *for every finite collection of line segments Ξ, if Ξ covers X, then the total length of Ξ exceeds c, and*

2. *for every $n > 0$ there is a finite collection of line segments Ξ such that Ξ covers X and the total length of Ξ is less than $c + 1/n$.*

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[26]The astute reader will have noticed that the potential infinity of intervals Ξ_2 deployed in our proof that the bisection point process lacks magnitude, already witnesses that the ancient-measure of the bisection point process, and by extension, any potential infinity of points, has ancient-measure at most $1/2$, which is already paradoxical on our interpretation. We would like to thank Henry Towsner for this observation.

[27][Borel, 1895]

[28]See [Stolz, 1884], [Cantor, 1884], and [Hawkins, 2001], pp. 61-66.

[29]Harnack uses the term "Inhalt" for the notion defined here (see [Harnack, 1885]). It is now generally referred to as outer-content in texts on analysis (see, for example, [Bressoud, 2008]) to distinguish it from related notions that were introduced during the development of the theory of measure and integration. Since we will make no use of these other notions, we retain the simplicity of Harnack's terminology.

The definition of content, so far as we are aware, represents the first attempt to introduce a concept that extends the notion of magnitude to collections of points more general than line segments. Of course, the concept of ancient-measure introduced above does just this; but this notion (though within the conceptual ambit of mathematicians of the ancient world, as argued above) was not actually articulated before 1884. Content is a simpler notion than ancient-measure, insofar as it is defined in terms of *finite*, rather than potentially infinite, covers. Readers may confirm their grasp of Definition 2 by establishing the following proposition.[30]

Proposition 2 *If X is a finite collection of points, then the content of X is 0.*

Throughout the 1870's and 1880's analysts struggled to determine the extent to which topological notions could explain phenomena in the theory of integration.[31] Part of this effort was devoted to determining the relationship between such topological notions and the notion of content, the progenitor of "measure-theoretic" concepts. The following result of Cantor[32] confirmed the intuition that collections of points that are "large" in a topological sense, should also be "large" in a measure-theoretic sense.

Definition 3 *A collection of points X is* dense *in a line segment* I *if and only for every sub-segment* J *of* I*, there is a point in X that lies on* J*.*

Proposition 3 *Let X be a collection of points that lie on the line segment* I*. If X is dense in* I*, then the content of X is equal the length of* I*.*

The reader can verify that the elementary poof of this proposition, given in Appendix A, would present no challenge to a student of mathematics in Plato's Academy.

Analysts of this period also sought to show that collections of points that are "negligible" in a topological sense are also "negligible" in terms of magnitude. An important concept of topological negligibility they studied was *nowhere-denseness*.

Definition 4 *A collection of points X is* nowhere-dense *in a line segment* I *if and only if for every sub-segment* J *of* I*, there is a sub-segment* K *of* J*, such that no point in X lies on* K*.*

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[30]The proof of Proposition 2 is essentially the same as that of Proposition 1.

[31]Chapter 4 of [Hawkins, 2001] presents a riveting account of this struggle.

[32]See [Cantor, 1883].

A result of Cantor lent some credence to the idea that nowhere-density might guarantee negligible magnitude. He constructed an example of a collection of points, $\mathcal{C} \subset [0, 1]$, now known as the Cantor discontinuum, that is nowhere-dense in $[0, 1]$; moreover, the content of $\mathcal{C}$ is 0, despite the fact that its cardinality is the same as that of the continuum.[33] This appeared to confirm the idea, popular at the time, that magnitude should be intimately connected to topological notions, and should have naught to do with cardinality. After all, the countable set of binary rational numbers is dense in $[0, 1]$, and therefore, by Proposition 3 has content 1, while the uncountable nowhere-dense set $\mathcal{C}$ has content 0.[34]

3.2   Modern-Measure

In the same paper in which he defined the notion of content, Harnack toyed with an alternative definition of a notion of magnitude for a collection of points X; in place of finite covers of X, he proposed to consider *countably infinite* covers of X. We formulate this notion as follows.

Definition 5 *Let X be a collection of points, and let c be a magnitude. We say X has* modern-measure c *if and only if*

1. *for every countable infinity of line segments Ξ, if Ξ covers X, then the total length of Ξ exceeds c, and*

2. *for every $n > 0$ there is a countable infinity of line segments Ξ such that Ξ covers X and the total length of Ξ is less than $c + 1/n$.*

The notion defined here is generally referred to as *outer-measure*.[35] Since we will have no occasion to refer to other notions of measure, we prefer the term modern-measure, to outer-measure, for the contrast with *ancient-*

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[33]Cantor attached considerable significance to this result. See, for example, his letter to Mittag-Leffler of November 26, 1883, [Meschkowski and Nilson, 1991], p. 151.

[34]Indeed, Hankel purported to prove that every nowhere-dense collection of points has content 0 (see [Hawkins, 2001], p. 167). Hankel's "result" had been anticipated by Dirichlet in 1829. Alas, this simple connection between topological notions and magnitude proved to be illusory. Smith, and then Volterra, constructed nowhere-dense sets with content greater than 0 ([Hawkins, 2001], p. 169).

[35]See [Oxtoby, 1996], p. 10.

measure.[36] To the best of our knowledge, Harnack[37] was the first to entertain the concept of modern-measure. Note that the definition of modern-measure is virtually identical to the definition of ancient-measure, except for the crucial detail that it enlists *countably infinite collections of line segments* rather than "potentially infinite collections of line segments." A countably infinite collection is one that can be enumerated by an arbitrary function with domain the positive integers. That is, the enumeration is regarded as an actually infinite extension and there is no requirement that it be given by a process of construction of any kind, or even that it be definable in any terms whatsoever. This notion of arbitrary enumeration was introduced into mathematics only in the nineteenth century and emerged slowly through work of Dirichlet, Riemann, Cantor, and Dedekind, among others.[38] As is clear from our explication of the notion of a potentially infinite collection, every such collection is countably infinite. The following proposition, which we record here for later use, is a corollary to this observation.

Proposition 4 *Let X be a collection of points. The modern-measure of X is no greater than the ancient-measure of X, and the ancient-measure of X is no greater than the content of X.*

3.3 The Persistence of the Paradox of Measure

As it happens, Harnack introduced the notion of modern-measure only to reject it immediately on the grounds that it was not a mathematically fruitful extension of the notion of magnitude to collections of points. Why? Because he recognized that the notion would yield paradoxical consequences, if thus deployed. Hawkins describes Harnack's assessment as follows: "...Harnack observed that if in the definition of outer content the restriction to a finite

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[36]It is worth remarking that the modern-measure of X is identical to the Lebesgue measure of X, for every set X that is Lebesgue measurable. (A collection of points X is *Lebesgue measurable* if and only if for every $n > 0$ there is a closed set C and an open set O such that $C \subseteq X \subseteq O$ and the difference between the modern-measure of O and the modern-measure of C is less than $1/n$.) It is also worth remarking that every collection of points that is relevant to our discussion of the Paradox of Measure, that is, line segments and potentially infinite collections, *is* Lebesgue measurable. From our point of view, the existence of non-Lebesgue-measurable sets is a twentieth-century curiosity that has no direct relevance to Zeno's Paradox of Measure or its resolution.

[37]See [Harnack, 1885].

[38]See [Ferreirós, 1999], Section V.1.

number of covering intervals is dropped, there is a remarkable, paradoxical consequence: every countable set ...would have zero 'outer content'."[39] Hawkins elaborates on Harnack's understanding of this "paradoxical consequence" and further suggests that Cantor was of like mind.

> To Harnack the implication of these observations was clear. They revealed the crucial importance of the restriction to a finite number of covering intervals in the definition of outer content. The idea that countable sets should have zero content appeared paradoxical to him because countable sets could be dense. For example, the set E of rational numbers in $[0, 1]$ is countable and dense. Because it is dense [its content is] 1, not 0. This seemed the appropriate measure of E by virtue of its ubiquitousness. It appeared absurd to regard a dense set as extensionless, as of negligible measure. Cantor, who had introduced the notion of a countable set, certainly shared Harnack's viewpoint.[40]

Both Harnack and Cantor rejected the concept of modern-measure as an explication of the notion of magnitude of a collection of points owing to the air of paradox surrounding it - how can a set dense in the linear continuum have content 0?! *This is exactly the heart of Zeno's Paradox of Measure as we interpret it.*

But there is more. Harnack finds yet another troubling difficulty with the concept of modern-measure - he purports to prove that with respect to this notion, the magnitude of the unit interval on the real line is strictly less than 1.[41] This is the *coup de grâce*. The modified notion of content could not possibly be a useful notion of magnitude, if it assigns a line segment a magnitude other than its length! Of course, this last conclusion is again at the heart of our interpretation of the Paradox of Measure. As we will see in Section 3.5, Harnack's "proof" that the modern-measure of the unit interval is less than 1 is fallacious. At this point, we wish to emphasize that the air of paradox surrounding the Paradox of Measure, as we interpret it, persisted into the 1880's; in the same way it had baffled thinkers of the ancient world,

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[39][Hawkins, 2001], p. 172. Harnack's argument for this conclusion is essentially the same as that given in the proof of Theorem 1, except that he had no need for the care we have taken to observe that the ever shrinking collections of covering segments for a potentially infinite collection of points can themselves be constructed to be potentially infinite.

[40][Hawkins, 2001], p. 172.

[41]See [Bressoud, 2008], p. 63.

it befuddled mathematicians who were entirely familiar with Cantor's proof that the linear continuum is uncountable.[42]

3.4 Enter Borel

The resolution of the Paradox of Measure hinges on a single fundamental result concerning the structure of the linear continuum established by Émile Borel in his doctoral dissertation of 1893. Though Borel is famous as a pioneer in establishing measure theory, his dissertation preceded that development, and focussed on the solution to a problem in complex analysis concerning the analytic continuation of a class of functions of a complex variable that had preoccupied Henri Poincaré and several other mathematicians for more than a decade.[43]

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[42]The following quotation from Cantor's letter to Paul Tannery dated October 5, 1888, [Meschkowski and Nilson, 1991], pp. 323–5, trans. A. Newstead, [Newstead, 2001], suggests the intriguing possibility that Cantor may have anticipated the Finiteness Theorem of Borel discussed below.

> You are right to point out that, I so to speak, renew the Pythagorean view, insofar as I teach that the geometrical continuum is a real compound of separate points, geometrical individuals, just as a forest is composed out of trees, but because the Pythagoreans understood the continuum as a sum of points, [a view] which is powerless against the demonstrations of Zeno of Elea, I take the continuum to be a point set (ensemble of points) of a more definite, precisely specified nature. My grasp of the geometrical (and temporal) continuum is one which harmoniously combines the advantages of the Aristotelian view with what is true in the Pythagorean way of understanding, so that there will be no Zeno waiting for me who will demonstrate any kind of contradiction whatsoever in my most well-considered concept of the continuum.

As mentioned earlier, Cantor's construction of a set of content zero equipollent to the linear continuum made it clear that the resolution of the Paradox of Measure did not lie in the cardinality of an interval. But his result, Proposition 3 above, tells us immediately that if one could establish that the measure of a line segment is equal to its content, then the Paradox would be overcome. We have no direct evidence that Cantor actually recognized this finiteness result, though he made implicit use of the compactness of Euclidean n-space in arguments advanced in [Cantor, 1884]; see [Hawkins, 2001], p. 62. Nonetheless, it is exactly this deep property of the linear continuum that allows for a harmonious combination of "the advantages of the Aristotelian view with what is true in the Pythagorean way of understanding."

[43]See [Hawkins, 2001], pp. 97-106 for a detailed description of the problem and Borel's contribution to its resolution. It is worthy of note that Poincaré himself was one of the

Borel's pivotal result made its appearance modestly, in a footnote to one of the main arguments in the thesis. In the course of this argument, Borel needed to show that a certain set of points X lying on a line segment I had a non-empty complement in I, that is, the collection of points $Y = \mathsf{I} - X \neq \emptyset$. In order to do so, he first established that the collection of points X could be covered by countably many intervals of total length strictly less than the length of I. Borel drew this conclusion

> based on a theorem interesting by itself . . . : If one has an infinity of subintervals on a line such that every point of the line is interior to at least one of them, a finite number of intervals chosen from among the given intervals can be effectively determined having the same property.[44]

3.4.1 Borel's Finiteness Theorem: "The First Fundamental Theorem of Measure Theory"

We formulate Borel's Finiteness Theorem as follows.[45]

Theorem 4 (Borel, 1898) *Suppose the countable infinity of line segments* $\mathsf{I}_1, \mathsf{I}_2, \ldots$ *covers the line segment* I*, that is, every point of* I*, including its endpoints, lies on the interior of at least one of these segments. Then there is a positive integer k, such that the finite collection of line segments* $\mathsf{I}_1, \ldots, \mathsf{I}_k$ *covers* I*.*

Borel's Finiteness Theorem is the key to the resolution of the Paradox of Measure. Two decades later, Borel referred to this result as the "First Fundamental Theorem of Measure Theory,"[46] because it has the immediate corollary that the modern-measure of a line segment is its length.[47] Though in the

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rapporteurs for Borel's thesis.

[44][Borel, 1895], p. 51 cited in [Hawkins, 2001], pp. 101, fn. 9.

[45]The result, and its generalizations, are often referred to as the Heine-Borel Theorem, though it is widely recognized that this is a misnomer, since Heine neither stated nor proved any such result.

[46]The appellation "Le premiere théorème fondamental" first appears in the second edition (1914) of *Leçons sur la Théorie des Fonctions* in a lengthy note to the first edition (1898) and is reprinted in [Borel, 1950], p. 223. In this note, Borel explains the significance of the result in establishing that his approach to assigning a measure to (what we now call) the Borel sets is well-defined.

[47]See Section 3.5, Corollary 1.

foregoing quotation from [Borel, 1895], Borel states his theorem for infinite covers, in point of fact, he only proved it for *countably* infinite covers, and in the first edition of his *Leçons sur la Théorie des Fonctions*, published only three years later, he states the result with this restriction.[48] The enormous significance of Borel's Finiteness Theorem was recognized by the mathematicians of his day, as is obvious from their efforts to produce novel proofs.[49] We include a proof of Borel's Finiteness Theorem, to highlight the mathematical resources upon which it draws. This will also prepare the way for the meta-mathematical reflections in Section 4.

3.4.2   A Proof of Borel's Finiteness Theorem

It is well-known that mathematicians of the nineteenth century achieved an understanding of the linear continuum that enabled a new level of rigor in arguments in analysis. In 1872 Cantor and Dedekind each presented constructions of the continuum of real numbers – Cantor's in terms of Cauchy sequences of rational numbers, and Dedekind's in terms of 'cuts' in the ordering of the rational numbers.[50] These constructions allowed mathematicians to present entirely rigorous treatments of the operations with limits that lay at the heart of the differential calculus. Many would credit these developments with depriving Zeno's paradoxes of motion of their force.[51]

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[48] See [Borel, 1898], p. 42. We adopt the formulation for countably infinite covers, since this is all that is required for the resolution of the Paradox of Measure. Borel's error in claiming the stronger result created some confusion, even among mathematicians of the stature of Lebesgue, who studied [Borel, 1895], and applied the Theorem in his 1901 thesis in an argument that required the result for uncountable covers. When Lebesgue realized that Borel had only established the result for countable covers, he gave a proof, published in 1904, for the case of arbitrary open covers. As it happens, Pierre Cousin had proved a version of the two-dimensional case of Borel's Finiteness Theorem for arbitrary open covers in 1895! Additional proofs of the Theorem were given by Schoenflies in 1900 and Young in 1902. It is Schoenflies who first, mistakenly, attributed the result to Heine, based on the similarity of methods of his own proof with Heine's proof of the significant result that a continuous function on a closed interval is uniformly continuous (a proof Heine apparently pirated from Dirichlet without attribution). *Cf.* [Lebesgue, 1904] and [Andre et al., 2013].

[49] See [Andre et al., 2013] and [Hildebrandt, 1926] for detailed discussions of the history and mathematics of Borel's Finiteness Theorem.

[50] See [Cantor, 1872] and [Dedekind, 1996].

[51] For example, [Salmon, 1980], p. 35:

> We should begin by noting that, although the calculus was developed in the seventeenth century, its foundations were beset with very serious logical

A central feature of Dedekind's approach was to provide an explicit understanding of the sense in which a line is *continuous*, and a construction of the linear continuum that would enable a perspicuous proof of its continuity in this sense. The ancients' understanding of continuity, though less explicit, appears quite close to Dedekind's. Aristotle captures the essence of the notion of continuity that characterizes the linear continuum as follows.

> I call something 'continuous' whenever the limit of both things at which they touch becomes one and the same.[52]

In comparison, the following property characterizes the "essence of continuity," according to Dedekind.

Dedekind Cut Property: "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."[53]

The following property, also with ancient pedigree, is a consequence of the Dedekind Cut Property; it will prove useful in the proof of Borel's Finiteness Theorem.

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difficulties until the nineteenth century – when Cauchy clarified such fundamental concepts as functions, limits, convergence of sequences and series, the derivative, and the integral; and when his successors Dedekind, Weierstrass, et al., provided a satisfactory analysis of the real number system and its connections with the calculus. I am firmly convinced that Zeno's various paradoxes constituted insuperable difficulties for the calculus in its pre-nineteenth-century form, but that the nineteenth-century achievements regarding the foundations of the calculus provide means which go far toward the resolution of Zeno's paradoxes [of motion].

[52]Aristotle, *Physics*, 227a11-12.

λέγω δ' εἶναι συνεχὲς ὅταν ταὐτὸ γένηται καὶ ἓν τὸ ἑκατέρου πέρας οἷς ἅπτονται.

Commentators recognize this passage as central to understanding Aristotle's conception of the linear continuum. See, for example, [White, 1988].

[53]See [Dedekind, 1996], Section 3. [Stein, 1990] emphasizes the similarity between Aristotle's and Dedekind's formulations.

Nested Interval Completeness: Suppose that $\{I_1, I_2, \ldots\}$ is a countably infinite collection of nested line segments, that is,

(N1) each succeeding segment is a subsegment of its predecessor, and

(N2) for every positive integer n, the length of I_n is at most $1/n$.

Then there is a unique point that lies on each of the segments I_n.[54] We are now ready to prove the Borel Finiteness Theorem.

Proof of Theorem 4:[55] Suppose the countably infinite collection of line segments $\{I_1, I_2, \ldots\}$ covers the segment I, and suppose for *reductio ad absurdum* that for no positive integer n, $\{I_1, \ldots, I_n\}$ covers I. We construct a collection of line segments $\{J_1, J_2, \ldots\}$, with $J_1 = I$ to satisfy conditions (N1-N2) of the Nested Interval Completeness Property, together with the condition that

(*) for every positive integer k, there is no positive integer n such that $\{I_1, \ldots, I_n\}$ covers J_k.

We begin our construction by setting $J_1 = I$; note that J_1 then satisfies (*) by hypothesis. Suppose for $k > 1$ we have constructed $\{J_1, J_2, \ldots, J_{k-1}, \}$ satisfying conditions (N1-N2) and (*), for all positive integers $j \leq k-1$. We construct J_k as follows. Let L and R be the left and right closed subintervals obtained by bisecting the closed interval J_{k-1}. Since J_{k-1} satisfies (*), it follows that at least one of L and R must satisfy (*). For otherwise, there would be positive integers n_l and n_r with $\{I_1, \ldots, I_{n_l}\}$ covering L and $\{I_1, \ldots, I_{n_r}\}$ covering R, and hence $\{I_1, \ldots, I_n\}$ covering J_{k-1}, where n is the larger of n_l

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[54]See [Propp, 2013] for a taxonomy of the logical relations among several continuity properties of the linear continuum, among them the Dedekind Cut Property, Nested Interval Completeness, and Order Completeness, also known as the Least Upper-Bound Principle. See [Sinkevich, 2015] for a history of the use of the Nested Interval Completeness Property from antiquity to the late nineteenth century. Note that our convention that, unless stated otherwise, line segments are understood to include their endpoints, remains in force in our statement of this property.

[55]The proof we present is essentially the same as that given by Borel in [Borel, 1898], pp. 42-43. The reader may observe a similarity between this proof and that of the König Infinity Lemma: the statement that a binary tree with infinitely many levels has an infinite path. Both involve an iterated application of the infinite pigeonhole principle - the statement that if you sort infinitely many objects into two pigeonholes, at least one of the holes will contain infinitely many objects - followed by inference of the existence of a sequence that witnesses the choice of an infinite hole at each stage. See [Stillwell, 2013], p. 75 for discussion of this point, and Section 4 below for an examination of the deeper connection between these results.

and n_r, contradicting the supposition that J_{k-1} satisfies (*). Let $J_k = L$, if L satisfies (*), and let $J_k = R$, otherwise. It is immediately clear from the choice of J_k that it satisfies condition (*). Moreover, since J_k is a subinterval of J_{k-1} of half its length, it is clear that the sequence $\{J_1, J_2, \ldots, J_k\}$ satisfies conditions (N1-N2), for all positive integers $j \leq k$.

This concludes our construction of the collection $\{J_1, J_2, \ldots\}$.[56] By the Nested Interval Completeness Property, there is a point p such that for every positive integer k, p lies on J_k. Since, by hypothesis, the collection of line segments $\{I_1, I_2, \ldots\}$ covers I, there is a positive integer m such that p lies on I_m. It follows at once from the Axiom of Archimedes, and the fact that $\{J_1, J_2, \ldots\}$ satisfies condition (N2), that for some positive integer n, for every point $q \in J_n$, q is a member of I_m. But this contradicts the fact that J_n satisfies (*). ∎

3.5   The Vindication of the Notion of Measure (Ancient and Modern)

The following corollary to Borel's Finiteness Theorem, drawn by Borel himself,[57] constitutes the resolution of the Paradox of Measure. It establishes that both ancient- and modern-measure assign the intuitively correct magnitude to a line segment, that is, its length.

Corollary 1 *Let I be a line segment. Both the modern-measure of I and the ancient-measure of I are equal to the length of I.*

Proof: It follows immediately from Borel's Finiteness Theorem that the modern-measure of a line segment I is equal to the content of I. And it is an immediate consequence of Proposition 3 that the content of a line segment is equal to its length. Hence, the modern-measure of a line segment is equal to its length, from which it follows, by Proposition 4, that the ancient-measure of a line segment is equal to its length. ∎

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[56]Insofar our interpretation of the the Paradox involves a sound mathematical argument that makes use only of notions entirely intelligible to mathematicians of the ancient world, the reader may legitimately wonder whether the resolution we propose is similarly accessible to such thinkers. It is reasonably clear to us that the sequence $\{J_1, J_2, \ldots\}$ constructed in the proof of Borel's Finiteness Theorem is not potentially infinite. Indeed, this may be the *only* point in the argument that lies beyond the grasp of ancient mathematicians. We reflect on this point briefly in Section 4.

[57][Borel, 1895].

3.6 A Reprise of the Paradox of Measure and its Resolution

Let us take stock of what we've accomplished to this point. We have provided a notion of magnitude, ancient-measure, which assigns magnitude 0 to all potential infinities of points (Theorem 3). As emphasized in Section 2.2, the understanding that every potential infinity of points has measure 0 was well within the grasp of ancient mathematicians. Insofar as the ancients would have found it difficult to conceive of a collection of points as neither finite nor potentially infinite, so far would they have rejected the claim that a line segment is constituted out of points. On the other hand, we have shown that the notion of ancient-measure assigns to every line segment, conceived as constituted out of a collection points, its intuitively correct magnitude, namely its length (Corollary 1). Borel called Theorem 4 the "First Fundamental Theorem of Measure Theory." It implies that the notions of ancient-measure and modern-measure accord with intuition when applied to the paradigmatic cases of magnitudes, that is, line segments. This resolves the Paradox in the sense that we can explain why the ancients were drawn to the conclusion that every collection of points lacks magnitude, while we can maintain, today, that every line segment, even if constituted out of points, has magnitude. Of course, Corollary 1 together with Theorem 3 imply that if a line segment is constituted out of a collection of points, then that collection is neither finite nor potentially infinite. Thus, it may be argued that from the perspective of the ancients, what we are calling a "resolution" of the Paradox, might well be regarded as an embrace of incoherence. As Quine aptly put the matter, "One man's antinomy is another man's falsidical paradox, give or take a couple of thousand years."[58]

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[58][Quine, 1976], p. 9. The well-known ancient dictum is that philosophy begins in wonder (Plato, *Theaetetus* 155d; Aristotle, *Metaphysics* 982b)—which is to say it begins with a feeling of puzzlement (ἀπορία) from which we recoil and flee (φεύγειν) in search of understanding (ἐπιστήμη). We believe that the history of thought about the Paradox of Measure is a testament to this philosophical trajectory, but we also acknowledge that, from the ancient point of view, what we call 'understanding' might yet constitute another encounter with ἀπορία. It is interesting to note that our advance in understanding about the Paradox has, over the centuries, proceeded at times accidentally, owing much of its progress to tangential inquiries in the footnotes and to the practically oriented applications of mathematical thinking to real-world problems (what Aristotle would call θεωρία and πρᾶξις respectively).

4   Beyond Borel

In this section we will reflect on the Paradox and its resolution from a meta-mathematical point of view. These reflections are made possible by advances in proof theory late in the last century. The astute reader will have noticed that we have made no mention, up to this point, of Cantor's result that the linear continuum is uncountable, except to say that mathematicians of the late-nineteenth century who were entirely familiar with this result remained flummoxed by the Paradox. In particular, there is no explicit use made in the proof of Borel's Finiteness Theorem of the fact that the linear continuum is uncountable, nor in the proofs of any of the other results, upon which Corollary 1 depends. Moreover, we have made no mention of the additivity properties of measure, ancient or modern, in our analysis of the Paradox or its resolution. Both these "lacunae" in our discussion may surprise readers who are familiar with what might justly be called the "Received Modern View" of the Paradox of Measure, as advanced by Adolf Grünbaum and elaborated by Brian Skyrms, which emphasizes the importance of both Cantor's result that the linear continuum is uncountable and the additivity properties of (Lebesgue) measure in the analysis and resolution of the Paradox of Measure.[59] This and the following section will cast light on these matters.

Before proceeding, it is worth noting that the resolution of the Paradox as we understand it yields as an immediate corollary the uncountability of the linear continuum.

Corollary 2 (Cantor, 1873) *The collection of points constituting a line segment is uncountable.*

Proof: By Corollary 1, the modern-measure of a line segment is its length, while, by Theorem 3 and Proposition 4, the measure of a countable collection of points is 0.   ∎

It is worthy of note that Borel's discovery of his Finiteness Theorem was motivated precisely by the need to argue for the uncountability of a point set in just this way.[60]

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[59]See   [Grünbaum, 1952],   [Grünbaum, 1967],   [Skyrms, 1983],   [Holden, 2004], [Friedman, 2012], and [Huggett, 2019] for some recent endorsements of this view.

[60]See [Borel, 1895]. This alternative proof of the uncountability of the linear continuum is well-known to students of analysis. *Cf.* [Oxtoby, 1996], pp. 1-4. Oxtoby presents this alternative "measure-theoretic" proof and contrasts it with a formulation of Cantor's original "topological" proof ([Cantor, 1996]) that makes use of only the Nested Interval Complete-

As we have seen in the sections above, the Borel Finiteness Theorem implies that the modern-measure of a line segment is its length, which, together with the fact that the modern-measure of a countable point set is 0, implies that a line segment (conceived as a point set) is uncountable (Cantor's Uncountability Theorem, [Cantor, 1996]). Insofar as all of these results are theorems of real analysis, what sense does it make to claim that one implies another – aren't they all obviously equivalent to one another from the point of view of real analysis?

In order to assess such claims, we need a framework with respect to which we can compare the strength of various theorems of real analysis. Reverse Mathematics, an area of proof theory that emerged in the 1970's, provides just such a framework.[61] Following in the grand tradition of Hilbert and Bernays' monumental work, *Grundlagen der Mathematik*, Reverse Mathematics provides formalizations of real analysis, and other areas of contemporary mathematics, in second-order arithmetic. The great discovery of contemporary research in this area, which gives the area its name, is that "in many particular cases, if a mathematical theorem is proved from appropriately weak set-existence axioms, then the axioms will be logically equivalent to the theorem."[62] It is exactly this aspect of Reverse Mathematics that will allow us to calibrate the strength of the various results at play in the resolution of the Paradox of Measure. The approach of Reverse Mathematics is to introduce a weak base theory, a fragment of second-order arithmetic, with respect to which the equivalence of theorems of analysis can be meaningfully assessed. This theory, $\mathsf{RCA_0}$, consists of a weak fragment of first-order arithmetic together with the recursive comprehension schema that asserts the existence of recursive sets of natural numbers.[63] It is noteworthy from our point of view that though $\mathsf{RCA_0}$ appears to be quite weak, it suffices to prove the nested interval completeness property, and Cantor's Uncountability Theorem, as well as the fact that the modern-measure of a countable point set is

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ness Property. We compare these two arguments from a meta-mathematical point of view in the following subsection.

[61][Hirschfeldt, 2014], pp. 3-5 emphasizes the interest of Reverse Mathematics in this connection.

[62][Simpson, 2009], pp. xxiii-xiv. This work remains the standard reference for Reverse Mathematics, though the field has developed rapidly in the little more than a decade since its publication.

[63]See [Simpson, 2009], Chapter II, for details. Some sense of the strength of $\mathsf{RCA_0}$ can be gleaned from the fact that its minimum ω-model consists of the recursive sets of natural numbers.

0.[64] On the other hand, RCA_0 *suffices to prove neither the Borel Finiteness Theorem, nor its corollary that the modern measure of a line segment is its length*, as we now explain.

The technique deployed by Borel in his 1898 proof of the Borel Finiteness Theorem (**BFT**) is similar to that used in standard proofs of the König Infinity Lemma.[65] The next result provides a deep explanation for this similarity. Here **WKL** is a formalization of the König Infinity Lemma in second-order arithmetic.[66]

Theorem 5 (Friedman) $\mathsf{RCA}_0 \vdash \mathbf{WKL} \iff \mathbf{BFT}$.

The next result, together with Theorem 5, provides a striking metamathematical analog to the historical progression we have traced toward the resolution of the Paradox of Measure.[67]

Theorem 6 $\mathsf{RCA}_0 \nvdash \mathbf{WKL}$.

Remarkably, Theorems 5 and 6 precisely identify a sense in which Borel's Finiteness Theorem is strictly stronger than those principles of analysis, as formalized by RCA_0, *that suffice to establish many other fundamental properties of the linear continuum, including nested interval completeness and uncountability.* Of course, we do not suggest that the relative strength of the Borel Finiteness Theorem from a metamathematical point of view explains the historical progression - some results established even earlier in the nineteenth century, such as the Bolzano-Weierstrass Theorem, require even stronger set existence principles to derive. But it does highlight the new depth of understanding of a fundamental feature of the linear continuum that the Borel Finiteness Theorem represents. It also complements the historical evidence that knowledge of the uncountability of the linear continuum was insufficient to dispel the mystery surrounding the Paradox of Measure. Moreover, it indicates the extent to which resolution of the Paradox of Measure outruns the mathematical resources sufficient to unravel the paradoxes of motion.[68]

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[64] See [Simpson, 2009], pp. 76-77.

[65] See especially footnote 55.

[66] See [Simpson, 2009], pp. 127-130 for discussion and a proof of Theorem 5 which was announced in [Friedman, 1974].

[67] See [Simpson, 2009], p. 31, for a proof of Theorem 6.

[68] See footnote 51. It is worth remarking that though $\mathsf{WKL}_0(= \mathsf{RCA}_0 + \mathbf{WKL})$ is far

The reader may legitimately complain that we have gotten a bit ahead of ourselves, insofar as it is the fact that *the modern-measure of a line segment is its length* (**MML**) that actually represents the resolution of the Paradox of Measure, and not the Borel Finiteness Theorem itself. As it turns out, Reverse Mathematics provides a fascinating characterization of the strength of **MML** in terms of a combinatorial principle weaker than **WKL**, yet still not derivable in $\mathsf{RCA_0}$. Let T be a binary tree and let T_n be the number of nodes of T at level n. **WKL** formalizes the principle that for every binary tree T, if $T_n > 1$, for every natural number n, then T has an infinite path. **WWKL** formalizes the principle that for every binary tree T, if there is a k such that $T_n/2^n > 1/k$, for every natural number n, then T has an infinite path.[69] It is evident that **WKL** implies **WWKL** over $\mathsf{RCA_0}$. The following remarkable results characterize the strength of the resolution of the Paradox of Measure.[70]

Theorem 7 (Yu-Simpson and Brown-Giusto-Simpson) .

1. $\mathsf{RCA_0} \vdash \mathbf{WWKL} \iff \mathbf{MML}$.

2. $\mathsf{RCA_0} \nvdash \mathbf{WWKL}$.

3. $\mathsf{RCA_0} + \mathbf{WWKL} \nvdash \mathbf{WKL}$.

The foregoing Theorem establishes that **MML**, which represents for us the resolution of the Paradox of Measure, though strictly weaker than the Borel Finiteness Theorem, is, nonetheless, independent of $\mathsf{RCA_0}$, a system strong enough to prove the uncountability of the linear continuum.

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richer than $\mathsf{RCA_0}$ from a *mathematical* point of view, it is nonetheless still comparatively weak from a *metamathematical* point of view – it is a conservative extension of primitive recursive arithmetic with respect to Π_2^0 sentences. This finititistic reduction is of great significance from the point of view of partial realizations of Hilbert's Program, *cf.* [Simpson, 1988] and [Simpson, 2009], pp. 377-378.

[69] That is, WWKL asserts that if there is a fixed positive lower bound on the density of the levels of a binary tree T, then T has an infinite path. *Cf.* [Simpson, 2009], p. 393.

[70] Part 1 of the theorem follows immediately from Theorem 1 of [Yu and Simpson, 1990], p. 175 and Theorem 3.3 of [Brown et al., 2002], p. 196, while parts 2 and 3 are established in [Yu and Simpson, 1990], p. 172. Denis Hirschfeldt (private communication) informs us that a further reversal of WWKL of interest in connection with the Paradox of Measure may be obtained via methods developed by [Brown et al., 2002] and [Dorais et al., 2016]. Namely, let **MNZ** be a formalization of the statement that the modern-measure of an interval is not zero; $\mathsf{RCA_0} \vdash \mathbf{WWKL} \iff \mathbf{MNZ}$.

5 Conclusion

In this paper we have presented a novel interpretation of Zeno's Paradox of Measure. The crux of the interpretation is the definition of a notion of magnitude, ancient-measure, with respect to which all potential infinities of points lack magnitude. The notion of ancient-measure itself, and the proof we give to show that the ancient-measure of potential infinities of points is zero, lie entirely within the conceptual repertoire of the mathematicians of antiquity. We observe that essentially the same argument remained a puzzle to mathematicians through the 1880's, especially insofar as it applies to collections of points dense in the linear continuum. The Paradox was finally resolved by Borel, who established that the ancient- (and modern-) measure of a line segment is equal to its length. This not only resolves the Paradox, but also buttresses the claim that ancient-measure is a suitable notion of magnitude in application to collections of points, especially in relation to the Paradox of Measure, where the only such collections in question are potential infinities and line segments. Indeed, Borel's name for his Finiteness Theorem – "The First Fundamental Theorem of Measure Theory" – signals his own recognition of its significance in this respect. We have also gone on to discuss the metamathematics of Borel's Finiteness Theorem, and the light this sheds on the extent to which the resolution of the Paradox outruns the resources of the mathematics of antiquity.

There are a number of respects in which our treatment of the Paradox of Measure diverges from prior discussions in the philosophical literature. First, we have emphasized that the resolution of the Paradox did not come about as a result of Cantor's proof that the linear continuum is uncountable: we have presented historical evidence that the Paradox remained puzzling through the 1880's, and we have observed that the resolution of the Paradox is independent of formal systems that are adequate to establish Cantor's result. This represents a major departure from earlier treatments of the resolution of the Paradox in the twentieth century. In particular, the highly influential works [Grünbaum, 1952] and [Grünbaum, 1967] advance the view that the crucial element in the resolution of the Paradox is exactly the fact that the linear continuum is uncountable. This view is echoed in [Skyrms, 1983], and is invoked in [Holden, 2004] and [Friedman, 2012], who discuss the Paradox in the context of the early and late modern period, and persists in what may be regarded as the standard contemporary treatment of Zeno's Paradoxes, [Huggett, 2019]. Insofar as our interpretation of the Paradox reveals

that it is not the uncountability of the continuum, but rather the Finiteness Theorem of Borel that leads to the resolution of the Paradox, scholarly understanding of the response of thinkers throughout the history of philosophy to the import of the Paradox for issues ranging from the reality of space to the possibility that intervals of time are composed of instants lacking duration will need to be reconsidered. Borel's Finiteness Theorem establishes a fundamental property of the linear continuum, the "compactness of line segments (closed intervals)", independent of the continuity of the line as understood from Aristotle to Dedekind; it is this fundamental property, first articulated and established by Borel, that lies at the heart of the resolution of the Paradox of Measure. As far as we are aware, philosophers have not been cognizant of the significance of Borel's Finiteness Theorem in this regard. We hope that directing attention to the fact that Borel's Finiteness Theorem represents a fundamental property of the linear continuum may have philosophical impact even beyond discussions of the Paradox of Measure. Second, we have refrained from deploying any measure-theoretic apparatus in our arguments, such as the "summation of zeroes" as would be legitimated by the countable additivity of Lebesgue measure. Borel's Finiteness Theorem is the *prolegomenon* to measure theory, and, from an intuitive point of view, it is at some conceptual remove from the apparatus he and Lebesgue built upon it. In particular, we have not appealed to two principles, clearly articulated in [Chen, 2021], that lie behind the prevailing contemporary interpretations of the Paradox:

> Additivity. The size of the whole is the sum of the sizes of its disjoint parts.
> Zeros‑Sum‑To‑Zero. Zeros, however many, always sum up to zero.[71]

As we have emphasized, we interpret the Paradox of Measure as a valid mathematical argument rooted in the practice of the mathematics of antiquity that does not appeal to either of these principles. Again, this represents a radical departure from other interpretations, and its impact on reevaluation of arguments concerning the nature of time and space may be even greater than the insight that cardinality plays no role in the resolution of the Paradox. This is connected with a third matter, worthy of note. We have not discussed issues concerning the existence of point-sets that are not Lebesgue measurable, nor

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[71][Chen, 2021], p. 4442.

speculated about their bearing on the resolution of the Paradox. We regard issues concerning Lebesgue measurability as entirely irrelevant to the resolution of Zeno's Paradox of Measure, insofar as the Paradox deals only with the magnitude of potential infinities of points and line segments, with respect to which ancient-measure and Lebesgue measure coincide. We do not, of course, mean to suggest that there is no philosophical interest in issues surrounding Lebesgue measure. Indeed, one might attach some metaphysical significance to questions in contemporary set theory dealing with "cardinal characteristics of the continuum," for example: "what is the least cardinal κ such that there is a subset of $\mathbb{R}$ of cardinality κ whose modern-measure is not zero?" We might see in this investigation the contemporary pursuit of a question distantly related to the Paradox of Measure: how small a point set (in the sense of cardinality) can have positive magnitude (in the sense of modern-measure)?[72] Remarkably, it is consistent with ZFC that there are sets of cardinality less than $\mathfrak{c}$ that have positive modern-measure.[73]

A   Proof of Proposition 3

Proof: Let X satisfy the hypothesis of the Proposition and suppose that X is dense in $[0, 1]$. We must show that

1. for every finite collection of intervals Ξ covering X, $\tau(\Xi) > 1$.

2. for every positive integer n, there is a finite collection of intervals Ξ such that Ξ covers X and $\tau(\Xi) < 1 + n^{-1}$.

(2): Fix n, and let $\Xi = \{(-(3n)^{-1}, 1 + (3n)^{-1})\}$. Ξ covers X and $\tau(\Xi) = 1 + 2 \cdot (3n)^{-1} < 1 + n^{-1}$.

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[72]Note that if a set X of cardinality less than $\mathfrak{c}$, the cardinality of $\mathbb{R}$, has positive modern-measure, then X is *not* Lebesgue-measurable. This follows from the fact that every Lebesgue-measurable set contains a closed set of positive modern-measure. But every set of positive modern-measure is uncountable, and every uncountable closed set has cardinality the continuum, by the Cantor-Bendixson Theorem.

[73][Jech, 2002], pp. 529-537. Indeed, let κ be the least cardinal such that there is a collection of points of cardinality κ that has positive modern-measure, and let λ be the least cardinal such that there is a family of cardinality λ of collections of points, each of modern-measure 0, whose union has positive modern-measure. It is consistent with ZFC that $\lambda < \kappa < \mathfrak{c}$. This suggests that the Paradox of Measure, even conceived in terms of cardinality of point-sets of positive measure, is a separate issue from questions of additivity.

(1): Suppose that $\Xi = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ is a finite collection of open intervals covering X. We may suppose, without loss of generality, that the intervals are ordered in such a way that $a_1 < a_2 < \ldots < a_k < 1$ and $b_1 < b_2 < \ldots < b_k$, for if that were impossible, one interval would be a subinterval of another and could thus be removed from Ξ and the remaining collection of intervals would cover X and be of smaller total length. To conclude the proof it suffices to show that

(1.1) $a_1 \leq 0$,

(1.2) $1 \leq b_k$, and

(1.3) for every $1 \leq j < k$, $a_{j+1} \leq b_j$.

(1.1): Suppose to the contrary that $0 < a_1$. Since X is dense in $[0, 1]$, it follows that for some $p \in X$, $0 < p < a_1$. Since $a_j > a_1$ for all $j > 1$, it follows that p is not contained in any interval in Ξ. This contradicts the hypothesis that Ξ covers X.

(1.2): The argument here is virtually identical to the that for (1.1).

(1.3): The argument again is very similar to that for (1.1). Suppose to the contrary that that for some $1 \leq j < k$, $b_j < a_{j+1}$. Since X is dense in $[0, 1]$, it follows that there is a $p \in X$ such that $b_j < p < a_{j+1}$. Therefore, contrary to hypothesis, Ξ fails to cover X. ∎

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