AGM Belief Revision in Monotone Modal Logics

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Abstract

Classical modal logics, based on the neighborhood semantics of Scott and Montague, provide a generalization of the familiar normal systems based on Kripke semantics. This paper defines AGM revision operators on several first-order monotonic modal correspondents, where each first-order correspondence language is defined by Marc Pauly's version of the van Benthem characterization theorem for monotone modal logic. A revision problem expressed in a monotone modal system is translated into first-order logic, the revision is performed, and the new belief set is translated back to the original modal system. An example is provided for the logic of Risky Knowledge that uses modal AGM contraction to construct counter-factual evidence sets in order to investigate robustness of a probability assignment given some evidence set. A proof of correctness is given.

1 Introduction

Within AGM Gärdenfors (1988), consistency maintenance is done within a supra-classical propositional logic. But the reliance on classical consistency paired with a propositional language presents a challenge for exporting AGM revision to non-classical logics in general, and to modal logic in particular.

A general technique for solving this problem is to translate a non-classical logic into classical logic, together with a specification of consistency particular to the non-classical logic, perform the operation of revision on this translation within classical logic, then translate the result back into the non-classical logic we started with Gabbay et al. (2008). In the case of normal modal logic, both the modal language and the semantic structure must be translated into first-order logic, and this translation for common normal systems will rely upon well-known frame-theoretic properties, expressed in first-order logic, of modally defined classes of (finite) Kripke frames Goldblatt (1993).

But there is a problem extending this technique to classical modal logics Chellas (1980), because there is generally no direct correspondence between neighborhood frames and first-order logic. This paper proposes to solve this problem by adapting Marc Pauly's Hansen (2003) technique of first simulating neighborhood structures by polymodal Kripke structures, then define a correspondence to first-order logic from the polymodal Kripke semantics wherein AGM revision can be defined.

In modal logic the technique of simulation was first used to construct counter-examples within polymodal modal logic to export back to monomodal systems of interest Thomason (1974, 1975). More recently the technique has been used to study the relationship between neighborhood semantics and Kripke's relational semantics, with a particular focus on supplemented neighborhood models Gasquet and Herzig (1996), Kracht and Wolter (1999), Hansen (2003). Supplemented neighborhood models underpin a variety of non-additive, monotone modal logics appearing in knowledge representation formalisms, including *Game Logic* Parikh (1985), *Concurrent Propositional Dynamic Logic* Goldblatt (1992), *Alternating-time logic* Alur et al. (1992), *Risky Knowledge* Kyburg and Teng (2002), and *Coalition Logic* Pauly (2002). Non-monotone classical modal logics have also been pressed into service, including *Local Reasoning* Fagin and Halpern (1988), and the logic of *Only Knowing* Humberstone (1987), Levesque (1990), which are based on the *Inaccessible Worlds* semantics of Humberstone (1983).

In addition to discovering properties of a logic by studying its simulation within a well-understood system, the techniques of simulation theory together with correspondence theory may be used to bring new capabilities to a classical modal logic. AGM belief revision is but one example, which is the subject of this paper.

Early work has focused on supplying a normal modal semantics for AGM Boutilier (1992). More recent work in modal belief revision focuses on specifically tailored classical modal logics, particularly dynamic epistemic logic van Benthem (2007), van Eijck (2009), and polymodal normal logics, such as branching time temporal logic Bonanno (2009). Each addresses particular issues that arise—principally updating and common knowledge, in the case of the former, and the interaction of temporal and epistemic operators on the standard interpretation in the latter. Also, Enqvist (2009) has focused on revising interrogative questions. But, as we see above, there are many interpretations of classical modal logics within knowledge representation, and one might like instead to have a general strategy for supplying AGM revision to systems of classical modal logic. The present paper aims to provide such a strategy.

2 Classical Modal Logic

To begin, we highlight the difference between neighborhood structures and standard Kripke structures. Whereas Kripke frames are characterized by a binary accessibility relation defined over a set of worlds, a **neighborhood frame** for the propositional modal language $\mathscr{L}_{\nabla}(\Phi)$ is a pair $\mathbb{F} = (W, \mathscr{N})$ where W is a non-empty set of worlds, and $\mathscr{N} : W \mapsto \wp(\wp(W))$ is a neighborhood function, i.e. $\mathscr{N}(w) \subseteq \wp(W)$, for each $w \in W$. If $\mathbb{F} = (W, \mathscr{N})$ is a neighborhood frame, Φ a countable set of propositional variables, and $V : \Phi \mapsto \wp(W)$ is a valuation on $\mathbb{F}$, then $\mathbb{M} = (W, \mathscr{N}, V)$ is a **neighborhood model** based on $\mathbb{F}$.

The satisfiability conditions for non-modal propositional formulas on neighborhood models are analogous to Kripke models, but modal necessity ($\nabla\varphi$) and possibility ($\Delta\varphi$) statements on neighborhood models are different. Like the normal modal logic (K) and its extensions, classical modal logics are based on the classical system (E) and the meaning of necessity statements in different classical systems is determined by the properties of neighborhood frames just as the meaning of necessity statements in different normal systems is determined by the properties of a Kripke frame. That said, there are four important classes of neighborhood models (minimal, supplemented, quasi-filters, augmented) that determine four modal systems (classical, monotone, regular, normal). The differences between these models can be reflected by the **truth conditions** for ($\nabla\varphi$). Let $\mathbb{M} = (W, \mathscr{N}, V)$ be a neighborhood model, w be a world in W, X a set of worlds, and $p \in \Phi$. Then:

Common Core

- $\Vdash_w^{\mathbb{M}} \bot$ iff never
- $\Vdash_w^{\mathbb{M}} p$ iff $w \in V(p)$, for $p \in \Phi$
- $\nVdash_w^{\mathbb{M}} p$ iff $w \notin V(p)$
- $\Vdash_w^{\mathbb{M}} \varphi \vee \psi$ iff $w \in V(\varphi)$ or $w \in V(\psi)$
- $\Vdash_w^{\mathbb{M}} \Delta\varphi$ iff $\Vdash_w^{\mathbb{M}} \neg\nabla\neg\varphi$

Minimal Models, 'e':

- $\Vdash_w^{\mathbb{M}^e} \nabla\varphi$ iff $\{w^* \mid \Vdash_{w^*}^{\mathbb{M}^e} \varphi\} \in \mathscr{N}(w)$
- $\Vdash_w^{\mathbb{M}^e} \Delta\varphi$ iff $\{W \setminus \{w^* \mid \Vdash_{w^*}^{\mathbb{M}^e} \varphi\}\} \notin \mathscr{N}(w)$

Supplemented Models, 'm':

- $\Vdash_w^{\mathbb{M}^m} \nabla\varphi$ iff $(\exists X \in \mathscr{N}(w), \forall w^* \in X) : \Vdash_{w^*}^{\mathbb{M}^m} \varphi$
- $\Vdash_w^{\mathbb{M}^m} \Delta\varphi$ iff $(\forall X \in \mathscr{N}(w), \exists w^* \in X) : \Vdash_{w^*}^{\mathbb{M}^m} \varphi$

  Quasi-filters, 'r':

- $\Vdash_w^{\mathbb{M}^r} \nabla\varphi$ iff $\mathscr{N}(w) \neq \{\emptyset\}$ and $\{\{w^* | \Vdash_{w^*}^{\mathbb{M}^r} \varphi\}\} = \mathscr{N}(w)$
- $\Vdash_w^{\mathbb{M}^r} \Delta\varphi$ iff $\mathscr{N}(w) \neq \{\emptyset\}$ and
  $\{\{W \setminus \{w^* | \Vdash_{w^*}^{\mathbb{M}^r} \varphi\}\}\} \neq \mathscr{N}(w)$

  Augmented, 'k':

- $\Vdash_w^{\mathbb{M}^k} \nabla\varphi$ iff $\{\{w^* | \Vdash_{w^*}^{\mathbb{M}^k} \varphi\}\} = \mathscr{N}(w)$
- $\Vdash_w^{\mathbb{M}^k} \Delta\varphi$ iff $\{\{W \setminus \{w^* | \Vdash_{w^*}^{\mathbb{M}^k} \varphi\}\}\} \neq \mathscr{N}(w)$.

The following are classical **modal schemata** and frame properties. (More soon on their relationship.) All instances of (**N**), (**C**), and (**M**) are theorems of any normal modal logic. However, none are theorems of classical modal logic. All instances of (**M**) are theorems of monotone logics, and all instances of (**M**) and (**C**) are theorems of regular logics.

To shorten notation, a neighborhood function $\mathscr{N}$ defines a map $N_m : \mathscr{P}(W) \mapsto \mathscr{P}(W)$ such that $N_m(X) = \{w \in W : X \in \mathscr{N}(w)\}$, so that $N_m(V(\varphi)) = V(\nabla\varphi)$. Consider now the following modal schemas and their corresponding neighborhood frame conditions.

(**N**)	$\nabla\top$	(n)	$\forall w \in W : W \in \mathscr{N}(w)$
(**P**)	$\neg\nabla\bot$	(p)	$\forall w \in W : \emptyset \notin \mathscr{N}(w)$
(**C**)	$\nabla\phi \wedge \nabla\psi \to \nabla(\phi \wedge \psi)$	(c)	$\forall w \in W,\ \forall X_1, X_2 \subseteq W :$ $(X_1 \in \mathscr{N}(w) \ \&\ X_2 \in \mathscr{N}(w)) \to X_1 \cap X_2 \in \mathscr{N}(w)$.
(**M**)	$\nabla(\phi \wedge \psi) \to \nabla\phi \wedge \nabla\psi$	(m)	$\forall w \in W,\ \forall X_1, X_2 \subseteq W :$ $(X_1 \subseteq X_2 \ \&\ X_1 \in \mathscr{N}(w)) \to X_2 \in \mathscr{N}(w)$.
(**D**)	$\nabla\phi \to \Delta\phi$	(d)	$\forall x \in W, \forall X \subseteq W : X \in \mathscr{N}(w) \to -X \notin \mathscr{N}(w)$.
(**T**)	$\nabla\phi \to \phi$	(t)	$\forall w \in W, \forall X \subseteq W : X \in \mathscr{N}(w) \to w \in X$.
(**B**)	$\phi \to \nabla\Delta\phi$	(b)	$\forall x \in W, \forall X \subseteq W :$ $w \in X \to \{W \setminus \{N_m(W \setminus X)\}\} \in \mathscr{N}(w)$
(**4**)	$\nabla\nabla\phi \to \nabla\phi$	(iv)	$\forall w \in W, \forall X, Y \subseteq W :$
(**4′**)	$\nabla\phi \to \nabla\nabla\phi$	(iv')	$\forall X, Y \subseteq W : X \in \mathscr{N}(w) \to N_m(X) \in \mathscr{N}(w)$. $(X \in \mathscr{N}(w) \ \&\ \forall x \in X : Y \in \mathscr{N}(w)) \to Y \in \mathscr{N}(w)$.
(**5**)	$\Delta\phi \to \nabla\Delta\phi$	(v)	$\forall x \in W, \forall X \subseteq W :$ $X \notin \mathscr{N}(w) \to \{W \setminus N_m(X)\} \in \mathscr{N}(w)$.

Define (**E**) as $\Delta\varphi \leftrightarrow \neg\nabla\neg\varphi$ and consider the following inference rules.

$$(RE)\ \frac{\varphi \leftrightarrow \psi}{\nabla\varphi \leftrightarrow \nabla\psi} \qquad (RM)\ \frac{\varphi \to \psi}{\nabla\varphi \to \nabla\psi} \qquad (RR)\ \frac{(\varphi_1 \wedge \varphi_2) \to \psi}{(\nabla\varphi_1 \wedge \nabla\varphi_2) \to \nabla\psi}$$

Classical modal systems contain (**E**) and are closed under (RE). **Monotone** modal systems are classical but contain all instances of (**M**); equivalently, they contain (**E**) and are closed under (RM). **Regular** modal systems are monotone but contain all instances of (**C**); equivalently, they contain (**E**) and are closed under (RR). **Normal** modal systems are regular and contain all instances of (**N**).

There are eight classical logics defined by combinations of the schemata **M**, **C**, and **N**, including the four just defined, each axiomatizable, determined by a class of finite neighborhood models, sound and strongly complete, and decidable Chellas (1980).

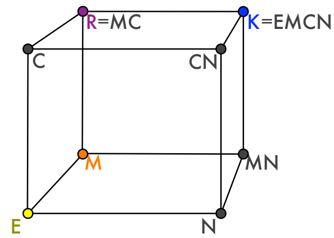

Figure 1: Basic systems of classical modal logic

3 Correspondence Languages

Recall the goal: to define AGM belief revision operators for systems of classical modal logic. The first step of our strategy involves translating classical modal formulas and the relevant neighborhood semantic structure into first-order logic. This section addresses the translation step by appealing to results from modal simulation theory, which identifies a class of neighborhood frames with some multi-modal Kripke frame, and correspondence theory, which here will characterize a polymodal Kripke frame by sentences of first-order logic. This requires specifying three languages: $\mathscr{L}_\nabla$, a classical propositional monomodal language; $\mathscr{L}_\Diamond$, a standard propositional polymodal language; and $\mathscr{L}^1_\nabla$, the final first-order translation language corresponding to $\mathscr{L}_\nabla$. This technique does not cover all classical modal systems, but it does cover many of them.

Let $p \in \Phi$ and pt be a unary modal operator. A **classical monomodal** grammar and a **standard polymodal** grammar are generated by the following, respectively:

- $\varphi := p \mid \neg\varphi \mid \varphi \vee \psi \mid \nabla\varphi$, for $\varphi \in \mathscr{L}_\nabla(\Phi)$;
- $\varphi := p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond_1\varphi \mid \Diamond_2\varphi \mid \mathsf{pt}$, for $\varphi \in \mathscr{L}_\Diamond(\Phi)$.

For the standard polymodal language $\mathscr{L}_\Diamond(\Phi)$, a **first-order correspondence language** $\mathscr{L}^1_\nabla(\Phi)$ is generated from first-order variables $x, y, z, \ldots$, unary predicates $P_0, P_1, \ldots$ for each propositional atom $p_0, p_1, \ldots \in \Phi$, binary relation symbol(s) R_1, R_2, and a unary relation symbol Q. The set of propositional atoms is constant, so we omit reference to Φ in the remainder.

3.1 The Common Core

First-order correspondence languages vary by the conditions imposed on the binary relations, and those conditions are determined by the interpretation supplied to $\Diamond_1$ and $\Diamond_2$ in $\mathscr{L}_\Diamond$ by a standard poly-modal Kripke frame. Similarly, a polymodal Kripke simulation within $\mathscr{L}_\Diamond$ of a monotonic modal logic expressed within $\mathscr{L}_\nabla$ will vary by the frame properties that determine $\Diamond_1$ and $\Diamond_2$ (and the nullary modalty, pt), conditions which are imposed by the monotonic classical system $\mathsf{M.S}_1, \ldots, \mathsf{S}_\mathsf{n}$. Otherwise, the translation operations are homomorphic for non-modal formulas which we refer to as the common core.

To focus on this common core, consider two translation functions τ and t. Let $\mathscr{F}^3$ be a polymodal Kripke frame, which will be fully defined later for different classes of neighborhood models, and $\mathscr{M} = (\mathscr{F}^3, V)$ a Kripke model based on $\mathscr{F}^3$. Then, a translation τ between $\mathscr{L}_\nabla$ and $\mathscr{L}_\Diamond$ on the (non-modal) common core is given on the left:

$$\begin{array}{rcl} & \mathscr{L}_\nabla \text{ to } \mathscr{L}_\Diamond & \\ \bot^\tau & = & \bot \\ p^\tau & = & p, \text{for } p \in \Phi \\ (\neg\varphi)^\tau & = & \neg(\varphi^\tau) \\ (\varphi \vee \psi)^\tau & = & (\varphi^\tau) \text{ or } (\varphi^\tau) \end{array} \qquad \begin{array}{rcl} & \mathscr{L}_\Diamond \text{ to } \mathscr{L}^1_\nabla & \\ (\bot)^t(w) & = & x \neq x \\ (p)^t(w) & = & P(w) \\ (\neg\varphi)^t(w) & = & \neg(\varphi^t(w)) \\ (\varphi \vee \psi)^t(w) & = & \varphi^t(w) \vee \psi^t(w). \end{array}$$

The right hand side of this table specifies the **local translation** t between $\mathscr{L}_\Diamond$ and $\mathscr{L}^1_\nabla$ for the common core. Here the unary predicates $P_i \in \mathscr{L}^1_\nabla$ are interpreted by their corresponding propositional variables $p_i \in \Phi$ as follows: $\Vdash^{\mathscr{M}}_w p^t = P(w)$ expresses that p is satisfied at world w in model $\mathbb{M}$, and this assertion is translated into first-order logic by $P(w)$. We write $p^t(w)$ to abbreviate $\Vdash^{\mathscr{M}}_w p^t = P(w)$, and $\neg(p^t(w))$ to abbreviate $\nVdash^{\mathscr{M}}_w p^t$.

Discussion: Intuitively, $(\varphi)^t$ translates the assertion that φ is satisfied at a world within a model. To translate that φ is valid with respect to a class of models, a **global translation** function translates the assertion that φ is satisfied at all worlds with respect to that class of models. We now consider conditions for constructing first-order correspondents for various common systems of monotonic modal logic, including conditions for global translations.

3.2 Supplemented models

Supplemented models have been studied extensively, and much is now known about simulation and correspondence for monotone modal logics Hansen (2003).

Expand the translation function τ between $\mathscr{L}_\nabla$ and $\mathscr{L}_\Diamond$ to cover modal formulas in supplemented models is achieved by defining a bi-frame for the language $\mathscr{L}_{\Diamond_2}$, following Gasquet and Herzig (1996), where here the modal indices are replaced by mnemonic symbols. Define a bi-modal Kripke frame $\mathscr{F}^3 = (W \cup \wp(W), R_{\mathscr{N}}, R_\ni, \mathsf{pt})$. The neighborhood function $\mathscr{N} \in \mathbb{F}$ is represented within $\mathscr{F}^2$ by:

$$\begin{aligned} R_{\mathscr{N}} &= \{(w,X) \in W \times \wp(W) \mid X \in \mathscr{N}(w)\}, \\ R_\ni &= \{(X,w) \in \wp(W) \times W \mid w \in X\}, \\ \mathsf{pt} &= W. \end{aligned}$$

Then, adding equation (1) to the τ-common core $\mathscr{L}_\Diamond$ (i.e., $\mathscr{L}_{\Diamond^m}$) yields a truth preserving translation between the class of supplemented models $\mathbb{M}^m$ and standard bi-modal modal Kripke models based on the class of frames $\mathscr{F}^2$ Gasquet and Herzig (1996), Kracht and Wolter (1999).

$$(\nabla\varphi)^\tau = \Diamond_{\mathscr{N}} \Box_\ni (\varphi)^\tau \tag{1}$$

A frame-validity preserving translation $\blacklozenge$ between $\mathscr{L}_{\nabla^e}$ and $\mathscr{L}_\Diamond$ is defined by $\varphi^\blacklozenge = \mathsf{pt} \rightarrow \varphi^\tau$.

Discussion: From the satisfiability conditions for ∇ and Δ, we can view the monotonic neighborhood function $\mathscr{N}$ to be comprised of two different types of relationships, each represented by a diamond modality. The first, $\Diamond_{\mathscr{N}}$, expresses when a set of worlds is within the neighborhood associated with a world w, and the second, $\Diamond_\ni$, expresses when w is within a set of worlds. Finally, since the accessibility relations for these two modalities range over worlds and sets of worlds, i.e., $W \cup \wp(W)$, the 0-arity modal constant pt is used to denote the worlds $w \in W$.

Turning to the translation function t between $\mathscr{L}_\Diamond$ and $\mathscr{L}^1_\nabla$, adding equation (2) to the t-common core $\mathscr{L}^1_\nabla$ (i.e., $\mathscr{L}^1_{\nabla^m}$) yields a local truth preserving translation between standard bi-modal Kripke models simulating the class of supplemented neighborhood models and the first-order correspondence language $\mathscr{L}^1_{\nabla^m}$.

$$(\nabla\varphi)^t(w) = \exists x(R_{\mathscr{N}}wx \wedge \forall y[R_{\ni}xy \rightarrow \varphi^t(y)]), \tag{2}$$

where R_iab abbreviates $(a,b) \in R_i$.

Finally, the global translation T between $\mathscr{L}_{\nabla^e}$ and $\mathscr{L}_{\Diamond}$ defined by $(\varphi)^T(w) = \forall w(Q(w) \rightarrow (\varphi)^t(w))$ preserves frame validity.

Discussion: By quantifying over subsets of worlds, frame validity expresses a second-order property which does not always admit expression by a first-order formula. In Kracht (1993), Kracht and Wolter (1999) it was observed that a particular class of classical modal formulas in language $\mathscr{L}_{\nabla}$, interpreted over bi-modal Kripke structures, correspond to Sahlqvist formulas, for which Salvqvist correspondence holds via the Sahlqvist-van Benthem algorithm. This technique was extended to monotonic modal logic by Marc Pauly in an unpublished manuscript, which is described in Hansen (2003). As will be seen in the discussion of minimal models, this result can be applied to some but not all classical modal systems.

3.3 Minimal models

Minimal models are the most general class of neighborhood models; this class determines system (E).

To expand the translation function τ between $\mathscr{L}_{\nabla}$ and $\mathscr{L}_{\Diamond}$ with respect to $\mathscr{F}^3$, adding equation (3) to the τ-common core yields a truth preserving translation between the class of minimal models $\mathbb{M}^e$ and standard polymodal modal Kripke models based on the class of frames $\mathscr{F}^3$ Gasquet and Herzig (1996).

$$(\nabla\varphi)^\tau = \Diamond_{\mathscr{N}}(\Box_{\ni}(\varphi)^\tau \wedge \Box_{\mathscr{N}}(\varphi)^\tau) \tag{3}$$

Turning to the translation function t between $\mathscr{L}_{\Diamond^e}$ and $\mathscr{L}^1_{\nabla^e}$, currently results are limited. This is because the language $\mathscr{L}_{\nabla}$ local translation, equation 4, is not a Sahlqvist formula.

$$(\nabla\varphi)^t(w) = \exists x(R_{\mathscr{N}}wx \wedge [\forall y(R_{\ni}xy \leftrightarrow \varphi^t)]) \tag{4}$$

Since supplemented models are just the class of minimal models in which all instances of (**M**) are valid, the correspondence results from monotonic modal logic apply. But it is an open question precisely what the classical modal fragment is beyond Pauly's identification of the monotonic modal fragment as the monotonic bisimulation invariant fragment mentioned above; the monotonicity condition of supplemented models is critical in the construction. Recent work has focused on developing an alternative correspondence theory based on a topological semantics ten Cate et al. (2009).

3.4 Quasi-filters & augmented models

The list of modal schemata in the previous section are divided into two families, each sound and strongly complete with respect to their associated frames. Let M+S be a propositional monotonic modal system, S a modal schema, then:

1. If $\mathsf{S} \subseteq \{\mathbf{N},\mathbf{C},\mathbf{T},\mathbf{4}',\mathbf{B},\mathbf{D}\}$, and $\mathsf{S}' \subseteq \{\mathbf{P},\mathbf{4},\mathbf{5}\}$, then MS and MS' is sound and strongly complete with respect to the class of monotonic $\mathscr{L}_{\Diamond}$ polyframes defined by S and S', respectively.
2. $\mathsf{MS} \cup \mathsf{S}'$ is not necessarily sound and strongly complete with respect to the class of monotonic $\mathscr{L}_{\Diamond}$ polyframes defined by all formulas in $\mathsf{S} \cup \mathsf{S}'$ Hansen (2003).

Finally, the class of monotonic $\mathscr{L}_{\nabla}$ polyframes satisfying condition (c) is defined by the class of supplemented models in which all instances of (**C**) are valid, and the class of $\mathscr{L}_{\nabla}$ bi-frames satisfying both (c) and (n) is defined by the class of supplemented models in which all instances of (**C**) and (**N**) are valid. The former are the class of quasi-filters; the latter the class of augmented models.

4 AGM

To define AGM revision on this family of correspondence languages we adapt a strategy for normal monomodal logic Gabbay et al. (2008) that requires (*i*) a sound and complete axiomatization of each classical modal system, (*ii*) a classical AGM revision operator.

Recall the AGM postulates for the revision operator, $*$, where $K = Cn(K)$, and φ, ψ are propositional formulas: (**K**$*$**1**) $K * \phi$ is a belief set; (**K**$*$**2**) $\phi \in (K * \phi)$; (**K**$*$**3**) $(K * \phi) \subseteq Cn(K \cup \{\phi\})$; (**K**$*$**4**) If $\neg\phi \notin K$, then $Cn(K \cup \{\phi\}) \subseteq (K * \phi)$; (**K**$*$**5**) $(K * \phi) = \mathscr{L}^{PL}$ only if $\phi \equiv \bot$; (**K**$*$**6**) If $\phi \equiv \psi$, then $(K * \phi) \equiv (K * \psi)$; (**K**$*$**7**) $K * (\phi \wedge \psi) \subseteq Cn((K * \phi) \cup \{\psi\})$; (**K**$*$**8**) If $\neg\psi \notin (K * \phi)$, then $Cn((K * \phi) \cup \{\psi\}) \subseteq K * (\phi \wedge \psi)$.

The correspondence language essentially maps the satisfiability conditions of modal formulas into corresponding first-order predicates along with additional first-order formulas that express the corresponding neighborhood frame conditions. Thus, the closure operator *Cn* is held constant: it is Tarski's classical consequence operator. The arguments are the first-order translations of modal formulae along with additional formulas needed to characterize the frame properties of a modal system.

Turn to the **definition of AGM revision** in EM. Let $\Lambda^t(w)$ be the first-order local translation into $\mathscr{L}^1_\nabla$ of a classical monotonic modal theory, $\phi^t(w), \psi^t(w)$ first-order local translations of classical monotonic modal formula, and $\mathfrak{N}_{\mathsf{M.S}}$ the (possibly empty) first-order characterization of classical monotonic modal system $\mathsf{M.S}$. Then:

$$\Lambda *_{\mathsf{m}} \psi = \{\phi : \Lambda^t(w) * (\psi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}}) \vdash \phi^t(w)\}.$$

We now have the following result. Proof is in the appendix.

Theorem 4.1. *The operator $*_{\mathsf{m}}$ is an AGM operator.*

This theorem states that AGM revision is definable on the most general monotonic modal fragment of first-order logic. There are two families of revision operators for monotone modal logic, which we may generalize.

Corollary 4.2. *For systems of monotonic modal logic $\mathsf{EM.S}$ or $\mathsf{EM.S'}$, for any $\mathsf{S} \subseteq \{\mathbf{N}, \mathbf{C}, \mathbf{T}, \mathbf{4'}, \mathbf{B}, \mathbf{D}\}$, and any $\mathsf{S'} \subseteq \{\mathbf{P}, \mathbf{4}, \mathbf{5}\}$, there exist operators $*_{\mathsf{m.s}}$ and $*_{\mathsf{m.s'}}$ that are AGM, but not necessarily for $*_{\mathsf{m.s \cup s'}}$.*

This corollary tells us that modal revision operators within the two families S and $\mathsf{S'}$ preserve the corresponding modal logic. This is because $\mathfrak{N}_{\mathsf{M.S}}$ is the first-order expression of the neighborhood frame properties that characterize the corresponding modal system $\mathsf{M.S}$, which remain in the revised modal theory by (**K**$*$**2**).

5 An Application

One application is to interpret the necessity operator ∇ as 'qualitative judgment of high likelihood' in system (EMN) Arló-Costa (2005) in general, or as qualitative judgments of high evidential probability in particular Kyburg and Teng (2002). In the case of evidential probability (EP), probability is assigned to a sentence based upon both logical and probabilistic information—and there is a small chance that each item of evidence is accepted in error. But, some evidence sets containing an error are better than others, which is to say that some evidence for a statement is more robust than other evidence. To assess robustness of an EP assignment to a statement, one needs to look at a set of counter-factual EP probability assignments Haenni et al. (2010) to measure the variation in probability assignments when various items of evidence are excluded because false. A counterfactual evidence set (relative to a statement) is determined by the contraction operator $\dot{-}_{\mathsf{mn}}$ defined by the Harper Identity Harper (1977), Gärdenfors (1988) with respect to $*_{\mathsf{mn}}$.

6 Limits to the Approach

The main bottleneck in this approach is the absence of a complete first-order correspondent for classical modal logic. But, even if we did know the first-order classical modal fragment, it is likely that this fragment would not cover all of classical modal logic. More is known about simulation, but even bimodal simulations would not yield a complete AGM theory for classical modal systems. Alternatively, AGM might apply directly to a suitably abstract notion of classical modal consequence, but this would require adapting AGM for a quantified language.

This said, the result should not be under-appreciated for it opens the study of belief change for a variety of non-adjunctive frameworks. In addition, the general technique holds promise for importing other capabilities into monotone modal systems.

7 Appendix

Proof of Theorem 4.1 The operator $*_{\mathsf{m}}$ is an AGM operator for the smallest monotonic logic, EM

Let Λ be an EM-consistent monotonic modal theory, and ϕ, ψ and γ sentences in $\mathscr{L}_{\nabla}$. We show that $*_{\mathsf{m}}$ satisfies the AGM postulates. First, observe that M is the smallest classical monotonic modal system, which is equivalent to $\mathsf{EM.S}$, where $\mathsf{S} = \emptyset$. Hence, $\mathfrak{N}_{\mathsf{M.S}} = \emptyset$.

1. ($\Lambda$$*$1): $\Lambda *_{\mathsf{m}} \phi$ is a belief set.

   Since $\Lambda^t(w) * (\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}}))$ is closed under $\vdash$ by (K$*$1), then $\Lambda *_{\mathsf{m}} \phi$ is closed under $\vdash_{\mathsf{EM}}$.

2. ($\Lambda$$*$2): $\phi \in (\Lambda *_{\mathsf{m}} \phi)$.

   From ($K$$*$2) we have $\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}} \in \Lambda^t(w) * (\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}}$. Since $\Lambda^t(w) * (\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}})$ is closed under $\vdash$, by (K$*$1), and $\vdash$ is reflexive, then $\Lambda^t(w) * (\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}}) \vdash \phi^t(w)$. So, $\phi \in (\Lambda *_{\mathsf{m}} \phi)$ by ($\Lambda$$*$2).

3. ($\Lambda$$*$3,4): If sentence ϕ is EM-consistent with Λ, then $\Lambda *_{\mathsf{m}} \phi$ is equal to the closure of $\{\Lambda \cup \{\phi\}\}$ under $\vdash_{\mathsf{EM}}$, written $C_{\mathsf{m}}(\Lambda \cup \{\phi\})$.

   First we make the following two observations.

   Observation 1. Recall that if Λ is an EM-consistent modal theory, then $\Lambda \nvdash_{\mathsf{EM}} \perp$ and there exists a monotone neighborhood model for Λ.

   Observation 2. If $\Lambda \cup \{\phi\}$ is consistent with respect to classical modal logic EM, then $\Lambda^t(w)$ is classically consistent with respect to its translation, $\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}}$. Since by hypothesis $\Lambda \cup \{\phi\}$ has a monotone neighborhood model, by Observation 1, there exists a classical first-order model of its translation, $\Lambda^t(w) \cup \{\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}}\}$.

   Suppose that Θ denotes the classical provability closure of the first-order translation from Observation 2, $\Lambda^t(w)*(\phi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}})$. We now show that if $\psi^t(w) \in \Theta$, then $\Lambda *_{\mathsf{m}} \phi \vdash \psi$.

   Suppose that $C_{\mathsf{m}}(\Lambda)$ is Λ closed under $\vdash_{\mathsf{EM}}$ and $\Lambda^t(w)$ is the first-order translation of Λ. We denote the corresponding $\mathfrak{N}_{\mathsf{M.S}}$-simulated closure in classical logic of the first-order translation by $Cn(\Lambda^t)$. There are two parts.

   (a) First, for any $\gamma \in \mathfrak{N}_{\mathsf{M.S}}$, if $\gamma^t \in Cn(\Lambda^t)$, then $\gamma \in \Lambda$. To see this, notice that $C_{\mathsf{m}}(\Lambda)$ is a maximally EM-consistent set, so $\gamma \in C_{\mathsf{m}}(\Lambda)$ iff $\Lambda \vdash_{\mathsf{EM}} \gamma$.
   Proof: Suppose that $\gamma \notin \Lambda$. Then, there is a classical monotone model satisfying $\Lambda \cup \{\neg\gamma\}$ and a translation of this into first-order logic. But on the first-order model for this translation $\gamma^t \notin Cn(\Lambda^t)$, which falsifies the hypothesis.

(b) Second, for a closed classical theory $Cn(\Lambda^t)$ s.t. $\mathfrak{N}_{\mathsf{M.S}} \subseteq Cn(\Lambda^t)$ and $\{\gamma : \gamma^t \in \Lambda^t\}$, then $\Lambda \vdash \gamma$ only if $\gamma^t \in Cn(\Lambda^t)$.
Proof: Suppose that $\gamma^t \notin Cn(\Lambda^t)$. Then there is a model of $\Lambda^t \cup \{\neg\gamma^t\}$, so there is classical monotone model satisfying $\Lambda \cup \{\neg\gamma\}$ which falsifies the hypothesis.

This concludes the proof of ($\blacksquare$$*$**3**,**4**).

4. ($\Lambda*5$): $\Lambda *_{\mathsf{m}} \phi = \mathscr{L}_{\nabla}$ only if $\phi \equiv \bot$.

   Since Λ is an EM-consistent modal theory, $\Lambda \neq \mathscr{L}_{\nabla}$. So $\Lambda^t(w) \neq \mathscr{L}^1_{\nabla}$. So if $\Lambda^t(w) * \phi^t(w) = \mathscr{L}^1_{\nabla}$, then $\phi^t(w) = \bot$; thus $\phi \equiv \bot$.

5. ($\Lambda*6$): If $\vdash_{\mathsf{EM}} \phi \equiv \psi$, then $\Lambda *_{\mathsf{m}} \phi \equiv \Lambda *_{\mathsf{m}} \psi$.

   If $\vdash_{\mathsf{EM}} \phi \equiv \psi$, then $\vdash \phi^t \wedge \mathfrak{N}_{\mathsf{M.S}}) \equiv \psi^t \wedge \mathfrak{N}_{\mathsf{M.S}}$. So, by (K$*$6), $\Lambda * (\phi^t \wedge \mathfrak{N}_{\mathsf{M.S}}) \equiv \Lambda * (\psi^t \wedge \mathfrak{N}_{\mathsf{M.S}})$. Therefore, $\Lambda *_{\mathsf{m}} \phi \equiv \Lambda *_{\mathsf{m}} \psi$.

6. ($\Lambda*7,8$): $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) = C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$, when ψ is EM-consistent with $\Lambda *_{\mathsf{m}} \phi$).

   Now we proceed in two parts.

   (a) $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) \subseteq C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$: By ($\Lambda*1$), $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) = C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} (\phi \wedge \psi))$. Suppose that $\gamma \in C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} (\phi \wedge \psi))$. Then by the correspondence theorem $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \psi^t \wedge \mathfrak{N}_{\mathsf{M.S}}))$. So $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \mathfrak{N}_{\mathsf{M.S}}) \cup \{\psi^t\})$, by (K$*$7), and $\gamma \in C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$, by correspondence. Since γ is an arbitrary modal formula, $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) \subseteq C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$.

   (b) $C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\}) \subseteq \Lambda *_{\mathsf{m}} (\phi \wedge \psi)$: Suppose that $\gamma \in C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} \phi)$. Since γ is EM-consistent with $\Lambda *_{\mathsf{m}} \phi)$, $\gamma \in C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$. Thus, $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \mathfrak{N}_{\mathsf{M.S}}) \cup \{\psi^t\})$, by the correspondence theorem, and $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \psi^t \wedge \mathfrak{N}_{\mathsf{M.S}}))$, by (K$*$8). So, $\gamma \in C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} (\phi \wedge \psi))$, by correspondence. Since γ is an arbitrary modal formula, $C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\}) \subseteq \Lambda *_{\mathsf{m}} (\phi \wedge \psi)$.

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