DEGREES OF MONOTONE COMPLEXITY

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Abstract. Levin and Schnorr (independently) introduced the monotone complexity, $K_m(\alpha)$, of a binary string α. We use monotone complexity to define the relative complexity (or relative randomness) of reals. We define a partial ordering $\leq_{K_m}$ on 2^ω by $\alpha \leq_{K_m} \beta$ iff there is a constant c such that $K_m(\alpha \upharpoonright n) \leq K_m(\beta \upharpoonright n) + c$ for all n. The monotone degree of α is the set of all β such that $\alpha \leq_{K_m} \beta$ and $\beta \leq_{K_m} \alpha$. We show the monotone degrees contain an antichain of size $2^{\aleph_0}$, a countable dense linear ordering (of degrees of cardinality $2^{\aleph_0}$), and a minimal pair.

Downey, Hirschfeldt, LaForte, Nies and others have studied a similar structure, the K-degrees, where K is the prefix-free Kolmogorov complexity. A minimal pair of K-degrees was constructed by Csima and Montalbán. Of particular interest are the noncomputable *trivial* reals, first constructed by Solovay. We define a real to be (K_m, K)-trivial if for some constant c, $K_m(\alpha \upharpoonright n) \leq K(n) + c$ for all n. It is not known whether there is a K_m-minimal real, but we show that any such real must be (K_m, K)-trivial.

Finally, we consider the monotone degrees of the computably enumerable (c.e.) and strongly computably enumerable (s.c.e.) reals. We show there is no minimal c.e. monotone degree and that Solovay reducibility does not imply monotone reducibility on the c.e. reals. We also show the s.c.e. monotone degrees contain an infinite antichain and a countable dense linear ordering.

§1. Background and Definitions. Several ways of defining the complexity of binary strings (i.e. members of $2^{<\omega}$) have been developed. Informally, the classical Kolmogorov complexity of a string σ (independently defined by Solomonoff [24], Kolmogorov [11] and Chaitin [2, 3]) is the length of the shortest program p that computes σ (using some fixed optimal abstract computer). Although this definition is beautifully simple and is useful for some purposes, it is not well suited for extending the definition of complexity to reals (i.e. members of 2^ω, binary sequences). In particular, there does not appear to be a simple definition of the random (1-random, Martin-Löf random) reals in terms of classical Kolmogorov complexity.[1] The problem is that the classical Kolmogorov complexity allows the lengths $|p|$ and $|\sigma|$ of p and σ to carry "extra" information. Loveland's definition [16] of uniform complexity removed the information in the length of σ. Informally the uniform complexity of σ is the length of the least program p that computes the nth bit of σ for all $n < |\sigma|$. However, uniform complexity does not

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[1] There has been some recent progress in this direction. Work of Miller [17] and Nies, Stephan and Terwijn [20] provides a natural definition of 2-randomness in terms of classical Kolmogorov complexity. Recently, Miller and Yu [18] have given a more complicated characterization of the 1-random reals in terms of classical Kolmogorov complexity.

deal with the extra information in the length of p and, like classical complexity, it is not well suited for defining the complexity of a real since it does not appear to provide a simple definition of the random reals.

Two definitions of the complexity of a string have been given that extend nicely to definitions of the random reals. Levin [13] defined monotone complexity in a way that eliminates the use of the length in both p and σ. Schnorr [22] independently defined a similar notion called process complexity. Schnorr's original definition of process complexity was not equivalent to monotone complexity, but he later introduced a slight change in the definition of process complexity [23] that makes the two complexities equivalent in the sense that their difference is bounded by a constant. The monotone complexity of σ, denoted $K_m(\sigma)$, will be the primary form of complexity used here, and we will give a formal definition below. The motivation for this research is that monotone complexity seems to be the most appropriate one for defining the complexity of a real. Not only does monotone complexity remove the hidden information in the lengths of both p and σ, it also characterizes the computable reals as the least complex reals. This makes sense since each computable real α can be described by a single finite program p. In the context of monotone complexity, it also makes sense to describe a noncomputable real as the limit of a sequence of computable reals.

The most commonly used complexity is prefix-free complexity. In contrast to monotone complexity, a prefix-free program can only describe a *finite* string. Prefix-free complexity deals with the "hidden" information in the length of p by requiring that the set of programs is prefix-free: no program can extend another program. This is sometimes described as requiring programs to be self-delimiting: each program must be able to compute its own length, so the length does not provide extra information. Although prefix-free complexity gives a nice definition of the random reals and has some technical advantages over monotone complexity, it is somewhat unnatural when applied to reals, since it does not provide a finite way to describe a computable real. We will denote the prefix-free complexity of σ by $K(\sigma)$. Prefix-free complexity was defined by Levin [14], Gács [10] and independently by Chaitin [4]. See the book by Li and Vitányi [15] for additional background on prefix-free complexity and the other complexities discussed above.

Each complexity also leads to a complexity of reals via the complexity functions on initial segments. We will consider two complexity functions to be equivalent if their difference is bounded. We use the notation $f \preceq g$ or $g \succeq f$ to mean there is a constant $c \in \omega$ such that $f(n) \leq g(n) + c$ for all $n \in \omega$. We use the notation $f \asymp g$ to mean that $f \preceq g$ and $g \preceq f$.

A monotone machine M is a computably enumerable (c.e.) set of pairs $\langle p, \sigma \rangle$ where $p, \sigma \in 2^{<\omega}$ and for every $\langle p, \sigma \rangle, \langle q, \tau \rangle \in M$, $p \subseteq q$ implies $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$. We define the monotone complexity of a string σ with respect to M to be ${K_m}^M(\sigma) = \min\{|p| : \langle p, \tau \rangle \in M \text{ for some } \tau \supseteq \sigma\}$. One can show there is a universal monotone machine U that can simulate any monotone machine. It follows that U is optimal: for any other monotone machine M, there is a constant c such that ${K_m}^U(\sigma) \leq {K_m}^M(\sigma) + c$ for all σ. For any string σ define $K_m(\sigma) = {K_m}^U(\sigma)$. In some constructions we will wish to approximate K_m by computable functions. Let ${K_m}^s(\sigma) = \min\{|p| : \langle p, \tau \rangle \in U_s \text{ and } \sigma \subseteq \tau\}$. For any program p and $n \in \omega$ define

$$\Phi_p^s(n) = \begin{cases} i \text{ if } \langle p, \sigma \rangle \in U_s \text{ and } \sigma(n) = i \\ \uparrow \text{ if there is no } \langle p, \sigma \rangle \in U_s \text{ with } |\sigma| > n \end{cases}.$$

We assume familiarity with prefix-free Kolmogorov complexity, K (also denoted H in the literature), and with standard notation and terminology for complexity and randomness (see [15] and [6]), but we note a few instances of special notation. We will use V to stand for the universal prefix-free machine used to define K. We adopt the convention that if n is an integer then $K(n)$ stands for $K(1^n)$. For any $\alpha \in 2^{<\omega}$ we define an *optimal monotone description* of α to be a program p of minimal length such that $\langle p, \beta \rangle \in U$ for some $\beta \supseteq \alpha$. An *optimal prefix-free description* of α is a program p of minimal length such that $\langle p, \alpha \rangle \in V$. Some of our constructions will involve the first place where two sequences differ. For $\alpha, \beta \in 2^{<\omega} \cup 2^{\omega}$ we define a partial function $\mu(\alpha, \beta) =$ the least initial segment of α (if any) that is incomparable with β. Finally, if α is a computable real we will extend the notation and define $K_m(\alpha)$ to be the least integer k such that $K_m(\alpha \upharpoonright n) \leq k$ for all $n \in \omega$.

§2. Basic Properties of K_m and K. This section contains properties of K_m and K that will be used in the following sections. We begin with well known results that are listed here without proof. (See [15].) The first proposition gives upper and lower bounds on $K(\alpha)$ in terms of the complexity of the length of α. (We use "log" to denote the binary logarithm with the convention that $\log x = 0$ for all $x < 1$.)

PROPOSITION 2.1. *For all $\alpha \in 2^{<\omega}$,*

$$K(|\alpha|) \leq K(\alpha) + O(1) \leq |\alpha| + K(|\alpha|) + O(1).$$

In particular, if $\alpha = 1^n$ then α can be described by a prefix-free encoding of the binary representation of n, so $K(n) \preceq \log n + K(\log n) \preceq 2 \log n$.

The second proposition gives an upper bound on $K_m(\alpha)$ in terms of the length of α. Since $K_m(1^n)$ is bounded by a constant, the corresponding lower bound is trivial.

PROPOSITION 2.2. *For all $\alpha \in 2^{<\omega}$, $K_m(\alpha) \leq |\alpha| + O(1)$.*

The next result shows that both K and K_m may be used to give simple definitions of Martin-Löf randomness.

PROPOSITION 2.3. *The following are equivalent (i) α is Martin-Löf random (ii) $K(\alpha \upharpoonright n) \succeq n$ (iii) $K_m(\alpha \upharpoonright n) \asymp n$.*

Note that the characterization of the random reals is, in a sense, simpler using K_m, since a real is random if and only if its monotone complexity is asymptotic to n. On the other hand, the prefix-free complexity of a random real always grows larger than n as indicated by the next proposition.

PROPOSITION 2.4. *If α is a random real then $\lim_{n \to \infty} K(\alpha \upharpoonright n) - n = \infty$.*

In fact, for most reals, the prefix-free complexity rises infinitely often to the maximum possible.

PROPOSITION 2.5. *The set of reals α such that for infinitely many n*

$$K(\alpha \upharpoonright n) \geq n + K(n) + O(1)$$

has uniform measure one.

The next two propositions record the fact that the prefix-free complexity of a real oscillates, but the monotone complexity does not. The result for K_m follows directly from the definition. Let D be a subset of $2^{<\omega}$. We will call a function $f : D \to \omega$ *monotone* if $\alpha \subseteq \beta$ implies $f(\alpha) \leq f(\beta)$ for all α and β in D.

PROPOSITION 2.6. *K_m is a monotone function.*

PROPOSITION 2.7. *K is not a monotone function.*

The next two propositions concern the subadditivity property. Although subadditivity does not hold for monotone complexity, K_m has a weaker form of subadditivity (involving K) that will be useful.

PROPOSITION 2.8. *K is subadditive, i.e. for all $\alpha, \beta \in 2^{<\omega}$,*

$$K(\alpha^\frown\beta) \leq K(\alpha) + K(\beta) + O(1).$$

PROPOSITION 2.9. *K_m is not subadditive, but for all $\alpha, \beta \in 2^{<\omega}$,*

$$K_m(\alpha^\frown\beta) \leq K(\alpha) + K_m(\beta) + O(1).$$

The next two results show that the difference between $K_m(\alpha)$ and $K(\alpha)$ is bounded by a logarithmic function of $|\alpha|$. (See [26] for a proof of Proposition 2.11.)

PROPOSITION 2.10. *For all $\alpha \in 2^{<\omega}$, $K_m(\alpha) \leq K(\alpha) + O(1)$.*

PROPOSITION 2.11. *For any $\alpha \in 2^{<\omega}$, real number $\epsilon > 0$ and integer $k \geq 1$, $K(\alpha) \leq K_m(\alpha) + \log|\alpha| + \log\log|\alpha| + \cdots (1+\epsilon)\log^k|\alpha| + O(1)$.*

We now prove several lemmas that will be of use in later sections. The first lemma establishes a bound on the prefix-free complexity of $\mu(\alpha, \beta)$ in terms of the monotone complexities of α and β.

LEMMA 2.12. *For incomparable $\alpha, \beta \in 2^{<\omega}$,*

$$K(\mu(\alpha,\beta)) \leq K_m(\alpha) + K_m(\beta) + 2\log K_m(\alpha) + 2\log K_m(\beta) + O(1).$$

PROOF. We define a prefix-free machine M by $\langle p^{*\frown}q^*, \sigma\rangle \in M$ iff there are $p, q, \alpha^*, \beta^* \in 2^{<\omega}$ such that $\langle p^*, p\rangle, \langle q^*, q\rangle \in V$, $\langle p, \alpha^*\rangle, \langle q, \beta^*\rangle \in U$ and $\sigma = \mu(\alpha^*, \beta^*)$. Note that the domain of M is prefix-free and M is single valued by the definition of monotone complexity. Let p and q be optimal monotone descriptions of α and β. Let p^* and q^* be optimal prefix-free descriptions of p and q. Then $\langle p^{*\frown}q^*, \mu(\alpha,\beta)\rangle \in M$. Hence $K(\mu(\alpha,\beta)) \leq K(p) + K(q) + O(1) \leq |p| + 2\log p + |q| + 2\log q + O(1)$. The lemma follows since $|p| = K_m(\alpha)$ and $|q| = K_m(\beta)$. ⊣

The next lemma provides a lower bound on the maximum of the monotone complexities of $\sigma^\frown 0$ and $\sigma^\frown 1$ in terms of the prefix-free complexity of their length. (The factor 1/3 in Lemma 2.13 is not best possible, but is sufficient for our purposes.)

LEMMA 2.13. *For any* $\sigma \in 2^{<\omega}$,

$$\max\{K_m(\sigma^\frown 0), K_m(\sigma^\frown 1)\} \geq K(|\sigma|+1)/3 - O(1).$$

PROOF. Let $\alpha = \sigma^\frown i$ and $\beta = \sigma^\frown(1-i)$, where $i \in \{0,1\}$ is chosen so that $K_m(\alpha) \geq K_m(\beta)$. By Lemma 2.12 there is a constant c such that $K(\alpha) \leq K_m(\alpha) + 2\log K_m(\alpha) + K_m(\beta) + 2\log K_m(\beta) + c$. Since $K_m(\alpha) \geq K_m(\beta)$ and $\log x \leq x/4$ for $x \geq 16$, we get $K(\alpha) \leq 3K_m(\alpha) + c$. Let $c_1 = c/3$. Then $\max\{K_m(\alpha), K_m(\beta)\} = K_m(\alpha) \geq K(\alpha)/3 - c_1 \geq K(|\sigma|+1)/3 - O(1)$. ⊣

The next lemma shows that when a string α is extended by concatenating a string β, the monotone complexity of β is bounded above by the monotone complexity of $\alpha^\frown\beta$ plus the prefix-free complexity of the length of α.

LEMMA 2.14. *For any* $\alpha, \beta \in 2^{<\omega}$, $K_m(\beta) \leq K_m(\alpha^\frown\beta) + K(|\alpha|) + O(1)$.

PROOF. We define a monotone machine M by $\langle p^\frown q, \tau\rangle \in M$ iff $\langle p, 1^k\rangle \in V$ for some k and $\langle q, \sigma^\frown\tau\rangle \in U$ for some σ with $|\sigma| = k$. The lemma follows by taking p to be an optimal prefix-free description of $1^{|\alpha|}$ and q to be an optimal monotone description of $\alpha^\frown\beta$. ⊣

The next lemma considers the case where the α in Lemma 2.14 is fixed. Then the term involving the length of α can be included in the constant.

LEMMA 2.15. *For any* $\alpha \in 2^{<\omega}$, *there is a constant c such that for any* $\beta \in 2^{<\omega}$, $K_m(\beta) \leq K_m(\alpha^\frown\beta) + c$.

PROOF. By Lemma 2.14, $K_m(\beta) \leq K_m(\alpha^\frown\beta) + K(|\alpha|) + c'$ for some constant c'. Let $c = K(|\alpha|) + c'$. ⊣

The final lemma of this section considers the situation where a string α is extended by a computable real β. Then the monotone complexity of $\alpha^\frown\beta \upharpoonright n$ is bounded by the prefix-free complexity of α plus a constant.

LEMMA 2.16. *For any computable* $\beta \in 2^\omega$, *there is a constant c_β such that for any* $\alpha \in 2^{<\omega}$ *and any* $n \in \omega$, $K_m(\alpha^\frown(\beta \upharpoonright n)) \leq K(\alpha) + c_\beta$.

PROOF. By Proposition 2.9 there is a constant c such that $K_m(\alpha^\frown(\beta \upharpoonright n)) \leq K(\alpha) + K_m(\beta \upharpoonright n) + c$. Since β is computable, there is a constant k such that $K_m(\beta \upharpoonright n) \leq k$ for all n. Let $c_\beta = c + k$. ⊣

§3. Monotone degrees. In this section we will study degrees of randomness defined in terms of monotone complexity. Although monotone complexity has been studied intensively, there does not appear to be much previous work on the monotone degrees. However, the following definitions as well as an extensive discussion of the K-degrees and other degree structures relating to randomness can be found in a forthcoming book by Downey and Hirschfeldt [6].

We define a partial ordering on reals by $\alpha \leq_{K_m} \beta$ iff $K_m(\alpha \upharpoonright n) \preceq K_m(\beta \upharpoonright n)$. We define $\alpha \equiv_{K_m} \beta$ iff $\alpha \leq_{K_m} \beta$ and $\beta \leq_{K_m} \alpha$. We define the monotone degree of a real α to be the set $\{\beta : \alpha \equiv_{K_m} \beta\}$. Note that the bottom monotone degree $\mathbf{0}_m$ is the (countable) set of all computable reals and the top monotone degree $\mathbf{1}_m$ is the set of all random reals. The top degree has the cardinality of the continuum since it is well known that there are $2^{\aleph_0}$ many random reals. We will show that there are $2^{\aleph_0}$ many intermediate monotone degrees (strictly between $\mathbf{0}_m$ and $\mathbf{1}_m$).

THEOREM 3.1. *For any real α, if α is neither random nor computable, then there is a real β such that the monotone degrees of α and β are incomparable.*

PROOF. Let γ be a random real. We will construct β in stages. Let $\beta_0 = \emptyset$. At an even stage $s > 0$ let $\beta_{s+1} = \beta_s \frown 0^k$ for some k such that $K_m(\beta_s \frown 0^k) < K_m(\alpha \upharpoonright (|\beta_s| + k)) - s$. Note that such a k exists since $K_m(\alpha \upharpoonright n) \to \infty$ and $K_m(\beta_s \frown 0^n)$ is bounded by Lemma 2.16. At odd stages s let $\beta_{s+1} = \beta_s \frown (\gamma \upharpoonright k)$ for some k such that $K_m(\beta_s \frown (\gamma \upharpoonright k)) > K_m(\alpha \upharpoonright (|\beta_s| + k)) + s$. To see that such a k exists, note that since γ is random, $\beta_s \frown \gamma$ is random by Lemma 2.15. Hence $K_m(\beta_s \frown (\gamma \upharpoonright n)) \asymp n$. Since α is not random, $K_m(\alpha \upharpoonright (|\beta_s| + n)) \prec n$, and the existence of k follows. This completes the construction. The theorem follows since the even stages of the construction ensure $K_m(\alpha \upharpoonright n) \not\preceq K_m(\beta \upharpoonright n)$ and the odd stages ensure $K_m(\beta \upharpoonright n) \not\preceq K_m(\alpha \upharpoonright n)$. ⊣

COROLLARY 3.2. *For any countable set of noncomputable, nonrandom reals A, there is a real β that is incomparable with each $\alpha \in A$.*

PROOF. Dovetail countably many strategies for making β incomparable with α_i as given in the proof of Theorem 3.1. ⊣

THEOREM 3.3. *There is an antichain of $2^{\aleph_0}$ monotone degrees.*

PROOF. We build a real γ_α for each $\alpha \in 2^\omega$ and ensure that if $\alpha \neq \beta$ then γ_α is incomparable with γ_β. At stage 0 let $\gamma_\emptyset = \emptyset$. At stage $s > 0$ we will define a binary string, γ_σ, for each $\sigma \in 2^s$. Let the strings in 2^s be labeled $\sigma_1, \sigma_2 \ldots \sigma_{2^s}$. Stage s involves $2^s + 1$ many steps. At step 0, let $\gamma_{\sigma_i,0} = \gamma_{\sigma \upharpoonright s-1}$ for all i. At step $j > 0$ choose each $\gamma_{\sigma_i,j}$ to be an extension of $\gamma_{\sigma_i,j-1}$ such that for all $i \neq j$, $K_m(\gamma_{\sigma_j,j}) > K_m(\gamma_{\sigma_i,j}) + s$. (This can be done by having γ_{σ_j} copy a random real and, for $i \neq j$, having γ_{σ_i} copy a computable real as in the strategies from the proof of Theorem 3.1.) To complete stage s, we let $\gamma_{\sigma_i} = \gamma_{\sigma_i,2^s}$ for all i. To complete the construction, for each α we define $\gamma_\alpha = \bigcup_{s \in \omega} \gamma_{\alpha \upharpoonright s}$. Now if $\alpha \neq \beta$ then there is a t such that for all $s \geq t$, $\alpha \upharpoonright s \neq \beta \upharpoonright s$. By the construction, for each $s \geq t$ there is an n and an m such that $K_m(\gamma_\alpha \upharpoonright n) > K_m(\gamma_\beta \upharpoonright n) + s$ and $K_m(\gamma_\beta \upharpoonright m) > K_m(\gamma_\alpha \upharpoonright m) + s$. Therefore, γ_α and γ_β are incomparable. ⊣

A *minimal pair* in a degree structure is a pair of distinct nonzero degrees $\mathbf{a}$ and $\mathbf{b}$ with the property that the only degree $\mathbf{d}$ such that $\mathbf{d} \leq \mathbf{a}$ and $\mathbf{d} \leq \mathbf{b}$ is the bottom degree $\mathbf{0}$. Csima and Montalbán [5] have shown that there is a minimal pair of K-degrees. The next theorem shows that there is a minimal pair of monotone degrees. To compare these two results, note that the $\mathbf{0}$ degree for the K-degrees is the set of K-trivial reals, while the $\mathbf{0}$ degree for the monotone degrees is the set of computable reals. The two results (which were obtained independently) are therefore different, although there are similarities in the method of proof.

THEOREM 3.4 (Minimal Pair). *There are noncomputable reals α and β such that for any real γ, $\alpha \geq_{K_m} \gamma$ and $\beta \geq_{K_m} \gamma$ implies γ is computable.*

PROOF. We will construct α and β in stages. Let c_1 be the constant in Lemma 2.13. Let $\rho = 0^\omega$ and let $c_2 = c_\rho$ as defined in Lemma 2.16. At each stage, α_s and β_s will be finite initial segments of α and β. Let $\alpha_0 = \beta_0 = \emptyset$. At each even stage $s > 0$, let α^*_{s-1} be an extension of α_{s-1} such that $K_m(\alpha^*_{s-1}) > K_m(\alpha_{s-1})$. Call $\sigma \in 2^{<\omega}$ *terminal* if there are is no proper extension $\tau \supset \sigma$ such that $K_m(\tau) = K_m(\sigma)$. Choose n large enough so that the following three conditions hold: (1) there is no terminal σ with $|\sigma| \geq n$ and $K_m(\sigma) < K(\alpha^*_{s-1}) + c_2 + s$, (2) $n \geq |\alpha^*_{s-1}|$ and (3) for all $m \geq n$, $K(m) \geq 3(K(\alpha^*_{s-1}) + c_1 + c_2 + s)$. Let $\beta_s = \beta_{s-1} \frown 0^{n-|\beta_{s-1}|}$ and let $\alpha_s = \alpha^*_{s-1} \frown 0^{n-|\alpha^*_{s-1}|}$. At an odd stage s, do the same as at an even s but with the roles of α and β reversed.

End of the construction

We now show that α and β form a minimal pair. Suppose $\gamma \leq_{K_m} \alpha$ and $\gamma \leq_{K_m} \beta$. We will call s a *change* stage if $K_m(\gamma \restriction n_{s+1}) > K_m(\gamma \restriction n_s)$.

Case 1: Suppose there are infinitely many change stages. There are either infinitely many even change stages or infinitely many odd ones. Assume there are infinitely many even ones. For each even change stage s, let x be such that $n_s < x \leq n_{s+1}$ and $K_m(\gamma \restriction x-1) < K_m(\gamma \restriction x)$. Since $x > n_s$, if $\gamma \restriction x-1$ is terminal, then $K_m(\gamma \restriction x) \geq K(\alpha^*_{s-1}) + c_2 + s$ by condition (1). On the other hand, if $\gamma \restriction x-1$ is not terminal, then $K_m(\gamma \restriction x) = \max\{K_m((\gamma \restriction x-1)\frown 0), K_m((\gamma \restriction x-1)\frown 1)\}$. By Lemma 2.13, $K_m(\gamma \restriction x) \geq K(x)/3 - c_1$. By condition (3), $K(m) \geq 3(K(\alpha^*_{s-1}) + c_1 + c_2 + s)$. Thus $K_m(\gamma \restriction x) \geq K(\alpha^*_{s-1}) + c_2 + s$ holds whether $\gamma \restriction x-1$ is terminal or not. Now by Lemma 2.16 we have $K_m(\alpha \restriction x) = K_m(\alpha^*_{s-1} \frown 0^{x-|\alpha^*_{s-1}|}) \leq K(\alpha^*_{s-1}) + c_2$. Substitution into the previous inequality yields $K_m(\gamma \restriction x) \geq K_m(\alpha \restriction x) + s$. Since this inequality holds for infinitely many even change stages s, we have that $\gamma \not\leq_{K_m} \alpha$. A similar argument shows that if there are infinitely many odd change stages then $\gamma \not\leq_{K_m} \beta$. Thus Case 1 cannot occur.

Case 2: There are finitely many change stages. Let s^* be the largest change stage. Then $K_m(\gamma) = K_m(\gamma \restriction n_{s^*}) \in \omega$. So γ is computable. ⊣

The existence of a minimal pair prompts one to ask whether there is a minimal (nonzero) degree. As discussed in the next section, it is known that there is no minimal K-degree in the computably enumerable reals and we will also show that there is no minimal monotone degree in the computably enumerable reals. However, in the full structures of the K-degrees and the monotone degrees the question is open. The next theorem and corollary shows that if a minimal monotone degree exists, it would have to be (K_m, K)-trivial as defined below. The definition is modeled after the definition of the K-trivial reals: α is *K-trivial* if $K(\alpha \restriction n) \preceq K(n)$.

DEFINITION 3.5. A real α is (K_m, K)-trivial if $K_m(\alpha \restriction n) \preceq K(n)$.

The first example of an K-trivial real was constructed by Solvay [25]. The K-trivial reals have many interesting properties and have been studied by (among others) Downey, Hirschfeldt, Nies and Stephan [9]. In particular Hirschfeldt and Nies [19] have shown a real α is K-trivial if and only if it is low for random (i.e. the set of reals that are random relative to α is exactly the set of random reals). Related forms of "triviality" have been studied by Kummer [12] and by Becher, Figueira, Nies and Picchi [1]. It is obvious that every K-trivial real is (K_m, K)-trivial. It is not known whether there are (K_m, K)-trivial reals that are not K-trivial. The following theorem shows that reals that are not (K_m, K)-trivial are "far" from the bottom monotone degree, in the sense that there are continuum-many reals below them.

THEOREM 3.6. *If a real α is not (K_m, K)-trivial then the set $\{\beta : \beta <_{K_m} \alpha\}$ has cardinality $2^{\aleph_0}$.*

PROOF. Assume α is not (K_m, K)-trivial. We define a set $A = \{a_0, a_1, \dots\}$ such that if $B \subseteq A$ and $\beta = \chi_B$ then $\alpha >_{K_m} \beta$. We will construct A in stages. For each $s \geq 0$, let $A_s = \{a_0, a_1, \dots a_{s-1}\}$ and for all $C \subseteq A_s$ let $\sigma_C = \chi_C \restriction a_{s-1} + 1$. We define a_s to be the least integer $a > a_{s-1}$ such that the following inequality holds for all $C \subseteq A_s$:

$$K_m(\sigma_C {}^\frown 0^\omega), K_m(\sigma_C {}^\frown 0^{a-a_{s-1}-1} {}^\frown 1 {}^\frown 0^\omega) \leq K_m(\alpha \restriction a+1) - s. \tag{1}$$

To see that such an a exists, first note that by Lemma 2.16 there is a constant k_1 such that $K_m(\sigma {}^\frown 0^\omega) \leq K(\sigma) + k_1$ for all $\sigma \in 2^{<\omega}$. Note also that for each $C \subseteq A_s$, $\sigma_C {}^\frown 0^{a-a_{s-1}-1} {}^\frown 1$ is determined by its length, $a+1$. Hence there is a constant k_C such that $K(\sigma_C {}^\frown 0^{a-a_{s-1}-1} {}^\frown 1) \leq K(a+1) + k_C$ for all $a > a_{s-1}$. Letting $k = k_1 + \max\{k_C : C \subseteq A_s\}$ we obtain

$$K_m(\sigma_C {}^\frown 0^\omega), K_m(\sigma_C {}^\frown 0^{a-a_{s-1}-1} {}^\frown 1 {}^\frown 0^\omega) \leq K(a+1) + k. \tag{2}$$

Since α is not (K_m, K)-trivial, there is an a such that $K_m(\alpha \restriction a+1) > K(a+1) + k + s$. Combining with inequality (2) we see that such an a satisfies inequality (1). This complete the construction of A.

It remains to show that A has the desired property. Let $B \subseteq A$ and $\beta = \chi_B$. We now show $\beta <_{K_m} \alpha$ by showing $K_m(\beta \restriction n) \prec K_m(\alpha \restriction n)$.

Case 1: $n > a_0$. Let s be largest such that $a_s < n$. Let $C = A_s \cap B$ and $\sigma_C = \chi_C \restriction a_{s-1} + 1$. If $a_s \in B$, then $\beta \restriction n \subseteq \sigma_C {}^\frown 0^{a_s - a_{s-1}+1} {}^\frown 1 {}^\frown 0^\omega$. If $a_s \notin B$, then $\beta \restriction n \subseteq \sigma_C {}^\frown 0^\omega$. Either way, the definition of a_s implies $K_m(\beta \restriction n) \leq K_m(\alpha \restriction a_s + 1) - s \leq K_m(\alpha \restriction n) - s$.

Case 2: $n \leq a_0$. Then $\beta \restriction n \subseteq 0^\omega$. Let $d = K_m(0^\omega)$. Then for all $n \leq a_0$ we have $K_m(\beta \restriction n) \leq K_m(\alpha \restriction n) + d$.

The two cases show $\beta \leq_{K_m} \alpha$. Since $K_m(\beta \restriction n) \leq K_m(\alpha \restriction n) - s$ for unboundedly large values of s in Case 1, $\beta <_{K_m} \alpha$. ⊣

COROLLARY 3.7. *If a real α is K_m-minimal, then α is (K_m, K)-trivial.*

PROOF. Suppose α is K_m-minimal but not (K_m, K)-trivial. By the previous theorem, $\{\beta : \beta <_{K_m} \alpha\}$ has cardinality $2^{\aleph_0}$. But since α is K_m-minimal this set is the countable set of computable reals, giving a contradiction. ⊣

We now introduce an operation, $\otimes$, that produces a real $\alpha \otimes f$ from a real α and a strictly increasing function f. The graph of the complexity function of $\alpha \otimes f$ is obtained by "horizontally stretching" the graph of the complexity function of α.

DEFINITION 3.8. Given any real α and a strictly increasing function $f : \omega \to \omega$, let $\alpha \otimes f$ be the real defined by

$$(\alpha \otimes f)(n) = \begin{cases} \alpha(f^{-1}(n)) & \text{if } n \in \text{range}(f) \\ 0 & \text{otherwise} \end{cases}$$

For any string $\tau \in 2^{<\omega}$ we define $\tau \otimes f$ as above except that $\tau \otimes f$ is a string with length $f(|\tau|)$.

THEOREM 3.9. *If α is any real and $f : \omega \to \omega$ is a strictly increasing computable function, then $K_m(\alpha \otimes f \upharpoonright n) \asymp K_m(\alpha \upharpoonright f^{-1}[n])$ where $f^{-1}[n] = \max\{k : f(k) \leq n\}$.*

PROOF. First we construct a monotone machine M_1 to show $K_m(\alpha \otimes f \upharpoonright n) \preceq K_m(\alpha \upharpoonright f^{-1}[n])$. We enumerate the axiom $\langle p, \sigma \rangle \in M_1$ iff $\sigma = (\tau \otimes f)^\frown 0^k$ for some $\tau \in 2^{<\omega}$ and $k \in \omega$, where $\langle p, \tau \rangle \in U$ and $k < f(|\tau| + 1) - f(|\tau|)$. Now we show that M_1 is a monotone machine. Suppose $\langle p, \sigma_1 \rangle, \langle q, \sigma_2 \rangle \in M_1$ and p and q are comparable. Then for some $\tau_1, \tau_2 \in 2^{<\omega}$ and some $k_1, k_2 \in \omega$, we have $\langle p_1, \tau_1 \rangle, \langle p_2, \tau_2 \rangle \in U$, $k_1 < f(|\tau_1|+1) - f(|\tau_1|)$ and $k_2 < f(|\tau_2|+1) - f(|\tau_2|)$. Since U is a monotone machine, τ_1 and τ_2 are comparable. Without loss of generality, assume $\tau_1 \subseteq \tau_2$. Then $\sigma_1, \sigma_2 \subseteq (\tau_2 \otimes f)^\frown 0^\omega$, so σ_1 and σ_2 are comparable and M_1 is monotone. Now if $\sigma = \alpha \otimes f \upharpoonright n$ then $\langle p, \sigma \rangle \in M_1$ where p is an optimal monotone description of $\alpha \upharpoonright f^{-1}[n]$. It follows that $K_m(\sigma) \preceq |p| = K_m(\alpha \upharpoonright f^{-1}[n])$.

Now we construct a monotone machine M_2 to show $K_m(\alpha \otimes f \upharpoonright n) \succeq K_m(\alpha \upharpoonright f^{-1}[n])$. We enumerate the axiom $\langle p, \sigma \rangle \in M_2$ iff $\langle p, \sigma \otimes f \rangle \in U$. Note that if p and q are comparable and $\langle p, \sigma_1 \rangle, \langle q, \sigma_2 \rangle \in M_2$ then we have $\langle p, \sigma_1 \otimes f \rangle, \langle q, \sigma_2 \otimes f \rangle \in U$. Since U is monotone, $\sigma_1 \otimes f$ and $\sigma_2 \otimes f$ are comparable. Without loss of generality, assume $\sigma_1 \otimes f \subseteq \sigma_2 \otimes f$. It clearly follows from the definition of $\otimes$ that $\sigma_1 \subseteq \sigma_2$, and so M_2 is monotone. Now let $\sigma = \alpha \upharpoonright f^{-1}[n]$. Then $\sigma \otimes f = (\alpha \otimes f) \upharpoonright f(f^{-1}[n]) \subseteq (\alpha \otimes f) \upharpoonright n$. Letting p be an optimal monotone description of $\sigma \otimes f$, we obtain $K_m(\sigma) \preceq |p| = K_m(\sigma \otimes f) \leq K_m(\alpha \otimes f \upharpoonright n)$. ⊣

The following corollary shows that for any fixed strictly increasing, computable function f, the operation $\alpha \otimes f$ induces a well-defined mapping on the K_m-degrees.

COROLLARY 3.10. *For any reals α and β and a strictly increasing, computable function f, if $K_m(\alpha \upharpoonright n) \asymp K_m(\beta \upharpoonright n)$ then $K_m(\alpha \otimes f \upharpoonright n) \asymp K_m(\beta \otimes f \upharpoonright n)$.*

PROOF. By Theorem 3.9, $K_m(\alpha \otimes f \upharpoonright n) \asymp K_m(\alpha \upharpoonright f^{-1}[n]) \asymp K_m(\beta \upharpoonright f^{-1}[n]) \asymp K_m(\beta \otimes f \upharpoonright n)$. ⊣

We now use Theorem 3.9 to show that there are infinitely many monotone degrees of cardinality $2^{\aleph_0}$.

COROLLARY 3.11. *There is an order-preserving embedding from the rationals to the monotone degrees such that each degree in the image of the embedding has cardinality $2^{\aleph_0}$.*

PROOF. It suffices to show that there is an order-preserving embedding from the interval $\mathbf{Q} \cap (0, 1)$ to the monotone degrees such that each degree in the image has cardinality $2^{\aleph_0}$. For any rational number $r \in (0, 1)$ let $f_r(n) = \lfloor n/r \rfloor$. Let α

be a random real. Then by Theorem 3.9, $K_m(\alpha \otimes f_r \upharpoonright n) \asymp K_m(\alpha \upharpoonright f_r^{-1}[n]) \asymp f_r^{-1}[n]$ since α is random. Now $f_r^{-1}[n] = \max\{k : \lfloor k/r \rfloor \ \leq n\} \asymp rn$. The map from r to $\alpha \otimes f_r$ induces the required embedding since $r < s$ implies $rn \prec sn$. Finally, note that if α and β are random reals and $\alpha \neq \beta$ then $\alpha \otimes f_r \asymp \beta \otimes f_r$ and $\alpha \otimes f_r \neq \beta \otimes f_r$. Since there are $2^{\aleph_0}$ random reals, this implies each degree in the image of the embedding has cardinality $2^{\aleph_0}$. ⊣

§4. Computably enumerable monotone degrees. A real α is said to be a *computably enumerable real* (or c.e. real) if it is the sum of a computably enumerable set of rationals. (In this context, we identify a binary string or sequence α with the real number with binary representation $0.\alpha$. Then $\alpha + \beta$ is interpreted as addition mod 1, $c\alpha$ is the usual multiplication by a real c, and $\alpha \leq \beta$ is the usual ordering of real numbers.) Solovay [25] defined a reducibility relation on the c.e. reals called *domination* or *Solvay reducibility.* We write $\beta \leq_S \alpha$ if there is a constant c and a partial computable function $\phi : \mathbf{Q} \to \mathbf{Q}$ such that for each rational $q < \alpha$ we have $\phi(q) \downarrow < \beta$ and $\beta - \phi(q) \leq c(\alpha - q)$. Solovay showed that if α and β are c.e. reals and $\alpha \leq_S \beta$ then $\alpha \leq_K \beta$. Downey, Hirschfeldt and Nies [8] showed that the c.e. reals are dense under Solovay reducibility and they state that their density proof can be adapted to show the K-degrees of c.e. reals are dense. Downey, Hirschfeldt and LaForte [7] defined another ordering, rH-reducibility, and noted that the same method of proof can be adapted to show the c.e. reals are dense under rH-reducibility. Downey and Hirschfeldt [6] generalized these results to show that the c.e. reals are dense under any Σ_3^0 measure of relative randomness such that the bottom degree includes the computable reals, the top degree includes Chaitin's Ω, and $+$ induces the least upper bound operation.

Returning to the monotone degrees, we define a computably enumerable (c.e.) monotone degree to be one that contains a c.e. real. Clearly, $\leq_{K_m}$ is Σ_3^0, the bottom c.e. monotone degree is the set of computable reals and the top c.e. monotone degree contains Chaitin's Ω. However, it is not known whether the monotone degree of $\alpha + \beta$ is the least upper bound for the monotone degrees of α and β in the c.e. monotone degrees (or even whether the least upper bound always exists). Furthermore, the following theorem shows that Solovay reducibility of c.e. reals does not imply monotone reducibility.

THEOREM 4.1. *There are c.e. reals α and β such that $\beta \leq_S \alpha$ but $\beta \not\leq_{K_m} \alpha$.*

PROOF. We will construct α so that $\alpha < 2^{-1}$ and define $\beta = 2\alpha$. Then for any rational $q < \alpha$, $\beta - 2q = 2(\alpha - q)$. Hence $\alpha \leq_S \beta$. We use a finite injury priority argument to construct α. At each stage s, we will have a finite string α_s approximating α. To insure $\alpha < 2^{-1}$ we set $\alpha_0 = 0$ (a one-bit string) and issue the (strongest priority) restraint $\alpha \supseteq \alpha_0$. For $k \in \omega$ we attempt to satisfy requirement R_k: for some n, $K_m(\alpha \upharpoonright n + 1) \geq K_m(\alpha \upharpoonright n) + k$. Before continuing with the construction, we note that the existence of an α satisfying the requirements proves the theorem. This follows since $K_m(\beta \upharpoonright n) \asymp K_m(\alpha \upharpoonright n + 1) \not\preceq K_m(\alpha \upharpoonright n)$.

We now describe stage s of the construction of α. For each k we have a number $n_{k,s}$, where $n_{k,0} = 0$ for all k. We will say that requirement R_k is *currently* *satisfied* at stage s if ${K_m}^s(\alpha_s \upharpoonright n_{k,s}+1) \geq {K_m}^s(\alpha_s \upharpoonright n_{k,s}) + k$. Let k be least such that R_k is not currently satisfied. We act for R_k, observing the restraints imposed by R_j for $j < k$ and reinitializing R_j for $j > k$. Let σ be the longest initial segment of α_s that is restrained by some R_j with $j < k$. Let $\gamma = 1^\infty$. Let u be least such that $\sigma^\frown 1^{u\frown} 0 \geq \alpha_s$ and ${K_m}^s(\sigma^\frown 1^{u\frown}0) \geq K^s(\sigma) + c_\gamma + k$, where c_γ is the constant in Lemma 2.16. To see that such a u exists, note that for large enough u, $K_m(\sigma^\frown 1^{u\frown}0) \geq K_m(\sigma^\frown 1^\infty) \geq K_m(\sigma^\frown 1^{u+1})$. Thus by Lemma 2.13 there is a constant c_1 such that $K_m(\sigma^\frown 1^{u\frown}0) \geq K(r)/3 - c_1$, where $r = |\sigma^\frown 1^{u\frown}0|$. Since $K(r) \to \infty$ as $u \to \infty$, we can choose u large enough so $K_m(\sigma^\frown 1^{u\frown}0) \geq K^s(\sigma) + c_\gamma + k$. Finally, note that ${K_m}^s(\sigma^\frown 1^{u\frown}0) \geq K_m(\sigma^\frown 1^{u\frown}0)$. Having established that a u exists with the desired property, set $\alpha_{s+1} = \sigma^\frown 1^{u\frown}0$, $n_{k,s+1} = r-1$, and issue the restraint $\alpha \supseteq \alpha_{s+1}$.

We now verify that each requirement is satisfied. We first note that it suffices to show that all requirements act finitely often. To see this, consider the last stage s at which any requirement R_j with $j \leq k$ acts. Let $n_k = n_{k,s+1}$. Then for all $t > s$ we have $n_{k,t} = n_k$ and R_k is currently satisfied at t. For large enough t, we have ${K_m}^t(\alpha_t \upharpoonright n_k+1) = K_m(\alpha \upharpoonright n_k+1)$ and ${K_m}^t(\alpha_t \upharpoonright n_k) = K_m(\alpha \upharpoonright n_k)$. Hence, R_k is satisfied. Now to see that all requirements act finitely often, suppose the opposite and let k be least such that R_k acts infinitely often. Let s be the least stage such that no requirement R_j with $j < k$ acts after s. Then the string σ of the previous paragraph is fixed for all stages $t > s$ at which R_k acts. Let $s_* > s$ be large enough so that $K^t(\sigma) = K(\sigma)$ for all $t > s_*$. We note that the value of u chosen increases each time R_k acts. Indeed, suppose R_k acts at t_1 with $u = u_1$ and R_k acts at t_2 with $u = u_2$ for some stages $t_2 > t_1 > s_*$. Then $u_2 > u_1$ since $\sigma^\frown 1^{u_2\frown}0 > \alpha_{t_1+1} = \sigma^\frown 1^{u_1\frown}0$. Using an argument similar to the one in the previous paragraph, it now follows that at some stage $t > s_*$ the value chosen for u will be large enough so that $K_m(\sigma^\frown 1^{u\frown}0) \geq K(\sigma) + c_\gamma + k$. Then R_k will not act after stage t, contradicting the assumption that R_k acts infinitely often. ⊣

Theorem 4.1 shows Solovay reducibility does not imply monotone reducibility for c.e. reals. The next lemma shows that a version of the *permitting* method (from computability theory) can be used in the contruction of two c.e. reals to ensure one is monotone reducible to the other.

LEMMA 4.2. *Suppose α and β are c.e. reals and α_s and β_s are nondecreasing computable sequences of computable reals converging to α and β respectively. If there is a computable function f such that $\lim_{x\to\infty} f(x) = \infty$ and for every $n, s, t \in \omega$, $\alpha_s \upharpoonright n = \alpha_t \upharpoonright n$ implies $\beta_{f(s)} \upharpoonright n = \beta_{f(t)} \upharpoonright n$, then $\alpha \geq_{K_m} \beta$.*

PROOF. The result follows easily if α or β is computable, so we may assume that α and β are irrational. We need to show that $K_m(\beta \upharpoonright n) \preceq K_m(\alpha \upharpoonright n)$. To do so, we define a monotone machine M. If $\langle p, \sigma\rangle \in U$ and $\sigma \subseteq \alpha_s$ for some s, enumerate $\langle p, \beta_{f(s)} \upharpoonright |\sigma|\rangle \in M$. To see that M is monotone, suppose p and q are comparable, $\langle p, \sigma\rangle, \langle q, \tau\rangle \in U$ and there are stages s and t such that $\sigma \subseteq \alpha_s$ and $\tau \subseteq \alpha_t$. Since U is monotone, σ and τ are comparable. Without loss of generality, assume $\sigma \subseteq \tau$. Let $n = |\sigma|$. Then $\alpha_s \upharpoonright n = \sigma = \tau \upharpoonright n = \alpha_t \upharpoonright n$. Thus $\beta_{f(s)} \upharpoonright n = \beta_{f(t)} \upharpoonright n \subseteq \beta_{f(t)} \upharpoonright |\tau|$. It follows that M is monotone. Now, for any n, let $\sigma = \alpha \upharpoonright n$. Let p be a minimal monotone description of σ. Since

α is irrational, there is a stage s_n such that for all $s \geq s_n$ we have $\alpha_s \upharpoonright n = \sigma$. Thus $\langle p, \beta_{f(s_n)} \upharpoonright n\rangle \in M$. By the hypothesis and definition of s_n, we have that $\beta_{f(s)} \upharpoonright n = \beta_{f(s_n)} \upharpoonright n$ for all $s \geq s_n$. Since β is irrational and $f(s) \to \infty$, this implies $\beta \upharpoonright n = \beta_{f(s_n)} \upharpoonright n$. Hence ${K_m}^M(\beta \upharpoonright n) \leq |p|$. The lemma follows. ⊣

Since it does not appear to be possible to apply Downey and Hirschfeldt's density theorem to the c.e. monotone degrees, we use Lemma 4.2 to give a direct proof that the computably enumerable monotone degrees are downward dense. (The author wishes to thank T. Slaman for suggesting the use of permitting in the proof of this theorem.) The proof of the theorem is a modification of the Sacks density theorem for the computably enumerable Turing degrees [21].

Theorem 4.3. *For any c.e. real* $\alpha >_{K_m} \mathbf{0}$ *there is a c.e. real* β *such that*

$$\mathbf{0} <_{K_m} \beta <_{K_m} \alpha.$$

Proof. The proof is a finite injury priority argument. Let α_s be a nondecreasing computable sequence of computable reals converging to α. We will build a nondecreasing computable sequence of computable reals β_s converging to the c.e. real β. We must satisfy three requirements: (i) $\beta \leq_{K_m} \alpha$, (ii) $\mathbf{0} \not\geq_{K_m} \beta$, (iii) $\beta \not\geq_{K_m} \alpha$.

We will use permitting to ensure (i). That is, we will build the sequence β_s so that for any n and any $t > s$, $\alpha_s \upharpoonright n = \alpha_t \upharpoonright n$ implies $\beta_s \upharpoonright n = \beta_t \upharpoonright n$. It follows from Lemma 4.2 that $\beta \leq_{K_m} \alpha$.

Requirement (ii) is equivalent to the assertion that β is not computable. We may break the requirement into infinitely many subrequirements. For each e we ensure that the eth partial computable function ϕ_e does not compute β. To do so, we will use the Sacks coding strategy. We define an increasing computable sequence of coding locations n_k. (We assume that all stronger strategies have finished acting. More precisely, if any stronger strategy acts, this strategy starts over, abandoning any coding locations it had chosen.) Coding locations are chosen so that $n_k > k, n_{k-1}$ and so that n_k is not restrained by any stronger strategy. Once a coding location n_k is chosen we ensure that at any subsequent stage s, $\alpha_s(k) = \beta_s(n_k)$ (unless n_k is abandoned). We may assume that at every stage s there are infinitely many n such that $\beta_s(n) = 0$ and infinitely many n such that $\beta_s(n) = 1$, allowing us to chose a new coding location satisfying the above constraints. We will then be able to preserve the relationship $\alpha_t(k) = \beta_t(n_k)$ at any subsequent stage t, since if the current approximation to $\alpha(k)$ changes at t then our current approximation to $\beta(n_k)$ is permitted to change at t. We will only choose a new coding location at a stage at which the initial segment of β (correctly) computed by ϕ_e is longer than at any previous stage. Note that the final outcome of this strategy is to chose and preserve a finite set of coding locations. Otherwise, the length of the initial segment of β_s computed by ϕ must be unbounded, and it follows that ϕ computes β and thus β is computable. Since there are is a computable infinite sequence of coding locations in β, it follows that β computes α. But then α is computable, contradicting the hypothesis that $\alpha >_{K_m} \mathbf{0}$.

We may break requirement (iii) into infinitely many subrequirements. For each $c \in \omega$ we will ensure that there is a witness $w \in \omega$ such that $K_m(\beta \upharpoonright w) + c <$

$K_m(\alpha \upharpoonright w)$. To ensure this, we restrain $\beta \upharpoonright n$ for increasing values of n. (Note, we are unable to restrain coding locations chosen by stronger priority strategies, but we may assume the strategy keeps starting over until we reach a stage at which none of the values at such coding locations will ever change again.) We continue increasing the constraint on β until we get a w and a stage s such that

$$K_m{}^s(\beta_s \upharpoonright w) + c < K_m{}^s(\alpha_s \upharpoonright w). \tag{3}$$

We now wait. If at some later stage t we have $K_m{}^t(\beta_t \upharpoonright n) + c \geq K_m{}^t(\alpha_t \upharpoonright n)$ for all $n \leq w$ then resume increasing the restraint on β until we find that (3) holds again (for a larger witness w at a later stage s). If the restraint on β grows infinitely often, β is computable, and for all n, $K_m(\beta \upharpoonright n) + c \geq K_m(\alpha \upharpoonright n)$, which implies $\beta \geq_{K_m} \alpha$. But then α is computable, contradicting the hypothesis that $\alpha >_{K_m} \mathbf{0}$. Therefore, the final outcome of this strategy is a finite restraint and (3) is permanently satisfied for some witness w (i.e. for a cofinal set of stages s).

⊣

§5. Strongly c.e. monotone degrees. A real is said to be strongly computably enumerable (strongly c.e.) if it is the characteristic function of a computably enumerable set. A monotone degree is said to be strongly c.e. if it contains a strongly c.e. real. The following lemma and corollary provide bounds on the complexity of a strongly c.e. real.

LEMMA 5.1. *If α is strongly c.e. then $K(\alpha \upharpoonright n) \preceq \log n + 2 \log \log n$.*

PROOF. We give an upper bound for $K(\alpha \upharpoonright n)$ by constructing a prefix-free machine M. Fix an index e such that $\alpha = \chi_{W_e}$. Let $\langle \overline{q}{}^\frown 01^\frown p, \sigma \rangle$ be in M if q is the binary representation of $|p|$, $\overline{q}$ is the string in which each bit of q is repeated twice, p is the binary representation of a number k, and $\sigma \subseteq \tau$, where τ is the characteristic function of $W_{e,s}$ and s is the stage at which k enters W_e. Note that M is prefix free since the position of the first 01 uniquely determines q, and q determines the length of the string. Note also that if k is the last number to enter $W_e \upharpoonright n$ then there will be a $\langle \overline{q}{}^\frown 01^\frown p, \sigma \rangle$ in M with p the binary representation of k and hence $\sigma = \alpha \upharpoonright n$. The result follows since $|p| \leq \log(k) + 1$ and $|\overline{q}| \leq 2(\log \log k + 1)$. ⊣

COROLLARY 5.2. *If α is strongly c.e. then $K_m(\alpha \upharpoonright n) \preceq \log n + 2 \log \log n$.*

PROOF. Obvious, since $K_m(\alpha \upharpoonright n) \preceq K(\alpha \upharpoonright n)$. ⊣

The term $2 \log \log n$ in the previous two results is sufficiently small for our purposes, but can be replaced by $\log \log n + \log \log \log n + \cdots + (1 + \epsilon) \log^k n$ for any real number $\epsilon > 0$ and integer $k \geq 2$.

DEFINITION 5.3. We will use the notation $f(n) \approx g(n)$ to mean there is a constant c such that $|f(n) - g(n)| \leq c \log \log n$ for all $n \in \omega$, $n \geq 4$.

LEMMA 5.4. *There is a strongly c.e. real α such that $K_m(\alpha \upharpoonright n) \approx \log n$.*

PROOF. Let $p_0, p_1, \ldots$ be an enumeration of $2^{<\omega}$ such that $i < 2^{|p_i|+1}$ for all $i \in \omega$. Let $\alpha(n) = 1$ if $\langle p_n, \sigma \rangle \in U$ for some σ where $\sigma(n) \downarrow = 0$. Otherwise, let $\alpha(n) = 0$. Clearly, α is strongly c.e.. By definition of α, $K_m(\alpha \upharpoonright n) \geq \min\{|p_i| : i \geq n\} \geq \log(n) - 1$. On the other hand, by Lemma 5.2 we have $K_m(\alpha \upharpoonright n) \preceq \log(n) + 2\log\log(n)$. ⊣

Theorem 5.5. *There are incomparable strongly c.e. monotone degrees.*

Proof. We construct α and β by stages in a recursive construction. At each stage s, α_s and β_s are computable reals and for each n the values of $\alpha_s(n)$ and $\beta_s(n)$ are nondecreasing in s. (That is a value can change from 0 to 1 at some stage, but not from 1 to 0.) We then let $\alpha(n) = \lim_{s\to\infty} \alpha_s(n)$ and $\beta(n) = \lim_{s\to\infty} \beta_s(n)$ for all n. Note that there is a computable function h such that for any $\sigma \in 2^{<\omega}$

$$(\forall n \in \omega)[h(|\sigma|) > K_m(\sigma^\frown 1^n)].$$

We ensure that α and β are incomparable by satisfying the following requirements.

Requirement R_j for j even: For some n, $K_m(\alpha \upharpoonright n) > K_m(\beta \upharpoonright n) + j$.

Requirement R_j for j odd: For some n, $K_m(\beta \upharpoonright n) > K_m(\alpha \upharpoonright n) + j$.

Requirement R_j will be satisfied on an interval $[m_j, m_{j+1})$, where $m_0 = 0$ and $m_{j+1} = m_j + 2^{h(m_j)+j} - 1$ for all j.

Strategy for R_j for j even: For all $n \in [m_j, m_{j+1})$ we set $\beta_s(n) = 1$. Let $l = m_{j+1} - m_j$ and let $p_0, p_1, \ldots, p_{l-1}$ be the set of strings of length less than $h(m_j) + j$. For all $i \in [0, l)$ let $n_i = m_j + i$ and set

$$\alpha_s(n_i) = \begin{cases} 0 \text{ if } \Phi^s_{p_i}(n_i) = 1 \text{ or } \Phi^s_{p_i}(n_i)\uparrow \\ 1 \text{ if } \Phi^s_{p_i}(n_i) = 0 \end{cases}.$$

Strategy for R_j for j odd: The same as for j even, but reverse the roles of α and β.

End of the Construction.

We will now see that R_j is satisfied for j even. (The same argument works for j odd by symmetry.) Using the notation from the construction, let $n = m_{j+1}$. We will show that $K_m(\alpha \upharpoonright n) > K_m(\beta \upharpoonright n) + j$. Suppose not. By definition of h, we have $K_m(\beta \upharpoonright n) = K_m((\beta \upharpoonright m_j)^\frown 1^l) < h(m_j)$. Thus $K_m(\alpha \upharpoonright n) < h(m_j) + j$. So there is some p with $|p| < h(m_j) + j$ such that $\Phi_p \upharpoonright n = \alpha \upharpoonright n$. But $p = p_i$ for some $i \in [0, l)$. Therefore, $\Phi_p(n_i) \neq \alpha(n_i)$, giving us a contradiction. ⊣

Corollary 5.6. *There is a countably infinite antichain of strongly computably enumerable monotone degrees.*

Proof. Dovetail the strategies of the previous proof for ω-many reals α_i. ⊣

Theorem 5.7. *There is an order-preserving embedding of the rationals into the strongly c.e. monotone degrees.*

Proof. By Lemma 5.4 let α be a strongly c.e. real such that $K_m(\alpha \upharpoonright n) \approx \log(n)$. For each rational r, with $0 < r < 1$ define a function f_r by $f_r(n) = (\lfloor n^{1/r} \rfloor)$. Note that since $1/r > 1$, f_r is strictly increasing. Let $\beta_r = \alpha \otimes f_r$. Then by Theorem 3.9, $K_m(\beta_r \upharpoonright n) \asymp K_m(\alpha \upharpoonright f^{-1}[n]) \approx \log(f^{-1}[n]) \asymp \log n^r = r \log n$. Thus $K_m(\beta_r \upharpoonright n) \approx r \log n$. It follows that if s and r are rational and $0 < s < r < 1$ then $\beta_s \prec \beta_r$. Thus we have an embedding of the rationals in the interval (0,1) (or equivalently, of all the rationals) into the strongly c.e. monotone degrees.

⊣

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