Stefan Wintein
Reinhard Muskens

Interpolation Methods for Dunn Logics and Their Extensions

Abstract. The semantic valuations of classical logic, strong Kleene logic, the logic of paradox and the logic of first-degree entailment, all respect the Dunn conditions: we call them Dunn logics. In this paper, we study the interpolation properties of the Dunn logics and extensions of these logics to more expressive languages. We do so by relying on the **Dunn***calculus*, a signed tableau calculus whose rules mirror the Dunn conditions syntactically and which characterizes the Dunn logics in a uniform way. In terms of the **Dunn** calculus, we first introduce two different interpolation *methods*, each of which *uniformly* shows that the Dunn logics have the interpolation property. One of the methods is closely related to *Maehara's method* but the other method, which we call *the pruned tableau method*, is novel to this paper. We provide various reasons to prefer the pruned tableau method to the Maehara-style method. We then turn our attention to extensions of Dunn logics with so-called *appropriate implication connectives*. Although these logics have been considered at various places in the literature, a study of the interpolation properties of these logics is lacking. We use the pruned tableau method to uniformly show that these extended Dunn logics have the interpolation property and explain that the same result cannot be obtained via the Maehara-style method. Finally, we show how the pruned tableau method constructs interpolants for functionally complete extensions of the Dunn logics.

Keywords: Interpolation methods, Dunn logics, First degree entailment, Logic of paradox, Strong Kleene logic, Exactly true logic, Tableau calculus.

1. Introduction

The following conditions were laid down by Dunn [11] to equip the logic of *first-degree entailment* FDE [8,9,11] with an intuitive semantics based on the values **T** (true and not false), **B** (both truth and false), **N** (neither true nor false) and **F** (false and not true).

i. $\neg\varphi$ is true if and only if φ is false,
   $\neg\varphi$ is false if and only if φ is true;

Special Issue: 40 years of FDE
Edited by **Hitoshi Omori and Heinrich Wansing**

Studia Logica
DOI: 10.1007/s11225-017-9720-5

Published online: 12 April 2017

ii. $\varphi \wedge \psi$ is true if and only if φ is true and ψ is true,
$\varphi \wedge \psi$ is false if and only if φ is false or ψ is false;

iii. $\varphi \vee \psi$ is true if and only if φ is true or ψ is true,
$\varphi \vee \psi$ is false if and only if φ is false and ψ is false.

To illustrate how Dunn's conditions fix the semantics of $\wedge$, $\vee$ and $\neg$ on $\mathbf{4} = \{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$, suppose that φ has the value $\mathbf{T}$, i.e. φ is true and not false. Then Dunn's condition i. tells us that $\neg\varphi$ is false and not true, i.e. has the value $\mathbf{F}$. Further reasoning along these lines leads to the following truth tables.

DEFINITION 1. (Truth tables for $\wedge$, $\vee$ and $\neg$)

$\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

$\neg$	
T	**F**
B	**B**
N	**N**
F	**T**

The entailment relation of FDE over the propositional language $\mathcal{L}$ that is based on $\wedge$, $\vee$ and $\neg$ is then obtained by stipulating that, in passing from premisses to conclusion, truth is to be preserved over all $\mathbf{4}$-valued valuations of $\mathcal{L}$'s sentences that respect the above truth tables.

Observe that Dunn's conditions do not only fix the semantics of $\mathcal{L}$'s connectives on $\mathbf{4}$, but also on $\mathbf{3b} = \{\mathbf{T}, \mathbf{B}, \mathbf{F}\}$, $\mathbf{3n} = \{\mathbf{T}, \mathbf{N}, \mathbf{F}\}$ and $\mathbf{2} = \{\mathbf{T}, \mathbf{F}\}$. We may thus define, for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$, the $\mathbf{z}$-entailment relation $\models_{\mathbf{z}}$ by stipulating that, in passing from premisses to conclusion, truth is to be preserved over all $\mathbf{z}$-valued valuations of $\mathcal{L}$'s sentences that respect the above truth tables. Indeed, $\models_{\mathbf{2}}$ comes down to classical logic CL, $\models_{\mathbf{3n}}$ gives us strong Kleene logic K3 [13], $\models_{\mathbf{3b}}$ yields the logic of paradox LP [18] and $\models_{\mathbf{4}}$ gives us, as already mentioned, FDE. As Dunn's conditions thus give us a uniform semantic approach to these four familiar logics, we will refer to them as *the Dunn logics*.

The Dunn conditions also motivated the construction of *the* **Dunn** *calculus* (cf. [24]), a tableau calculus that gives us a uniform *syntactic* approach to the Dunn logics. The **Dunn** calculus is a signed tableau calculus with signs coding for truth ($\mathsf{1}$), falsity ($\mathsf{0}$), non-truth ($\hat{\mathsf{1}}$) and non-falsity ($\hat{\mathsf{0}}$). To explain the use of these four signs, note that the Dunn conditions may also be phrased in terms of non-truth and non-falsity. For instance, the following is equivalent to Dunn condition ii:

ii′. $\varphi \wedge \psi$ is not true if and only if φ is not true or ψ is not true, $\varphi \wedge \psi$ is not false if and only if φ is not false and ψ is not false.

The **Dunn** calculus translates the original Dunn conditions and their formulations in terms of non-truth and non-falsity as tableau rules. For instance, the tableau rules corresponding to Dunn conditions *ii* and *ii′* are as follows:

$$\frac{x : \varphi \wedge \psi}{x : \varphi,\ x : \psi}\,(\wedge_x),\ \text{ if } x \in \{\mathsf{1}, \hat{\mathsf{0}}\} \qquad \frac{x : \varphi \wedge \psi}{x : \varphi \ \mid\ x : \psi}\,(\wedge_x),\ \text{ if } x \in \{\hat{\mathsf{1}}, \mathsf{0}\}$$

The **Dunn** calculus recognizes four distinct closure conditions: one for each value of $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$. These closure conditions have a straightforward rationale. For instance, $\{\mathsf{1} : \varphi, \mathsf{0} : \varphi\}$ is **2**-closed and **3n**-closed as no sentence can be true and false according to a **2**- or **3n**-valuation. On the other hand, **3b**- and **4**-valuations do allow for sentences that are both true and false and, accordingly, $\{\mathsf{1} : \varphi, \mathsf{0} : \varphi\}$ is neither **3b**- nor **4**-closed. The **Dunn** calculus allows us to capture the Dunn logics in a uniform manner, as we have the following result:

$$\Gamma \models_{\mathbf{z}} \varphi \iff \{\mathsf{1} : \gamma \mid \gamma \in \Gamma\} \cup \{\hat{\mathsf{1}} : \varphi\} \text{ has a } \mathbf{z}\text{-closed tableau.} \tag{1}$$

In this paper, we will invoke the **Dunn** calculus to study the *interpolation property* of the Dunn logics.

Interpolation property If, according to a logic $\mathbf{L}$, α entails β, β is not a tautology and α is not an anti-tautology,[1] we say that the $\mathbf{L}$-*interpolation condition* for α and β is satisfied. A logic $\mathbf{L}$ is said to have the *interpolation property* if, whenever the $\mathbf{L}$-interpolation condition for α and β is satisfied, there is a sentence γ, called the $\mathbf{L}$-*interpolant* of α and β, such that every propositional variable[2] that occurs in γ occurs both in α and in β and such that, according to $\mathbf{L}$, α entails γ and γ entails β.

It is known that all four Dunn logics have the interpolation property. However, in the literature this information is stored as four separate facts which are proved in a highly non-uniform manner. For instance, Takeuti [23, p. 33] shows that CL has the interpolation property via what he calls *Maehara's method* and that proceeds via an induction on proof length in a sequent calculus for CL. On the other hand, Anderson and Belnap [1, p161] prove that FDE has the interpolation property by showing that their Hilbert-style calculus $\mathbf{E_{fde}}$ for FDE only proves so-called *tautological entailments*.

[1] β is a tautology just in case it is entailed by any sentence whatsoever and α is an anti-tautology just in case it entails any sentence whatsoever.

[2] In the present paper, we will restrict ourselves to the propositional case.

And both Bendova [10] and Milne [14] give a semantic proof to establish that K3 has the interpolation property.[3]

In the first part of this paper, we will introduce two distinct *uniform* interpolation methods for the Dunn logics: both methods come with a single constructive proof that establishes that all four Dunn logics have the interpolation property. Both methods construct interpolants from closed tableaux of the **Dunn** calculus, but they do so in rather different ways. Our first method we call the *Maehara-style method*. The reason for doing so is that, although there are notable differences, this method generalizes Maehara's interpolation method for CL to quite some extent. Our second method, *the pruned tableau method*, does not have classical (or other) ancestors, but originates in this paper. As we will see, the pruned tableau method is preferable to the Maehara-style method for, amongst others, the following two reasons:

- The pruned tableau method does not, in contrast to the Maehara-style method, rely on a proof by induction but constructs the interpolant directly from a closed tableau.
- Interpolants obtained via the pruned tableau method have lower sentential complexity than those obtained with the Maehara-style method.

In a nutshell then, the pruned tableau method gives us a simpler and more direct way to obtain interpolants for the Dunn logics than the Maehara-style method does.

In a recent paper, Pietz and Rivieccio [17] presented *Exactly True Logic* (ETL), an interesting variation upon FDE that is obtained by preserving *exact truth*, i.e. the value $\mathbf{T}$, over all $\mathbf{4}$-valued valuations of $\mathcal{L}$. Although both [17] and Wintein and Muskens [25] study ETL to quite some extent, no investigation of its interpolation property is to be found in the literature. We will first explain that the Maehara-style method cannot be invoked to construct interpolants for ETL. However, we will also show that ETL has the interpolation property by using the pruned tableau method, which is yet another reason to prefer the pruned tableau method over the Maehara-style method.

The pruned tableau method is not only interesting of itself but also has a noteworthy corollary. In a recent article, Milne [14] shows that CL's interpolation property allows for the following non-classical refinement: whenever the CL-interpolation condition for α and β is satisfied, there is a sentence γ

[3] We are not aware of an explicit proof which establishes that LP has the interpolation property. However, as φ entails ψ according to K3 just in case $\neg\psi$ entails $\neg\varphi$ according to LP, one readily shows that LP has the interpolation property on the basis of K3 having the property.

such that (i) every propositional variable that occurs in γ occurs both in α and β (ii) α entails γ *according to* $\mathsf{K3}$ and (iii) β entails γ *according to* LP. The CL-interpolant that is obtained via the pruned tableau method is readily shown to satisfy conditions (i), (ii) and (iii) and so Milne's result is an immediate corollary of the pruned tableau method. We will also show that Milne's result can be invoked to characterize CL as the transitive closure of the union of $\mathsf{K3}$ and LP which, so we think, is a novel result of independent interest.

The language $\mathcal{L}$ does not have a connective reserved for expressing implication. For CL, this is not a problem, as one may define material implication $\rightarrow$ in terms of $\vee$ and $\neg$ as usual. Material implication is an *appropriate implication connective* [2] for classical logic, which is to say that, according to CL, $\Gamma \cup \{\varphi\}$ entails ψ just in case Γ entails $\varphi \rightarrow \psi$. However, it is well-known that no appropriate implication connective for $\mathsf{K3}$, LP or FDE is definable in the language $\mathcal{L}$. This motivates us to consider the language $\mathcal{L}_{\supset}$, which extends $\mathcal{L}$ with a connective $\supset$ that has the following Dunn condition.

iv. $\varphi \supset \psi$ is true if and only if φ is not true or ψ is true,
$\varphi \supset \psi$ is false if and only if φ is true and ψ is false.

As this Dunn condition fixes the semantics of $\supset$ on **2**, **3b**, **3n** and **4**, we may extend the Dunn logics to $\mathcal{L}_{\supset}$ and one then readily shows that $\supset$ is an appropriate implication connective for each of the extended Dunn logics thus obtained. Although quite some authors[4] have studied the Dunn logics over $\mathcal{L}_{\supset}$, the interpolation properties of these logics have, to the best of our knowledge, not been investigated before. We extend our **Dunn** calculus with tableau rules for $\supset$ and then invoke the pruned tableau method to prove, in a uniform way, that the extended Dunn logics over $\mathcal{L}_{\supset}$ all have the interpolation property. We will also explain that attempts to obtain this result via the Maehara-style method are bound to fail.

Classical logic is functionally complete, which is to say that $\mathcal{L}$'s connectives (restricted to **2**) allow us to express every truth function on **2**. However, none of the other three Dunn logics, nor their extensions to $\mathcal{L}_{\supset}$, are functionally complete: the connectives of $\mathcal{L}_{\supset}$ do not allow us to express all truth functions on **3b**, **3n** or **4**. It is thus natural to consider functionally complete

[4]In Avron [4], the extension of $\mathsf{K3}$ to $\mathcal{L}_{\supset}$ is studied. Likewise, [4], studies the extension of LP to $\mathcal{L}_{\supset}$ and the same logic is also studied by, for instance Batens [6] and Batens & de Clerq [7]. Finally, the extension of FDE to $\mathcal{L}_{\supset}$ is studied in, amongst others, Arieli & Avron [3].

extensions of the Dunn logics and quite some authors[5] have studied the resulting logics. Takano [22] shows that any functionally complete many-valued logic has the interpolation property. Although it thus follows from Takano's results that the functionally complete extensions of the Dunn logics have the interpolation property, to actually construct an interpolant by the general and semantic means provided in [22] is quite cumbersome. As we will show however, the pruned tableau method constructs these interpolants in a simple and *informative* manner.

The paper is structured as follows. Section 2 states preliminaries. In Sections 3.1 and 3.3, we introduce our two interpolation methods and show how they can be invoked to prove, in a uniform manner, that the Dunn logics have the interpolation property. In Section 3.2, we explain in which sense the Maehara-style method resembles and in which sense it differs from Maehara's interpolation method for CL. In Section 3.4 we use the pruned tableau method to prove that ETL has the interpolation property and explain that this result cannot be obtained via the Maehara-style method. In Section 3.5 we show that Milne's result is an immediate corollary of the pruned tableau method and invoke Milne's result to characterize CL in terms of $\mathsf{K3}$ and LP. In Section 4 we use the pruned tableau method to show that the Dunn logics as defined over $\mathcal{L}_{\supset}$ have the interpolation property and explain that this result cannot be obtained via the Maehara-style method. In Section 5 we use the pruned tableau method to show that the functionally complete extensions of the Dunn logics have the interpolation property. Section 6 concludes.

2. Preliminaries

2.1. Uniform Notation for Dunn Logics

Throughout the paper, **2**, **3b**, **3n** and **4** are defined as in the introduction and we will use **Z** to denote the set that consist of these four subsets of **4**: $\mathbf{Z} = \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$. In addition, the following notation for certain subsets of **4** will be in force.

$$\mathsf{1} := \{\mathbf{T}, \mathbf{B}\}, \qquad \mathsf{0} := \{\mathbf{F}, \mathbf{B}\}, \qquad \hat{\mathsf{1}} := \{\mathbf{F}, \mathbf{N}\}, \qquad \hat{\mathsf{0}} := \{\mathbf{T}, \mathbf{N}\}. \tag{2}$$

And so $\mathsf{1}$, $\mathsf{0}$, $\hat{\mathsf{1}}$ and $\hat{\mathsf{0}}$ code for, respectively, truth, falsity, non-truth and non-falsity. The elements of $\{\mathsf{1}, \mathsf{0}, \hat{\mathsf{1}}, \hat{\mathsf{0}}\}$ will both be used semantically, as

[5] For functionally complete extensions of FDE, see e.g. Muskens [15], Arieli and Avron [3], Ruet [20], Pynko [19], or Omori and Sano [16]. Muskens [15] also studies functionally complete extensions of LP and $\mathsf{K3}$.

abbreviating a subset of **4** as indicated by (2), but also syntactically, as signs of the **Dunn** calculus. It will always be clear from context which usage is at stake.

We consider the propositional language $\mathcal{L}$ that is based on $\{\wedge, \vee, \neg\}$ and define a **z**-*valuation for* $\mathcal{L}$ to be to a function from the sentences of this language to $\mathbf{z} \in \mathbf{Z}$ that respects the truth-tables of Definition 1. Also, $\mathbf{V_z}$ will denote the set of all **z**-valuations for $\mathcal{L}$. The $\mathcal{L}$ entailment relation $\models_{\mathbf{z}}$ preserves truth over all **z**-valuations:

$$\Gamma \models_{\mathbf{z}} \varphi \Leftrightarrow \text{if } V(\gamma) \in \mathsf{1} \text{ for all } \gamma \in \Gamma \text{ then } V(\varphi) \in \mathsf{1}, \text{ for all } V \in \mathbf{V_z} \tag{3}$$

Per definition, $\models_{\mathbf{2}}$, $\models_{\mathbf{3n}}$, $\models_{\mathbf{3b}}$ and $\models_{\mathbf{4}}$ are equal to, respectively, CL, $\mathsf{K3}$, LP and FDE and so we may also write:

$$\models_{\mathbf{2}} = \models_{CL}, \qquad \models_{\mathbf{3n}} = \models_{K3}, \qquad \models_{\mathbf{3b}} = \models_{LP}, \qquad \models_{\mathbf{4}} = \models_{FDE}$$

2.2. The Dunn Tableau Calculus

A *signed sentence of* $\mathcal{L}$ is an object of form $x : \varphi$ with sign $x \in \{\mathsf{1}, \mathsf{0}, \hat{\mathsf{1}}, \hat{\mathsf{0}}\}$ and with φ a sentence of $\mathcal{L}$. Tableaux in the **Dunn** calculus will be certain sets of *branches*, which are sets of signed sentences of $\mathcal{L}$. The following definition specifies what it means for a valuation to *satisfy* a branch.

Definition 2. (Satisfaction for branches) Let $\mathcal{B}$ be a branch and let $\mathbf{z} \in \mathbf{Z}$. A valuation $V \in \mathbf{V_z}$ *satisfies* $\mathcal{B}$ iff every $x : \varphi \in \mathcal{B}$ is such that:

$$x = \mathsf{1} \implies V(\varphi) \in \{\mathbf{T}, \mathbf{B}\} \qquad x = \mathsf{0} \implies V(\varphi) \in \{\mathbf{F}, \mathbf{B}\}$$
$$x = \hat{\mathsf{1}} \implies V(\varphi) \in \{\mathbf{F}, \mathbf{N}\} \qquad x = \hat{\mathsf{0}} \implies V(\varphi) \in \{\mathbf{T}, \mathbf{N}\}$$

We say that $\mathcal{B}$ is **z**-*satisfiable* if there is a $V \in \mathbf{V_z}$ that satisfies $\mathcal{B}$ and that $\mathcal{B}$ is **z**-*unsatisfiable* if there is no such V.

The tableau rules of the **Dunn** calculus are displayed below.

Definition 3. (Tableau rules of the **Dunn** calculus)

$$\frac{x : \varphi \wedge \psi}{x : \varphi,\ x : \psi} (\wedge_x) \quad \text{if } x \in \{\mathsf{1}, \hat{\mathsf{0}}\} \qquad \frac{x : \varphi \wedge \psi}{x : \varphi \mid x : \psi} (\wedge_x) \quad \text{if } x \in \{\hat{\mathsf{1}}, \mathsf{0}\}$$

$$\frac{x : \varphi \vee \psi}{x : \varphi \mid x : \psi} (\vee_x) \quad \text{if } x \in \{\mathsf{1}, \hat{\mathsf{0}}\} \qquad \frac{x : \varphi \vee \psi}{x : \varphi, x : \psi} (\vee_x) \quad \text{if } x \in \{\hat{\mathsf{1}}, \mathsf{0}\}$$

$$\frac{x : \neg\varphi}{y : \varphi} (\neg_x) \quad \text{if } \langle x, y\rangle \in \{\langle \mathsf{1}, \mathsf{0}\rangle, \langle \hat{\mathsf{0}}, \hat{\mathsf{1}}\rangle\} \qquad \frac{x : \neg\varphi}{y : \varphi} (\neg_x) \quad \text{if } \langle x, y\rangle \in \{\langle \hat{\mathsf{1}}, \hat{\mathsf{0}}\rangle, \langle \mathsf{0}, \mathsf{1}\rangle\}$$

For formal considerations it will be useful to have a general form for rules, for which we choose $x : \varphi / B_1, \ldots, B_n$, where $x : \varphi$ is a signed sentence called the *top formula* of the rule and each B_i is a set of signed sentences called a *set of bottom formulas* of the rule. For example, one instantiation of $(\wedge_1)$ could formally be written as $1 : \varphi \wedge \psi / \{1 : \varphi, 1 : \psi\}$ and one instantiation of $(\wedge_0)$ could be expressed as $0 : \varphi \wedge \psi / \{0 : \varphi\}, \{0 : \psi\}$. This general form is useful, even though neither the number of sets of bottom formulas nor their cardinality ever exceeds 2. Here is our official definition of a tableau.

Definition 4. (Tableaux) Let $\mathcal{T}$ and $\mathcal{T}'$be sets of branches. We say that $\mathcal{T}'$ is a *one-step expansion* of $\mathcal{T}$ if, for some $\mathcal{B} \in \mathcal{T}$, $x : \varphi \in \mathcal{B}$, and rule $x : \varphi / B_1, \ldots, B_n$, $\mathcal{T}' = (\mathcal{T} \backslash \{\mathcal{B}\}) \cup \{\mathcal{B} \cup B_1, \ldots, \mathcal{B} \cup B_n\}$.

Let $\mathcal{B}$ be a finite branch. A set of branches $\mathcal{T}$ is a *tableau with initial branch* $\mathcal{B}$ if there is a sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$ such that $\mathcal{T}_0 = \{\mathcal{B}\}$, $\mathcal{T}_n = \mathcal{T}$, and each $\mathcal{T}_{i+1}$ is a one-step expansion of $\mathcal{T}_i$ $(0 \leq i < n)$. We also say that a finite $\mathcal{B}$ has tableau $\mathcal{T}$ if $\mathcal{T}$is a tableau with initial branch $\mathcal{B}$.

The **Dunn** calculus recognizes four closure conditions, one for each value of $\mathbf{z} \in \mathbf{Z}$.

Definition 5. (Closure conditions) Let $\mathcal{B}$ be a branch. We say that:

$\mathcal{B}$ is $\mathbf{4}$-*closed* $\iff$ $\{1 : \varphi, \hat{1} : \varphi\} \subseteq \mathcal{B}$ *or* $\{0 : \varphi, \hat{0} : \varphi\} \subseteq \mathcal{B}$,
$\mathcal{B}$ is $\mathbf{3b}$-*closed* $\iff$ $\mathcal{B}$ *is* $\mathbf{4}$-*closed or* $\{\hat{1} : \varphi, \hat{0} : \varphi\} \subseteq \mathcal{B}$,
$\mathcal{B}$ is $\mathbf{3n}$-*closed* $\iff$ $\mathcal{B}$ *is* $\mathbf{4}$-*closed or* $\{1 : \varphi, 0 : \varphi\} \subseteq \mathcal{B}$,
$\mathcal{B}$ is $\mathbf{2}$-*closed* $\iff$ $\mathcal{B}$ *is* $\mathbf{3b}$-*closed or* $\mathcal{B}$ *is* $\mathbf{3n}$-*closed.*

A branch that is not $\mathbf{z}$-closed is called $\mathbf{z}$-*open.* When, for some propositional atom p, $\mathcal{B}$ contains some $\mathbf{z}$-closed subset $\{x : p,\ y : p\}$, we say that $\mathcal{B}$ *is atomically* $\mathbf{z}$-*closed.* A tableau is (atomically) $\mathbf{z}$-*closed* just in case all its branches are (atomically) $\mathbf{z}$-closed; if not, the tableau is (atomically) $\mathbf{z}$-*open.*

The following theorem attests that the **Dunn** calculus is sound and complete with respect to $\mathbf{z}$-unsatisfiable branches.

Theorem 1. *A finite branch* $\mathcal{B}$ *is* $\mathbf{z}$-*unsatisfiable iff* $\mathcal{B}$ *has a* $\mathbf{z}$-*closed tableau.*

Proof. Proof: See [24]. ∎

An immediate corollary of the above theorem is that the **Dunn** calculus allows us to capture the Dunn logics in a uniform way. For, with $1 : \Gamma := \{1 : \gamma \mid \gamma \in \Gamma\}$, it readily follows that:

$$\Gamma \models_{\mathbf{z}} \varphi \iff 1 : \Gamma \cup \{\hat{1} : \varphi\} \text{ has a } \mathbf{z}\text{-closed tableau} \tag{4}$$

The following example illustrates the convenience of the uniform approach to the Dunn logics that is provided by the **Dunn** calculus.

EXAMPLE 1. Let $\alpha = ((p \wedge \neg p) \wedge r) \vee (q \wedge t)$ and let $\beta = ((s \vee \neg s) \vee t) \wedge (q \vee r)$. Consider the following tableaux for $\{1 : \alpha\}$ and $\{\hat{1} : \beta\}$ that may be depicted as follows: Thus, the depicted tableaux of $\{1 : \alpha\}$ and $\{\hat{1} : \beta\}$ have branches X_1, X_2 and Y_1, Y_2 respectively, where:

$$\begin{aligned} X_1 &= \{1 : \alpha, 1 : (p \wedge \neg p) \wedge r, 1 : p \wedge \neg p, 1 : r, 1 : p, 1 : \neg p, 0 : p\} \\ X_2 &= \{1 : \alpha, 1 : q \wedge t, 1 : q, 1 : t\} \\ Y_1 &= \{\hat{1} : \beta, \hat{1} : (s \vee \neg s) \vee t, \hat{1} : s \vee \neg s, \hat{1} : t, \hat{1} : s, \hat{0} : s\} \\ Y_2 &= \{\hat{1} : \beta, \hat{1} : q \vee r, \hat{1} : q, \hat{1} : r\} \end{aligned}$$

Note that, with $\mathcal{B}_1 = X_1 \cup Y_1$, $\mathcal{B}_2 = X_1 \cup Y_2$, $\mathcal{B}_3 = X_2 \cup Y_1$ and $\mathcal{B}_4 = X_2 \cup Y_2$, $\mathcal{T}^{\mathbf{3}} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$ is a tableau for $\{1 : \alpha, \hat{1} : \beta\}$. As $\mathcal{T}^{\mathbf{3}}$ is $\mathbf{z}$-closed for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ it follows from (4) that $\alpha \models_{\mathbf{z}} \beta$ whenever $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$.

The two tableaux that are depicted in Example 1, as well as the tableau $\mathcal{T}^{\mathbf{3}}$ for $\{1 : \alpha, \hat{1} : \beta\}$, are *fulfilled*, where a tableau $\mathcal{T}$ is fulfilled iff, for each one-step expansion $\mathcal{T}'$ of $\mathcal{T}$, we have $\mathcal{T}' = \mathcal{T}$. It is readily established[6] that a finite branch $\mathcal{B}$ has a *unique* fulfilled tableau; we will use square brackets to denote the fulfilled tableau $[\mathcal{B}]$ of $\mathcal{B}$. If $\mathcal{B}$ has a $\mathbf{z}$-closed tableau, then in particular $[\mathcal{B}]$ is $\mathbf{z}$-closed. Moreover, $[\mathcal{B}]$ is then not only $\mathbf{z}$-closed but also *atomically* $\mathbf{z}$*-closed*[7].

2.3. Interpolation: Notation and a Useful Lemma

In this section we first define the interpolation property, and some associated notions, for an arbitrary propositional language L. Then we state a lemma pertaining to the Dunn logics that will be useful for showing that these logics have the interpolation property.

Let L be an arbitrary propositional language and let $Sen(L)$ be its set of sentences. We write $\mathsf{Voc}(\varphi)$ (the *vocabulary* of φ) to denote the set of propositional variables that occur in $\varphi \in Sen(L)$. With Σ a set of sentences of L, and with Θ a signed set of such sentences, $\mathsf{Voc}(\Sigma) = \bigcup_{\varphi \in \Sigma} \mathsf{Voc}(\varphi)$ and $\mathsf{Voc}(\Theta) = \bigcup_{x:\varphi \in \Theta} \mathsf{Voc}(\varphi)$.

Let $\models$ be any relation between sets of sentences and sentences of L. A sentence φ of L is said to be a *tautology of* $\models$ when $\emptyset \models \varphi$ and φ is called

[6] Do a straightforward induction on the number of logical connectives occurring in $\mathcal{B}$.

[7] A proof can be given similar to Smullyan's [21] proof of the corresponding fact for classical logic.

an *anti-tautology of* $\models$ when $\varphi \models \psi$ for any $\psi \in Sen(L)$. With α and β sentences of L, we say that the $\models$-*interpolation condition* for α and β is satisfied just in case $\alpha \models \beta$, α is not an anti-tautology of $\models$ and β is not a tautology of $\models$. Also, we say that γ is a $\models$-*interpolant for* α and β iff $\alpha \models \gamma$, $\gamma \models \beta$ and γ is a sentence in the joint vocabulary of α and β: $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$. The *interpolation property* for $\models$ is then defined as follows.

Definition 6. (The Interpolation Property) $\models$ has the *interpolation property* iff whenever the $\models$-interpolation condition for α and β is satisfied, there is a $\models$-interpolant for α and β.

The following lemma, pertaining to the Dunn logics, will turn out to be useful.

Lemma 1. *Let* $\mathbf{z} \in \mathbf{Z}$ *and suppose that the* $\models_{\mathbf{z}}$*-interpolation condition for* α *and* β *is satisfied. Then* $\mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta) \neq \emptyset$.

Proof. Suppose, for a reductio ad absurdum, that the $\models_{\mathbf{z}}$-interpolation condition for α and β is satisfied whereas $\mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta) = \emptyset$. As α is not an anti-tautology and as β is not a tautology, there have to be valuations $V', V'' \in \mathbf{V_z}$ such that $V'(\alpha) \in \mathsf{1}$ and $V''(\beta) \notin \mathsf{1}$. Let $V \in \mathbf{V_z}$ be the (unique) valuation that valuates the propositional atoms of $\mathcal{L}$ as follows.

$$V(p) = \begin{cases} V'(p) & \text{if } p \in \mathsf{Voc}(\alpha) \\ V''(p) & \text{otherwise.} \end{cases}$$

As $\mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta) = \emptyset$, it readily follows from the definition of V that $V(\alpha) = V'(\alpha) \in \mathsf{1}$ and that $V(\beta) = V''(\beta) \notin \mathsf{1}$. Hence, V testifies that $\alpha \not\models_{\mathbf{z}} \beta$ so that the $\models_{\mathbf{z}}$-interpolation condition for α and β is *not* satisfied. ■

3. Two Interpolation Methods for Dunn Logics

3.1. The Maehara-Style Interpolation Method

In this section we first present our Maehara-style interpolation method which is then used to show, in one fell swoop, that all four Dunn logics have the interpolation property.

By the *bifurcation* of a set of signed sentences Θ, we mean the ordered pair $\langle \Theta_L, \Theta_R \rangle$ such that $\{\Theta_L, \Theta_R\}$ is the partition of Θ in which all elements of Θ_L have their sign in $\{\mathsf{1}, \mathsf{0}\}$ and all elements of Θ_R have their sign in $\{\hat{\mathsf{1}}, \hat{\mathsf{0}}\}$. To show that the Dunn logics have the interpolation property via our Maehara-style method, the following lemma is crucial.

Lemma 2. (Bifurcation lemma) *Let Θ be a finite set of signed sentences of $\mathcal{L}$ and let $\langle \Theta_L, \Theta_R \rangle$ be the bifurcation of Θ. If Θ has a* **z***-closed tableau and if* $\mathsf{Voc}(\Theta_L) \cap \mathsf{Voc}(\Theta_R) \neq \emptyset$ *there is a $\gamma \in Sen(\mathcal{L})$ such that: (i)* $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\Theta_L) \cap \mathsf{Voc}(\Theta_R)$ *and such that (ii) both $\Theta_L \cup \{\hat{1} : \gamma\}$ and $\{1 : \gamma\} \cup \Theta_R$ have a* **z***-closed tableau.*

Proof. We will prove the Bifurcation lemma via induction on the minimal number k of one-step expansions that are needed to convert $\mathcal{T}_0 = \{\Theta\}$ into a closed tableau of Θ.

Induction base If $k = 0$, Θ is $\mathbf{z}$-closed. It suffices to consider the following four cases, which are easily established. Below, p is a propositional atom that is contained in $\mathsf{Voc}(\Theta_L) \cap \mathsf{Voc}(\Theta_R)$.

For $\{1 : \varphi, \hat{1} : \varphi\} \subseteq \Theta$, $\mathbf{z} \in \mathbf{Z}$: $\gamma = \varphi$ satisfies (i) and (ii).
For $\{0 : \varphi, \hat{0} : \varphi\} \subseteq \Theta$, $\mathbf{z} \in \mathbf{Z}$: $\gamma = \neg\varphi$ satisfies (i) and (ii).
For $\{1 : \varphi, 0 : \varphi\} \subseteq \Theta$, $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}\}$: $\gamma = p \wedge \neg p$ satisfies (i) and (ii).
For $\{\hat{1} : \varphi, \hat{0} : \varphi\} \subseteq \Theta$, $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}\}$: $\gamma = p \vee \neg p$ satisfies (i) and (ii).

Induction step Suppose that Θ has a $\mathbf{z}$-closed tableau $\mathcal{T}_{k+1}$ that results from $\mathcal{T}_0 = \{\Theta\}$ in $k + 1$ one-step expansions. Let $(\circ_x)$ be the tableau rule that is used in the one-step expansion from $\mathcal{T}_0$ to $\mathcal{T}_1$.

Suppose that $(\circ_x) = (\wedge_1)$. Then Θ has form $\Sigma \cup \{1 : \varphi \wedge \psi\}$ and, with $\Theta_1 = \Sigma \cup \{1 : \varphi, 1 : \psi\}$, we have $\mathcal{T}_1 = \{\Theta_1\}$. Let $\langle \Sigma_L, \Sigma_R \rangle$ be the bifurcation of Σ. As Θ_1 has a $\mathbf{z}$-closed tableau in k one-step expansions it follows from the induction hypothesis that there is a sentence γ that satisfies (i) and (ii) with respect to the bifurcation $\langle \Sigma_L \cup \{1 : \varphi, 1 : \psi\}, \Sigma_R \rangle$ of Θ_1. It readily follows that γ also satisfies (i) and (ii) with respect to the bifurcation $\langle \Sigma_L \cup \{1 : \varphi \wedge \psi\}, \Sigma_R \rangle$ of Θ, which is what we had to show.

Suppose that $(\circ_x) = (\wedge_0)$. Then Θ has form $\Sigma \cup \{0 : \varphi \wedge \psi\}$ and, with $\Theta_1 = \Sigma \cup \{0 : \varphi\}$ and $\Theta_2 = \Sigma \cup \{0 : \psi\}$, we have $\mathcal{T}_1 = \{\Theta_1, \Theta_2\}$. Let $\langle \Sigma_L, \Sigma_R \rangle$ be the bifurcation of Σ. As both Θ_1 and Θ_2 have $\mathbf{z}$-closed tableaux in $\leq k$ one-step expansions, it follows from the induction hypothesis that there are sentences γ_1 and γ_2 that satisfy (i) and (ii) with respect to the bifurcations $\langle \Sigma_L, \Sigma_R \cup \{0 : \varphi\} \rangle$ and $\langle \Sigma_L, \Sigma_R \cup \{0 : \psi\} \rangle$ of Θ_1 and Θ_2 respectively. It readily follows that $\gamma = \gamma_1 \wedge \gamma_2$ satisfies (i) and (ii) with respect to the bifurcation $\langle \Sigma_L, \Sigma_R \cup \{0 : \varphi \wedge \psi\} \rangle$ of Θ, which is what we had to show.

For tableau rules other than $(\wedge_1)$ and $(\wedge_0)$, the proof is completely similar. For the actual construction of interpolants however, it will be convenient to explicitly write down the remaining cases. To do so, let us abbreviate the statement 'Θ has a $\mathbf{z}$-closed tableau and γ is a sentence of $\mathcal{L}$ that satisfies conditions (i) and (ii) with respect to the bifurcation $\langle \Theta_L, \Theta_R \rangle$ of Θ' as $\Theta \rhd_{\mathbf{z}} \gamma$.

Using this notation the results from above, together with the remaining cases, can be stated as follows:

$$\Sigma \cup \{x : \varphi, x : \psi\} \rhd_{\mathbf{z}} \gamma \Rightarrow \Sigma \cup \{x : \varphi \wedge \psi\} \rhd_{\mathbf{z}} \gamma \qquad x \in \{1, \hat{0}\}$$

$$\Sigma \cup \{x : \varphi\} \rhd_{\mathbf{z}} \gamma_1 \ , \ \Sigma \cup \{x : \psi\} \rhd_{\mathbf{z}} \gamma_2 \Rightarrow \Sigma \cup \{x : \varphi \wedge \psi\} \rhd_{\mathbf{z}} \gamma_1 \wedge \gamma_2 \quad x \in \{\hat{1}, 0\}$$

$$\Sigma \cup \{x : \varphi, x : \psi\} \rhd_{\mathbf{z}} \gamma \Rightarrow \Sigma \cup \{x : \varphi \vee \psi\} \rhd_{\mathbf{z}} \gamma \qquad x \in \{\hat{1}, 0\}$$

$$\Sigma \cup \{x : \varphi\} \rhd_{\mathbf{z}} \gamma_1 \ , \ \Sigma \cup \{x : \psi\} \rhd_{\mathbf{z}} \gamma_2 \Rightarrow \Sigma \cup \{x : \varphi \wedge \psi\} \rhd_{\mathbf{z}} \gamma_1 \vee \gamma_2 \quad x \in \{1, \hat{0}\}$$

$$\Sigma \cup \{x : \varphi\} \rhd_{\mathbf{z}} \gamma \Rightarrow \Sigma \cup \{y : \neg\varphi\} \rhd_{\mathbf{z}} \gamma \qquad \langle x, y\rangle \text{ or } \langle y, x\rangle \in \{\langle 1, 0\rangle, \langle \hat{0}, \hat{1}\rangle\}$$

■

Here is our uniform Maehara-style proof which establishes that all the Dunn logics have the interpolation property.

THEOREM 2. *The Dunn logics have the interpolation property.*

PROOF. Suppose that the $\models_{\mathbf{z}}$-interpolation condition for α and β is satisfied. Then, $\{1 : \alpha, \hat{1} : \beta\}$ has a **z**-closed tableau and it follows from Lemma 1 that $\mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta) \neq \emptyset$. Hence, in virtue of the Bifurcation lemma, there is a $\gamma \in Sen(\mathcal{L})$ such that $\{1 : \alpha, \hat{1} : \beta\} \rhd_{\mathbf{z}} \gamma$. It immediately follows that γ is a $\models_{\mathbf{z}}$-interpolant for α and β. ■

It readily follows from Theorem 2 that FDE has the *perfect interpolation property* (cf. [1]) which is to say that an FDE-interpolant of α and β exists whenever α entails β according to FDE, as recorded by the following corollary.

COROLLARY 1. *If* $\alpha \models_{FDE} \beta$ *then there exists an* FDE*-interpolant of* α *and* β*.*

PROOF. An inspection of the **4**-closure conditions and the tableau rules of the **Dunn** calculus reveals that neither $\{1 : \alpha\}$ nor $\{\hat{1} : \beta\}$ can have a **4**-closed tableau: FDE has no (anti-)tautologies. Hence, the result follows from Theorem 2. ■

The example below, which continues Example 1, illustrates how one constructs interpolants for the Dunn logics via our Maehara-style interpolation method.

EXAMPLE 2. (Interpolants via the Maehara-style method) Let α, β, X_1, X_2, Y_1, Y_2 and $\mathcal{T}^{\mathbf{3}} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$ all be as in Example 1. For $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ the $\models_{\mathbf{z}}$-interpolation condition for α and β is satisfied and we will show how the Maehara-style method constructs a $\models_{\mathbf{z}}$-interpolant for α and β for these three values of $\mathbf{z}$.

If $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}\}$ then $\mathcal{T}^{\mathbf{3n}} = \{X_1 \cup \{\hat{1} : \beta\}, \mathcal{B}_3, \mathcal{B}_4\}$ is a **3n**-closed and hence **2**-closed tableau of $\{1 : \alpha, \hat{1} : \beta\}$. Hence, it follows from the (induction base of the) Bifurcation lemma that for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}\}$:

$$X_1 \cup \{\hat{1} : \beta\} \rhd_{\mathbf{z}} r \wedge \neg r, \qquad \mathcal{B}_3 \rhd_{\mathbf{z}} t, \qquad \mathcal{B}_4 \rhd_{\mathbf{z}} q. \tag{5}$$

Note that $X_1 \cup \{\hat{1} : \beta\}$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are obtained by applying tableau rules to $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_4$ respectively, where:

$$\begin{aligned}
\mathcal{C}_1 &= \{1 : \alpha,\ \hat{1} : \beta,\ 1 : (p \wedge \neg p) \wedge r\} \\
\mathcal{C}_3 &= \{1 : \alpha,\ \hat{1} : \beta,\ 1 : q \wedge t,\ \hat{1} : (s \vee \neg s) \vee t\} \\
\mathcal{C}_4 &= \{1 : \alpha,\ \hat{1} : \beta,\ 1 : q \wedge t,\ \hat{1} : q \vee r\}
\end{aligned}$$

Moreover, as $X_1 \cup \{\hat{1} : \beta\}$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are obtained by applying only tableau rules *that have a single set of bottom formulas* to $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_4$, it follows from (5) and the (induction step of the) Bifurcation lemma that $\mathcal{C}_1 \rhd_{\mathbf{z}} r \wedge \neg r$, that $\mathcal{C}_3 \rhd_{\mathbf{z}} t$ and that $\mathcal{C}_4 \rhd_{\mathbf{z}} q$. As $\mathcal{C}_3$ and $\mathcal{C}_4$ are obtained by applying $(\wedge_{\hat{1}})$ to $\mathcal{C}_{34} = \{1 : \alpha,\ \hat{1} : \beta,\ 1 : q \wedge t\}$ it follows that $\mathcal{C}_{34} \rhd_{\mathbf{z}} t \wedge q$. Further, as $\{1 : \alpha,\ \hat{1} : \beta\}$ is obtained from $\mathcal{C}_1$ and $\mathcal{C}_{34}$ by applying $(\vee_1)$, it follows that $\{1 : \alpha,\ \hat{1} : \beta\} \rhd_{\mathbf{z}} (r \wedge \neg r) \vee (t \wedge q)$. And so it follows that, for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}\}$, $\gamma := (r \wedge \neg r) \vee (t \wedge q)$ is a $\models_{\mathbf{z}}$-interpolant for α and β.

If $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}\}$ then $\mathcal{T}^{\mathbf{3b}} = \{Y_1 \cup \{1 : \alpha\}, \mathcal{B}_2, \mathcal{B}_4\}$ is a **3b**-closed and hence **2**-closed tableau of $\{1 : \alpha, \hat{1} : \beta\}$. An argument similar to the one above reveals that for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}\}$, we have $\{1 : \alpha,\ \hat{1} : \beta\} \rhd_{\mathbf{z}} (t \vee \neg t) \wedge (r \vee q)$ so that $\delta := (t \vee \neg t) \wedge (r \vee q)$ is a $\models_{\mathbf{z}}$-interpolant for α and β.

As $\mathcal{T}^{\mathbf{3}}$ as defined in Example 1 is a tableau of $\{1 : \alpha, \hat{1} : \beta\}$ that is $\mathbf{z}$-closed for all $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ one may also use the Maehara-style method to define $\models_{\mathbf{z}}$-interpolants based on this tableau. However, doing so results in interpolants with higher sentential complexity. As the reader may care to verify, by applying the Maehara-style method to $\mathcal{T}^{\mathbf{3}}$ we get that:

$\gamma' := ((r \wedge \neg r) \wedge r) \vee (q \wedge t)$ is a $\models_{\mathbf{z}}$-interpolant for α and β for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}\}$

$\delta' := ((r \vee \neg r) \wedge r) \vee (q \wedge t)$ is a $\models_{\mathbf{z}}$-interpolant for α and β for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}\}$

3.2. Maehara-Style Interpolation Versus Maehara's Method

In this section we sketch Maehara's interpolation method for CL. Doing so is instructive as it allows us to point out an important difference between our Maehara-style method and Maehara's method that, as we will see later on, has some interesting consequences.

Although Maehara's method (cf. [23, p 33]) is presented in terms of a sequent calculus for CL, the method is readily translated in terms of a tableau calculus. To do so, let us observe that the **Dunn** calculus contains a traditional signed tableau calculus for CL, which we'll call *the* CL *calculus*, as a subcalculus. Tableaux of the CL calculus are sets of *positive $\mathcal{L}$-branches*, i.e. sets of sentences of $\mathcal{L}$ that are signed with $\mathsf{1}$ or $\mathsf{0}$. The tableau rules of the CL calculus are the positive rule of the **Dunn** calculus, i.e. the rules $(\circ_x)$ with $\circ \in \{\neg, \wedge, \vee\}$ and $x \in \{\mathsf{1}, \mathsf{0}\}$. The closure conditions of the CL calculus are as follows: a positive branch $\mathcal{B}$ is *closed* iff $\{\mathsf{1} : \varphi, \mathsf{0} : \varphi\} \subseteq \mathcal{B}$ for some sentence φ. The CL calculus is a notational variant of the calculus presented by Smullyan [21] and so it readily follows that

$$\Gamma \models_{CL} \varphi \iff \mathsf{1} : \Gamma \;\cup\; \{\mathsf{0} : \varphi\} \text{ has a closed tableau (in the } \mathsf{CL} \text{ calculus)}$$

To apply Maehara's classical interpolation method, one extends the language $\mathcal{L}$ with the propositional constant symbol $\perp$ to obtain the language $\mathcal{L}^{\perp}$ and one then defines a positive $\mathcal{L}^{\perp}$-branch $\mathcal{B}$ to be *closed*$^{\perp}$ just in case $\mathcal{B}$ is closed or $\mathsf{1} : \perp \in \mathcal{B}$. The essential ingredient that is needed to prove that CL has the interpolation property by Maehara's method is the following lemma.

LEMMA 3. (Partition lemma) *Let Θ be a finite positive $\mathcal{L}$-branch and let $\{\Theta_L, \Theta_R\}$ be an arbitrary partition of Θ. If Θ has a closed tableau there is a $\gamma \in Sen(\mathcal{L}^{\perp})$ such that: (i) $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\Theta_L) \cap \mathsf{Voc}(\Theta_R)$ and such that (ii) both $\Theta_L \cup \{\mathsf{0} : \gamma\}$ and $\{\mathsf{1} : \gamma\} \cup \Theta_R$ have a closed$^{\perp}$ tableau.*

PROOF. By induction on the number of one-step expansions needed to convert $\{\Theta\}$ into a closed tableau of Θ. ■

It then readily follows from the Partition lemma that CL has the interpolation property,as attested by the following theorem.

THEOREM 3. CL *has the interpolation property.*

PROOF. If the CL-interpolation conditions are satisfied, $\mathsf{1} : \alpha, \mathsf{0} : \beta$ has a closed tableau so that there is a sentence γ of $\mathcal{L}^{\perp}$ that satisfies condition (i) and (ii) of the Partition lemma. By replacing all occurrences of $\perp$ in γ with $p \wedge \neg p$ for some $p \in \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$ (which exists in virtue of Lemma 1) one obtains a sentence of $\mathcal{L}$ that is a CL-interpolant for α and β. ■

The Partition lemma of Maehara's method plays a similar role as the Bifurcation lemma of our Maehara-style method. A notable difference between the two methods is that our Maehara-style method does not, in contrast to Maehara's method, involve a detour through the extended language

$\mathcal{L}^{\perp}$. The reason that our Maehara-style method does not need such a detour may be accounted for by the fact that the Bifurcation lemma is formulated in terms of the *unique bifurcation* of a branch whereas the Partition lemma is formulated in terms of *all the partitions* of a branch. The latter difference is an essential one, as the following two remarks purport to illustrate.

First, observe that the Bifurcation lemma cannot be phrased in terms of arbitrary partitions. To see this, note that $\Theta = \{\mathsf{1} : p, \hat{\mathsf{1}} : p,\}$ has a $\mathbf{4}$-closed tableau so that, when phrased in terms of arbitrary partitions, the "Bifurcation lemma" would require the existence of a sentence γ of $\mathcal{L}$ such that $\mathsf{Voc}(\gamma) \subseteq \{p\}$ and such that both $\{\hat{\mathsf{1}} : p, \hat{\mathsf{1}} : \gamma\}$ and $\{\mathsf{1} : p, \mathsf{1} : \gamma\}$ have a $\mathbf{4}$-closed tableau. One readily shows that such a γ cannot exist. Hence, we may say that bifurcations are an essential ingredient of our Maehara style method.

Second, one may consider getting rid of the arbitrary partitions of the Partition lemma by phrasing that lemma in terms of *separated partitions*, where the separated partition $\langle \Theta_L, \Theta_R \rangle$ of a positive branch Θ is such that Θ_L contains all elements of Θ with sign $\mathsf{1}$ and Θ_R contains all elements of Θ with sign $\mathsf{0}$. Call the lemma that results from rephrasing the Partition lemma in terms of separated partitions the *Separated partition lemma*. The *truth* of the Separated partition lemma immediately follows from the truth of the Partition lemma. However, the inductive proof that is underlying the Partition lemma breaks down for the Separated partition lemma, as the reader may care to verify by trying to prove the inductive step associated with a tableau rule for negation. Without such an inductive proof, the Separated partition lemma does not tell us how to *construct* a classical interpolant. Hence, we may say that arbitrary partitions are an essential ingredient of Maehara's method.

In Sections 3.4 and 4, we will see some interesting consequences of the fact that the Maehara-style method essentially relies on bifurcations.

3.3. The Pruned Tableau Method

The next theorem shows how to construct interpolants for the Dunn logics via the pruned tableau method.

THEOREM 4. *The Dunn logics have the interpolation property.*

PROOF. Let $\mathbf{z} \in \mathbf{Z}$ and suppose that the $\models_{\mathbf{z}}$-interpolation condition for α and β is satisfied. It suffices to show that there is a sentence $\gamma \in Sen(\mathcal{L})$ with $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$ and such that (i) $\{\mathsf{1} : \alpha, \hat{\mathsf{1}} : \gamma\}$ has a $\mathbf{z}$-closed tableau and (ii) $\{\mathsf{1} : \gamma, \hat{\mathsf{1}} : \beta\}$ has a $\mathbf{z}$-closed tableau.

Let $[1 : \alpha]$ and $[\hat{1} : \beta]$ be the fulfilled tableaux of $\{1 : \alpha\}$ and $\{\hat{1} : \beta\}$ respectively and note that these tableau are **z**-open as α is not an anti-tautology and β is not a tautology of $\models_{\mathbf{z}}$. Also, note that for any $A \in [1 : \alpha]$ and $B \in [\hat{1} : \beta]$, $A \cup B$ is **z**-closed as $\alpha \models_{\mathbf{z}} \beta$. Let A be a **z**-open branch of $[1 : \alpha]$ and define the sets A^1 and A^0 as follows, where p is a propositional variable:

$$\begin{aligned} A^1 &:= \{p \mid 1 : p \in A \text{ and } \hat{1} : p \in B \text{ for some } \mathbf{z}\text{-open } B \in [\hat{1} : \beta]\} \\ A^0 &:= \{\neg p \mid 0 : p \in A \text{ and } \hat{0} : p \in B \text{ for some } \mathbf{z}\text{-open } B \in [\hat{1} : \beta]\} \end{aligned} \tag{6}$$

As A is **z**-open, and as $A \cup B$ is **z**-closed for any $B \in [\hat{1} : \beta]$, it follows that $A^1 \cup A^0$ is non-empty and so the following sentence is well-defined:

$$\gamma(A) := \bigwedge \left(A^1 \cup A^0\right) \tag{7}$$

In terms of the sentences $\gamma(A)$, we define the sentence γ as follows.

$$\gamma := \bigvee\{\gamma(A) \mid A \text{ is a } \mathbf{z}\text{-open branch of } [1 : \alpha]\} \tag{8}$$

Clearly, $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$ and so it remains to be shown that γ satisfies (i) and (ii).

(i) Let $[\hat{1} : \gamma]$ be the fulfilled tableau of $\{\hat{1} : \gamma\}$. With $A \in [1 : \alpha]$, $X \in [\hat{1} : \gamma]$, it suffices to show that $A \cup X$ is **z**-closed. If A is **z**-closed, we are done, so suppose A is **z**-open. Then, from the definition of γ and the tableau rule $(\vee_{\hat{1}})$, it follows that $\hat{1} : \gamma(A)$ is an element of every branch of $[\hat{1} : \gamma]$ and so in particular of X. Further, from the definition of $\gamma(A)$ and the tableau rule $(\wedge_{\hat{1}})$ it follows that there is some atomic p for which:

$$(1 : p \in A \text{ and } \hat{1} : p \in X) \text{ or } (0 : p \in A \text{ and } \hat{1} : \neg p \in X)$$

from which it readily follows that $A \cup X$ is **4**-closed and hence **z**-closed.

(ii) Let $[1 : \gamma]$ be the fulfilled tableau of $\{1 : \gamma\}$. With $X \in [1 : \gamma]$ and $B \in [\hat{1} : \beta]$, it suffices to show that $X \cup B$ is **z**-closed. If B is **z**-closed, we are done. So suppose B is **z**-open. Per definition of γ and the tableau rule $(\vee_1)$ it follows that there is a **z**-open branch A of $[1 : \alpha]$ such that $1 : \gamma(A) \in X$. As A and B are atomically **z**-open and as $A \cup B$ is atomically **z**-closed, it follows that there is an atomic p such that

$$(1 : p \in A \text{ and } \hat{1} : p \in B) \text{ or } (0 : p \in A \text{ and } \hat{0} : p \in B)$$

Suppose, without loss of generality, that $1 : p \in A$ and $\hat{1} : p \in B$. Then p is one of the conjuncts of $\gamma(A)$ and hence, as $1 : \gamma(A) \in X$, it follows that $1 : p \in X$. But then $X \cup B$ is **4**-closed and hence **z**-closed. ∎

Table 1. $\models_{\mathbf{z}}$-interpolants via the pruned tableau and the Maehra-style method

	2	**3n**	**3b**
Pr. tab.	q	$t \wedge q$	$r \vee q$
$\mathbf{T^{3n}}$	γ	$\gamma = (r \wedge \neg r) \vee (t \wedge q)$	-
$\mathbf{T^{3b}}$	δ	-	$\delta = (t \vee \neg t) \wedge (r \vee q)$
$\mathbf{T^{3}}$	γ'/δ'	$\gamma' = ((r \wedge \neg r) \wedge r) \vee (q \wedge t)$	$\delta' = ((r \vee \neg r) \wedge r) \vee (q \wedge t)$

Thus, in contrast to the Maehara-style method, constructing interpolants via the pruned tableau method does not rely on a construction by induction but reads off the interpolant directly from a closed tableau. The pruned tableau method is not only simpler and more direct in this sense, but also the obtained interpolants typically have lower sentential complexity than those obtained with the Maeharae-style method, as illustrated by the following example.

Example 3. (Interpolants via the pruned tableau method) With α, β, X_1, X_2, Y_1, Y_2 as in Example 1, that example showed that $[\mathbf{1} : \alpha] = \{X_1, X_2\}$ and $[\hat{\mathbf{1}} : \beta] = \{Y_1, Y_2\}$ are the fulfilled tableau of $\{\mathbf{1} : \alpha\}$ and $\{\hat{\mathbf{1}} : \beta\}$ respectively. Let us illustrate how, for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}, \mathbf{3b}\}$ the pruned tableau method obtains a $\models_{\mathbf{z}}$-interpolant for α and β. For $\mathbf{z} = \mathbf{3b}$, X_1, X_2 and Y_2 are **3b**-open whereas Y_1 is **3b**-closed. And so, with $\gamma(X_1)$, $\gamma(X_2)$ and the $\models_{\mathbf{3b}}$-interpolant γ as defined by (7) and (8) respectively, we have $\gamma(A_1) = r$, $\gamma(A_2) = r$ and $\gamma = r \vee q$. For $\mathbf{z} = \mathbf{3n}$, X_2, Y_1 and Y_2 are **3n**-open whereas X_1 is **3n**-closed. And so $\gamma(X_2)$ and the $\models_{\mathbf{3n}}$-interpolant are equal to $q \wedge t$. Finally, for $\mathbf{z} = \mathbf{2}$, X_2 and Y_2 are **2**-open whereas X_1 and Y_1 are **2**-closed. And so $\gamma(X_2)$ and the $\models_{\mathbf{2}}$-interpolant are equal to q.

The pruned tableau method delivers a unique $\models_{\mathbf{z}}$-interpolant for α and β whenever their $\models_{\mathbf{z}}$-interpolation condition is satisfied. In contrast, $\{\mathbf{1} : \alpha, \hat{\mathbf{1}} : \beta\}$ may have several **z**-closed tableaux, which ensures that the $\models_{\mathbf{z}}$-interpolants that are obtained via the Maehara-style method are not unique, as witnessed by (the discussion following) Example 2. With α and β as defined in Example 2 and Example 3, the below table displays the **z**-interpolants—for $\mathbf{z} = \mathbf{2}$, **3n** and **3n**— for α and β that we obtained in those examples by the pruned tableau method and by applying the Maehara-style method to the tableaux $\mathbf{T^3}$, $\mathbf{T^{3n}}$ and $\mathbf{T^{3b}}$ as defined in Example 2.

3.4. The Pruned Tableau Method and ETL

In a recent paper, Pietz and Rivieccio introduce and study *exactly true logic* (ETL), whose semantic definition is as follows.

$$\Gamma \models_{ETL} \varphi \Leftrightarrow \text{if } V(\gamma) = \mathbf{T} \text{ for all } \gamma \in \Gamma \text{ then } V(\varphi) = \mathbf{T}, \text{ for all } V \in \mathbf{V_4}$$

In order to capture $\models_{ETL}$ via the **Dunn** calculus, we define $\vdash_{ETL}$ as follows.

$$\Gamma \vdash_{ETL} \varphi \Leftrightarrow 1 : \Gamma \ \cup \ \hat{0} : \Gamma \ \cup \ \{\hat{1} : \varphi\} \text{ has a } \mathbf{4}\text{-closed tableau} \tag{9}$$

Although perhaps a bit surprising at first sight, $\vdash_{ETL}$ coincides with $\models_{ETL}$, as the proof of the following proposition explains.

Proposition 1. $\Gamma \models_{ETL} \varphi \iff \Gamma \vdash_{ETL} \varphi$.

Proof. First observe that it readily follows from the definition of $\models_{ETL}$ and Theorem 1 that $\Gamma \models_{ETL} \varphi$ iff

$1 : \Gamma \ \cup \ \hat{0} : \Gamma \ \cup \ \{\hat{1} : \varphi\}$ has a $\mathbf{4}$-closed tableau and
$1 : \Gamma \ \cup \ \hat{0} : \Gamma \ \cup \ \{0 : \varphi\}$ has a $\mathbf{4}$-closed tableau.

However, an inspection of the tableau rules readily verifies that a branch $\mathcal{B}$ has a $\mathbf{4}$-closed tableau iff its *counterpart* has a $\mathbf{4}$-closed tableau, where the counterpart of $\mathcal{B}$ is defined as follows:

$$\{1 : \varphi \mid \hat{0} : \varphi \in \mathcal{B}\} \cup \{0 : \varphi \mid \hat{1} : \varphi \in \mathcal{B}\} \cup \{\hat{1} : \varphi \mid 0 : \varphi \in \mathcal{B}\} \cup \{\hat{0} : \varphi \mid 1 : \varphi \in \mathcal{B}\}$$

As $1 : \Gamma \ \cup \ \hat{0} : \Gamma \ \cup \ \{\hat{1} : \varphi\}$ is the counterpart of $1 : \Gamma \ \cup \ \hat{0} : \Gamma \ \cup \ \{0 : \varphi\}$ it follows that $\Gamma \models_{ETL} \varphi$ just in case $\Gamma \vdash_{ETL} \varphi$. ■

To establish that ETL has the interpolation property in terms of the **Dunn** calculus, we thus need to show the following: whenever the ETL-interpolation condition for α and β is satisfied, there is a sentence γ in the joint vocabulary of α and β such that both $\{1 : \alpha, \hat{0} : \alpha, \hat{1} : \gamma\}$ and $\{1 : \gamma, \hat{0} : \gamma, \hat{1} : \beta\}$ have a $\mathbf{4}$-closed tableau. Observe that, as $\langle\{1 : \alpha\}, \{\hat{0} : \alpha, \hat{1} : \beta\}\rangle$ is the bifurcation of $\{1 : \alpha, \hat{0} : \alpha, \hat{1} : \beta\}$, the Bifurcation lemma and hence the Maehara-style method cannot be used to establish this. Moreover, given the problems with rephrasing the Bifurcation lemma in terms of arbitrary partitions as discussed in Section 3.2, to show that ETL has the interpolation property via a Maehara-inspired method seems to be problematic in the current setting. However, the pruned tableau method can be readily used to construct ETL-interpolants, as attested by the following proposition.

PROPOSITION 2. ETL *has the interpolation property.*

PROOF. Suppose that the ETL-interpolation condition for α and β is satisfied. It suffices to show that there is a sentence $\gamma \in Sen(\mathcal{L})$ with $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$ and such that (i) $\{\mathsf{1} : \alpha, \hat{\mathsf{0}} : \alpha, \hat{\mathsf{1}} : \gamma\}$ has a $\mathbf{4}$-closed tableau and (ii) $\{\mathsf{1} : \gamma, \hat{\mathsf{0}} : \gamma, \hat{\mathsf{1}} : \beta\}$ has a $\mathbf{4}$-closed tableau.

Let $[\mathsf{1} : \alpha, \hat{\mathsf{0}} : \alpha]$ and $[\hat{\mathsf{1}} : \beta]$ be the fulfilled tableaux for $\{\mathsf{1} : \alpha, \hat{\mathsf{0}} : \alpha\}$ and $\{\hat{\mathsf{1}} : \beta\}$ respectively, which are $\mathbf{4}$-open in virtue of the the ETL-interpolation condition. Let A and B be $\mathbf{4}$-open branches of $[\mathsf{1} : \alpha, \hat{\mathsf{0}} : \alpha]$ and $[\hat{\mathsf{1}} : \beta]$ respectively and note that $A \cup B$ is $\mathbf{4}$-closed. Define A^{1} and A^{0} as in (6), letting $\mathbf{z} = \mathbf{4}$. For the following three reasons $A^{\mathsf{1}} \cup A^{\mathsf{0}}$ is non-empty: (1) A is $\mathbf{4}$-open, (2) $A \cup B$ is $\mathbf{4}$-closed (3) B only contains sentences that are signed with $\hat{\mathsf{1}}$ or $\hat{\mathsf{0}}$, which means that the (atomic) $\mathbf{4}$-closure of $A \cup B$ must be due to the occurrence of an element of form $\mathsf{0} : p$ or $\mathsf{1} : p$ in A. Thus one may define $\gamma(A)$ as in (7) and γ as in (8), i.e.:

$$\gamma := \bigvee \{\gamma(A) \mid A \text{ is a } \mathbf{4}\text{-open branch of } [\mathsf{1} : \alpha, \hat{\mathsf{0}} : \alpha]\}$$

Clearly, $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$ and entirely similar as in the proof of Theorem 4 one shows that γ satisfies (i) and (ii). ■

The pruned tableau method thus readily establishes that ETL has the interpolation property whereas the Maehara-style method cannot be used to establish this fact. We take it that this is an additional (to the reasons mentioned in Section 3.3) reason to prefer the pruned tableau method over the Maehara-style method.

3.5. Milne's Result and a Novel Characterization of CL

In a recent paper, Peter Milne [14] establishes what he calls 'a non-classical refinement of the interpolation property for classical logic'. Milne's refinement tells us that, whenever the $\models_{CL}$-interpolation condition for α and β is satisfied, there is a sentence γ in the joint vocabulary of α and β such that:

$$\alpha \models_{K3} \gamma \quad \text{and} \quad \gamma \models_{LP} \beta \tag{10}$$

Milne presents a semantic proof of (10) but his result also follows readily from our proof of Theorem 4. This is attested by the following corollary, which shows that the CL-interpolant that is obtained via the pruned tableau method satisfies (10).

Corollary 2. *Suppose that the* $\models_{CL}$*-interpolation condition for* α *and* β *is satisfied. Let* γ *be the* $\models_{CL}$*-interpolant for* α *and* β *as defined in the proof of Theorem 4. Then* $\alpha \models_{K3} \gamma$ *and* $\gamma \models_{LP} \beta$*.*

Proof. Suppose that the $\models_{CL}$-interpolation condition for α and β is satisfied and let γ be as indicated above. Observe that in the proof of Theorem 4 it is shown that for any **2**-open branch A of $[1 : \alpha]$ and for any $X \in [\hat{1} : \gamma]$, $A \cup X$ is **4**-closed. Now a branch A of $[1 : \alpha]$ is **2**-open if and only if A is **3n**-open and so it follows that for any **3n**-open branch A of $[1 : \alpha]$, $A \cup X$ is **4**-closed and hence **3n**-closed. This establishes that $\alpha \models_{K3} \gamma$. Similarly, one shows that $\gamma \models_{LP} \beta$. ■

Although Milne's result is interesting in itself, we also feel that his has an even more interesting consequence: CL can be characterized in terms of $\mathsf{K3}$ and LP, as the following propositions attests.

Proposition 3. (Characterizing CL in terms *of* $\mathsf{K3}$ and LP*)* $\alpha \models_{CL} \beta \iff$ *there is a sentence* χ *such that* $\alpha \models_{K3} \chi$ *and* $\chi \models_{LP} \beta$*.*

Proof. For the left-to-right direction, we distinguish 3 cases. First, if α is an anti-tautology of CL, setting $\chi = \beta$ establishes the claim. Second, if β is a tautology of CL, setting $\chi = \alpha$ establishes the claim. Third, if α is not an anti-tautology of CL and β is not a tautology of CL, the claim is established by Corollary 2. The right-to-left direction follows from the fact that CL extends both $\mathsf{K3}$ and LP and the transitivity of classical consequence. ■

4. Interpolation and Appropriate Implication

As was discussed in the introduction, $\mathsf{K3}$, LP and FDE as defined over $\mathcal{L}$ do not enjoy an appropriate implication connective, which motivates an extension of $\mathcal{L}$ with $\supset$. The Dunn conditions of $\supset$, which were given in the introduction, determine the following truth table.

$\supset$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**T**	**B**	**N**	**F**
N	**T**	**T**	**T**	**T**
F	**T**	**T**	**T**	**T**

We consider the propositional language $\mathcal{L}_{\supset}$ that is based on $\{\wedge, \vee, \neg, \supset\}$ and define a **z**-*valuation for* $\mathcal{L}_{\supset}$ to be to a function from the sentences of this language to $\mathbf{z} \in \mathbf{Z}$ that respects the truth tables of Definition 1 and the above truth table of $\supset$. Also, $\mathbf{V}^{\supset}_{\mathbf{z}}$ will denote the set of all $\mathbf{z}$-valuation for $\mathcal{L}_{\supset}$. We define the $\mathcal{L}_{\supset}$ entailment relations $\models^{\supset}_{\mathbf{z}}$ which preserve truth over all $\mathbf{z}$-valuations of $\mathcal{L}_{\supset}$:

$$\Gamma \models^{\supset}_{\mathbf{z}} \varphi \Leftrightarrow \text{if } V(\gamma) \in \mathsf{1} \text{ for all } \gamma \in \Gamma \text{ then } V(\varphi) \in \mathsf{1}, \text{ for all } V \in \mathbf{V}^{\supset}_{\mathbf{z}}$$

Although $\models^{\supset}_{\mathbf{2}}$ is just classical logic with $\supset$ denoting material implication, the other $\models^{\supset}_{\mathbf{z}}$ logics are genuine extensions of $\mathsf{K3}$, LP and FDE for which $\supset$ is an appropriate implication connective, as the following proposition attests.

Proposition 4. *For* $\mathbf{z} \in \mathbf{Z}$*:* $\Gamma \cup \varphi \models^{\supset}_{\mathbf{z}} \psi \iff \Gamma \models^{\supset}_{\mathbf{z}} \varphi \supset \psi$.

Proof. By inspection. ■

The Dunn conditions for $\supset$ do no only determine its truth table, but they also give rise to the following tableau rules.

$$\frac{x : \varphi \supset \psi}{\hat{\mathsf{1}} : \varphi \mid x : \psi}\,(\supset_x) \quad \text{if } x \in \{\mathsf{1}, \hat{\mathsf{0}}\} \qquad\qquad \frac{x : \varphi \supset \psi}{\mathsf{1} : \varphi,\ x : \psi}\,(\supset_x) \quad \text{if } x \in \{\hat{\mathsf{1}}, \mathsf{0}\}$$

The tableau rules of the $\mathbf{Dunn}^{\supset}$ *calculus* are obtained by adding the above rules for $\supset$ to those of the **Dunn** calculus. The $\mathbf{Dunn}^{\supset}$ calculus has the same $\mathbf{z}$-closure conditions as the **Dunn** calculus and the following proposition will not come as a surprise.

Proposition 5. $\Gamma \models^{\supset}_{\mathbf{z}} \varphi \iff \mathsf{1} : \Gamma \cup \{\hat{\mathsf{1}} : \varphi\}$ *has a* $\mathbf{z}$*-closed tableau.*

Proof. See [24]. ■

In Section 3.4, we explained that the Maehara-style interpolation method cannot be used to show that ETL has the interpolation property. It turns out that showing that the $\models^{\supset}_{\mathbf{z}}$-logics have the interpolation property via the Maehara-style method is also problematic. To see the problem, note that all the rules of the **Dunn** calculus are *bifurcation-neutral*, which means that: whenever the sign x the top formula $x : \varphi$ of a rule $x : \varphi / B_1, \ldots, B_n$ is positive (i.e. $x \in \{\mathsf{1}, \mathsf{0}\}$) so are the signs of the formulas that occur in $B_1, \ldots, B_n$, and whenever x is negative (i.e. $x \in \{\hat{\mathsf{1}}, \hat{\mathsf{0}}\}$), so are the signs of the formulas that occur in $B_1, \ldots, B_n$. In contrast to the rules of the **Dunn** calculus, $(\supset_{\mathsf{1}})$ and $(\supset_{\hat{\mathsf{1}}})$ are *not* bifurcation-neutral. As a consequence, the proof of the Bifurcation lemma breaks down when it is phrased in terms of the $\mathbf{Dunn}^{\supset}$ calculus, as the reader may care to verify.

However, by invoking the pruned tableau method, we will show, in a uniform way, that the logics $\models^{\supset}_{\mathbf{z}}$ all have the interpolation property. In order to present the proof in a neat way, we first have the following definition.

Definition 7. ($\mathbf{z}$-closers) With A *and* B sets of signed sentences, a $\mathbf{z}$-*closer of* $\langle A, B\rangle$ is a pair $\langle x : \varphi, y : \varphi\rangle$ with $x : \varphi \in A$, $y : \varphi \in B$ and such that $\{x : \varphi, y : \varphi\}$ is $\mathbf{z}$-closed. We write $\mathsf{Cl}^{\mathbf{z}}(A, B)$ for the set of all $\mathbf{z}$-closers of $\langle A, B\rangle$ and define $\mathsf{Cl}^{\mathbf{z}}_A(A, B)$ and $\mathsf{Cl}^{\mathbf{z}}_B(A, B)$ as follows.

$$\mathsf{Cl}^{\mathbf{z}}_A(A, B) := \{x : \varphi \mid \langle x : \varphi, y : \varphi\rangle \in \mathsf{Cl}^{\mathbf{z}}(A, B)\}$$
$$\mathsf{Cl}^{\mathbf{z}}_B(A, B) := \{y : \varphi \mid \langle x : \varphi, y : \varphi\rangle \in \mathsf{Cl}^{\mathbf{z}}(A, B)\}$$

To show that the logics $\models^{\supset}_{\mathbf{z}}$ have the interpolation property, we will rely on the following lemma

Lemma 4. *For any* $\varphi \in Sen(\mathcal{L}_{\supset})$ *and* $x \in \{1, 0\}$, *any tableau of* $\{x : \varphi\}$ *has at least one positive branch* $\mathcal{B}$, *i.e. at least one branch* $\mathcal{B}$ *such that* $x \in \{1, 0\}$ *whenever* $x : \psi \in \mathcal{B}$.

Proof. By an induction on the sentential complexity of φ that can be left to the reader. ■

Theorem 5. *The logics* $\models^{\supset}_{\mathbf{z}}$ *have the interpolation property.*

Proof. Suppose that the interpolation condition for $\models^{\supset}_{\mathbf{z}}$ is satisfied. It suffices to show that there is a sentence $\gamma \in Sen(\mathcal{L}_{\supset})$ such that $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$ and such that (i) $\{1 : \alpha, \hat{1} : \gamma\}$ has a $\mathbf{z}$-closed tableau and (ii) $\{1 : \gamma, \hat{1} : \beta\}$ has a $\mathbf{z}$-closed tableau.

Let $[1 : \alpha]$ and $[\hat{1} : \beta]$ be the fulfilled tableau of $\{1 : \alpha\}$ and $\{\hat{1} : \beta\}$ respectively. For each sign $x \in \{1, 0, \hat{1}, \hat{0}\}$ and $\mathbf{z}$-open branch A of $[1 : \alpha]$, we define the set A_x—consisting of propositional atoms signed with x— as follows:

$$A_x = \{x : p \mid x : p \in \mathsf{Cl}^{\mathbf{z}}_A(A, B) \text{ for some } \mathbf{z}\text{-open } B \in [\hat{1} : \beta]\}$$

We use the sets of signed sentences A_x to define their "unsigned counterparts" A^x. First, we set:

$$A^1 := \mathsf{Voc}(A_1) \qquad\qquad A^0 := \{\neg p \mid p \in \mathsf{Voc}(A_0)\} \tag{11}$$

If $[1 : \alpha]$ has a positive branch that is $\mathbf{z}$-open, let $\mathcal{A}$ be an arbitrary such branch and let $\chi := \bigwedge(\mathcal{A}^1 \cup \mathcal{A}^0)$, where the definition of $\mathcal{A}^1$ and $\mathcal{A}^0$ is given by (11). If $[1 : \alpha]$ does not have a positive branch that is $\mathbf{z}$-open, let $q \in \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$—such a q exists as the interpolation condition is satisfied—and let $\chi := q \wedge \neg q$. We use the sentence χ to define the sets $A^{\hat{1}}$ and $A^{\hat{0}}$ as follows.

$$A^{\hat{1}} := \{p \supset \chi \mid p \in \mathsf{Voc}(A_{\hat{1}})\} \qquad A^{\hat{0}} := \{\neg p \supset \chi \mid p \in \mathsf{Voc}(A_{\hat{0}})\}$$

Let A be a $\mathbf{z}$-open branch of $[1 : \alpha]$. As $A \cup B$ is $\mathbf{z}$-closed for any $B \in [\hat{1} : \beta]$, it follows that $A^1 \cup A^0 \cup A^{\hat{1}} \cup A^{\hat{0}}$ is non-empty and so the following sentence is well-defined:

$$\gamma(A) := \bigwedge \left(A^1 \cup A^0 \cup A^{\hat{1}} \cup A^{\hat{0}} \right)$$

In terms of the sentences $\gamma(A)$, we define the sentence γ as follows.

$$\gamma := \bigvee \{\gamma(A) \mid A \text{ is a } \mathbf{z}\text{-open branch of } [1 : \alpha]\} \tag{12}$$

Clearly, $\mathsf{Voc}(\gamma) \subseteq \mathsf{Voc}(\alpha) \cap \mathsf{Voc}(\beta)$ and so it remains to be shown that γ satisfies (i) and (ii).

(i) Let $[\hat{1} : \gamma]$ be the fulfilled tableau of $\hat{1} : \gamma$. With $A \in [1 : \alpha]$, $X \in [\hat{1} : \gamma]$, it suffices to show that $A \cup X$ is $\mathbf{z}$-closed. If A is $\mathbf{z}$-closed we are done, so suppose A is $\mathbf{z}$-open. Then, from the definition of γ and the tableau rule $(\vee_{\hat{1}})$, it follows that $\hat{1} : \gamma(A)$ is an element of every branch of $[\hat{1} : \gamma]$ and so in particular of X. Further, from the definition of $\gamma(A)$ and the tableau rule $(\wedge_{\hat{1}})$ it follows that for some propositional atom p:

$$(1 : p \in A \text{ and } \hat{1} : p \in X) \text{ or } (0 : p \in A \text{ and } \hat{1} : \neg p \in X) \text{ or}$$
$$(\hat{1} : p \in A \text{ and } \hat{1} : p \supset \chi \in X) \text{ or } (\hat{0} : p \in A \text{ and } \hat{1} : \neg p \supset \chi \in X)$$

From an inspection of the tableau rules pertaining to $\neg$ and $\supset$ it readily follows that $A \cup X$ is $\mathbf{4}$-closed and hence $\mathbf{z}$-closed.

(ii) Let $[1 : \gamma]$ be the fulfilled tableau of $1 : \gamma$. With $X \in [1 : \gamma]$, $B \in [\hat{1} : \beta]$, it suffices to show that $X \cup B$ is $\mathbf{z}$-closed. If B is $\mathbf{z}$-closed, we are done. So suppose B is $\mathbf{z}$-open. Per definition of γ and the tableau rule $(\vee_1)$ it follows that there is a $\mathbf{z}$-open branch A of $[1 : \alpha]$ such that $1 : \gamma(A) \in X$. As A and B are $\mathbf{z}$-open and as $A \cup B$ is $\mathbf{z}$-closed, it follows that $\mathsf{Cl}^{\mathbf{z}}(A, B)$ is not empty. Moreover, $A \cup B$ is atomically closed. So let $\langle x : p, y : p \rangle \in \mathsf{Cl}^{\mathbf{z}}(A, B)$ where p is a propositional variable.

If $x = 1$ it follows that p is one of the conjuncts of $\gamma(A)$, that $1 : p \in X$ and hence that $X \cup B$ is $\mathbf{z}$-closed.

If $x = 0$ it follows that $\neg p$ is one of the conjuncts of $\gamma(A)$, that $0 : p \in X$ and hence that $X \cup B$ is $\mathbf{z}$-closed.

If $x = \hat{1}$ it follows that $p \supset \chi$ is one of the conjuncts of $\gamma(A)$ and hence $\hat{1} : p \in X$ or $1 : \chi \in X$. If $\hat{1} : p \in X$ then $X \cup B$ is $\mathbf{z}$-closed, so suppose that $1 : \chi \in X$. An inspection of the definition of χ reveals that we must consider the following two cases.

(1) $\chi = q \wedge \neg q$. Then, every positive branch of $[1 : \alpha]$ is $\mathbf{z}$-closed. But then, as $[1 : \alpha]$ has at least one positive branch it follows that $\mathbf{z}$ is either $\mathbf{2}$ or $\mathbf{3n}$. But if $\mathbf{z}$ is $\mathbf{2}$ or $\mathbf{3n}$, any branch which contains $1 : q \wedge \neg q$ is $\mathbf{z}$-closed and so $X \cup B$ in particular.

(2) $\chi = \bigwedge(\mathcal{A}^1 \cup \mathcal{A}^0)$ for some positive $\mathbf{z}$-open branch $\mathcal{A}$ of $[1 : \alpha]$. As $\mathcal{A}$ is $\mathbf{z}$-open and as $\mathcal{A} \cup B$ is $\mathbf{z}$-closed it follows that there is some propositional atom r such that ($1 : r \in \mathcal{A}$ and $\hat{1} : r \in B$) or ($0 : r \in \mathcal{A}$ and $\hat{0} : r \in B$). If $1 : r \in \mathcal{A}$ it follows that r is one of the conjuncts of χ, that $1 : r \in X$ and hence that $X \cup B$ is $\mathbf{z}$-closed. If $0 : r \in \mathcal{A}$ it follows that $\neg r$ is one of the conjuncts of χ, that $0 : r \in X$ and hence that $X \cup B$ is $\mathbf{z}$-closed.

If $x = \hat{0}$ it follows that $\neg p \supset \chi$ is one of the conjuncts of $\gamma(A)$ and hence that $\hat{1} : \neg p \in X$ or $1 : \chi \in X$. If $\hat{1} : \neg p \in X$ then $\hat{0} : p \in X$ and hence $X \cup B$ is $\mathbf{z}$-closed. If $1 : \chi \in X$, an argument entirely similar to that given above reveals that $X \cup B$ is $\mathbf{z}$-closed. ■

5. Interpolation and Functional Completeness

In order to define functionally complete extensions of the Dunn logics, we consider the propositional constant symbols f, b and n, which we interpret as constant functions denoting the values $\mathbf{F}$, $\mathbf{B}$ and $\mathbf{N}$ respectively. Also, we consider the unary connective $-$ that will denote Fitting's [12] *conflation operator*, i.e. the following truth function on $\mathbf{4}$:

$$-\mathbf{T} = \mathbf{T} \qquad -\mathbf{B} = \mathbf{N} \qquad -\mathbf{N} = \mathbf{B} \qquad -\mathbf{F} = \mathbf{F} \tag{13}$$

Let $\mathcal{L}_\mathbf{4}$ be the language that is obtained by extending $\mathcal{L}_\supset$ with f, b, n and $-$.

Proposition 6. *$\mathcal{L}_\mathbf{4}$ (with its logical vocabulary interpreted as indicated above) is functionally complete with respect to $\mathbf{4}$.*

Proof. See Avron [5]. ■

By an *$\mathcal{L}_\mathbf{4}$-valuation*, we mean a function from the sentences of $\mathcal{L}_\mathbf{4}$ to $\mathbf{4}$ that respects the above indicated interpretation of $\neg, \wedge, \vee \supset, -, \mathsf{f}, \mathsf{b}$ and n. We will use $\mathbf{V}_\mathbf{4}^\#$ to denote the set of all $\mathcal{L}_\mathbf{4}$-valuations and define the $\mathcal{L}_\mathbf{4}$ entailment relation $\models_\mathbf{4}^\#$ as follows:

$$\Gamma \models_\mathbf{4}^\# \varphi \Leftrightarrow \text{ if } V(\gamma) \in 1 \text{ for all } \gamma \in \Gamma \text{ then } V(\varphi) \in 1, \text{ for all } V \in \mathbf{V}_\mathbf{4}^\#$$

Takano [22] shows that any functionally complete many-valued logic, and so in particular $\models_\mathbf{4}^\#$, has the interpolation property. However, to actually construct interpolants for $\models^{\#}_{\mathbf{4}}$ by the general and semantic means provided in [22] is quite cumbersome. As we will show below, the pruned tableau method constructs these interpolants in a simple and informative manner. To do so, let us define tableau rules for the conflation operator:

$$\frac{x : -\varphi}{y : \varphi}\,(-_x) \qquad\qquad \text{if } \langle x, y\rangle \text{ or } \langle y, x\rangle \in \{\langle \mathsf{1}, \hat{\mathsf{0}}\rangle, \langle \hat{\mathsf{1}}, \mathsf{0}\rangle\}$$

The tableau rules of the $\mathbf{Dunn}^{\#}_{\mathbf{4}}$ *calculus* are obtained by adding the above rules for the conflation operator to those of the $\mathbf{Dunn}^{\supset}$ calculus. Tableaux in the $\mathbf{Dunn}^{\#}_{\mathbf{4}}$ calculus are sets of $\mathcal{L}_{\mathbf{4}}$-branches, i.e. sets of signed sentences of $\mathcal{L}_{\mathbf{4}}$. The closure conditions of the $\mathbf{Dunn}^{\#}_{\mathbf{4}}$ calculus are as expected: an $\mathcal{L}_{\mathbf{4}}$-branch $\mathcal{B}$ is $\mathbf{4}^{\#}$*-closed* just in case

$$\mathcal{B} \text{ is } \mathbf{4}\text{-closed or } \{\mathsf{1} : \mathsf{f}, \hat{\mathsf{0}} : \mathsf{f}, \hat{\mathsf{1}} : \mathsf{b}, \hat{\mathsf{0}} : \mathsf{b}, \mathsf{1} : \mathsf{n}, \mathsf{0} : \mathsf{n}\} \cap \mathcal{B} \neq \emptyset$$

One then readily shows that the $\mathbf{Dunn}^{\#}_{\mathbf{4}}$-calculus captures $\models^{\#}_{\mathbf{4}}$.

Proposition 7. *$\Gamma \models^{\#}_{\mathbf{4}} \varphi \iff \mathsf{1} : \Gamma \cup \{\hat{\mathsf{1}} : \varphi\}$ has a $\mathbf{4}^{\#}$-closed tableau.*

Proof. Via a straightforward modification of the proof of Theorem 1. ∎

The proof of the below proposition shows how the pruned tableau method constructs $\models^{\#}_{\mathbf{4}}$-interpolants.

Proposition 8. *$\models^{\#}_{\mathbf{4}}$ has the interpolation property.*

Proof. Suppose that the $\models^{\#}_{\mathbf{4}}$-interpolation condition for α and β is satisfied. For any $\mathbf{4}^{\#}$-open branch A of $[\mathsf{1} : \alpha]$ and any sign x, define the set A^x as follows:

$$A^{\mathsf{1}} := \{p \mid \mathsf{1} : p \in A \text{ and } \hat{\mathsf{1}} : p \in B \text{ for some } \mathbf{4}^{\#}\text{-open } B \in [\hat{\mathsf{1}} : \beta]\}$$
$$A^{\mathsf{0}} := \{\neg p \mid \mathsf{0} : p \in A \text{ and } \hat{\mathsf{0}} : p \in B \text{ for some } 4^{\#}\text{-open } B \in [\hat{\mathsf{1}} : \beta]\}$$
$$A^{\hat{\mathsf{1}}} := \{-\neg p \mid \hat{\mathsf{1}} : p \in A \text{ and } \mathsf{1} : p \in B \text{ for some } \mathbf{4}^{\#}\text{-open } B \in [\hat{\mathsf{1}} : \beta]\}$$
$$A^{\hat{\mathsf{0}}} := \{-p \mid \hat{\mathsf{0}} : p \in A \text{ and } \mathsf{0} : p \in B \text{ for some } \mathbf{4}^{\#}\text{-open } B \in [\hat{\mathsf{1}} : \beta]\}$$

As the union of the sets A^x thus defined is non-empty, the following sentence is well-defined:

$$\gamma(A) := \bigwedge \left(A^{\mathsf{1}} \cup A^{\mathsf{0}} \cup A^{\hat{\mathsf{1}}} \cup A^{\hat{\mathsf{0}}}\right) \tag{14}$$

We now define the sentence γ as follows:

$$\gamma := \bigvee \{\gamma(A) \mid A \text{ is a } \mathbf{4}^{\#}\text{-open branch of } [\mathsf{1} : \alpha]\} \tag{15}$$

By an argument similar to the one used in the proof of Theorem 4, it follows that γ is a $\models^{\#}_{\mathbf{4}}$-interpolant for α and β. ■

Indeed, the proof of proposition 8 is a straightforward generalization of the proof of Theorem 4, which ensures that the pruned tableau construction of $\models^{\#}_{\mathbf{4}}$-interpolants is just as simple as the construction of interpolants for the Dunn logics over $\mathcal{L}$. The proof of proposition 8 is also informative, as it shows that $\models^{\#}_{\mathbf{4}}$-interpolants can always be found $\mathcal{L}^{-}$, the sublanguage of $\mathcal{L}_{\mathbf{4}}$ that is based on $\{\neg, -, \wedge, \vee\}$. That is, we have the following corollary to proposition 8.

Corollary 3. *If the $\models^{\#}_{\mathbf{4}}$-interpolation condition for α and β is satisfied, then there is a $\gamma \in Sen(\mathcal{L}^{-})$ that is a $\models^{\#}_{\mathbf{4}}$-interpolant for α and β.*

Proof. By inspecting the proof of Proposition 8. ■

Let us, for sake of completeness, also briefly discuss interpolation for functionally complete extensions of $\models_{\mathbf{3b}}$ and $\models_{\mathbf{3n}}$. Avron [5] shows that the language $\mathcal{L}_{\mathbf{3b}}$, obtained by extending $\mathcal{L}_{\supset}$ with f and b, is functionally complete with respect to $\mathbf{3b}$. From this result, it is not hard to show that $\mathcal{L}_{\mathbf{3n}}$, obtained by extending $\mathcal{L}_{\supset}$ with f and n, is functionally complete with respect to $\mathbf{3n}$. Completely similar to the above, one defines the notion of an $\mathcal{L}_{\mathbf{3b}}$- and $\mathcal{L}_{\mathbf{3n}}$-valuation and by preserving truth over these valuations one defines the $\mathcal{L}_{\mathbf{3b}}$- and $\mathcal{L}_{\mathbf{3n}}$-entailment relations $\models^{\#}_{\mathbf{3b}}$ and $\models^{\#}_{\mathbf{3n}}$ respectively. We may show that $\models^{\#}_{\mathbf{3b}}$ and $\models^{\#}_{\mathbf{3n}}$ have the interpolation property by invoking the $\mathbf{Dunn}^{\supset}$ calculus in combination with the $\mathbf{3b}^{\#}$- and $\mathbf{3n}^{\#}$ closure conditions, which are defined as follows.

$$\mathcal{B} \text{ is } \mathbf{3b}^{\#}\text{-closed} \Leftrightarrow \mathcal{B} \text{ is } \mathbf{3b}\text{-closed or } \{1:\mathsf{f}, \hat{0}:\mathsf{f}, \hat{1}:\mathsf{b}, \hat{0}:\mathsf{b}\} \cap \mathcal{B} \neq \emptyset$$
$$\mathcal{B} \text{ is } \mathbf{3n}^{\#}\text{-closed} \Leftrightarrow \mathcal{B} \text{ is } \mathbf{3n}\text{-closed or } \{1:\mathsf{f}, \hat{0}:\mathsf{f}, 1:\mathsf{n}, 0:\mathsf{n}\} \cap \mathcal{B} \neq \emptyset$$

One may show that, for $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}\}$:

$$\Gamma \models^{\#}_{\mathbf{3b}} \varphi \iff 1:\Gamma \,\cup\, \{\hat{1}:\varphi\} \text{ has a } \mathbf{z}^{\#}\text{-closed tableau}$$

By invoking the pruned tableau method one readily shows that $\models^{\#}_{\mathbf{3b}}$ and $\models^{\#}_{\mathbf{3n}}$ have the interpolation property and that their interpolants can always be found in $\mathcal{L}_{\supset}$:

Proposition 9. *For* $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}\}$*: (i)* $\models^{\#}_{\mathbf{z}}$ *has the interpolation property and (ii) whenever the* $\models^{\#}_{\mathbf{z}}$*-interpolantion condition for* α *and* β *is satisfied, there is a* $\gamma \in Sen(\mathcal{L}_{\supset})$ *that is a* $\models^{\#}_{\mathbf{z}}$*-interpolant for* α *and* β*.*

Proof. The proof of (i) is entirely similar to the proof of Theorem 5, and (ii) follows from this similarity. ■

Theorem 5 shows that $\models^{\supset}_{\mathbf{3b}}$, $\models^{\supset}_{\mathbf{3n}}$ and $\models^{\supset}_{\mathbf{4}}$ have the interpolation property and the proof of this theorem immediately delivers a proof (cf. Proposition 9) which shows that $\models^{\#}_{\mathbf{3b}}$ and $\models^{\#}_{\mathbf{3n}}$ have the interpolation property. However, as the proof of Theorem 5 crucially involves Lemma 4 and that lemma is no longer true with the tableau rules pertaining to the conflation operator $-$ in force, the proof strategy of Theorem 5 cannot be invoked to show that $\models^{\#}_{\mathbf{4}}$ has the interpolation property.

6. Concluding Remarks

We first explained how the Dunn conditions give us a uniform semantic approach to CL, K3, LP and FDE and also, in terms of the **Dunn** calculus, a uniform syntactic one. We then used the **Dunn** calculus to define two distinct *uniform* interpolation methods for Dunn logics and their extensions: the Maehara-style method, which is inspired by Maehara's interpolation method for CL and the pruned tableau method, which is novel to this paper. The pruned tableau method is simpler than the Maehara-style method as it does not construct interpolants via an inductive argument and as the constructed interpolants typically have lower sentential complexity than those of the Maehara-style method. Moreover, whereas the Maehara-style method cannot be used to construct interpolants for ETL and for extensions of the Dunn logics with an appropriate implication connective, we showed that the pruned tableau method can be unproblematically invoked to do so. We also showed that Milne's recent "non-classical refinement of CL's interpolation property" is an immediate corollary of the pruned tableau method and we used this corollary to characterize CL as the transitive closure of the union of K3 and LP. We concluded the paper by showing how the pruned tableau method constructs interpolants for functionally complete extensions of the Dunn logics.

Acknowledgements. We would like to thank the referee for helpful comments. S.W. wants to thank the Netherlands Organisation for Scientific Research (NWO) for funding the project The Structure of Reality and the Reality of Structure (project leader: F. A. Muller), in which he is employed.

References

[1] Anderson, A.R., and N.D. Belnap, *Entailment: The Logic of Relevance and Necessity, Volume I*, Princeton University Press, 1975.

[2] Arieli, O., and A. Avron, Reasoning with Logical Bilattices, *Journal of Logic Language and Information* 5: 25–63, 1996.

[3] Arieli, O., and A. Avron, The Value of the Four Values, *Artificial Intelligence* 102: 97–141, 1998.

[4] Avron, A., Natural 3-Valued Logics–Characterization and Proof Theory, *Symbolic Logic* 56(1): 276–294, 1991.

[5] Avron, A., On the expressive power of three-valued and four-valued languages, *Journal of Logic and Computation* 9: 977–994, 1999.

[6] Batens, D., Paraconsistent Extensional Propositional Logics, *Logique et Analyse* 90: 195–234, 1980.

[7] Batens, D., and K. de Clerq, A rich paraconsistent eextension of full positive logic, *Logique et Analyse* 47: 185–188, 220–257, 2004.

[8] Belnap, N.D., How a Computer Should Think, in G. Ryle, (ed.), *Contemporary Aspects of Philosophy*, Oriel Press, Stocksfield 1976, pp. 30–56.

[9] Belnap, N.D., A useful four-valued logic, in J.M. Dunn, and G. Epstein, (eds.), *Modern Uses of Multiple-Valued Logic* Springer, Dordrecht, 1977.

[10] Bendova, K., Interpolation and three-valued logics, *Reports on Mathematical Logic* 39: 127–131, 2005.

[11] Dunn, J.M., Intuitive Semantics for First-Degree Entailments and 'Coupled Trees', *Philosophical Studies* 29: 149–168, 1976.

[12] Fitting, M., Bilattices and the Semantics of Logic Programming, *Journal of Logic Programming* 11: 91–116, 1991.

[13] Kleene, S.C., *Introduction to Metamathematics*, D. Van Nostrand Company, New York and Toronto, 1952.

[14] Milne, P., A non-classical refinement of the interpolation property for classical propositional logic, *Logique and Analyse* 59(235): 273–281, 2016.

[15] Muskens, R.A., On Partial and Paraconsistent Logics, *Notre Dame Journal of Formal Logic* 40: 352–373, 1999.

[16] Omori, H., and H. Sano, Generalizing Functional Completeness in Belnap-Dunn Logic, *Studia Logica* 103(5): 883–917, 2015.

[17] Pietz, A., and U. Rivieccio, Nothing but the Truth, *Journal of Philosophical Logic* 42: 125–135, 2013.

[18] Priest, G., The Logic of Paradox, *Journal of Philosophical Logic* 8: 219–241, 1979.

[19] Pynko, A.P., Functional Completeness and Axiomatizability within Belnap's Logic, *Journal of Applied Non-classical Logics* 9(1): 61–105, 1999.

[20] Ruet, P., Complete sets of connectives and complete sequent calculus for Belnap's Logic, Tech. rep., Ecole Normal Supérieure, Logic Colloquium 96, 1996.

[21] Smullyan, R., *First-order Logic*, Dover, New York, 1995.

[22] Takano, M., Interpolation in many-valued logics with designated values, *Kodai Mathematical Journal* 12: 125–131, 1989.

[23] Takeuti, G., *Proof Theory*, North-Holland, Amsterdam, 1987.

[24] Wintein, S., On all Strong Kleene generalizations of classical logic, *Studia Logica* 104(3): 503–545, 2016.

[25] Wintein, S., and R.A. Muskens, A gentzen calculus for nothing but the truth, *Journal of Philosophical Logic* 45(4): 451–465, 2016.

S. Wintein
Faculty of Philosophy
Erasmus University Rotterdam
Rotterdam
The Netherlands
stefanwintein@gmail.com

R. Muskens
Tilburg Center for Logic, Ethics, and Philosophy of Science (TiLPS)
Tilburg University
Tilburg
The Netherlands
r.a.muskens@gmail.com