



Theories of Fairness and Aggregation

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Abstract

We investigate the issue of aggregativity in fair division problems from the perspective of cooperative game theory and Broomean theories of fairness. Paseau and Saunders (Utilitas 27:460–469, 2015) proved that no non-trivial theory of fairness can be aggregative and conclude that theories of fairness are therefore problematic, or at least incomplete. We observe that there are theories of fairness, particularly those that are based on cooperative game theory, that do not face the problem of non-aggregativity. We use this observation to argue that the universal claim that no non-trivial theory of fairness can guarantee aggregativity is false. Paseau and Saunders's mistaken assertion can be understood as arising from a neglect of the (cooperative) games approach to fair division. Our treatment has two further pay-offs: for one, we give an accessible introduction to the (cooperative) games approach to fair division, whose significance has hitherto not been appreciated by philosophers working on fairness. For another, our discussion explores the issue of aggregativity in fair division problems in a comprehensive fashion.

Keywords Fairness · John Broome · Aggregation · Claims problems · Cooperative game theory · Fair division

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1 Introduction

When our collective needs exceed the resources available, or when what is there is less than what is demanded, a fair division problem arises: how, in order to be fair, should a scarce good be divided? Take, for instance, the following fair division problem.

Problem I John owes £80 to Ann and £40 to Bob but has only £60 left.

How then, in order to be fair, must John divide the £60 that he has left between Ann and Bob? A popular answer, advocated for by John Broome (1990), is that John must divide the £60 proportional to the claims of Ann and Bob. As Ann and Bob have claims of £80 and £40 respectively, a proportional division of £60 results in the allocation (40, 20) in which Ann receives £40 and Bob £20. Now suppose that Ann and Bob are also involved in a further fair division problem:

Problem II Jack owes £40 to Ann and £80 to Bob but has only £90 left.

Again, let us suppose that Jack divides the £90 that he has left proportional to the claims of Ann and Bob, so that the allocation (30, 60) results, in which Ann receives £30 and Bob £60.

Thus, by applying the proportional rule to Problem I and Problem II, Ann receives an aggregate amount of $(40 + 30 =)$ £70 whereas Bob receives an aggregate amount of $(20 + 60 =)$ £80: the aggregated allocation that results from applying the proportional rule to both problems is (70, 80).

To the extent to which dividing fairly comes down to applying the proportional rule, the allocations (40, 20) and (30, 60) that result from applying the proportional rule to Problem I and Problem II are fair allocations. But by applying the proportional rule to Problem I and Problem II one also realises the aggregated allocation (70, 80). And so when dividing fairly comes down to applying the proportional rule, by extension the aggregated allocation (70, 80) must also be considered fair.

In a recent paper, Paseau and Saunders (2015) have argued that the aggregated allocation (70, 80) that results from applying the proportional rule to the above two problems is *unfair*. In a nutshell, their argument runs as follows. The aggregated amount that has to be divided is $(60 + 90 =)$ £150 with aggregated claims of Ann and Bob equal to $(80 + 40 =)$ £120 and $(40 + 80 =)$ £120, respectively. As Ann and Bob have equal (aggregated) claims, fairness in general, and the proportional rule in particular, require that they receive equal (aggregated) amounts. As the aggregated allocation that results from applying the proportional rule to Problem I and Problem II is (70, 80), that allocation is unfair: Ann is getting too little and Bob is getting too much. The above example illustrates that the proportional rule is not *aggregative*. As Paseau and Saunders (2015: 460) put it, this means that the proportional rule is subject to *the problem of non-aggregativity*: ‘Two transactions, each of which is fair in isolation, may produce an aggregate result which would be judged as unfair had it resulted from a single distribution.’

To hold that fair division comes down to applying the proportional rule is to adopt a particular *theory of fairness*. Thus, someone who seeks to escape the

problem of non-aggregativity may attempt to do so by adopting another theory of fairness. According to Paseau and Saunders, any such attempt is bound to fail as *any* (non-trivial)¹ theory of fairness is non-aggregative: the pivotal claim that supports their philosophical discussion of aggregativity is **NAT**.

NAT There is no (non-trivial) aggregative theory of fairness.

According to Paseau and Saunders, **NAT** means that any theory of fairness is subject to the problem of non-aggregativity. Hence, such theories ‘are all to that extent problematic, or at least incomplete’ (2015: 468). Moreover, they hold that **NAT**’s ‘significance has not hitherto been appreciated by philosophers working on fairness’ (2015: 461).

In this paper, we show that **NAT** is false. We do so by exploring different kinds of theories of fairness. We introduce the *claims approach* and the *games approach* to fair division and explain that both approaches can be used to model fair division problems such as Problem I and II. Ever since Broome’s seminal paper on fairness (Broome 1990), fair division problems are modelled as claims problems in the philosophical literature (see Hooker 2005; Saunders 2010; Tomlin 2012; Curtis 2014 or Piller 2017 for an overview). However, by drawing on O’Neill (1982), we observe that the very same fair division problems can also be modelled as cooperative games. By doing so, solution values in cooperative game theory become available to analyse fair division [e.g. the value of Shapley (1953), the nucleolus (Schmeidler 1969), and the τ -value of (Tijds 1981)].

Using the games approach, fair division problems can indeed be analysed respecting aggregativity. Whereas the claims approach does not harbour any aggregative theory of fairness, there are such theories on the games approach: as a categorical statement, **NAT** is simply false. Paseau and Saunders’s mistaken assertion of **NAT** can thus be understood as arising from a neglect of the games approach to fair division. Our exploration of aggregativity also has broader implications related to the role of the games approach to fair division. We show that the games approach can model any fair division problem that the claims approach can model, but not vice versa. Moreover, not all division rules from the claims approach can be translated into solution values in the games approach. These facts about the games and the claims approach imply that fairness theorists make important methodological choices when modeling fair division problems.

The paper is structured as follows.

In Sects. 2.1 and 2.2 we introduce the claims and the games approach to fair division, respectively. We explain, in Sect. 2.3, that the theories of fairness associated with these approaches act on different *fairness structures* and explain, in terms of these fairness structures, what it means for a theory of fairness to be aggregative. In Sect. 2.4 we discuss the *Shapley value*, which is an aggregative theory of fairness that is associated with the games approach. Hence, the Shapley value testifies that, as a categorical statement, **NAT** is false.

¹ A theory of fairness is trivial if it allots nothing to each agent in each fair division problem.

In Sect. 3.1 we introduce the *run-to-the-bank (RTB)* rule, which is a theory of fairness associated with the claims approach. We then show, in Sect. 3.2, that with respect to fair division problems such as Problem I and II above, the Shapley value gives the same recommendations as the *RTB* rule. These theories of fairness thus coincide on the recommendations, whereas they come apart with regards to aggregativity. We explain, in Sect. 3.3, that this does *not* give rise to a *paradox of aggregativity*, contrary to what one may think at first glance.

In Sect. 4 we explore in what way aggregativity is relevant for theories of fair division. In Sect. 4.1 we revisit Problem I and II both in terms of the *RTB* rule and the Shapley value and explain that certain conclusions with respect to these problems are an artefact from adopting the claims approach to fair division. In Sect. 4.2 we argue—without relying on the games approach - that even though no theory of fairness associated with the claims approach is aggregative, that does not, *pace* Paseau and Saunders, render such theories problematic. In Sect. 4.3 we comment on the trade-offs related to adopting the games or claims approach to fair division and on the role that aggregativity plays in that trade-off.

Finally, Sect. 5 contains some concluding remarks.

2 There are Aggregative Theories of Fairness

2.1 The Claims Approach and Aggregativity

In analysing fair division problems such as Problem I and Problem II, Paseau and Saunders adopt what we will call the *claims approach to fair division*. In the philosophical literature, this approach has arguably started with John Broome's seminal paper on fairness (Broome 1990). Broomean theories of fairness focus on those fair division problems in which some agents have *claims* to a good that is to be distributed. Roughly, a claim is a specific type of reason, owed to the agent herself, as to why she should have some of the good that is to be divided. A thorough analysis of what is a claim is not to be found in the literature, but need, desert, and promises are typically taken to induce claims. In the introduction, we also (implicitly) analysed Problem I and Problem II as claims problems, the official definition of which is as follows.

Claims problems A *claims problem* $\mathcal{C} := (E, N, c)$ consists of an amount of good $E \geq 0$, also called the *estate*, a set of receiving agents N and a claims vector c specifying the amount of the estate that agent i has a claim to ($c_i \geq 0$), and which is such that together the claims exceed the amount of the good available ($\sum_{i \in N} c_i \geq E$).

Hence, on the claims approach, Problem I and Problem II are analysed as claims problems \mathcal{C}^I and \mathcal{C}^{II} , respectively, where:

$$\mathcal{C}^I = (60, \{A, B\}, (80, 40)) \quad \mathcal{C}^{II} = (90, \{A, B\}, (40, 80))$$

We already discussed the proportional rule P , which is an example of a *division rule*. More generally, a division rule is defined as follows.

Division rules A *division rule* r is a function that maps each claims problem (E, N, c) to an *allocation* $x \in \mathbb{R}^N$, with the property that each agent receives a non-negative amount that does not exceed his or her claim ($0 \leq x_i \leq c_i$), and the sum of what is allocated does not exceed the estate ($\sum_{i \in N} x_i \leq E$).

This definition permits a multitude of different division rules, as it does make very little specific demands on what the allocation should look like. Notably, this general definition of a division rule does not even prescribe that such a rule is *efficient*, which means it is not required that a division rule proposes to allocate all of the estate. Indeed, the rule which allots 0 to each agent in each claims problem, call this *the trivial rule*, respects the definition of a division rule. Clearly, the trivial rule is, in sharp contrast to the proportional rule, a rather uninteresting division rule.

In the economic literature on fair division, claims problems and division rules are studied extensively: see Thomson (2003) for an overview. Here, we just mention one further example of a non-trivial division rule, the so-called *constrained equal losses rule* (*CEL rule*). Given a claims problem (E, N, c) , the *CEL rule* proposes an efficient division of the estate E in such a way that each agent *loses* an *equal* amount with respect to her claim, subject to the *constraint* that no agent loses more than her claim.² Applying the *CEL rule* to claims problem C^I results in the allocation (50, 10), so that both A and B lose an equal amount of 30 with respect to their claims of 80 and 40, respectively. Also, applying the *CEL rule* to claims problem C^{II} results in the allocation (25, 65) so that both agents lose 15 with respect to their respective claims.

Given any two claims problems $C = (E, N, c)$ and $C' = (E', N, c')$ that involve the same set of agents N , by aggregating C and C' , i.e. by adding the estates and claims vectors of C and C' , one obtains a further claims problem $C + C' := (E + E', N, c + c')$, which is then called *the aggregated problem* of C and C' . For example, by aggregating claims problems C^I and C^{II} one obtains the aggregated problem $C^I + C^{II}$ which we will also denote as C^{I+II} :

$$C^{I+II} = (150, \{A, B\}, (120, 120))$$

In the introduction, we informally demonstrated that the proportional rule P is not *aggregative with respect to C^I and C^{II}* , meaning that:

$$P(C^I) + P(C^{II}) \neq P(C^{I+II})$$

In contrast, the *CEL rule* is *aggregative with respect to C^I and C^{II}* , as

$$CEL(C^I) + CEL(C^{II}) = CEL(C^{I+II})$$

In order for a division rule r to be *aggregative*, it has to be aggregative with respect to *any* two claims problems $C = (E, N, c)$ and $C' = (E', N, c')$ that involve the same set of agents. That is, r is aggregative just in case for *any* such C and C' we have:

² Thus according to the *CEL rule*, agent i receives $x_i = \max\{0, c_i - \lambda\}$, where λ is chosen such that $\sum_{i \in N} x_i = E$.

$$r(\mathcal{C}) + r(\mathcal{C}') = r(\mathcal{C} + \mathcal{C}') \quad (1)$$

The trivial rule that was discussed above is clearly an aggregative, albeit an uninteresting, division rule. Claims problems \mathcal{C}^I and \mathcal{C}^{II} testify that the proportional rule is not aggregative. But what about the *CEL* rule, is it aggregative? It follows from Theorem 1 below that the answer is ‘no’, as *there are no non-trivial aggregative division rules*.

Theorem 1 (There are no non-trivial aggregative division rules)

Proof See “Appendix”. □

Paseau and Saunders (2015) prove Theorem 1 themselves but remark that ‘there is an extensive and sophisticated economics literature in this area which appears to imply [Theorem 1]’. Their presumption is definitely correct as a proof of Theorem 1 can also be found in Bergantiños and Méndez-Naya (2001: 227).³ Bergantiños and Méndez-Naya’s result has led to some interesting studies of ‘aggregativity in claims problems’ in the economic literature⁴, which is not directly relevant for our purposes here (but see Sect. 4).

What is directly relevant for our purposes here is the philosophical upshot of Theorem 1. According to Paseau and Saunders, Theorem 1 implies **NAT**.

NAT There is no non-trivial aggregative theory of fairness.

The reason that Paseau and Saunders take Theorem 1 to imply **NAT** is simply that they equate theories of fairness with division rules. Although it makes sense to do so *on the claims approach to fair division*, that is not the only approach. In the next section, we will discuss another framework for fair division, which we call the *games approach*, that can also be used to analyse fair division problems such as Problem I and Problem II. In order to analyse such problems, the games approach does not model them as claims problems, but rather as *cooperative games*. On the claims approach, theories of fairness are division rules that act on claims problem but, as we will see, on the games approach theories of fairness are *solution values* that act on cooperative games: theories of fairness on the claims and games approach act on different *fairness structures*. On the claims approach, there are no (non-trivial) aggregative theories of fairness, but on the games approach there are such theories. Indeed, as a categorical statement, **NAT** is simply false, as we will explain in the remainder of this section.

2.2 A First Look at the Games Approach

Let us consider Problem I once more and observe the following. If John would fully repay Bob there is still $60 - 40 = \text{£}20$ left for Ann, which is to say that Ann can guarantee herself $\text{£}20$. When John does his utmost to fully reimburse Ann, he has to

³ Actually, our proof of Theorem 1 is adapted from Bergantiños and Méndez-Naya (2001).

⁴ See e.g. Bergantiños and Méndez-Naya (2001); Bergantiños and Vidal-Puga (2006), and Alcalde et al. (2014)

give her £60 leaving nothing for Bob, which is to say that Bob can guarantee himself £0. Now consider the group consisting of Ann and Bob. If all receiving agents other than Ann and Bob (of which there are none) are fully reimbursed, there is £60 left for *Ann and Bob together*: the group consisting of Ann and Bob can guarantee itself £60. We have now implicitly analysed Problem I as a *cooperative game*, the definition of which is as follows.

Cooperative games A *cooperative game* is a pair (N, v) , with N a set of agents and with $v: \mathcal{P}(N) \rightarrow \mathbb{R}_+$, $v(\emptyset) = 0$ the *characteristic function* of the game which specifies the value that each group of agents (or coalition) can guarantee itself.⁵ In particular, $v(N)$ represents the value that the *grand coalition* N can guarantee itself.

Thus, the cooperative games that are associated with Problem I and Problem II are given by $(\{A, B\}, v^I)$ and $(\{A, B\}, v^{II})$ respectively, where:

$$\begin{aligned} v^I(\emptyset) &= 0 & v^I(\{A\}) &= 20 & v^I(\{B\}) &= 0 & v^I(\{A, B\}) &= 60 \\ v^{II}(\emptyset) &= 0 & v^{II}(\{A\}) &= 10 & v^{II}(\{B\}) &= 50 & v^{II}(\{A, B\}) &= 90 \end{aligned}$$

One central question that is studied by *cooperative game theory* is the following.

Given a game (N, v) , how to divide $v(N)$ amongst the agents in N ? (2)

Note that the money that is to be divided in Problem I and Problem II is, per definition, the value of the ‘grand coalition’ (the group of all agents, i.e. in this case, Ann and Bob together) in the associated games v^I and v^{II} , respectively. Hence, an answer to question (2) can resolve fair division problems such as Problem I and Problem II, as discussed in more detail in Sect. 2.3. In the literature on cooperative game theory, question (2) is answered by specifying a *solution value*.

Solution values A *solution value* φ is a function that maps each cooperative game (N, v) to an *allocation* $x \in \mathbb{R}^N$ with the property that $\sum_{i \in N} x_i \leq v(N)$.

There is an extensive literature in cooperative game theory that proposes and compares different solution values and all of them can be—and typically are—understood as proposals to divide the value of the grand coalition *fairly*. Prominent solution values that have been proposed in the literature are the *Shapley value* (cf. Shapley 1953), the *nucleolus* (cf. Schmeidler 1969), and the τ -*value* (cf. Tijs 1981). We will revisit the Shapley value in some detail later on and apply it to v^I and v^{II} . However, it will be instructive to first spend a few words on the notions of a *theory of fairness* and a *fairness structure*.

2.3 Fairness: Theories, Structures, and Aggregation

Let us revisit Problem I once more, in which John owes £80 to Ann, £40 to Bob but has only £60 left. We have seen that there are two different ways to divide the £60 in this fair division problem. One way is to analyse Problem I as claims problem C^I

⁵ $\mathcal{P}(N)$ is the powerset of N , i.e. the set of all subsets of N .

and then to apply a division rule to C^I in order to divide the £60. The other way is to analyse Problem I as cooperative game v^I and then to apply a solution value to v^I in order to do so. To be sure, the availability of these two approaches is not confined to Problem I. Indeed, *any fair division problem that can be analysed as a claims problem $C = (E, N, c)$ can also be analysed as a cooperative game⁶ (N, v^C)* , where

$$v^C(S) = \max\{0, E - \sum_{i \notin S} c_i\} \quad \text{for each } S \subseteq N. \quad (\star)$$

Note that the estate E in claims problem C coincides with the value of the grand coalition $v^C(N)$ in the game that is associated with C via (\star) . Hence, division rules and solution values provide two different ways to divide $E = v^C(N)$.

Paseau and Saunders equate theories of fairness with division rules, which makes sense from the perspective of the claims approach. However, from the perspective of the games approach, it makes just as much sense to equate a theory of fairness with a solution value. By a *theory of fairness*, we mean a function that assigns an allocation of the good-to-be-divided for each *fairness structure* that is within its domain. A fairness structure is obtained by *modelling* a fair division problem, that is by extracting the characteristics of the problem on the basis of which, according to the model, fair division should proceed. Thus, the fairness structures associated with the claims approach are claims problems whereas the fairness structures associated with the games approach are cooperative games. Both division rules and solution values are theories of fairness, albeit theories that take different fairness structures as their input.

Although our notions of *theory of fairness* and of *fairness structure* are abstract, they are theoretically fruitful. For one thing, they allow us to spell out the notion of an *aggregative theory of fairness* in a way that does not privilege the claims or games approach to fairness.

Aggregative Theories of Fairness A theory of fairness **ToF** is called *aggregative* when the following holds. Given any two fairness structures S_1 and S_2 that involve the same set of receiving agents, the sum of the allocations that result from applying **ToF** to S_1 and S_2 is equal to the allocation that results from applying **ToF** to the fairness structure that results from aggregating S_1 and S_2 .

We have seen how aggregation works on the claim approach in Sect. 2.1. To aggregate claims problems $C = (E, N, c)$ and $C' = (E', N, c')$ one adds the estates and claims vectors of both problems and thus obtains the aggregated claims problem $C + C' = (E + E', N, c + c')$. When we abstract away from the particular fairness structures that are exploited by the claims approach, we may say that to aggregate two fairness structures (that involve the same set of receiving agents) one adds—component wise—all information of the two structures. From the more abstract notion of aggregation thus arrived at, it readily follows how to aggregate fairness structures on the games approach: to aggregate games (N, v) and (N, v') one adds,

⁶ The converse statement does not hold as shown in Heilmann and Wintein (2017). Hence in an important sense, the scope of the games approach to fair division is broader than the scope of the claims approach. For more details, see also Sect. 4.3.

for each coalition $S \subseteq N$, its value in both problems and thus obtains the aggregated game $(N, v + v')$. As an example, by aggregating games (N, v^I) and (N, v^{II}) that are associated with Problem I and II, respectively, we obtain⁷ the aggregated game (N, v^{I+II}) , where:

$$v^{I+II}(\emptyset) = 0 \quad v^{I+II}(\{A\}) = 30 \quad v^{I+II}(\{B\}) = 50 \quad v^{I+II}(\{A, B\}) = 150$$

The definition of an aggregative solution value now readily follows from the general definition of an aggregative theory of fairness. A solution value φ is said to be aggregative with respect to games (N, v) and (N, v') when:

$$\varphi(N, v) + \varphi(N, v') = \varphi(N, v + v') \quad (3)$$

A solution value φ is *aggregative* when φ is aggregative with respect to *any* pair of cooperative games that involve the same set of agents N , i.e. φ is aggregative when (3) holds for any games (N, v) and (N, v') .

The *trivial solution value*, i.e. the solution value that assigns 0 to each agent in each cooperative game, is a rather uninteresting example of an aggregative solution value. The question arises whether there are also non-trivial aggregative solution values, i.e. whether the games approach harbours non-trivial aggregative theories of fairness. As we will explain in the next section, the Shapley value testifies that the answer to that question is ‘yes’.

2.4 The Shapley Value and the Failure of NAT

Let us illustrate the Shapley value by showing how the Shapley values for v^I and v^{II} , the games associated with Problem I and Problem II as described in Sect. 2.2, are obtained. For the sake of convenience, let us first display these games once more.

$$\begin{aligned} v^I(\emptyset) &= 0 & v^I(\{A\}) &= 20 & v^I(\{B\}) &= 0 & v^I(\{A, B\}) &= 60 \\ v^{II}(\emptyset) &= 0 & v^{II}(\{A\}) &= 10 & v^{II}(\{B\}) &= 50 & v^{II}(\{A, B\}) &= 90 \end{aligned}$$

Given a cooperative game (N, v) , the Shapley value considers all *orders* of the agents in N . As v^I and v^{II} only involve two agents, A and B , there are just two such orders in these games: $\langle A, B \rangle$ and $\langle B, A \rangle$ respectively. Agent orders may be thought of as possible manners in which the grand coalition, consisting of all agents, can be formed. So in the order $\langle A, B \rangle$, agent A is the first to arrive and thereby realises the singleton coalition $\{A\}$. Next B arrives and he joins A to form the grand coalition $\{A, B\}$. The Shapley value records, for each order, the *marginal contributions* that the agents make with respect to the coalitions that are formed upon their arrival. Consider the order $\langle A, B \rangle$ for v^I . When A arrives, she realises a marginal contribution of $v^I(\{A\}) - v^I(\emptyset) = 20 - 0 = 20$ with respect to the empty coalition. Then B arrives and he realises a marginal contribution of $v^I(\{A, B\}) - v^I(\{A\}) = 60 - 20 = 40$ with respect to coalition $\{A\}$. So, in v^I , the marginal contributions of A and B induced by $\langle A, B \rangle$ are 20 and 40, respectively. *According to the Shapley value, agents receive their*

⁷ We will write v^{I+II} as convenient shorthand for $v^I + v^{II}$.

Table 1 Shapley values for v^I and v^{II}

Order	MC_A	MC_B	Order	MC_A	MC_B
$\langle A, B \rangle$	20	40	$\langle A, B \rangle$	10	80
$\langle B, A \rangle$	60	0	$\langle B, A \rangle$	40	50
$Sh(v^I)$	40	20	$Sh(v^{II})$	25	65

average marginal contribution over all agent orders. As the reader may care to verify, the order $\langle B, A \rangle$ in v^I induces marginal contributions for A and B of 60 and 0, respectively. Hence the Shapley value for v^I allots $\frac{20+60}{2} = 40$ to A and $\frac{40+0}{2} = 20$ to B : $Sh(v^I) = (40, 20)$. Table 1 below conveniently summarises the computation of the Shapley value for v^I and also presents this computation for v^{II} .

Given an arbitrary cooperative game (N, v) , let us write $\Pi(N)$ to denote the set of all orders of the agents in N and, given such an order π , let $MC^v(\pi)$ denote the vector that records the marginal contributions that the agents realize in order π on the basis of v . The general definition of the Shapley value can then be stated as follows.⁸

$$Sh(N, v) = \frac{\sum_{\pi \in \Pi(N)} MC^v(\pi)}{|N|!} \tag{4}$$

From this definition of the Shapley value, it can readily be shown that the Shapley value is aggregative, as recorded by the following theorem.

Theorem 2 Sh is aggregative: $Sh(N, v) + Sh(N, v') = Sh(N, v + v')$

Proof See appendix. □

Theorem 2 testifies that **NAT**, when taken as a categorical statement, is simply false. Indeed, the Shapley value is a (non-trivial) theory of fairness that is aggregative.

2.5 Aggregativity After the Failure of NAT

In demonstrating that **NAT** is false, *pace* Paseau and Saunders (2015), we have achieved a narrow, but important, argumentative goal. In the remainder of the paper, we turn to discussing questions of broader significance that are implied by what we have demonstrated. These questions are, on the one hand, raised by the *strategy* we adopted to show that **NAT** is false. On the other hand, the very fact that **NAT** is false needs evaluation in terms of its implications for Paseau and Saunders (2015) and beyond. There are two issues.

⁸ Axiomatically, the Shapley value can be characterised as the only solution value that satisfies the following four properties (cf. Shapley 1953): *Efficiency* (for each game (N, v) , the Shapley value proposes an allocation x such that $\sum x_i = v(N)$), *Null player* (for each game (N, v) , the Shapley value proposes an allocation x that allots 0 to each player whose marginal contribution to each coalition in which she is not contained (including \emptyset) is 0), *Symmetry* (for each game (N, v) , the Shapley value proposes an allocation x that gives the same amount to each pair of players whose marginal contribution to each coalition which they are not contained in is the same), and *Aggregativity*.

Firstly, we have introduced the claims and the games approach to modelling and analysing fair division. These two approaches clearly differ in a number of important respects, not least that there are aggregative theories of fairness on the games approach, but not on the claims approach. Yet, we have also seen that the two approaches are closely related in some other respects. Recall the equation (★) which shows how any fair division problem that can be modelled as a claims problem can also be modelled as a game. This suggests that there could be more similarities between the two approaches. Indeed, as we will see, there exist division rules and solution values that give the *very same recommendations*. We will explore the potential tension between these kinds of similarities and differences between the claims and games approach. This, in turn, allows us to make the nature and scope of the aggregativity condition much more precise. We will do so in Sect. 3.

Secondly, we will investigate what kind of role aggregativity should play in theorising about fair division. We will argue that aggregativity is not a property of fairness, and that non-aggregative theories of fairness are not problematic. That is, the proponent of the claims approach might face the choice between giving up aggregativity or the claims approach in favour of the games approach. Contrary to what Paseau and Saunders (2015) say, non-aggregativity should thus not be viewed as a problem, or so we argue in Sect. 4.

Together, exploring these issues will yield a comprehensive treatment of aggregativity in the claims approach and in the games approach.

3 Aggregativity in the Claims and Games Approach: No Paradox

We have already seen that there are two approaches to fair division. But what exactly is their relation? Are they mutually exclusive or complementary? Do they always offer differing analyses of aggregativity? Here, we will turn to answering these questions. Sect. 3.1 introduces the *run-to-the-bank (RTB)* rule, a theory of fairness associated with the claims approach, i.e. a division rule. In Sect. 3.2 we show that the *RTB*-rule coincides with the Shapley value in the following sense: for each claims problem \mathcal{C} , the *RTB* rule prescribes the same allocation as the Shapley value does for the game $v^{\mathcal{C}}$ that is associated with \mathcal{C} via (★). According to Theorem 1, there are no aggregative division rules, and so the *RTB* rule is not aggregative. Hence, the *RTB* rule and Shapley value are two theories of fairness that coincide in their recommendations, yet come apart in terms of aggregativity. Theories that both coincide and come apart...does that mean there is a paradox of aggregativity? In Sect. 3.3 we explain why there is no such paradox. Moreover, later in Sect. 4, we will show that the close relation between the *RTB* rule and Shapley value can be invoked in a fruitful way: to explain *why* the claims and games approach fare differently with respect to aggregativity.

3.1 The Run-to-the-Bank Rule

We will illustrate the *run-to-the-bank (RTB)* rule by showing how the *RTB* allocations for $\mathcal{C}^I = (60, \{A, B\}, (80, 40))$ and $\mathcal{C}^{II} = (90, \{A, B\}, (40, 80))$, the

claims problems associated with Problem I and Problem II respectively, are obtained.

Given a claims problem $C = (E, N, c)$, the *RTB* rule considers all *orders* of the agents in N . Such an order is thought of as corresponding with a *bank run*, where all claimants ‘run’ to an institution that is responsible for allocating the estate. When an agent arrives at the bank in a given run he receives the minimum of (i) his claim and (ii) the estate that is left after reimbursement of the agents that arrived earlier in the run. Consider run $\langle A, B \rangle$ in claims problem C^I . The first agent to arrive at the bank is A . Although A ’s claim is 80, when she arrives the (remaining) estate is only 60 so this 60 is what she receives. When B arrives next, there is nothing left so that B receives 0 at his arrival. Thus, the run $\langle A, B \rangle$ in C^I results in pay-offs to A and B of 60 and 0, respectively. Similarly, as the reader may care to verify, in C^I the run $\langle B, A \rangle$ results in pay-offs to A and B of 20 and 40, respectively. *According to the RTB rule, agents receive their average pay-offs (PO) over all runs.* Thus, the *RTB* allocation for C^I allots $\frac{60+20}{2} = 40$ to A and $\frac{0+40}{2} = 20$ to B . Table I below conveniently summarises the computation of the *RTB* allocation⁹ for C^I and also presents this computation for C^{II} .

A comparison of Tables 1 and 2 reveals that the *RTB* rule and Shapley value recommend the same allocations for both Problem I and Problem II. In the next subsection we explain that and why this is no coincidence.

3.2 How the RTB Rule and Shapley Value Coincide

In Sect. 2.3 we explained that any fair division problem that can be analysed as a claims problem C can also be analysed as a cooperative game v^C , where

$$v^C(S) = \max\{0, E - \sum_{i \notin S} c_i\} \quad \text{for each } S \subseteq N. \tag{★}$$

It may have struck the reader that this means that a solution value φ gives rise to a division rule. For the estate in a claims problem C can be divided by applying φ to the game v^C that is associated with C . Hence any solution value φ specifies, albeit via a detour through v^C , how to divide the estate in a claims problem C . Solution values implicitly define division rules. And so, in particular the Shapley value implicitly defines a division rule which, as the following theorem attests, coincides with the *RTB* rule.

Theorem 3 For any claims problem C : $RTB(C) = \text{Sh}(v^C)$.

Proof See O’Neill (1982). □

Remember, from Sect. 3.1, that the *RTB* rule and Shapley value recommend the same allocations for both Problem I and II:

⁹ Given an arbitrary claims problem (E, N, c) , let us write $\Pi(N)$ to denote the set of all orders of the agents in N and, given such an order π , let $PO(\pi)$ denote the vector that records the pay-offs that the agents receive in that order. Then $RTB(E, N, c) = \frac{\sum_{\pi \in \Pi(N)} PO(\pi)}{|\Pi(N)|}$.

Table 2 *RTB* allocations for C^I and C^{II}

Run	PO_A	PO_B	Run	PO_A	PO_B
$\langle A, B \rangle$	60	0	$\langle A, B \rangle$	40	50
$\langle B, A \rangle$	20	40	$\langle B, A \rangle$	10	80
$RTB(C^I)$	40	20	$RTB(C^{II})$	25	65

$$RTB(C^I) = Sh(v^I), \quad RTB(C^{II}) = Sh(v^{II}) \tag{5}$$

How are Theorem 3 and equation (5) related? Consider Problem I (the case of Problem II is completely similar in this respect). Now, Theorem 3 indeed establishes $RTB(C^I) = Sh(v^{C^I})$. But that does not establish what is said in (5). In particular, Theorem 3 only relates C^I to v^{C^I} , but (5) relates C^I directly to v^I . What is needed for Theorem 3 to establish (5) is the additional assumption that $v^{C^I} = v^I$. And, indeed, that is the case: for Problem I, we can obtain the same game with two methods: one, by modelling the fair division problem *directly* as the game v^I ; two, by first modelling the fair division problem as the claims problem C^I , and then inducing the game v^{C^I} from the claims problem, via (\star) . Now, although $v^{C^I} = v^I$, it is important to realise that the methods by which they are obtained are conceptually different. To wit, method two does presuppose the claims approach, whereas method one does not.

As we have just seen, v^I is not induced from C^I via (\star) , but it is identical to the game that is induced as such, viz. $v^I = v^{C^I}$. It will be fruitful—as will be apparent later—to call this relation between C^I and v^I that of *relatedness* \sim_\star . In general, a claims problem C and a cooperative game w are *related* \sim_\star , denoted $C \sim_\star w$, just in case w is identical to the game that is induced by C in accordance with (\star) , i.e. just in case $w = v^C$. Trivially then, $C \sim_\star v^C$ for any claims problem C . Figure 1 conveniently summarizes the present discussion.

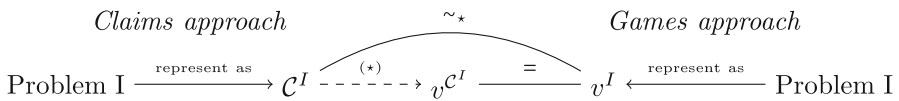


Fig. 1 Problem I and the relations between the claims and the games approach.

Figure 1 shows that C^I and v^I are representations of Problem I associated with the claims and games approach respectively, that v^{c^I} is induced by C^I via (\star) , that v^I (is not so induced but) is identical to v^{c^I} , and hence that $C^I \sim_{\star} v^I$.

The *RTB* rule and Shapley value thus coincide in the sense of Theorem 3. At the same time, they come apart in terms of aggregativity. *Prima facie*, these relations between the *RTB* rule and the Shapley value may seem strange or even paradoxical. In the next section we will explain, by relying on the notion of *relatedness* \sim_{\star} , that in fact they are not: there is no paradox of aggregativity.

3.3 No Paradox of Aggregativity

Why is there no paradox of aggregativity? That is, why is there no conflict between Theorem 3 and the fact that the Shapley value is, whereas the *RTB* rule is not, aggregative? In a nutshell, the answer is that *relatedness* \sim_{\star} is not preserved under aggregation. Let us unpack this dense answer. We say that *relatedness* \sim_{\star} is preserved under aggregation, just in case, for any claims problems C and C' and games w and w' , (6) is true:

$$\text{If } C \sim_{\star} w \text{ and } C' \sim_{\star} w' \text{ then } C + C' \sim_{\star} w + w' \quad (6)$$

Now if *relatedness* \sim_{\star} were preserved under aggregation, there would be a paradox of aggregativity. For if *relatedness* \sim_{\star} were preserved under aggregation, we could easily establish that, as the Shapley value is aggregative and coincides with the *RTB* rule (in the sense of Theorem 3), the *RTB* rule must be aggregative as well. The argument for the aggregativity of the *RTB* rule would then run as follows, where C and C' are two arbitrary claims problems:

- i. $RTB(C) + RTB(C') = \text{Sh}(v^C) + \text{Sh}(v^{C'})$ (Theorem 3)
 - ii. $\text{Sh}(v^C) + \text{Sh}(v^{C'}) = \text{Sh}(v^C + v^{C'})$ (aggregativity of Sh)
 - iii. $C + C' \sim_{\star} v^C + v^{C'}$ (Theorem 3 and (6))
 - iv. $\text{Sh}(v^C + v^{C'}) = RTB(C + C')$ (iii, Theorem 3)¹⁰
- $\therefore RTB(C) + RTB(C') = RTB(C + C')$

The above argument makes precise the thought that “as the *RTB* rule coincides with the Shapley value and as the Shapley value is aggregative, the *RTB* rule must be aggregative as well”, i.e. it makes precise the thought that there is a paradox of aggregativity. By doing so, it also makes explicit the flaw inherent in that thought: the assumption that *relatedness* \sim_{\star} is preserved under aggregation, used in step iii in the above argument. The fairness structures associated with Problem I and II illustrate *in concreto* that *relatedness* \sim_{\star} is not preserved under aggregation, as indicated in the below figure.

¹⁰ Note that it immediately follows from Theorem 3 that $RTB(C) = \text{Sh}(w)$ whenever $w \sim_{\star} C$.

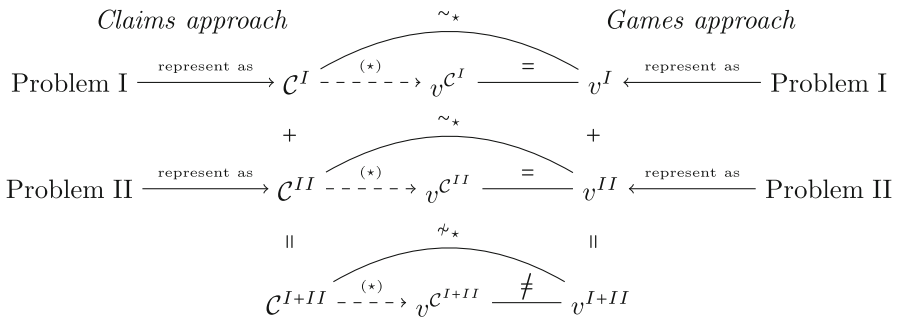


Fig. 2 Relatedness \sim^* is not preserved under aggregation

The first lines of Fig. 2 were discussed in Sect. 3.2 and the interpretation of the second line is completely similar to that of the first. The first two lines illustrate respectively that C^I and v^I and that C^{II} and v^{II} are related \sim^* . The third line displays the associated aggregated fairness structures C^{I+II} and v^{I+II} and shows that these are *not* related \sim^* , as $v^{C^{I+II}} \neq v^{I+II}$. Indeed, we have that:

$$\begin{aligned}
 v^{C^{I+II}}(\emptyset) &= 0 & v^{C^{I+II}}(\{A\}) &= 30 & v^{C^{I+II}}(\{B\}) &= 30 & v^{C^{I+II}}(\{A, B\}) &= 150 \\
 v^{I+II}(\emptyset) &= 0 & v^{I+II}(\{A\}) &= 30 & v^{I+II}(\{B\}) &= 50 & v^{I+II}(\{A, B\}) &= 150
 \end{aligned}$$

There is thus no tension between Theorem 3 and the fact that the *RTB* rule and the Shapley value come apart in terms of aggregativity: there is no paradox of aggregativity. The tension is removed once it is realised that relatedness \sim^* is not preserved under aggregation, as illustrated by Figure 2.

4 What Now for Aggregativity in Fair Division?

Aggregativity has proven to be a useful workhorse to compare different approaches to fair division and, in particular, to learn about the games approach to fair division. We now turn to re-focus the discussion on aggregativity. After having shown that **NAT** is false and comparing the claims and the games approach, where do we stand on the issue of aggregativity in fair division? We will argue that aggregativity is not a property of fairness, and that non-aggregative theories of fairness are not problematic. So, in light of the different approaches to fair division, it is best to conceive of aggregativity as a condition that may or may not be applicable to certain types of fair division problems.

4.1 Aggregativity on Different Fairness Structures

In the introduction, we saw that Paseau and Saunders use Problem I and II to illustrate that the proportional rule is subject to the problem of non-aggregativity. They argued that the aggregated allocation that results from applying the

proportional rule to those problems¹¹ is unfair: according to this aggregated allocation, Ann receives less than Bob whereas fairness requires that they receive equal amounts. In this section, we will explain that the thought that Ann and Bob should receive equal aggregated amounts is an artefact of the claims approach.

Consider two fairness theorists, let us call them Saul and Melvin. Our theorists agree that a theory of fairness should, ideally, be aggregative. However, they disagree on how fair division problems should be modelled. Saul is strongly in favour of the claims approach and his favourite theory of fairness is the *RTB* rule. Melvin, however, is a proponent of the games approach and an advocate of the Shapley value.

Now, Saul will analyse Problem I and II by adopting the claims approach to fair division. For him, the fairness structures of Problem I and II are given by C^I and C^{II} respectively. As he wants to respect aggregativity, the aggregated allocation that results from applying his favourite division rule to Problem I and II must be equal to the allocation that results from applying this rule to C^{I+II} . Remember that Ann and Bob have equal claims in the aggregated claims problem:

$$C^{I+II} = (150, \{A, B\}, (120, 120))$$

Fairness requires equal treatment of equals and Ann and Bob are equal from the perspective of C^{I+II} . Hence, any sensible division rule, thus also the *RTB* rule, must recommend that Ann and Bob receive equal amounts on the basis of C^{I+II} . And so it follows from the above that Ann and Bob should receive equal amounts in the aggregated allocation that results from applying the *RTB* rule (or any sensible division rule) to Problem I and II. An aggregated allocation in which they do not receive equal amounts is unfair. Hence, the aggregated allocation (65, 85) that results from applying the *RTB* rule to Problem I and II is unfair.

Melvin, who adopts the games approach, analyses Problem I and II as follows. The fairness structures associated with Problem I and II are given by v^I and v^{II} , respectively. Since he also assumes that a theory of fairness should be aggregative, the aggregated allocation that results from applying his favourite solution value to Problem I and II must be equal to the allocation that results from applying the solution value to v^{I+II} :

$$v^{I+II}(\emptyset) = 0 \quad v^{I+II}(\{A\}) = 30 \quad v^{I+II}(\{B\}) = 50 \quad v^{I+II}(\{A, B\}) = 150$$

Observe that v^{I+II} lays bare the fact that Bob has a larger aggregated guarantee value than Ann does: $v^{I+II}(\{B\}) > v^{I+II}(\{A\})$. And so, from the perspective of v^{I+II} , Ann and Bob are *not* equals, which is reflected accordingly in the allocation of the Shapley value. More generally, theories of fairness on the games approach take guarantee values, as recorded by a cooperative game, as their input. For such a theory it makes sense to allot Bob, who has the larger aggregated guarantee value, a larger aggregated amount than Ann. From the perspective of the games approach it makes perfect sense that Bob receives more than Ann. In particular, there is nothing

¹¹ More precisely: from applying the proportional rule to the claims problems associated with those problems.

unfair about the aggregated allocation (65, 85) that results from applying the Shapley value to Problem I and II.

Hence, the thought that Ann and Bob should receive equal aggregated amounts is an artefact of the claims approach: only on the claims approach are Ann and Bob “equal” according to the structure that is obtained by aggregating the fairness structures representing Problem I and II respectively. From Melvin’s perspective, the aggregated allocation (65, 85), in which Ann and Bob receive unequal amounts, is perfectly fair. But Saul is in a dilemma. For he should either give up his thought that theories of fairness must be aggregative, or, if not, he should follow Melvin and trade in the claims approach for the games approach. In Sect. 4.2 we will argue, *pace* Paseau and Saunders, that even on the claims approach it is far from clear that a non-aggregative theory of fairness is problematic. Section 4.3 is concerned with the trade-off between the claims and games approach and the role that aggregativity plays in that trade-off. Hence, Sects. 4.2 and 4.3 will guide Saul in resolving his dilemma.

4.2 On the Problem of Non-aggregativity

Paseau and Saunders (2015: 460) maintain that a theory that is not aggregative is subject to the *problem* of non-aggregativity: ‘Two transactions, each of which is fair in isolation, may produce an aggregate result which would be judged as unfair had it resulted from a single distribution.’ We will argue that it is far from clear that a non-aggregative theory of fairness is problematic. We think that *even on the claims approach*, Paseau and Saunders’s analysis of Problem I and II fails as an illustration of the problematic character of non-aggregativity.

As we take the claims approach for granted here, there is no denying that Problem I and II induce claims problems C^I and C^{II} , respectively. Further, there is no denying the mathematical fact that $C^I + C^{II} = C^{I+II}$. Yet, we will show that C^{I+II} is not induced by any relevant fair division problem. Hence, the recommendations of one’s favourite theory of fairness for C^{I+II} do not have any normative import for the aggregated allocation that is realised by applying that theory to C^I and C^{II} , respectively. Hence, there is no problem of non-aggregativity.

To recap, the supposed problem of non-aggregativity is that the aggregated recommendations for C^I and C^{II} are different from those for C^{I+II} . Paseau and Saunders maintain that this should not be so. They consider the fact that Ann and Bob are both owed £120 whereas, in total, there is £150 left. We record this fact in the following summary.

Summary I+II In total, Ann and Bob are both owed £120. In total, John and Jack have £150 left.

Summary I+II is definitely a convenient global summary of some of the information that is given in Problem I and II. But it is doubtful whether it makes sense to represent Summary I+II as a genuine *claims problem*. Remember that a claims problem consists of an estate with agents that have claims to certain amounts of *that*

estate. Now John and Jack have, in total, £150 left and Ann and Bob are both owed £120, but nothing in Summary I+II (nor in its underlying problems I and II) prescribes that we have to interpret the £150 as a single estate, with Ann and Bob having claims to receive £120 *of that very estate*. Whereas C^I and C^{II} clearly are faithful models of Problem I and II, respectively, C^{I+II} does not model any relevant fair division problem. Hence, an appeal to C^{I+II} in order to demonstrate the problematic character of the aggregated allocation that results from applying one's favourite theory of fairness to C^I and C^{II} , is unsuccessful.

Interestingly, Paseau and Saunders (2015: 463) discuss and disqualify a response to the problem of non-aggregativity that is related to the above discussion:

A response might be that the nature of the claims determines how they should be met. If [Ann] is owed money by [John] and [Jack] separately, then they should repay separately, whereas if the money is owed by [John] and [Jack] together then they should repay collectively. But this response appears unsatisfactory.¹²

Paseau and Saunders provide three reasons as to why they consider the response to be unsatisfactory. We will discuss these reasons in turn and dismiss all of them. Reason 1 is as follows:

First, it is intuitively unfair if Ann receives greater satisfaction than Bob though they have equally strong claims to the same amount (albeit against different debtors), so an adequate theory of fairness should be able to account for this. Paseau and Saunders (2015: 463)

We submit that, *pace* Paseau and Saunders, fairness does *not*, neither intuitively nor theoretically, dictate that two agents with equally strong claims to the same amount should receive equal satisfaction. To see this, suppose that John has borrowed £120 from Ann and that Jack has borrowed £120 from Bob. When payment is due, John has only £30 left whereas Jack has only £60 left. John and Jack use the money they have left to pay off their creditors: John pays £30 to Ann so that Ann's claim receives $\frac{30}{120} \cdot 100\% = 25\%$ satisfaction and Jack pays £60 to Bob so that Bob's claim receives $\frac{60}{120} \cdot 100\% = 50\%$ satisfaction. Ann and Bob both have a claim to £120, i.e. they have claims to the same amount, but there is nothing unfair about these claims receiving unequal satisfaction. There is nothing fair about it either. For, given the claims approach, the issue of fairness only arises when there are agents with *competing claims*, i.e. claims of different agents to receive certain amounts *of the very same estate*. In other words, the fact that agents have claims to the same amount does not guarantee that these claims are part of a fair division problem. And if they are not part of a fair division problem, it is not clear what to think of assertions concerning the (un)fairness of the satisfaction of these claims. And so, Reason 1 fails.

Reason 2 reads as follows.

¹² We have adapted this quotation, and a further one below, to the slightly different notation in our article. In Paseau and Saunders (2015), Ann = C_1 , and John = D and Jack = D^* .

Second, Broome (1990: 92) defined claims as duties *owed to* particular individuals, but he did not specify that they must be *owed by* particular individuals. Thus, if Ann has a claim to £80 and a claim to £40, then Ann does have a claim to £120, even if no single agent owes this amount to her. Paseau and Saunders (2015: 463)

Now, Broome's account of fairness deals with situations where 'there are several candidates to receive a good, but the good cannot be divided up to go round them all'. The good in question may be very important, as when 'not enough kidneys are available for everyone who needs one' (Broome 1990: 87). With respect to such fair division problems, Broome (1990: 92) introduces the notion of a claim as follows.

...I shall first draw a distinction of a different sort amongst the reasons why a candidate should get the good: some of these reasons are duties *owed to the candidate herself*, and others are not. I shall call the former *claims* that the candidate has to the good.

Thus Broome's account of fairness deals with *fair division problems* in which some scarce good (such as a number of kidneys or an amount of money) has to be divided amongst several candidates. There are several types of reasons as to why an agent should have some of the good (such as teleological reasons or side-constraints) in such a problem and Broome argues that fairness is concerned with a specific type of such reasons: *claims*. For Broome, a claim is *thus defined relative to a fair division problem*, as a specific reason to give a candidate some of the good-to-be-divided in the fair division problem under consideration. Paseau and Saunders neglect this crucial feature of Broome's account of fairness: they assume that claims are defined in an absolute sense, irrespective of the specific fair division problems with respect to which they are defined. Ann has a claim to £80 in fair division Problem I and a claim to £40 in fair division Problem II but nothing in Broome (1990) or in his other work on fairness mandates that we ascribe Ann a claim to receive £120 in a further fair division problem.

Now, Paseau and Saunders observe, in Reason 2, that the claims approach does not specify that claims are *owed by* particular individuals. We agree and note that their observation immediately follows from an inspection of the definition of a claims problem: whether or not there are individuals who own the estate is simply not present in the definition of a claims problem. But what *is* present in the definition of a claims problem, is that there are agents which all have claims to receive some amount of *the very same* estate. Without competing claims that are defined with respect to the same estate, there is, on the claims approach, no fair division problem. As nothing in Summary I+II (nor in the underlying Problem I and II) warrants that the amounts of £120 have to be interpreted as claims to a single estate, Reason II also fails.

The above also answers Reason 3 by Paseau and Saunders:

Third, nothing in our schematic example specifies whether [John] = [Jack] or [John] ≠ [Jack]. Paseau and Saunders (2015: 463)

It may very well be the case that the estate of C^I and the estate of C^{II} are owned by the same debtor, say John. The point is that C^I and C^{II} are models of different fair division problems and that the claims of Ann and Bob are defined relative to these fair division problems. Nothing that Paseau and Saunders (or Broome) tell us warrants that claims that are defined relative to these fair division problems induce claims that are defined relative to a further fair division problem.

In summary, our discussion shows that Saul, or any proponent of the claims approach, may question the extent of the significance of the problem of non-aggregativity. Whereas Paseau and Saunders are definitely correct that, on the claims approach, there are no aggregative theories of fairness, we are not convinced that this is problematic.

4.3 Aggregativity and the Trade-Off Between Claims and Games

In the previous section we showed that the arguments of Paseau and Saunders, which purport to establish that the claims approach harbours a *problem* of non-aggregativity by analysing Problem I and II, fail. Now, there could be *other* arguments, spelled out in terms of fair division problems *other* than Problem I and II, that establish that, after all, there is such a problem. Although that may be the case, Sect. 4.2 establishes that Theorem 1 as such, i.e. the mathematical fact that there are no (non-trivial) aggregative division rules, is far from sufficient to establish a *problem* of non-aggregativity. Now, in the economic literature, where Theorem 1 is already known since Bergantiños and Méndez-Naya (2001), Theorem 1 is not taken as reporting a problematic fact. When discussing the aggregativity property for division rules, Bergantiños and Méndez-Naya (2001: 225) describe the property of aggregativity as guaranteeing that two different procedures for dividing an estate coincide. They do not describe aggregativity as being required by fairness, as being intuitively appealing or desirable: although it may be convenient to have two procedures available that give the same result, fairness as such does not require that such procedures are available. Although we concur with Bergantiños and Méndez-Naya's assessment of Theorem 1, we realize that there *might be* further arguments that establish that Theorem 1 points to a problematic fact about theories of fairness on the claims approach. For argument's sake, let's assume that this 'might be' turns out to be an 'are'. Then, should Saul, or any proponent of the claims approach who thinks that theories of fairness should be aggregative, trade in the claims approach for the games approach? Of course, given Saul's take on aggregativity, the fact that the games approach harbours aggregative theories of fairness provides *a* reason to favour the games over the claims approach. But Saul may wonder whether there are there further reasons for, or against, trading in the claims for the games approach to fairness. The best advice we can give Saul at this point is to read (Heilmann and Wintein 2017), where the trade-off between the claims and games approach is discussed in detail. Here, we can only mention two important considerations that we think Saul should be aware of.

*R*₁. *In favour of the games approach: broader scope.* We already explained, in Sect. 2.3, that the games approach has a scope that is *at least as broad* as the claims approach: any fair division problem that can be analysed as a claims problem can also be analysed as a cooperative game. In (Heilmann and Wintein 2017) however, we show that there are fair division problems that cannot be properly analysed using the claims approach whereas they are naturally represented as cooperative games. Hence, the games approach to fairness has a scope that is strictly broader than that of the claims approach, which may be a reason to favour the former over the latter.

According to Theorem 3, the *RTB* rule coincides with the Shapley value. This means that for Saul, whose preferred theory of fairness is the *RTB* rule, trading in the *RTB* rule for the Shapley value, and so the claims approach for the games approach, may not be such a dramatic change. In fact, reason *R*₁ and Saul's take on aggregativity, jointly constitute a clear rationale for Saul to adopt the Shapley value rather than the *RTB* rule. But the preferred theory of fairness of a typical proponent of the claims approach, say John Broome, is not the *RTB* rule but rather the proportional rule *P*. For such a proponent, trading in the claims for the games approach is a more dramatic change, as reason *R*₂ explains.

*R*₂. *Against the games approach: unavailability of the proportional rule*¹³ The proportional rule *P* is unavailable on the games approach: there is no solution value φ whatsoever that coincides with the proportional rule. That is, for no solution value φ we have that $P(C) = \varphi(v^C)$ for any claims problem C .¹⁴ And so, the costs of trading in the claims for the games approach are definitely higher for a proponent of *P* than for Saul, who favours the *RTB* rule.¹⁵

Whereas *R*₁ and *R*₂ state motivations to adopt the claims or games approach *tout court*, an “all or nothing choice” for one of the approaches may not be the only viable option for a fairness theorist. Indeed, her choice to adopt a particular approach may be dependent on the nature of the fair division problem at hand. See Heilmann and Wintein (2017) for more details.

5 Concluding Remarks

We discussed the claims and games approach to fair division and explained that these approaches harbour theories of fairness that act on different fairness structures. We explained that whereas there are no aggregative theories of fairness on the claims approach, there are such theories on the games approach. More specifically,

¹³ Another context where the proportional rule is unavailable is when the good-to-be-divided comes in indivisible units. What should a “Broomean fairness theorist”, who is sympathetic to the slogan that fairness consists of the proportional satisfaction of claims, do in such a context? See Wintein and Heilmann (2018) for an elaborate answer to that question..

¹⁴ See Proposition 4 of Heilmann and Wintein (2017)

¹⁵ See Heilmann and Wintein (2017) for a detailed discussion of the relation between fairness and proportionality on the games approach

Theorem 2 testifies that the Shapley value is aggregative. Hence Peseau and Saunders's assertion of **NAT**—there are no aggregative theories of fairness—is false. We revisited the problem of non-aggregativity in light of the failure of **NAT**, by analysing fair division problems both on the claims approach (via the *RTB* rule) and on the games approach (via the Shapley value). Even though the *RTB* rule coincides with the Shapley value in the sense of Theorem 3 there is, perhaps surprisingly, no paradox of aggregativity, as we explained in Sect. 3.3.

Our discussion has also allowed us to give an accessible introduction to the games approach to fair division, whose significance has not hitherto been appreciated by philosophers working on fairness. We have shown that the games approach can model any fair division problem that the claims approach can model, but not vice versa. (cf. equation (★) in Sect. 2.3). Moreover, not all division rules from the claims approach can be translated into solution values in the games approach, the proportional rule being a case in point (cf. Reason R_2 in Sect. 4.3). Fairness theorists thus make important methodological choices when they decide in which approach to model fair division problems. Indeed, we also explained that certain conclusions that Peseau and Saunders draw are an artefact of their adopting the claims approach to fair division. Finally, we argue that even upon adopting the claims approach, the fact that one does not have access to aggregative theories of fairness is not something that should be considered problematic.

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Appendix

Theorem 1 (There are no non-trivial aggregative division rules)

Proof Let (E, N, c) be an arbitrary claims problem and let r be an aggregative division rule. We will first establish the following claim:

(A) Let $i \in N$ be an agent for which $\sum_{j \neq i} c_j \geq E$. Then $r_i(E, N, c) = 0$, i.e. r allots 0 to such an agent i .

To do so, let $i \in N$ be an agent for which $\sum_{j \neq i} c_j \geq E$. Let c' be the claims vector that is just like c except that $c_i = 0$ and let c'' be the claims vector that is just like c except that $c_j = 0$ whenever $j \neq i$. Indeed, $c = c' + c''$ from which it follows

that $(E, N, c) = (E, N, c') + (0, N, c'')$. Note that in particular (E, N, c') satisfies the definition of a claims problem: as $\sum_{j \neq i} c_j \geq E$ we also have that $\sum_j c'_j \geq E$. As r is an aggregative division rule, we have $r_i(E, N, c) = r_i(E, N, c') + r_i(0, N, c'') = 0 + 0 = 0$, which establishes (A). In order to establish our theorem, it suffices to establish (B):

(B) Let (E, N, c) be a claims problem and let $i \in N$ be an arbitrary agent. Then $r_i(E, N, c) = 0$, i.e. r allots 0 to each agent i .

In order to establish (B), let c_j be the claim of some agent other than i and let $\mathbf{0}$ be a vector of n zeros. As r is aggregative, it follows that:

$$r_i(E, N, c) = r_i(c_j, N, c) + r_i(E - c_j, N, \mathbf{0}) \quad (7)$$

We have that $r_i(E - c_j, N, \mathbf{0}) = 0$ as agent i 's claim is zero in $(E - c_j, N, \mathbf{0})$. Further, as (c_j, N, c) is a claims problem in which the sum of claims of all agents other than i is greater-than-or-equal to the estate, it follows from (A) that $r_i(c_j, N, c) = 0$ so that it follows from (7) that $r_i(E, N, c) = 0$. As i was arbitrary, r is trivial, which is what we had to show. \square

Theorem 2 Sh is aggregative: $\text{Sh}(N, v) + \text{Sh}(N, v') = \text{Sh}(N, v + v')$

Proof It follows from the definition of the Shapley value (4) that it suffices to show that, for any order of agents $\pi \in \Pi(N)$:

$$MC^v(\pi) + MC^{v'}(\pi) = MC^{v+v'}(\pi) \quad (8)$$

Let π be an order and let its k th element be agent j : $\pi_k = j$. Let S be the coalition consisting of all agents who arrive before j according to π , i.e. $S = \{\pi_m \mid m < k\}$. It immediately follows from the definition of the vector of marginal contributions that (i) $MC^v(\pi)_k = v(S \cup j) - v(S)$, that (ii) $MC^{v'}(\pi)_k = v'(S \cup j) - v'(S)$ and that (iii) $MC^{v+v'}(\pi)_k = (v + v')(S \cup j) - (v + v')(S)$. Thus, by adding $MC^v(\pi)_k$ and $MC^{v'}(\pi)_k$ and rearranging terms, we get:

$$MC^v(\pi)_k + MC^{v'}(\pi)_k = v(S \cup j) + v'(S \cup j) - (v(S) + v'(S)) \quad (9)$$

Per definition of the aggregated game of v and v' we have that $(v + v')(U) = v(U) + v'(U)$ for any $U \subseteq N$. Hence, it follows from this definition that the right-hand side of (9) is equal to $(v + v')(S \cup j) - (v + v')(S)$. And hence it follows from (9) and (iii) that $MC^v(\pi)_k + MC^{v'}(\pi)_k = MC^{v+v'}(\pi)_k$, so that (8) holds true, which is what we had to show. \square

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