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Bolzano's Argument for the Existence of Substances: a Formalization with Two Types of Predication

Kordula Świętorzecka[1] (ORCID)

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Abstract The topic of our analysis is the argument for the existence of substances given by Bernard Bolzano in *Athanasia* (1827), where he essentially employs two ontological categories: *substance* and *adherence*. Bolzano considers the real and conditioned *Inbegriff* of all adherences, which are *wirklich* and *nicht selbst bestehen*. He claims that the formed collection is dependent on something external and non-adherential, which therefore is a substance. Bolzano's argumentation turns out to be structurally similar to his argument for the existence of God from *Lehrbuch der Religionswissenschaft* (1834), but in each of these reasonings, we find different plausible interpretations of the key concept "Inbegriff". The latter argumentation refers to the mereological totality of existentially conditioned objects. We propose the explication of the Bolzanian *Inbegriff* of all adherences using two types of predication: we consider its extension as composed of certain intensional counterparts of adherences. In our approach, we use a fragment of the theory of abstract objects formulated by E. Zalta (1983), describing two different relations between individuals and properties: extensional *exemplification* and intensional *encoding*. We put our reconstruction in a wider context of Bolzano's ontology, formulating the needed axioms with two primitive predicates of second order ... *is an adherence*, ... *is conditioned by something real* as well as the conditionally introduced first order predicate constant ℑn for *Inbegriff of all adherential ideas*. Finally, we sketch a model for our theory.

Keywords Formal ontology · Abstract objects · Argument for the existence of substances · Bernard Bolzano

✉ Kordula Świętorzecka
  k.swietorzecka@uksw.edu.pl

1   Cardinal Stefan Wyszyński University in Warsaw, Warsaw, Poland

We take into consideration a fragment of B. Bolzano's ontology, which essentially employs two ontological categories: *substance* (*Substanz*) and *adherence* (*Adhärenz*). The notion of adherence is involved in a formulation of the argument for the existence of substances given by Bolzano in *Athanasia* (1827) and this reasoning is formalized in the present paper.

A key concept of the considered argumentation is the notion of *Inbegriff* of all adherences — real attributes of real objects (possibly other real attributes). The totality of them is also real and conditioned. However, its conditions cannot be adherences, which are already included into it. All reals are either adherences or substances, so the formed *Inbegriff* must be conditioned by (at least one) substance.

In our attempt to reconstruct Bolzano's discourse, we are inspired by its similarity to the argument for the existence of God given by Bolzano in *Lehrbuch der Religionswissenschaft* (1834, 177–179). In the latter case, Bolzano also used the notion of *Inbegriff* to come to the main existential conclusion. He formed it out of all conditioned beings and argued that the reality of this *Inbegriff* is dependent on something real but external, in conclusion: on some unconditioned being, which he assumed to be God. A reconstruction showing some plausible path of this reasoning can be done in frame of a certain extension of mereology based on Zermelo-Fraenkel set theory (ZF) as Świętorzecka (2014). We will propose a few similar steps in the present approach. However, the argument for the existence of substances has also its specificity connected with the nature of *Inbegriff* of all adherences. The totality of all conditioned beings in the argument for the existence of God can be identified with their mereological collection, but this does not seem to be an adequate manner of speaking when we collect adherences understood by Bolzano as some special properties of beings. Actually, properties may be described in an extensional or intensional way, and we suppose that just the combination of these two approaches is involved in the analyzed reasoning. Speaking of adherences, Bolzano perhaps followed his distinction which allowed him to consider them as predicated to some real individuals but also as represented by a sort of *ideas in themselves* (*Ideen an sich*). Ideas were components of his ontology and some of them could be mentioned just as intensional counterparts of adherences. Our proposal is to focus on these intensional characterizations and to recognize them as exemplifications of Bolzanian *Inbegriff*. We base our description on some part of the theory of abstract object by E. Zalta (1983) which allows us to express the distinction between extensional and intensional predication. We are going to show that the apparatus of Zalta's theory gives the possibility to trace the subsequent steps of Bolzano's argumentation and to notice their similarity to the structure of his argument for the existence of God.

1 The Original Argument and a Few Preformal Supplements

One might ask why the existence of substances was not evident enough for Bolzano, and why he formulated arguments for their existence. Actually, he used the term "substance" in two meanings, speaking of simple and complex substances (Bolzano 1851, §50–65). Simple substances were understood as everlasting, eternal individuals, similar to Leibnizian monads but possessing *windows*: they stayed in a causal

relationship with other substances.[1] Complex substances were regarded by Bolzano as *aggregates* of an infinite number of simple substances. In fact, it could be said that arguing for the existence of the latter is not trivial, because simple substances are not visible and cannot be any subject of our direct experience. What can be perceived by our senses are only *aggregates* and *qualities*. Bolzano tried to come to the considered conclusion in two ways: taking as a starting point the existence of aggregates or the existence of qualities. He formulated the argumentation for the existence of (simple) substances based on the first assumption in his *Paradoxes* (Bolzano 1851, 155). The shift from the existence of aggregates — perhaps: mereological units — to the existence of their *atoms* — simple substances — was validated by a sort of a foundation principle. The other alternative argument referring to the nature of substantial qualities which "cannot exist and be actual apart from the presence of a substance to which they can belong" was not analyzed and exploited there. However, the track dealing with the pair *substance–quality* (or its certain kind named "adherence") could be the only one possibly successful solution if the considered universe of metaphysical discourse would be composed only of simple substances (and not aggregates). This perspective is present in *Athanasia*, where Bolzano undertakes the problem of the immortality of souls mentioned just as simple substances. He formulates the following reasoning:

> Gibt es nun irgend `Ein` Wirkliches, so ist offenbar, daß es auch `Eine` oder `einige Substanzen` geben müsse. Denn ein Wirkliches, das nicht Substanz ist, muß eine Adhärenz, also `Beschaffenheit` sein und setzet folglich noch irgendein anderes Wirkliche, an dem es sich befindet, voraus. Dies andere Wirkliche kann zwar abermals nur eine Adhärenz oder Beschaffenheit sein, weil auch Beschaffenheiten noch ihre Beschaffenheiten haben, und wenn nur jene wirklich sind, so sind es auch diese. Allein, wenn wir uns alle wirklichen Dinge, die bloße Adhärenzen sind, in Einen Inbegriff vereinigt vorstellen, so leuchtet bald ein, daß auch dieser nicht für sich selbst bestehen könne, sondern erst eines oder einiger anderer wirklichen Dinge, an denen er sich befindet, bedürfe. Dies E i n e nun, oder auch diese m e h r e n wirklichen Dinge sind nicht mehr Adhärenzen, weil sie sonst mit zu jenem Inbegriff gezählt werden müßten; sie sind daher Substanzen. So gewiß es also irgend Ein Wirkliches gibt, so gewiß gibt es auch Substanzen. (Bolzano 1827, 22)

> [1] If there is `something real` [*Wirkliches*], it is evident that there must be `one` or `more substances`. [2] Because anything real which is not a substance must be an adherence, therefore, it is a property [*Beschaffenheit*], and therefore, it presupposes another real in which it occurs [*befindet an*]. [3] This other real can be again only an adherence or a `property`, because properties can also have their properties, and if these are real, then also the other ones. [4] Now, if we imagine that we put together all real things, which are merely

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[1] The very adequate comparison of Bolzanian substances with monads was proposed by R. Zimmermann in his *Leibnitz' Monadologie. Deutsch mit einer Abhandlung über Leibnitz' und Herbart's Theorieen des wirklichen Geschehens* (Vienna, 1847). At that time, it was more "convenable" to refer to Herbart than to Bolzano. Herbart actually understood the substances in the same way as Bolzano did. (I owe these historical remarks to Professor E. Morscher.)

adherences, in one collection [*Inbegriff*], it is clear that this [collection] cannot exist [*bestehen*] by itself, but it needs one or `more` other things in which it occurs. [5] This `one` or `more` real things now are not adherences because they must not belong to this collection; so they are substances. [6] Thus, if it is certain that there is something real, it is also certain that there are substances.

This argument will be the subject of our analysis.

To give some intuitions grounding the proposed reconstruction, let us note a number of explanations of the concepts *substance* and *adherence* combined together, given by Bolzano in different places of his work.[2]

The pair *substance–adherence* is listed in *Wissenschaftslehre* as categories of *relation* together with the pair: *cause–effect* and *community* (Bolzano 1837, vol. 1/2014, 403). Interestingly, the term "adherence" is supposed to be the original expression of Bolzanian philosophy.[3] According to W. Künne's conjecture, Bolzano could introduce "adherence" instead of the medieval term "accidence" probably to keep the right opposition to the meaning of the term "substance" (Künne 2015). The essential feature of any substance is to exist *in se*, in contrast to adherences existing *in alio* (although some adherences are attached to substances by necessity, for instance, omniscience and omnipotence as attributes of God, so they are not accidental). Actually, the distinction between substances and adherences followed the Aristotelian line, although it was refereed to the universe of Platonic individuals.

Bolzano notes the following basic features of discussed categories:

1. daß alles `Wirkliche` zu einer von folgenden zwei Arten gehöre, daß es entweder `Substanz` oder `Adhärenz` sey, daß Adhärenz bloßein solches Wirkliche heiße, das sich an einem Andern als eine Beschaffenheit desselben befindet. Was sich `nicht` an einem Andern befindet, sondern, wie man dieß auszudrücken pflegt, `für sich bestehet`, das heißt `Substanz` oder `Wesen`. [...], daß alles, was anfängt oder aufhört, nur eine Adhärenz sey, während jede Substanz etwas Beständiges ist, das weder anfangen, noch vergehen kann, sondern, so ferne es einmal ist, zu aller Zeit seyn muß. (Bolzano 1834, §70, 185)

1. that `reals` belong to one of the following two kinds, that it is `substance` or `adherence`, that adherence is called a real, which is attached to another thing as property. That which is not attached to another, but, as one likes to express it, exists by `itself`, is called a `substance` or `essence`, [...], that everything which begins or ends is only an adherence, whereas every substance is something stable which cannot begin or end but, insofar as it exists once, has to be forever.

Actually, the most extensive explanations of the considered pair are formulated in *Athanasia*. The above-quoted passage is a repetition of ideas expressed there. From

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[2]The extensive philosophical discussion with numerous references is presented by Schnieder (2002). Our description is dictated by a formal theory given later.

[3]Schnieder (2006, 12) notes that later Meinong used the transitive verb "to adhere" to speak about attributes attached to things, and just like Bolzano, he accepted that they are as real as their subjects.

Athanasia, we also learn that adherences are real attributes (*Beschaffenheiten*) of real individuals[4] and that they are located on (attached to) substances or on (to) other adherences. The relation between adherence and its subject is not transitive in the sense that the adherence of another adherence located in a certain substance (or an adherence) is not an adherence of the latter one. However, adherences may be subject to a sort of importation. If we take any adherence a_i, adherence a_j, and substance s such that a_j is attached to $s - a_j(s)$ and if the adherence a_i is attached to a_j, then s possesses the complex adherence $a_i(a_j)$ (although a_i is of course not attached to s).[5] Finally, Bolzano seems to claim that adherences are particulars in a sense that they are unique attributes of particular substances (*trops*, cf. (Morscher 2013)).[6]

The relation between substance and adherence is expressed in subject-predicate sentences of the shape *A is B*. When we say that *A is B*, the connection between *A* and *B* should be understood in such a way that *B* represents one of the attributes of an object falling under *A* and substantial terms cannot be instantiated under *B*:

> [...] dass wenn der Gegensatz zwischen Substanz u. Adhärenz scharf aufgefasst u. festgehalten werden soll, so müsste man unter Substanz lediglich nur das-jenige Wirkliche (Seyende) verstehen, was immer nur als Subject - niemals als Prädicat gedacht werden könne. Dieses vorausgesetzt, ist es unrichtig zu sagen, dass auf die Frage: Was (ist dies)? je die Substanz zur Antwort komme. Denn was auf diese Frage zur Antwort kommt, ist eine Prädikatvorstellung: das-ist-A (hat die Beschaffenheit A), also eine Adhärenz. (Bolzano 1835, 117)

> [...] if the difference between substance and adherence is precisely defined and established, one has to understand by substance only that real which can be thought always only as subject, never as predicate. This assumed, it is wrong to say that for the question: What (is that)? a substance can be an answer. Because what can be an answer to this question is a predicative idea: that - is - A (has the Beschaffenheit A), therefore an adherence.

The intended restriction of an instantiation of *B* is kept by the canonical sentential structure to which all subject-predicate sentences are reducible. Bolzano assumes that every sentence *A is B* may be equivalently reformulated to the sentence *A has b*, where "*b* represents the abstractum that belongs to the concretum *B*" (Bolzano 1837, 1973, 172). If *b* is an adherence, we say in the sentence: *A has b*, that *b* is "some property to be found in a certain object" *A*, and where actually the term "has" does

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[4]It is not clear if there are any real attributes which are not adherences (explanations given in the text are not consequent). Bolzano sometimes uses the term "Beschaffenheit" instead of "Adhärenz" (Bolzano 1827, 21).

[5]Bolzano gives an example of *unusual power* (a_i) and *memory* (a_j) of any given person (s) (Bolzano 1827, 23).

[6]B. Schnieder notes that particularity is not considered as a defining feature of adherences, although it may apply to adherences in view of the fact that actually, they are subjects of many observable changes ("the colour of the flower, for example, may be paler today than it was yesterday", cf. the whole argument of Bolzano (1827, 23f) cited by Schnieder (2006, 12), where are also considered some philosophical consequences of treating adherences as particular properties). However, Bolzano also gives universals as examples of real attributes (Morscher 2013). In any event, this problem will not influence our reconstruction of the analyzed argument.

not mean a "possession (i.e. the capacity to make some use)" (*ibid*, 124). Searching for some semantic explanation for such a vision, we should refer to specific Bolzanian ontology. Its domain, in addition to reality of material objects and psychical acts, contained the special (independent of others) "realm" of *propositions in themselves* and *ideas in themselves*. *Propositions in themselves* are ideal counterparts of sentential expressions, and *ideas in themselves* are the ideal objects corresponding to components of sentences: the terms and copula "has". Looking for some intensional interpretation of adherential terms, we would place the ideal objects corresponding to them somewhere in the intersection of *real* and *concrete* ideas. Actually, Bolzano divided ideas in many different kinds (Bolzano 1837, 1973, 75–163) but we focus just on these two mentioned types, following some explanations and examples given by Bolzano. When we consider a situation in which person *A* has (the adherence or the property of) *animality*, then the *concretum* would be an *animal* which would be a real and concrete idea. It is real (as opposed to an *imaginary* idea, (*ibid*, 109–112)) in this sense that it is non-contradictory and concrete (in distinction to an *abstract* idea (*ibid*, 92–93)), because "an animal [is] the idea of something that has the property of animality" (*ibid*, 92).[7] Finally we can also say: *A* is an *animal*. On the other side, an *animal in itself* does not have the property of *animality*, but we would rather say that the former represents the latter. In fact, this kind of predication will play an important role in our formalization of Bolzano's argument.

2 Formalization

As we have already declared, we base our proposal on a certain fragment of Zalta's theory of abstract objects (Zalta 1983). The author of the chosen tool originally designed it as a formal basis for the Meinongian ontology of fictional objects. He used it (and some extensions of it) to analyze fragments of Plato's theory of forms, the Leibnizian monadology, and Frege's theory of sense and denotations. Some of the ideas involved in Zalta's formalism were applied in his analysis of the Leibnizian theory of concepts (Zalta 2000). Now, we take this frame to express a part of Bolzano's theory of substance and adherence, which, in fact, also has clear Platonic roots.

The specificity of Zalta's theory is that it describes two types of predication: extensional *exemplification* and intensional *encoding* (*representation*). They are noted by two different concatenations of predicate and individual terms. Let us take an individual term t and a predicate term τ. The expression τt is used to speak about the exemplification of the property named τ with the individual named t. We can say that τt is true just when the object symbolized by t belongs to the extension of the property named τ. The intensional way of speaking is captured by the concatenation $t\tau$. In this case, the individual symbolized by t represents (encodes) the property named τ and it belongs to the intension of this property (being its *coder*). Such

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[7]There are also examples formulated by Bolzano in which abstract nouns are used as adherential terms (*wisdom*, *courage*). We will not interpret them as individual names of properties (this idea is discussed by Schnieder (2006, 15–17)) but as predicate terms having their extensions in the semantics of Zalta's system.

"backward" expressions are used by Zalta to speak about connections between Meinongian abstract objects or Platonian forms and corresponding properties. He also characterizes Leibnizian monads as coders which are "blueprints" of individuals belonging to one-element extensions of some properties. Now, we bring this approach into play and consider ideas representing adherences and forming the extension of Bolzano's *Inbegriff*.

We take a fragment of Zalta's system theory, named here AOT.

The system AOT is expressed in a language consisting of individual terms (Tm) – variables: x, y, z, ...; unary predicate terms (TM) – variables: F, G, H, ..., primitive constant $E!$ to be read: ... *exists* (*concretely*); binary constant $=_E$ used in contexts with individual terms; logical symbols: $\neg$, $\rightarrow$, $\forall$, λ and parentheses.

For $t \in Tm$ and $\tau \in TM$, the expressions: τt are read: *individual t exemplifies the property τ* or *t has τ*, and expressions: $t\tau$ – *individual t encodes the property τ* or *property τ determines t*.

The set of all formulas consists of atomic formulas – expressions of the shape: τt, $t\tau$, where τ may be λ term (λ forms a predicate term out of a formula; $[\lambda x.\varphi]$ has as its intended meaning the predicate of fulfilling the formula φ), formulas of shapes: $t =_E t'$ and all other expressions formed with the help of logical symbols in a standard way.

Following Zalta's terminology, by an *encoding formula,* we understand any expression of the form $t\tau$ and by a *propositional formula* – every formula with no encoding subformula and with no $\forall$ for predicate variables (but possibly with quantifiers for individual variables).

We take α, β as representing individual or predicate variables. Symbol t represents individual or predicate terms.

The theory AOT is based on the logic characterized by the following:

- classical propositional tautologies
- all formulas of the shape:

($\forall 1$) $\forall\alpha\varphi \rightarrow \varphi_\alpha^t$, t is substitutable for α
($\forall 2$) $\forall\alpha(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall\alpha\psi)$, α is not free in φ
($Comp\lambda$) $[\lambda x.\varphi]\, y \leftrightarrow \varphi_x^y$, where φ is a propositional formula.

Primitive rules are the following: $(MP) \vdash A \rightarrow B, \vdash A \Longrightarrow \vdash B$
$(R\forall) \vdash \varphi \Longrightarrow \vdash \forall\alpha\varphi$

Comprehension schema $Comp\lambda$ allows us to introduce λ terms, but with the restriction on formulas occurring in the scope of λ, they should not contain encoding subformulas and quantifiers bounding predicate variables.[8]

We adopt the following definitions of Zalta for *the property of being abstract object* and for an *intensional identity*:

(D1) $A!x =_{df} [\lambda y.\neg E!y]x$
(D2) $F = G =_{df} \forall x(xF \leftrightarrow xG)$

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[8]The restriction is sufficient to block Clark's antinomy described by Zalta (1983, 158–159): the possibility of introducing $[\lambda x.\exists F(xF \land \neg Fx)]$ and $[\lambda x.\forall F(xF \rightarrow Fx)]$ would lead to contradiction in view of the later introduced AOT proper axiom A3.

The proper axioms of AOT are as follows:

(A1) $x =_E y \leftrightarrow E!x \wedge E!y \wedge \forall F(Fx \leftrightarrow Fy)$ (*identity of concretes*)
(A2) $E!x \rightarrow \neg \exists F(xF)$ (*no-coder*, concretes are no-coders)
(A3) $\exists x(A!x \wedge \forall F(xF \leftrightarrow \varphi))$, x is not free in φ (*being-so* of *A-objects*)
(A4) $\alpha = \beta \rightarrow (\varphi\alpha \leftrightarrow \varphi[^{\beta}_{\alpha}])$ (substantiation of identicals)

where the symbol $=$ is understood as

(D3) $x = y =_{df} x =_E y \vee (A!x \wedge A!y \wedge \forall F(xF \leftrightarrow yF))$ or as in D2.

Let us focus on the axiom A3 in the context of a certain difference between Meinong's and Bolzano's ontology. Actually, the intention of Zalta was to formulate an intensional calculus which would contain a counterpart of a basic ontological theorem accepted by Meinong, according to which "for every describable set of properties, there is an object which exemplifies just members of the set" (Zalta 1983, 6).[9] This postulate expressed just in A3 is strictly connected with (or perhaps it is simply a version of) the known "principle of the independence of being from being-so" which is an ontological radicalization of Twardowski's conviction that it is possible to speak about non-existent objects of representations.[10] Twardowski formulated his theory of the contents and objects of representations being sure that he followed Bolzano's fundamental distinction of *subjective ideas*, *objective ideas*, and their *objects*. He treated mental acts and their contents as counterparts of the Bolzanian subjective and objective ideas. Mental acts may have empty content but still they have their objects. Actually, this was a neuralgic point of an incorrect interpretation of Bolzano's approach. Bolzano did not open the door for the Meinongian multitude of fictional objects but claimed that if ideas in themselves have any objects, these are real; otherwise, the idea is *objectless* and there are no fictional entities corresponding to it (Künne 2003). These interpretative remarks show that Zalta's theory in its full power probably decides to much compared with original Bolzano's approach, but actually, this fact does not influence our research. In our context — we are focused on the existence of real individuals — the question of the existence of inconsistent ideal objects is irrelevant. Moreover, as we will see, it is possible to formulate the analyzed argument without full axiom A3.[11] Actually, the schema A3 leads to the following:

(T1) $\exists !x(A!x \wedge \forall F \neg xF)$ (A3)

so we can introduce a constant for this abstract object which does not encode anything:

(D4) $x = \mathfrak{o} =_{df} (A!x \wedge \forall F \neg xF)$ (*null*)

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[9]As it is shown by Zalta, a plausible formalization of this principle added to a sufficiently rich version of the second-order classical logic results in a contradiction.

[10]In these explanations, we follow some interesting remarks of Künne (2003).

[11]We would say that A3 gives a logical way out of Meinong's overabundance into which we do not get standing on the ground of Bolzano's ontology.

We notice that

(T2) $\forall F \neg x F \wedge \neg\, x = \mathfrak{o} \to E!x$ (D4, D1)

T2 states that every *no-coder* different from $\mathfrak{o}$ is a concrete.

Analogously, we can introduce a name for the all-encoding abstract individual.

Interestingly, although in AOT, it is decided that both universes of properties and of abstract objects are not empty, it is not settled if the universe of concretes is non-empty.[12] This fact fits our approach: Bolzano starts his argumentation with the existence of adherences and argues for the existence of substances which we identify with concrete objects in our later analysis.

In our formalization, we use the following definition of *characterization of a property*:

(D5) $x con F =_{df} \forall G(xG \leftrightarrow G = F)$ (*x characterizes F*)

D5 is used by Zalta in his reconstruction of Plato's theory of forms as a definition of *participation of ideas* (Zalta 1983, 41).

As a theorem in AOT, we get

(T3) $x con F \wedge x con G \to F = G$ (D2, D5)

Thus, there is no individual which characterizes — is a *characteristic coder* of — two (intensionally) different properties.

We can also note that for every property F, there is a *characteristic coder*

(T4a) $\forall F \exists x (x con F)$ (A3: $\forall F \exists x \forall G(xG \leftrightarrow G = F)$) and D5)

To show the uniqueness of a characteristic coder of every property, we take the following definition of $\exists!$:

(D6) $\exists! x \varphi =_{df} \exists x\Big(\varphi \wedge \forall y(\varphi y \to x = y)\Big)$

Now, we have also

(T4b) $\forall F \exists! x (x con F)$

Proof

1.	$x con F \to \forall G(xG \leftrightarrow G = F)$	D5
2.	$x con F \wedge y con F \to \forall G(xG \leftrightarrow yG)$	1
3.	$x con F \to A!x$	A2, D1
4.	$x con F \wedge y con F \to A!x \wedge A!y$	3
5.	$x con F \wedge y con F \to \forall G(xG \leftrightarrow yG) \wedge A!x \wedge A!y$	2, 4
6.	$x con F \wedge y con F \to x = y$	D3, 5
7.	$\forall F \exists! x (x con F)$	T4a, 6

$\square$

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[12]The assumed logic already guarantees that sets of properties and individuals are not empty, but it is not determined if any individual is abstract or concrete.

Let us introduce the following definition of *inherence*:

(D7) $x \varepsilon y =_{df} \exists F(Fx \land \neg x = \mathfrak{o} \land ycon F)$ (*is*)

It can be proved that

(T5) $x \varepsilon y \land ycon F \rightarrow Fx$

Proof

1. $ycon F \land ycon G \rightarrow (Gx \rightarrow Fx)$ T3, A4
2. $Gx \land ycon G \rightarrow (ycon F \rightarrow Fx)$ 1
3. $\exists G(Gx \land ycon G) \rightarrow (ycon F \rightarrow Fx)$ 2
4. $x = \mathfrak{o} \land \exists G(Gx \land ycon G) \rightarrow (ycon F \rightarrow Fx)$ 3
5. $x \varepsilon y \land ycon F \rightarrow Fx$ D7, 4

$\square$

T5 corresponds to the already mentioned Bolzano's idea of reformulation of subject-predicate sentences into sentences of the shape *A has b*. In our case, a sentence $x \varepsilon y$ (*x is y*) may be transformed to Fx (*x has F*) if *y* is a characteristic coder of *F* (Bolzano would say that if *y* is *concretum* of *abstractum F*).

Let us now come back to the main approach.

AOT is an attractive frame for our analysis due to the morphology of its language. We remind that Bolzano considered the schema *A has b* as a canonical form of all propositions. If we considered Zalta's "gerundive versions" of predicative adjectives like "the property of being red" and "gerundive versions" of predicative nouns like "the property of being an animal" ("the property of animality") as permissible instantiations of *b*, then we would come to some examples given by Bolzano (cf. ft. 7). Now, "red object" and "animal" are names of concretes of the considered abstracts (in Bolzano's sense) and we name some of them "adherential ideas".

To express our version of Bolzano's argument, we add to the language of AOT two second-order constants *B* and *Ad* to be read *... is conditioned by something real*, and *... is an adherence*. We also extend our vocabulary by the specific predicate term $\mathfrak{In}$ for *the property of being an intensional characterization of any adherence* (or *adherential idea*). In connection with the intended meaning of $\mathfrak{In}$, we have to recall the restriction occurring in (*Compλ*). We want to use the name $\mathfrak{In}$ for a certain property possessed by *characteristic coders* of all adherences, that is the property described by a coding formula with quantification over properties. In this situation, we weaken all $\mathfrak{In}$ substitutions of logical tautologies and schemata A3, A4 preceding them by the formula

(*IN*) $\exists ! Z \forall x (Zx \leftrightarrow \exists F(Ad F \land xcon F))$

which we name *the condition of consistency of the Inbegriff of all adherences*.

For example, instead of ($\forall$1), in case of $\mathfrak{In}$ substitutions, we have

($\forall 1_{IN}$) $IN \rightarrow (\forall F\varphi \rightarrow \varphi_F^{\mathfrak{In}})$

We interpret the universe of $E!$ individuals as the set of substances and so we accept the following definition:

(D8) $Sx =_{df} E!x$ (*substance*)

We remind that the term $E!$ in Zalta's approach has a different meaning than the term "concrete" used by Bolzano. Bolzano was speaking about concrete ideas (not objects) and concrete existence of adherences, complex substances, and their *Inbegriffe*.

   Now, we propose the following proper axioms for our reconstruction of Bolzano's argumentation together with their paraphrases:

(AB0) $IN \to \forall x(\Im nx \leftrightarrow \exists F(AdF \wedge xconF))$
        If the Inbegriff of all adherences is consistent, then x has this property ($\Im n$)
        iff x characterizes some adherence.
(AB1) $\exists x Sx \vee \exists F AdF$
        There exists something real: a substance or an adherence.
(AB2) $IN \to (\exists x\Im nx \to B\Im n)$
        If the Inbegriff of all adherences is consistent, then it is conditioned by
        something real, if it is attributed to anything.
(AB3) $\forall F(BF \to \forall x(Fx \to Sx) \vee \exists x(Fx \wedge \exists y(y\varepsilon x \wedge \neg Fy)))$
        Every property F which is conditioned by something real, is attributed to a
        substance or to something which is an attribute of y which is not F.
(AB4) $IN \to (\Im nx \wedge \neg xconS \wedge \neg\Im ny \wedge y\varepsilon x \to \exists z(y\varepsilon z \wedge zconS))$
        If the Inbegriff of all adherences is consistent, then if it is attributed to indi-
        vidual x which does not characterize a substance, but which is an attribute
        of some individual y which is not an adherential idea, then y is a subject of
        the idea characterizing a substance.

The main theorem, which we consider as the subject of the proof, is

ThS. $IN \to \exists x Sx$
        If there exists the collection of all adherential ideas, then there exists at least
   *one substance.*

Proof

1.	$\exists x Sx \vee \exists F AdF$	AB1 ($=1a \vee 1b$)
1b.	$\exists F AdF$	ass
2.	$\exists F\exists x(AdF \wedge xconF)$	T4a, 1b
3.	$IN \to \exists x\Im nx$	AB0, 2
4.	$IN \to B\Im n$	AB2, 3
5.	$IN \to (B\Im n \to \forall x(\Im nx \to Sx) \vee \exists x(\Im nx \wedge \exists y(y\varepsilon x \wedge \neg\Im ny)))$ AB3, $\forall 1_{IN}$
6.	$IN \to \forall x(\Im nx \to Sx) \vee \exists x(\Im nx \wedge \exists y(y\varepsilon x \wedge \neg\Im ny))$	4, 5
6a.	$IN \to \forall x(\Im nx \to Sx)$	ass
7.	$IN \to \forall x(\Im nx \to \exists FxconF)$	AB0
8.	$IN \to \forall x(\Im nx \to \exists FxF)$	D5, 7
9.	$IN \to \forall x(\Im nx \to \neg Sx)$	8, A2, D8
10.	$IN \to \exists x(\Im nx \wedge \exists y(y\varepsilon x \wedge \Im ny))$	6, 6a, 9
11.	$IN \to \exists x(\Im nx \wedge \exists y(y\varepsilon x \wedge \Im ny) \wedge (xconS \vee \neg xconS))$	10

11a.	$IN \rightarrow \exists x(\mathfrak{I}\mathfrak{n}x \wedge \exists y(y\varepsilon x \wedge \mathfrak{I}\mathfrak{n}y) \wedge xconS))$	*ass*
12.	$y\varepsilon x \wedge xconS \rightarrow Sy$	T5
13.	$IN \rightarrow \exists x Sx$	11a, 12 ($IN \rightarrow 1a$)
11b.	$IN \rightarrow \exists x(\mathfrak{I}\mathfrak{n}x \wedge \exists y(y\varepsilon x \wedge \mathfrak{I}\mathfrak{n}y) \wedge \neg xconS))$	*ass*
14.	$IN \rightarrow \exists y \exists z(y\varepsilon z \wedge zconS)$	AB4, 11b
15.	$IN \rightarrow \exists y Sy$	14, 12 ($IN \rightarrow 1a$)

Let us picture the structure of the proof:

$\square$
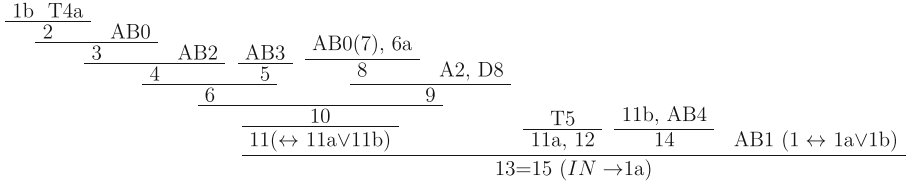
To notice the connections between the original text of Bolzano and our reconstruction, first, we pay attention to axiom AB0. It is the conditional definition of $\mathfrak{I}\mathfrak{n}$ intended to describe the meaning of *Inbegriff* considered by Bolzano in sentence [4] from the analyzed fragment of *Athanasia*. Axiom AB1 is a formalization of the antecedents of implication [1] which is then repeated in [6] as a premise. The mentioned separation of two types of reals substances and adherences is already prejudiced by our language: we use for them variables of different sorts. AB2 is meant to express the statement from [4]: the *Inbegriff* composed of adherences is conditioned by something real. AB3 is a weaker version of the second part of sentence [2] and sentence [3]. Bolzano claims that the conditionality of adherences (so: real properties) presupposes other reals, but these may be substances or again other (real) properties. The possibility of speaking about properties of properties is expressed in our theory by the fact that some properties may be attached to characteristic coders of other properties. In AB3, we express only a necessary (and not sufficient) requirement for being a property conditioned by something real, which is to be a property founded on substances or (at least) to be attached to a coder characterizing another property which is attached to individuals not belonging to the extension of this first property. Finally, AB4 — which may be called *principle of substantial foundation of* $\mathfrak{I}\mathfrak{n}$ — validates [5] and allows us to obtain the main conclusion. The important point is that in our reconstruction, we reach a weaker conclusion that of Bolzano. Indeed, by the restriction in $Comp\lambda$, the formula IN is not an AOT thesis. We take it as the antecedents of these formulas in which the constant $\mathfrak{I}\mathfrak{n}$ is used: AB0, AB2, and AB4. Finally, IN is also the antecedents of implication **ThS**.[13]

Commenting the deductive power needed to obtain our argument, we can note that in our derivation, axiom A2 and theorems T4a and T5 are used essentially. Axiom A2 would be accepted by Bolzano: certainly, substances are not ideas. T5

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[13]To show the consistency of our formalism, we sketch a model of it at the end of our considerations.

follows the Bolzanian idea of transformation of subject-predicate sentences into the canonical form. The acceptance of T4a may be problematic because of its inferential dependency on the powerful axiom A3. In any case, if we would be inclined to reduce this power, we could introduce the weakening of T4a as a special axiom: $\forall F(AdF \rightarrow \exists x(x\,con\,F))$ and the proof of **ThS** would work.

Now, let us return to the already mentioned inspiration and few interesting similarities between the analyzed argumentation from *Athanasia* and the argument for the existence of God from *Lehrbuch der Religionwissenschaft* (1834, 177–179). We take in account the exposition of the latter given by Świętorzecka (2014). The formal basis of this approach is the system expressed in the language of the ZF theory extended by predicate $\sqsubset$ for the part relation. For convenience, in the following footnote, we draft the assumed theory originally formulated by A. Pietruszczak (2000, 172–181).[14] The considered formalization uses two primitive one-place predicates: W and B to be read: *is real* and *is conditioned*, as well a two-place predicate R — *is a condition of*.

Specific axioms are as follows:

(AβG)	$\forall x(Gx \leftrightarrow Wx \wedge \neg Bx)$	(*x is God*)
(Aβ0)	$Zb \wedge \forall x(x\epsilon b \leftrightarrow Wx \wedge Bx)$	
	All real conditioned things form a set	
(Aβ1)	$\exists x Wx$	
	There is something real	
(Aβ2)	$\forall v \forall y(v\sigma y \wedge \forall z(z\epsilon y \rightarrow Wz) \rightarrow Wv)$[15]	
	Each mereological sum of real objects is real	
(Aβ3)	$\forall x \forall y(Wx \wedge yRx \rightarrow Wy)$	
	Conditions of real objects are real	
(Aβ4)	$\forall x(Bx \rightarrow \exists y(yRx \wedge (y\wr x \vee \exists z(z\wr x \wedge zRy))))$[16]	
	If x is conditioned, then x has some condition which is disjoint with x or this condition is conditioned by an object which is also disjoint with x.	

The proof of $\exists x Gx$ goes in a similar way to our derivation of **ThS**. At first, we assume that b is an empty set, then because of Aβ1 and AβG, we get $\exists x Gx$. If b is not empty, then it may be formed the mereological sum v of b: $\forall y(y\epsilon b \rightarrow y \sqsubseteq v) \wedge \forall y(y \sqsubseteq v \rightarrow \exists u(u\epsilon b \wedge u \circ y))$ (on the ground of $\forall z(\exists y(y\epsilon z) \rightarrow \exists x(x\sigma z)))$,

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[14] We obtain it by adding the following mereological axioms and definitions to ZF:

(*asym*)	$\forall x \forall y(x \sqsubset y \rightarrow \neg(y \sqsubset x))$	(asymmetry of part relation)
(*trans*)	$\forall x \forall y \forall z(x \sqsubset y \wedge y \sqsubset z \rightarrow x \sqsubset z)$	(transitivity of the part relation)
(*ing*)	$x \sqsubseteq y \leftrightarrow x \sqsubset y \vee x = y$	(ingrediens)
(*overlap*)	$x \circ y \leftrightarrow \exists z(z \sqsubseteq x \wedge z \sqsubseteq y)$	(overlapping)
(*dis*)	$x \wr y \leftrightarrow \neg(x \circ y)$	(disjointness)
(σ)	$x\sigma z \leftrightarrow Zz \wedge \forall y(y\epsilon z \rightarrow y \sqsubseteq x) \wedge \forall y(y \sqsubseteq x \rightarrow \exists u(u\epsilon z \wedge u \circ y))$	(the mereological sum)
(*ext*)	$\forall x \forall y \forall z(x\sigma z \wedge y\sigma z \rightarrow x = y)$	(extensionality)
(*)	$\forall z(\exists y(y\epsilon z) \rightarrow \exists x(x\sigma z))$	(there are only mereological sums of something)
(*at*)	$\forall z(Zz \rightarrow \neg\exists x(x \sqsubset z))$	(sets do not have parts)

[15] Cf. (σ) in ft. 14.
[16] Cf. (*dis*) in ft. 14.

which is the formal counterpart of *Inbegriff* of all conditioned beings. From specific axioms Aβ2–Aβ4, we get again $\exists x \mathsf{G}x$.

We start our comparison with a comment on two central collections: the *Inbegriff* of all adherences and the *Inbegriff* of all real conditioned individuals (*totum reale*). The first one is defined by AB0 and the second one is a mereological sum formed from set b described by Aβ0. Both totalities have different ontological status — the former is conceived of as a certain property (with some extension which is a distributive set) and the latter is an individual (perhaps it is meant as a complex substance). However, they are considered in frame of quite similar theories from the formal point of view. Our AB1 expresses essentially the same content as Aβ1. The notion of *being conditioned by something real* symbolized here by B has similar properties as a combination of two primitive notions from the latter reconstruction noted by predicates *is real* and *is conditioned*. Axioms Aβ2 and Aβ3 express the requirement that should be met by the property of being real, and AB2 and AB3 play the same role concerning the property of being conditioned by something real. It may be noted that Aβ2 is more general than the corresponding AB2 in this sense that in Aβ2, any mereological sum of real objects (and so also this one formed from b) is described, and in AB2, we do not speak about any collection of adherential ideas, but only about the maximal one named $\mathfrak{In}$. Finally, we compare the crucial axiom Aβ4 whith AB4. The first one is a foundation principle applied to any conditioned individual x (thus also to *totum reale*), which guarantees the reality of existential conditions disjoint with x. AB4 is "a more modest" version of Aβ4 because it speaks only about the substantial foundation of $\mathfrak{In}$ and again not about any collection of adherential ideas. Despite all these similarities, we should also note certain important difference which comes from the theories lying behind the compared reconstructions. The mereological theory of sets and individuals used by Świętorzecka (2014) is strong enough to guarantee the existence of the set of all real conditioned things, from which the mereological whole can be formed (every non-empty set has its mereological sum). Zalta's theory does not give us a corresponding principle for properties: the possibility of introducing names for properties is restricted in $Comp\lambda$, and in connection with this, we had to take in account our condition IN. Our cautious formalization may be now interpreted in the semantics designed by D. Scott for the monadic fragment of Zalta's theory.

We sketch Scott's approach following Zalta (1983, 160–162).

For any set of concrete objects $\mathcal{E}$ (possibly empty), Scott considers the set of properties $R = \{r : r =: \underline{+}\alpha, \text{ for } \alpha \subseteq 2^{\mathcal{E}}\}$. The set of all subsets of R is a domain of abstract objects and it forms with the set of $\mathcal{E}$ the set of all individuals: $D = 2^R \cup \mathcal{E}$. There are defined two functions $extr \colon R \to 2^D$ and $int \colon R \to 2^D$ which assign to every property its extension and intension according to the following conditions:

1. $o \in extr(r) \Leftrightarrow$ (i) $\exists_{e \in \mathcal{E}}(o = e \text{ and } \exists_{\alpha}(r = \underline{+}\alpha \text{ and } e \in \alpha))$
   or (ii) $\exists_{a \in 2^R}(o = a \text{ and } \exists_{\alpha}(r = +\alpha))$,
2. $o \in int(r) \Leftrightarrow$ (i) $\exists_{e \in \mathcal{E}}(o = e \text{ and } r \neq r)$ or (ii) $\exists_{a \in 2^R}(o = a \text{ and } r \in a)$.

For any given $< \mathcal{E}, R, D, extr, int >$, we consider valuation function v which takes as arguments individual terms or predicate terms and assigns them respectively elements of D and R, the constant predicate $E!$ has constant valuation $-\mathcal{E}$. The formula τt is true under valuation v iff $v(t) \in extr(v(\tau))$ and the formula $t\tau$ is true under

valuation v iff $v(t) \in int(v(\tau))$. Further truth conditions are described in a standard way. A formula φ is valid if it is true for every valuation. Scott's semantic validates the monadic fragment of Zalta theory.[17]

In frame of this semantics, we sketch a model for our axioms. We take the set of concrete objects $\mathcal{E}^{\star} = \{s\}$. In this case, the set of relations is $R^* = \{+\emptyset, -\emptyset, +\{s\}, -\{s\}\}$. The set of all objects is $D = 2^R \cup \mathcal{E}$, so here $D^* = 2^{R^*} \cup \{s\}$. We interpret our symbols Ad and B as sets: $Ad^* = \{-\{s\}\}$ and $B^* = Ad^* \cup \{+\emptyset\}$. In our case, the semantic counterpart of the term S is $-\{s\}$ ($-\{s\} \in Ad^*$), $ext(-\{s\}) = \{s\}$ and $int(-\{s\}) = \{X : X = \{-\{s\}\}$ or $\{-\{s\}\} \subseteq X\}$. Now, axioms AB1 and AB3 are true. It is also true that $\exists F BF$. Axioms AB0, AB2, and AB4 are also true just because IN is not fulfilled. It is easy to check that none of the four elements of R^* is a candidate for making our IN true. Probably, the general construction of Scott's semantics does not allow us to consider such properties as formulated in IN. But of course, that does not mean that IN causes a contradiction. Actually, if we found another semantics with models of a relatively weak fragment of AOT (required for our derivation of **ThS**) verifying AB1, AB3, and IN ˘ so, showing that the *Inbegriff of all adherences* understood as ℑn is consistent ˘ then it would be possible to strengthen our axioms AB0, AB2, and AB4 by deleting IN and constructing the argument for the main thesis without the special assumption IN. However, this issue remains open in our present discussion.

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[17]It validates more than we have in the interpreted system. For example a formula $\forall x(Fx \leftrightarrow Gx) \rightarrow \forall x(xF \leftrightarrow xG)$ is valid.

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