A New Way to Block a Dutch Book Argument

or

The Stubborn Non-probabilist

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We point out a yet unnoticed flaw in Dutch Book arguments that relates to a link between degrees of belief and betting quotients. We offer a set of precise conditions governing when a nonprobabilist is immune to the classical Dutch Book argument. We suggest that diachronic Dutch Book arguments are also affected.

Suppose you decide that your first task on a sunny Tuesday morning is to convince your friend who does not subscribe to probabilism (that is, he claims his degrees of belief need not be classical probabilities) of the error of his ways[1]. You decide to try the classical Dutch Book argument first. To your surprise you discover that your friend is not worried about the somewhat pragmatic nature of the argument, allows you to set all the stakes to 1 for convenience, and, while claiming that the set of propositions about which he holds some degree of belief is finite, he is eager to contemplate betting on virtually anything. He also considers a bet to be fair if its expected profit both for the buyer and seller is null, and even accepts the package principle, that is, believes a set of bets to be fair if each of the bets in that set is fair. Knowing all that, when telling your friend about how fair betting quotients are connected with the Kolmogorov axioms, and then about the identification of fair betting quotients with degrees of belief, you expect him to be immediately convinced. To your surprise he shakes his head in opposition, saying 'I agree that fair betting quotients are exactly those which satisfy the axioms of classical probability. Still, even when we set all stakes to 1, I don't believe that these quotients are my degrees of belief.'

'But Alan', you say, 'this is standard. We went through this. We agreed that if your degree of

* Apologies to all Readers for not having set this note in LaTeX. This because I submitted it to a journal which only accepts very brief Word documents. Now that it's been promptly rejected, I am thinking of someday expanding it into a full paper; still, introducing all notions properly and adding the philosophical context will take much work. As it stands, the note assumes familiarity with the basic formal epistemology literature. I would be very grateful for any remarks on the quality and clarity, or lack thereof. I can't believe I haven't found a paper on this, so it's probable I'm making a fool of myself. If so, then maybe my mistakes will be instructive for someone!

1 Why Tuesday? See Hájek 2008.

belief in *A* is *b(A)*, and your degree of belief in *~A* is *b(~A)*, then your betting quotient for the bet for *A* is that particular *q* for which the expression

$$b(A)\cdot(1-q)+b(\sim A)\cdot(-q),$$

that is, the expected value of the bet, equals 0. And it's a matter of mundane calculation that *q* is exactly *b(A)*. In general this means that betting quotients *are* your degrees of belief.'

'Still, look' – your friend responds – 'you're missing one thing. It's just that in my case *b(A)+b(~A)* is in general not equal to 1. My degrees of belief are such that for each proposition *A* there is a non-zero number *r_A* for which it holds that *b(A)+b(~A)=r_A*; some of those numbers may be equal to 1, but none need be. And so my betting quotient for the bet for a proposition *A* is in general *b(A)/r_A*. Can you run your argument using such quotients?'

Well, *can you*? It turns out that sometimes you can – but sometimes not. It all depends on the particulars of your friend's belief state. In what follows I will specify the formal details. Notice that the way the story is set up, our friend has granted you the assumptions needed to overcome the well known flaws of the Dutch Book argument (discussed e.g. in Vineberg 2011 and Bradley 2015). Still, it seems that even then he needs not be persuaded by the reasoning. This suggests that we have here a new problem for Dutch Book arguments.

To reflect for a moment on the nature of the issue, notice that assuming that in general *b(A)+b(~A)=*1 does not amount to assuming the probabilist thesis, that is, the problem is not that of pure petitio principii. Still, by doing so we are assuming something with which a nonprobabilist may by no means agree. We just know that by denying it, he has to hold that the additivity axiom or the 'normalization' axiom (stating that the probabilities of tautologies equals 1) is not satisfied by his degrees of belief.

We can arrive at the problem from another direction. The traditional way of looking at the Dutch Book argument for probabilism would have it imply that possessing degrees of belief which violate classical probability axioms is a mark of irrationality. This should be puzzling if we think about the particular form of the 'normalization' axiom used in the classical axiomatization of probability. If we believe tautologies to a degree different from 1, we can apparently be Dutch-booked. Surely there's a mistake here: the choice of the number 1 as the probability of tautologies is purely conventional. The number 2 (say) would do just as well. But if we are careful about setting the betting quotients the way with which our nonprobabilistic friend would agree, then if his degree of belief in countertautologies is 0 and his degree of belief in tautologies is 2, his betting quotient for tautologies is 1, exactly the same as in the classical case.

Let us continue towards the theorem specifying the class of cases in which a nonprobabilist is not Dutch-bookable. A *belief space* is a tuple *(W,Prop,b)*, where *W* is a nonempty finite set, *Prop* is a Boolean algebra of subsets of *W* ('propositions'), and *b* is a function from *Prop* to the real numbers, called the *belief function*. We will write *T* for the tautological proposition *W* and *F* for the countertautological proposition, that is for the empty set. We will say that a function *b* from *Prop* to the real numbers is a *classical probability function* iff it satisfies the following three axioms:

(1) *b(T)*=1 (the normalization axiom);

(2) for any *A* in *Prop* *b(A)* ≥ 0;

(3) for any *A* and *B* whose intersection is empty *b(A v B)=b(A)+b(B)*.

Notice that as mentioned above, (1) and (3) imply that for any *A* in *Prop* *b(A)+b(~A)*=1, that is, they imply the assumption we need for the 'classical' connection between degrees of belief and betting quotients. In our case we wish to play by our friend's rules, that is, for any *A*, we want to set the betting quotient for *A* to *b(A)/(b(A)+b(~A))*: this way we will make sure that according to our friend the expected value of the bet for *A* is 0 (remember that all the stakes are set to 1; nothing important in the argument depends on this). We will say a belief space *B=(W,Prop,b) induces a betting quotient function q* from *Prop* to the reals (which may turn out to be a classical probability function, but may not) if

(B1) for any *A* in *Prop* *b(A)+b(~A)* is not equal to 0;

(B2) for any *A* in *Prop* *q(A)=b(A)/(b(A)+b(~A))*.

It follows that if a belief space induces a betting quotient function, that is, if (B1) holds, then that function is unique. For a belief space of a probabilist, the induced betting quotient function q just the original degree of belief function b. We can state the classical Dutch Book theorem (see for example Kemeny 1955 and Lehman 1955) as follows: a belief space *(B,W,b)* is not Dutch-bookable if and only if the betting quotient function induced by it is a classical probability function. The question is now: which belief spaces induce such betting quotient functions? The answer lies in the following simple theorem. In its formulation we refer by '*At*' to the set of atomic propositions of a given belief space *(B,W,b)*; they are minimally small nonempty subsets of *W*, which exist since *W* is finite.

Theorem. Suppose a belief space *(B,W,b)* induces a betting quotient function *q*. *q* is a classical probability function if and only if the following conditions are true:

(Q1) *b(F)*=0;

(Q2) for any *A* in *Prop*, if *b(A)* and *b(~A)* are both not equal to 0, then they have the same sign;

(Q3) for any nonempty *A* in *Prop* not equal to *T*, *b(A) = Σ_{B ε At: B ε A} b(B)*.

Proof sketch. (1) is equivalent to (Q1). (2) is equivalent to (Q2). To show that (Q1) together with (Q3) imply (3), in the context of the assumption of *(B,W,b)* inducing a betting quotient function, we proceed by cases. Case 1: both *A* and *B* are equal to *F*. (3) follows from from (Q1). Case 2: *A* is equal to *T* and *B* is equal to *F*. (3) follows from (Q1) and the assumption that *(B,W,b)* induces a betting quotient function, since from that it follows that *b(A)* is different from 0. Case 3: None of *{A,B}* is equal to any of *{T,F}*. In that case using (Q3) we can quickly show that *q(A)+q(B) = q(A v B)*. The last thing to notice is that (3) implies (Q3), since *q(A) = q(UB | B ε At: B ε A}* (where *U* is the set-theoretical union), to which (3) is applicable since our structures are finite. □

Notice that (Q3), saying that the degree of belief in a non-tautological proposition is equal to the sum of the degrees of belief in atomic propositions which imply it, is strictly weaker than additivity of *b*. If we agree with the convention that the sum over an empty set is 0, we could omit (Q1) and the word 'nonempty' from (Q3).

It follows immediately from the above Theorem that if your belief space is a measure space with the total measure being different from 1 – that is, not a *probability* space – you will not be Dutch-bookable. But there are different cases. For a non-Dutch Bookable belief space which violates all three classical probability axioms, take *B=(W,Prop,b)* with *W={1,2}*, *Prop = {T,F,{1},{2}}*, and *b* defined as follows: *b(F)*=0, *b({1})*=-1/3=*b({2})*, *b(T)*=-1. Notice that the induced betting quotient function *q* is a classical probability function: *q(F)*=0, *q({1})*=1/2=*q({2})*, *q(T)*=1, so the belief space is not Dutch-bookable. By simple transformations of *B* one will quickly find examples of non-Dutch-bookable belief spaces which violate other combinations of the classical probability axioms.[2]

All this should not suggest that a nonprobabilist is rational; it shows that if we proceed with the Dutch Book argument in the way that is fair to our nonprobabilistic interlocutor, that is, if we do not impute our way of calculating the expected value of a bet on him, it may turn out that he is actually not susceptible to a Dutch Book. But this does not mean that a nonprobabilist is always safe, and the above Theorem provides a recipe for presenting a belief space which will be Dutch Bookable. Consider for example *B'=(W',Prop',b')* with *W'={1,2,3}*, *Prop'* being the power set of *W'* and *b'* defined as follows: *b'(F)*=0, *b'(X)*=1/3 if *X* is a singleton, *b'(Y)*=1/2 if *Y* is a doubleton, *b'(T)*=1. We see that *b'* violates additivity. Notice that if *q'* is the induced betting quotient function, *q'({1})=q'({2})*=2/5, but *q'({1,2})*=3/5. This means that *q'*, violating additivity, is not a classical

2 Looking at the Theorem again, it would seem that to arrive at a non-Dutch-bookable belief space one has to take a classical probability space, multiply all degrees of belief apart from the one in T by the same non-zero real number (possibly negative), and then put in an arbitrary non-zero degree of belief in T. If this is correct, then I owe this observation to Joanna Luc.

probability function, so by the above Theorem and the classical results mentioned earlier *B′* is Dutch-bookable.

There are of course belief spaces *(W,Prop,b)* which do not induce a betting quotient function because for some proposition *A* *b(A)+b(~A)* equals 0. In that case we could define a partial betting quotient function in the obvious way, which would lead us to some examples of Dutch Books without the need to figure out what the subject's betting quotient for *A* should be. A different direction would be to propose in such cases a different method for calculating betting quotients; one option is to bite the bullet and say that such a subject should accept any price for bets for *A* and for ~*A* (since for any price he will calculate the expected value of the bet as 0), which will of course make him instantly Dutch-bookable.

A similar line of argument can be offered to the effect that a nonprobabilist may avoid the classical diachronic Dutch Book argument for conditionalization (succintly presented e.g. in Briggs 2009).[3] That is to say, the particular way of arguing for conditionalization which goes back to D. Lewis and P. Teller sometimes, but not always, fails. We leave the details, which are too lengthy to include in a short paper like this one, to the Reader. We have not ruled out that other diachronic Dutch Book arguments for conditionalization may exist which would be fully nonprobabilist-proof.

In conclusion: even if we forget about all the problems of Dutch Book arguments which are usually mentioned in the formal epistemology literature, it turns out that another one lurks in the basic step of connecting degrees of belief with betting quotients. I have specified the class of belief spaces which, while nonprobabilistic, is not susceptible to a Dutch Book if we try to avoid that problem. We can expect a similar situation with arguments based on scoring rules – but this is a topic for a longer paper.

References

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3 Why would we even want to convince a nonprobabilist that he should update by conditionalization? I don't have an answer to this. Maybe we wouldn't. Even so, it may be interesting to realise that some additional assumptions are needed for the classical arguments. Let me just point out that Skyrms 1987 explicitly assumes for the proof of the Converse Diachronic Dutch Book theorem that the agent is probabilistic.

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