Hindawi
Complexity
Volume 2018, Article ID 9430637, 8 pages
https://doi.org/10.1155/2018/9430637

Research Article

Hyperchaos in a Conservative System with Nonhyperbolic Fixed Points

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Received 8 November 2017; Revised 21 February 2018; Accepted 12 March 2018; Published 22 April 2018

Academic Editor: José Ángel Acosta

Chaotic dynamics exists in many natural systems, such as weather and climate, and there are many applications in different disciplines. However, there are few research results about chaotic conservative systems especially the smooth hyperchaotic conservative system in both theory and application. This paper proposes a five-dimensional (5D) smooth autonomous hyperchaotic system with nonhyperbolic fixed points. Although the proposed system includes four linear terms and four quadratic terms, the new system shows complicated dynamics which has been proven by the theoretical analysis. Several notable properties related to conservative systems and the existence of perpetual points are investigated for the proposed system. Moreover, its conservative hyperchaotic behavior is illustrated by numerical techniques including phase portraits and Lyapunov exponents.

1. Introduction

As a research hotspot in the field of nonlinear science over the last five decades, chaos theory has achieved great development since the Ueda attractor [1], the Lorenz attractor [2], and Li-Yoke chaos [3] were discovered. So far, chaos theory has been successfully applied in many fields, such as electronic engineering [4], computer science [5], communication systems [6, 7], complex networks [8], chemical engineering [9], and economic models [10]. In the recent decade, hundreds of physical chaotic models [11–14] and artificial chaotic systems [15–19] have been investigated in theory and by numerical simulations due to the potential applications of chaotic system in various chaos-based technologies [20, 21]. By now, numerous dissipative-chaos-based encryption algorithms have been developed to ensure the safety of information, but these algorithms are not strong enough because the dissipative chaotic attractors can be reconstructed by delay embedding method based on the sampled data. Besides the general properties of chaos such as ergodicity, aperiodic, uncorrelated, broadband, and white-noise-like, the conservative chaos has no chaotic attractor, which means that the conservative chaos when applied in information security has distinct advantages over dissipative chaos.

It is well known that the nonlinear dynamical systems can produce various dynamical behaviors, such as periodic motion, quasiperiodic motion, chaos, and hyperchaos. Generally, hyperchaos is defined as its dynamical behavior with at least two positive Lyapunov exponents (LEs), and the minimal dimension for an autonomous continuous hyperchaotic system is four. In comparison with chaos, hyperchaos, especially conservative hyperchaos, is preferable for those applications that require the chaotic systems showing complicated dynamics, such as network security and data encryption. The first hyperchaotic system was proposed in 1979 by Rössler [22]. Since then, various hyperchaotic systems have been found [23–26], but almost all of these hyperchaotic systems are dissipative and it is difficult to find odd-dimensional systems that can generate conservative hyperchaotic flows. In 1994, Sprott found a 3D nonequilibrium non-Hamiltonian system with conservative chaos, lately known as the Sprott A system [15], which is described as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= -y^2 + 1,\end{aligned} \quad (1)$$

where x, y, and z are the system states. It has been proven that system (1) is a special case of the Nosé-Hoover system [27]. Moreover, there exist coexisting quasiperiodic and conservative chaotic flows under different initial conditions. In view of the fact that these systems with conservative hyperchaos are rare, thus, it is very interesting to find hyperchaotic systems with conservative flows.

Through sorting out the published papers about chaotic systems of recent years, it is found that hidden attractors and perpetual points become two important new research topics. The properties of hidden attractor are different from that of self-excited chaotic attractor [28, 29]. It has been shown that the dynamical systems having hidden attractors include these systems without equilibrium, with no unstable equilibrium, with one stable equilibrium, and with lines, curves, planes, and surfaces of equilibria [30–35]. Recently, perpetual points were introduced as another structural feature of nonlinear systems [36]. In some cases, the perpetual points are useful to locate hidden attractors and to find coexisting attractors in multistable systems [36, 37].

The main contribution of this paper is that a rare five-dimensional (5D) autonomous dynamical system is proposed and investigated. In comparison with the known chaotic systems, the system has the following four characteristics:

(i) The system with nonzero initial values has two non-hyperbolic fixed points.

(ii) The system is conservative, which can be theoretically verified by the existence of a conserved quantity, but cannot be confirmed by the trace of its Jacobian matrix, which means that the flows generated by the system are compressible.

(iii) Numerically, the symmetric LEs spectrum shows that the system has conservative chaos.

(iv) Hyperchaotic motion can be found under specific conditions in this conservative system.

The rest of this paper is organized as follows. In Section 2, we introduce a new 5D dynamical system and analyze its basic dynamics. In Section 3, the existence of perpetual point of the new system is investigated. Section 4 illustrates the dynamical behaviors of the proposed system with the help of LEs and phase portraits. The conclusion is presented in the last section.

2. System Model

Consider the following 5D continuous dynamical system:

$$\begin{aligned}\dot{x} &= ay + w^2 \\ \dot{y} &= -ax \\ \dot{z} &= xv \\ \dot{v} &= bw - xz \\ \dot{w} &= -bv - xw,\end{aligned} \quad (2)$$

where x, y, z, v, and w are the system variables and a and $b \in \mathbb{R}^+$ are the system's constant parameters. Obviously, system (2) does not meet the requirements for classic Hamiltonian systems. Although system (2) is not Hamiltonian, it does have a conserved quantity, which will be discussed in the following sections.

2.1. Invariance and Time-Reversibility. Many of chaotic trajectories are symmetric; if $t \to +\infty$, it is easy to get the invariance of the system (2) under the coordinate transformation $(x, y, z, v, w) \to (x, y, -z, -v, -w)$, which persists for all values of a and b. However, the invariance of the solution of system (2) in the time-reversible direction can also be achieved under the transformation $(t, x, y, z, v, w) \to (-t, -x, y, z, v, -w)$. Generally, a dynamical system is reversible if there is an involution in phase space which reverses the direction of time. It is a very common thing to find the time-reversibility for conservative systems, but there are exceptions [38, 39].

2.2. Conservation

Definition 1. Dynamical systems, whose Hamiltonian (i.e., conserved quantity) does not vary in time, are called conservative systems; otherwise they are called dissipative systems [40].

Remark 2. There is difference between the conservative system and the conservative motion. Generally, the conservative motion is most often encountered in conservative systems, especially in Hamiltonian systems, but it can be also found in nonconservative systems, such as the Sprott A system; in other words, the conservative motion exists not only in conservative systems but also in nonconservative systems, which can be verified by numerical techniques, such as LE spectra.

By the definition, if we can find a Hamiltonian for system (2) and prove the time derivative of the conserved quantity equals zero, system (2) is conservative.

Theorem 3. *One of the Hamiltonians of system (2) is* $H(x, y, z, v, w) = (1/2)(x^2 + y^2 + z^2 + v^2 + w^2)$, *which satisfies* $\dot{H}(x, y, z, v, w) = 0$.

Proof. The linear first-order PDE of system (2) can be expressed as [41, 42]

$$\dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} + \dot{z}\frac{\partial H}{\partial z} + \dot{v}\frac{\partial H}{\partial v} + \dot{w}\frac{\partial H}{\partial w} = 0. \quad (3)$$

In order to get the general solutions of (3), we choose (4) as the associated system.

$$\frac{dx}{ay + w^2} = \frac{dy}{-ax} = \frac{dz}{xv} = \frac{dv}{bw - xz} = \frac{dw}{-bv - xw}. \quad (4)$$

According to (4), for simplicity, we choose a particular solution of the following smooth positive-definite standard quadratic form:

$$H(x, y, z, v, w) = \frac{1}{2}\left(x^2 + y^2 + z^2 + v^2 + w^2\right). \quad (5)$$

Based on the Theorem 3, we can conclude that system (2) is conservative. □

In addition, we can also discuss the time evolution of volumes under a flow described by Liouville [43].

Lemma 4. *It is assumed that system (2) can generate a flow $\phi_t(x, y, z, v, w, t)$. Let D_{t_0} denote a domain in $\mathbb{R}^5$ and D_t denote the evolution of D_{t_0} under the flow $\phi_t(x, y, z, v, w, t)$. If $V(t)$ represents the volume of D_t, system (2) is dissipative if $\nabla \cdot V < 0$ and conservative if $\nabla \cdot V = 0$.*

According to Lemma 4, we might determine whether system (2) is dissipative or conservative by $\nabla \cdot V$. We note that the divergence of flow of system (2) is

$$\nabla \cdot V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{v}}{\partial v} + \frac{\partial \dot{w}}{\partial w} = -x, \quad (6)$$

which shows that it is difficult to directly discern whether system (2) is dissipative or conservative since $\nabla \cdot V$ depends on the system variable x.

2.3. Fixed Points and Stability. In order to determine the fixed points (FPs) and stability of system (2), it is necessary to calculate $\dot{x} = 0$, $\dot{y} = 0$, $\dot{z} = 0$, $\dot{v} = 0$, and $\dot{w} = 0$; then one obtains

$$\begin{aligned} ay + w^2 &= 0 \\ -ax &= 0 \\ xv &= 0 \\ bw - xz &= 0 \\ -bv - xw &= 0. \end{aligned} \quad (7)$$

Solving (7) yields the FPs of system (2) as follows: $(x_e, y_e, z_e, v_e, w_e) = (0, 0, r, 0, 0)$, where $r \in \mathbb{R}$ is a parameter and denotes any point of the z-axis. Hence, the z-axis is the line of FPs of system (2).

The Jacobian matrix of system (2) is

$$J(\cdot) = \begin{pmatrix} 0 & a & 0 & 0 & 2w \\ -a & 0 & 0 & 0 & 0 \\ v & 0 & 0 & x & 0 \\ -z & 0 & -x & 0 & b \\ -w & 0 & 0 & -b & -x \end{pmatrix}. \quad (8)$$

For any FP of system (2), we can obtain the corresponding characteristic equation of (8) by calculating $f(\lambda) = |\lambda I - J| = 0$:

$$f(\lambda) = |\lambda I - J| = \lambda\left(\lambda^2 + a^2\right)\left(\lambda^2 + b^2\right) = 0. \quad (9)$$

Obviously, all FPs of system (2) are center FPs (nonhyperbolic FPs) since there exist one real root $\lambda_1 = 0$ and two pairs of purely imaginary roots $\lambda_{2,3} = \pm ai$ and $\lambda_{4,5} = \pm bi$, which can effectively exhibit that there are no asymptotically stable equilibria or limit cycles in conservative systems in phase space.

Theorem 5. *For system (2), there are two center FPs under any initial conditions $(x_0, y_0, z_0, v_0, w_0)$ with $H_0 \neq 0$.*

Proof. If the initial value of system (2) is $(x_0, y_0, z_0, v_0, w_0)$, we obtain

$$H_0 = \frac{1}{2}\left(x_0^2 + y_0^2 + z_0^2 + v_0^2 + w_0^2\right), \quad (10)$$

which means that $H(x_e, y_e, z_e, v_e, w_e) = H_0$. Considering $(x_e, y_e, z_e, v_e, w_e) = (0, 0, r, 0, 0)$, we have

$$r = \pm\sqrt{x_0^2 + y_0^2 + z_0^2 + v_0^2 + w_0^2}, \quad (11)$$

which shows that system (2) has two center FPs under any initial conditions except $H_0 = 0$. Actually, these two center FPs are the intersections of z-axis and the hypersphere $H(x, y, z, v, w) = (1/2)(x^2 + y^2 + z^2 + v^2 + w^2) = (1/2)r^2$ in phase space. □

Due to the existence of nonhyperbolic FPs, system (2) is different from most published chaotic systems with hyperbolic FP, such as the Lorenz and Lorenz-like systems, the Rössler system, and Chua's circuit [2, 16, 44, 45]. Chaotic systems with nonhyperbolic FPs are rare, but they can be found not only in conservative systems but also in dissipative systems.

In 1994, Sprott proposed the following three dissipative chaotic systems [15]:

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= x - y \\ \dot{z} &= 1 - x^2, \end{aligned} \quad (12)$$

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + z \\ \dot{z} &= xz + 3y^2, \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= x^2 - y \\ \dot{z} &= 1 - 4x. \end{aligned} \quad (14)$$

Obviously, there are two FPs in (12) and one FP in (13) and (14). There exist zero real parts in the eigenvalues of these FPs; thus these three systems belong to the kind of chaotic systems with nonhyperbolic FPs. In 2005, Wei et al. presented the following chaotic jerk system with one single nonhyperbolic FP [46, 47]:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= z\\ \dot{z} &= h(x, y, z),\end{aligned} \tag{15}$$

where $h(x, y, z) = a + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz$ $(a, a_i \in \mathbb{R},\ i = 1, 2, \ldots, 9)$ is the most general quadratic function. If $a_4 = 0$ or $a_4 \neq 0$, $a = a_1^2/4a_4$, system (15) has only one nonhyperbolic FP. So far, these chaotic systems with nonhyperbolic FP(s) are unusual relative to these chaotic system with hyperbolic FP(s). For a given set of initial values and system parameters, system (2) has two nonhyperbolic FPs, and moreover, the hyperchaotic motion can also be found in (2), which will be verified in Section 4.

3. Analysis of Perpetual Point

Perpetual point (PP) as a new research topic in the field of nonlinear dynamics has aroused the researchers' interest. Nazarimehr et al. gave the category of flows from the viewpoint of FP and PP, which includes flows with FPs and PPs, with FPs but without any PPs, without any FPs but with PPs, and without any FPs and PPs [48].

Generally, FPs are constant discrete points for ordinary differential equations (ODEs), which are always discussed with the corresponding Jacobian matrix when we study the dynamical behaviors of nonlinear systems. Now, we consider the following generalized nonlinear autonomous dynamical system:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n)\\ \dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, x_2, \ldots, x_n),\end{aligned} \tag{16}$$

where x_i are system variables and $f_i(\cdot)$ are smooth nonlinear functions, with $i = 1, 2, \ldots, n$. By calculating the FPs of the system (16), one obtains

$$E_{\mathrm{FP}} = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid f_i(x_1, x_2, \ldots, x_n) = 0\right\}. \tag{17}$$

Therefore, the point(s) with zero velocity of these state variables of the system (16) can be obtained. Similarly, we can calculate the zero acceleration of these state variables of system (2) as follows:

$$E_{\mathrm{AP}} = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid f_i(\cdot) \sum_{j=1}^{n} \frac{\partial f_i(\cdot)}{\partial x_j} = 0\right\}, \tag{18}$$

where $i, j = 1, 2, \ldots, n$ and the subscript "AP" denotes the acceleration of these state variables at any point. Through the comparison of (17) and (18), it is found that the solutions to (17) must be the solutions to (18), but there may exist other solutions to (18). PPs are these points where all the acceleration of these state variables of the system (16) is zero but the velocities are not [36], which can be denoted by

$$E_{\mathrm{PP}} = \left(E_{\mathrm{AP}} - E_{\mathrm{AP}} \cap E_{\mathrm{FP}}\right). \tag{19}$$

Note that FPs and PPs are preserved under the transformations of the invariance and time-reversibility of system (2) because these transformations are linear [49]. At first, PP can be used to distinguish whether a dynamical system is dissipative or conservative [36] and locate hidden attractors [37]. Based on the recent work, Jafari et al. pointed out the limitation of PP for confirming conservation in dynamical systems, which is supported by the Sprott A system [50], and the insufficiency of PP for locating hidden attractors in dynamical systems, which is also supported by three examples [51]. Therefore, the existence of PPs to distinguish dissipative systems from conservative systems and locate hidden attractors is not applicable for all dynamical systems.

According to (7), the FPs of system (2) are

$$E_{\mathrm{PP}} = \left\{(x, y, z, v, w) \in \mathbb{R}^5 \mid x = y = v = w = 0,\ z = r\right\}, \tag{20}$$

where $r \in \mathbb{R}$. Based on (18), we can get

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\\ \ddot{v}\\ \ddot{w}\end{pmatrix} = \begin{pmatrix}-a^2x - 2bvw - 2xw^2\\ -a^2y - aw^2\\ bxw + ayv - x^2z + vw^2\\ -b^2v - bxw - ayz - x^2v - zw^2\\ -b^2w + bx(z + v) - ayw + x^2w - w^3\end{pmatrix} = 0. \tag{21}$$

Solving (21) yields

$$E_{\mathrm{AP}} = \left\{(x, y, z, v, w) \in \mathbb{R}^5 \mid x = y = v = w = 0,\ z = m\right\}, \tag{22}$$

where parameter $m \in \mathbb{R}$. Obviously, $E_{\mathrm{FP}} = E_{\mathrm{AP}}$. Then, we conclude

$$E_{\mathrm{PP}} = \left(E_{\mathrm{AP}} - E_{\mathrm{AP}} \cap E_{\mathrm{FP}}\right) = \emptyset, \tag{23}$$

which shows that system (2) belongs to the class of flows with FPs but without any PPs. In this case, the conclusion obtained by Prasad [36] is valid to determine the conservation of system (2) due to the inexistence of PP; that is, $E_{\mathrm{PP}} = \emptyset$.

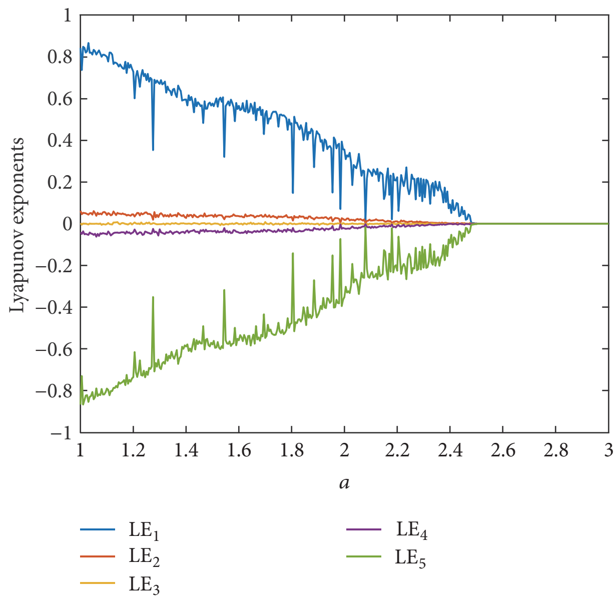

Figure 1: The dependence of finite-time local LEs of the system (2) on parameter a, with $b = 1$.

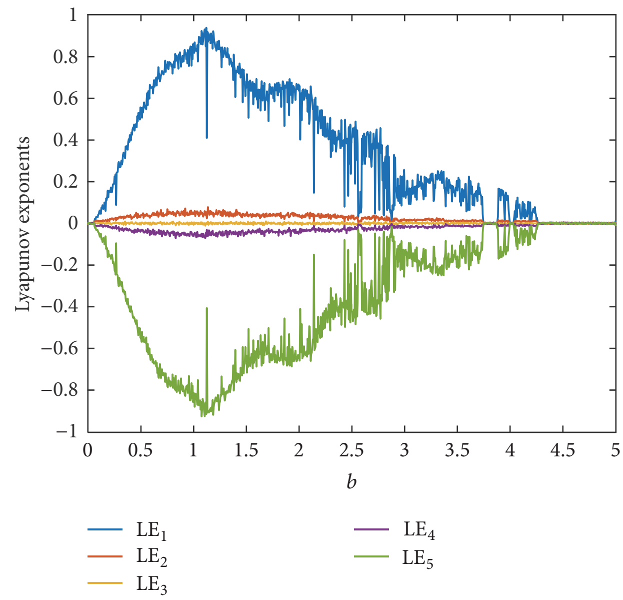

Figure 2: The dependence of finite-time local LEs of the system (2) on parameter b, with $a = 1$.

4. Dynamical Behaviors of the Conservative Hyperchaotic System

In this section, some numerical simulations for system (2) are presented to investigate the dynamical behaviors of system (2). Generally, to analyze the dynamics of nonlinear systems, there are many methods such as Lyapunov exponent spectrum and phase portrait, which can be used to analyze system (2). Based on the fourth-order Runge-Kutta method, we keep the absolute and relative error 10^{-9} and adopt variable step to solve system (2) in MATLAB.

The dynamics of classical chaotic systems include periodic motion, quasiperiodic motion, and chaos and hyperchaos, which can be confirmed by the calculation of Lyapunov exponents (LEs) [52]. If an autonomous dynamical system is chaotic, its dimension must be equal to or more than three and at least one positive LE is greater than zero. If there is one positive LE, the system is chaotic; while there are two and more positive LEs, the system is hyperchaotic [53]. For a dynamical system with dissipative flows, the sum of all LEs must be less than zero, while for a dynamical system with conservative flows, the sum of all LEs equals zero [53]. In Sections 2 and 3, the theoretical analysis shows that system (2) is conservative, but system (2) with conservative hyperchaos needs to be further verified by the sum of its LEs.

Remark 6. Generally, the dimension for integer-order dissipative dynamical systems with hyperchaos is greater than or equal to FOUR, but the dimension for integer-order conservative systems with hyperchaos is at least FIVE because there exists linear dependence among of all system variables which are reflected in the conserved quantity $H(\cdot)$ = constant.

If the system parameter $b = 1$ and the initial values $(x_0, y_0, z_0, v_0, w_0) = (1, 1, 1, 1, 1)$, we can get the LE spectra of system (2) by varying a, as shown in Figure 1. It can be observed that the dynamics of system (2) switches among hyperchaos, chaos, and quasiperiodic motion with increasing parameter a, and system (2) has a rather wide range of parameter a in which the generated flows are chaotic. When $a \in [1, 2.505]$, the largest LE is greater than zero and the second largest LE is greater than or equal to zero, which means system (2) is chaotic or hyperchaotic. When $a \in (2.505, 3]$, all LEs equal zero, which implies system (2) undergoes a quasiperiodic motion.

Similarly, if the system parameter $a = 1$ and the initial values $(x_0, y_0, z_0, v_0, w_0) = (1, 1, 1, 1, 1)$, Figure 2 illustrates the dependence of LEs of system (2) on parameter b. It is found that the curves reflect the change of dynamics between hyperchaos and quasiperiodic motion with increasing parameter b. When $b \in [0.055, 3.75] \cup [3.825, 4.265]$, the largest LE and the second largest LE are greater than zero, which means system (2) is hyperchaotic. When $b \in [0, 0.055) \cup (3.75, 3.875) \cup (4.265, 5]$, all LEs equal zero, which implies system (2) undergoes a quasiperiodic motion.

The LE spectra in Figures 1 and 2 are finite-time local LE spectra [54–56], which are obtained by the Wolf algorithm in MATLAB [52], with the time step of 0.1 second and the runtime of 1000 seconds. Note that, for transient chaos, the local finite-time LEs may be positive for a very long time, but the final values of LEs will be negative or zero.

Moreover, from Figures 1 and 2 we can see that the nonzero LEs of system (2) are symmetric to zero LE, which can account for the conservative motion of the system (2) since the sum of all LEs is zero. Owing to the LEs $(LE_1, LE_2, LE_3, LE_4, LE_5) = (0.836, 0.045, 0, -0.045, -0.836)$ when parameters $a = b = 1$ and the initial conditions $(x_0, y_0, z_0, v_0, w_0) = (1, 1, 1, 1, 1)$, the flows generated from system (2) are hyperchaotic. The plots of the flow starting at $(x_0, y_0, z_0, v_0, w_0)$ are shown in Figure 3.

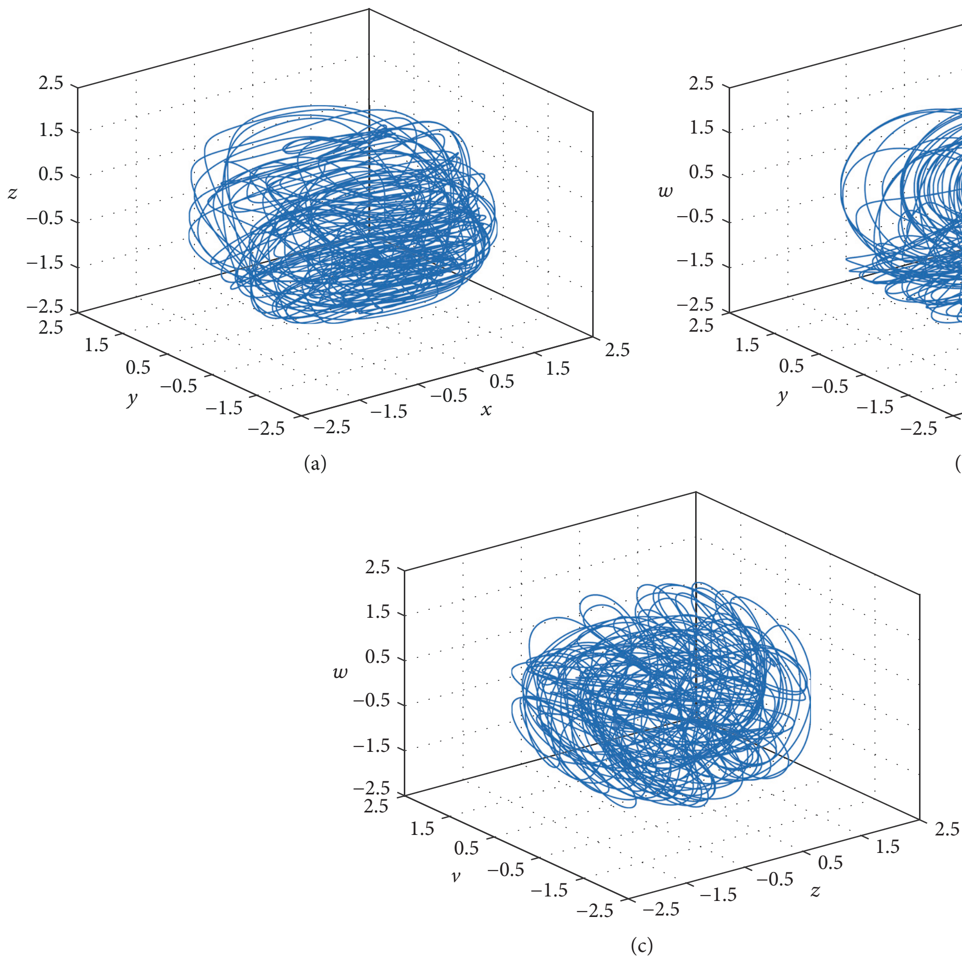

Figure 3: Phase portraits of the system (2) with parameters $a = 1$, $b = 1$, and the initial conditions $(x_0, y_0, z_0, v_0, w_0) = (1, 1, 1, 1, 1)$: (a) $x - y - z$ space; (b) $x - y - w$ space; (c) $z - v - w$ space.

Note that system (2) is conservative; as shown by the existence of the invariant $H(\cdot)$, different initial conditions may lead to different dynamical behaviors. Therefore, system (2) is a multistable system. Visually, the hyperchaotic motion in Figure 3 forms a strange attractor, but in fact, there is no chaotic attractor in conservative systems [57]. Since the focus of this paper is investigating the hyperchaotic behavior of system (2), other dynamical behaviors including chaos and quasiperiodic motion are not studied.

Remark 7. A 5D integer-order dissipative dynamical system has strange attractors with fractal dimension between two and five, while a 5D integer-order conservative dynamical system fills a 5D volume, and therefore, the fractal dimension of system (2) is five. Generally, the fractal dimension can be expressed by the so-called Kaplan–Yorke dimension D_{KY}, which is always substituted by Lyapunov dimension and is defined as

$$D_{\mathrm{KY}} = D_L = j + \frac{\sum_{i=1}^{j} \mathrm{LE}_i}{\left|\mathrm{LE}_{j+1}\right|}, \qquad (24)$$

where j is the maximum value of k such that $\xi_k = \mathrm{LE}_1 + \cdots + \mathrm{LE}_k \geq 0$. Obviously, the fractal dimension of system (2) is five according to (24), which means system (2) is conservative.

5. Conclusion

This paper proposed a new conservative system with two nonhyperbolic FPs. The conservation of the system has been verified theoretically, and the numerical results in MATLAB also showed there exist conservative hyperchaotic flows in the proposed system. Compared with other chaotic systems, the dynamics of the proposed hyperchaotic system are messier and more complex; hence the proposed hyperchaotic system may have a good potential application value in the field of information technology such as secure communication and encryption.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is partly supported by the National Natural Science Foundation of China (Grant nos. 61773282, 61573199) and South African National Research Foundation (no. 112142).

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