Logic and Sense

Urszula Wybraniec-Skardowska

Cardinal Wyszyński University in Warsaw

In the paper, original formal-logical conception of syntactic and semantic: intensional and extensional senses of expressions of any language *L* is outlined. Syntax and bi-level intensional and extensional semantics of language *L* are characterized categorically: in the spirit of some Husserl's ideas of pure grammar, Leśniewski-Ajukiewicz's theory syntactic/semantic categories and in accordance with Frege's ontological canons, Bocheński's famous motto—*syntax mirrors ontology* and some ideas of Suszko: *language should be a linguistic scheme of ontological reality and simultaneously a tool of its cognition*. In the logical conception of language *L*, its expressions should satisfy some general conditions of language adequacy. The adequacy ensures their unambiguous syntactic and semantic senses and mutual, syntactic, and semantic compatibility, correspondence guaranteed by the acceptance of a postulate of categorial compatibility syntactic and semantic (extensional and intensional) categories of expressions of *L*. From this postulate, three principles of compositionality follow: one syntactic and two semantic already known to Frege. They are treated as conditions of homomorphism partial algebra of *L* into algebraic models of *L*: syntactic, intensional, and extensional. In the paper, they are applied to some expressions with quantifiers. Language adequacy connected with the logical senses described in the logical conception of language *L* is, of course, an idealization, but only expressions with high degrees of precision of their senses, after due justification, may become theorems of science.

Keywords: logic of language, categorial language, syntactic and semantic senses, intensional semantics, meaning, constituent of knowledge, extensional semantics, ontological object, denotation, categorization, syntactic and semantic compatibility, algebraic models, truth, structural compatibility, compositionality

1. Introduction

What are the competences of logic in response to the questions about the notion of sense which are so often and difficult to answer? The questions about the notion of sense are somehow difficult because of the possibility of different ways of forming them.

It happens so because the word "sense" has many meanings and it appeals to us in many ways. On the base of philosophy (or/and theology) for centuries, we have been trying to grasp and understand what is the sense of our lives, the sense of existence, the sense of our action and endeavor, and what is the sense of the world in general. For discovering its rational justifications, logical knowledge is needed. Is logic sufficient here? Of course it is not. It cannot justify the opinion about the sense of life and our activity. From the point of view of philosophy, there are various visions and many theories regarding the sense of the world, the sense of life etc.

Urszula Wybraniec-Skardowska, full professor, Ph.D., DSc., Department of Philosophy, Cardinal Wyszyński University in Warsaw, Poland; main research fields: Logic, Philosophy and Logic of Language, Information Sciences, Mathematics.

In the above mentioned philosophical meaning, the word "sense" was used as a certain property of extra-linguistic objects, when it was said that something has or has not sense in referring to such objects. This meaning of the word "sense" must be clearly distinguished from the logical, semiotic one. It is transferring from the basic, semiotic meaning of this word, the meaning referring to linguistic objects: verbal words, expressions, texts and—generally—to conventional signs. Both the semiotic usage of the word "sense" and the non-semiotic one are not homogenous. Therefore, one can speak of many notions of sense.

In this paper, we would like to characterize and formalize various notions of semiotic sense; from the viewpoint of logic, only such notions of sense can be of interest to us. The contemporary logic, logic of language (logical semiotics) can define the semiotic sense, *logical sense* strictly with regards to some general aspects of developing of cognition of the world and, at the same time, contributing to an explication of one of the most important traditional philosophical problems: *Language adequacy of our knowledge in relation to cognition of reality*—briefly: *language adequacy*. It is connected with the mutual relations between the three elements of the triad (see Diagram 1):
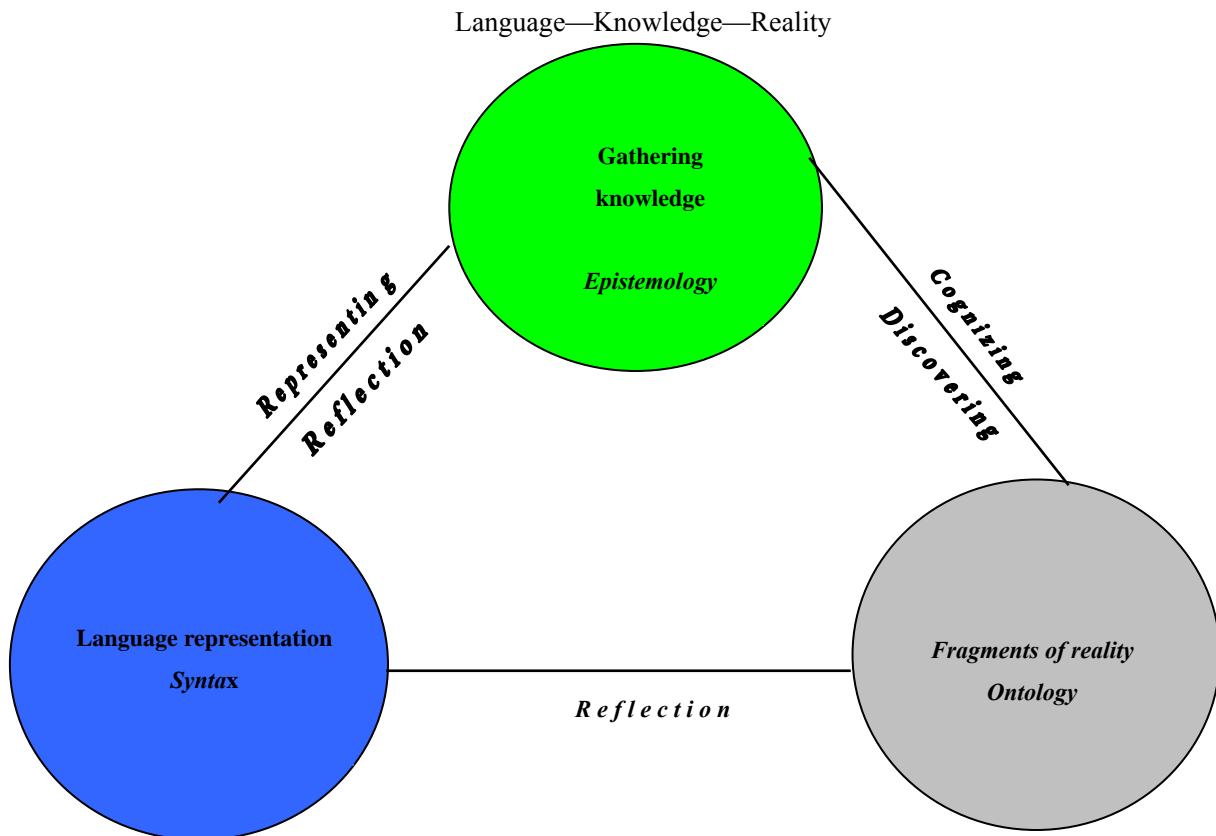

Diagram 1. Language adequacy of knowledge in relation to cognition of reality.

Language serves for representation of human knowledge gained in the process of cognition of reality. It is simultaneously a means of description of cognizing reality and a tool of communication. Operating language by means of logic and thinking enables to transform and enrich the knowledge to know or discover the world better. Language should also be a reflection of some defined fragments of reality and simultaneously a reflection of gathered and enlarging knowledge about it. As we see, language and its syntax are related both with ontology, thus with all that exists, and with epistemology dealing with cognition of the world.

For any language *L*, the adequacy could be achieved first of all if some general conditions of logical meaningfulness of the language *L* are satisfied. They are immediately connected with the logical sense of its expressions.

2. Three Kinds of Sense of Expressions of the Language *L*

In logic, we can distinguish three kinds of sense of expressions of the language *L* (see [62] and [68]):

• *syntactic sense*, when expressions of *L* are well-formed; it is defined in syntax of *L* and, in accordance with Carnap's distinction [21]: *intension-extension* or Frege's differentiation [23]: *Sinn-Bedeutung*, two kinds of *semantic sense*,

• *intensional sense*, when expressions of *L* have a meaning, *intension*; it is defined in intensional semantics of *L*,

• *extensional sense*, when expressions of *L* have a denotation, *extension*; it is defined in extensional semantics of *L*.

The syntactic and semantic notions of sense must be differentiated and explicated. This is possible through such a conceptualization of these notions that will lead to formal-logical theory of syntax and semantics of language *L*, which specifies and describes these notions.

3. Main Ideas of Formalization of Language *L*

In the paper, formal-logical considerations relate to syntax and bi-level intensional and extensional semantics of language *L* characterized *categorially*:

• in the spirit of some ideas of Husserl [30] and Leśniewski-Ajdukiewicz's theory of syntactic/semantic categories (see [36], [37] and [3], [4]),

• in accordance with Frege's ontological canons [22],

• in accordance with Bocheński's motto [13]: *syntax mirrors ontology* and some ideas of Suszko [50-53]: *language should be a linguistic scheme of ontological reality and simultaneously a tool of its cognition.*

The paper includes developing and some explications of these authors' ideas. It also presents, in a synthetic form, some ideas presented in my papers published in years 1991-2015. Language *L* is there defined, if the set *S* of all *well-formed expressions* (briefly *wfes*), also called *sensible expressions* of *L*, is determined. These expressions must satisfy requirements of categorial syntax and categorial semantics. The categorial syntax is connected with generating the set *S* by the classical categorial grammar the idea of which originated from Ajdukiewicz[1] and belonging *wfes* of *S* to appropriate categories. A characteristic feature of categorial syntax is that each composed *wfe* of the set *S* has a *functor-argument* structure, that it is possible to distinguish in it the main part—the so-called *main functor*, and the other parts—called *arguments* of this functor, yet each constituent of the *wfe* has determined syntactic category. Categorial intensional and extensional semantics is connected with *meaning* and *denotation* of *wfes* of *S* and with their belonging to appropriate semantic categories: intensional and extensional, respectively (see [25], [61], [63-66]). Each constituent of the composed *wfe* has determined semantic (intesional and extensional) category, can have a *meaning* assigned to it, and thus also a *category of knowledge* (the category of constituents of knowledge) and also *denotation*, and thus—an *ontological category* (the *category of ontological objects*).

The *meanings* (*intensions*) od *wfes* of *L* are treated as certain constituents of inter-subjective knowledge: logical notions, logical judgments, operations on such notions or judgments, on the former and the latter, on

other operations.

    Object references (*references*) of *wfes* of *L*, and also constituents of knowledge, are objects of the cognized reality: individuals, states of things, operation on the indicated objects, and the like. *Denotations* (*extensions*) of *wfes* of *L* and constituents of knowledge are sets of such objects. The compatibility of these denotations is called *semantic compatability* of *L* (see Diagram 2).
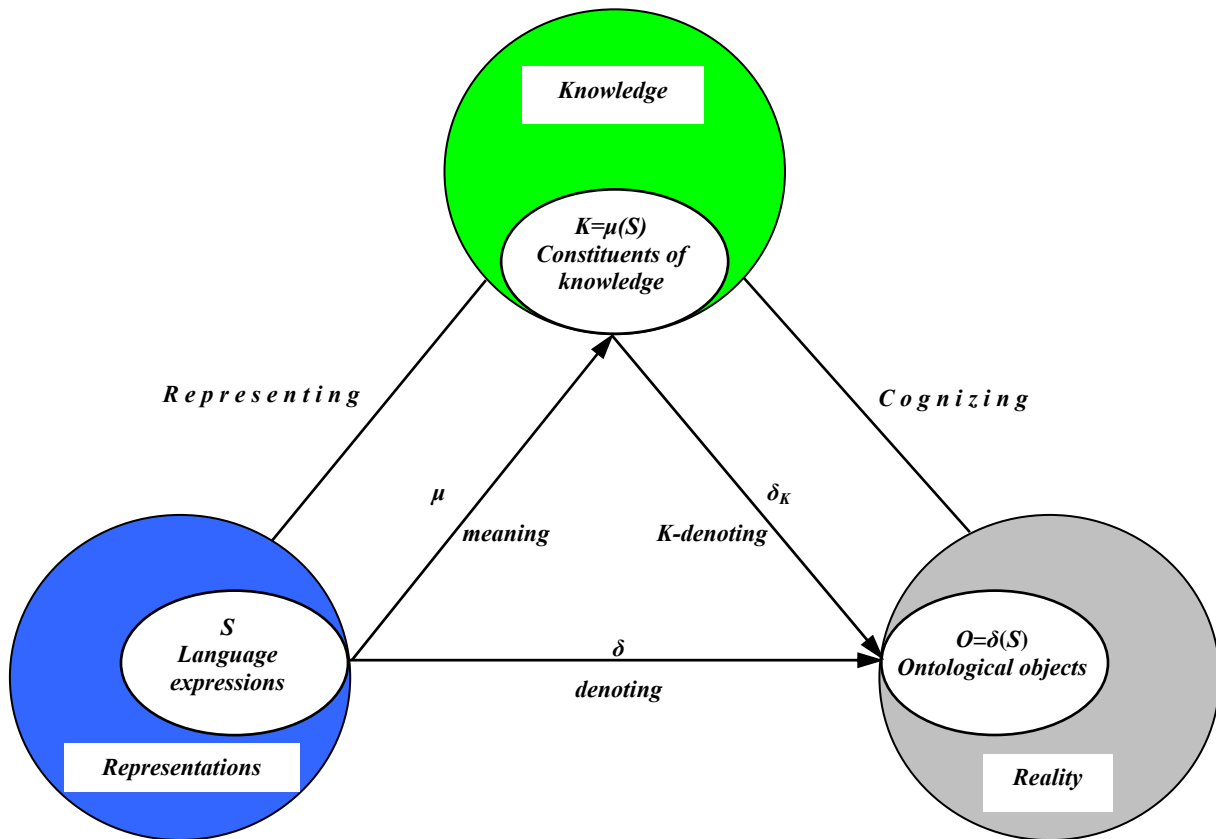

Diagram 2. Semantic compatibility.

4. Some General Conditions of Logical Sense of Expressions of the Language *L*

    In the logical conception of the language *L* and semiotic senses outlined in the paper, expressions of *L* have syntactic, intensional, and extensional senses and satisfy some general conditions of logical sense of its expressions and the language adequacy.

    The requirements imposed on the language *L* by logic and logical senses of its expressions are sharp, strict and formal. Baseline conditions apply to syntactic and semantic unambiguity expressions of the language *L* and the subsequent—relate to categorial compatibility and structural compatibility.

4.1. Syntactic and Semantic Unambiguity

    Baseline conditions of logical sense of expressions of the language *L* are the following:
- expressions of *L* should be syntactically connective, should be *wfes* of the set *S*,
- structurally unambiguous: have one syntactic sense, i.e., do not contain amphiboly and have one functor-argument structure,

- semantically unambiguous: have one intensional and one extensional sense, i.e., have one meaning, *intension*, and at a fixed meaning have one denotation, *extension*.

Commentary: Syntactic and semantic unambiguity is not, of course, a feature of natural languages and often languages of non-exact sciences but every syntactically and semantically ambiguous expression of such languages may be treated as a schema representing all of its interpretations that are unambiguous expressions (with exactly one syntactic or/and one semantic sense) and which are suitable for an adequate description of specified fragments of reality.

The sentence "*Teachers are tired because they teach students in various schools and they have a lot of them*" is structurally ambiguous (contains amphiboly) but it can be treated as scheme of two unambiguous sentences: "*Teachers are tired because they teach students in various schools and they have a lot of students*" and "*Teachers are tired because they teach students in various schools and they have a lot of schools.*"

On the other hand the sentence "*John's eyes were full of sweets*" has no meaning and no intensional and extensional sense; it is a *semantic nonsense*.

In categorial approach to the language L generated by the classical categorial grammar to every *wfe e* of the set S is unambiguously assigned a categorial index (type) $i(e)$ of a certain set I and every composed *wfe* of S has functor-argument structure. Categorial indices were introduced by Ajdukiewicz [3] into logical semiotics with the aim to determine the syntactic role of expressions and to examine their *syntactic connection*, in compliance with the *principle of syntactic connection* (*Sc*) discussed below.

The set S of all *wfes* of L is then defined as the smallest set including the vocabulary of L and closed with respect to the principle (*Sc*) which in free formulation says that

(*Sc*) the categorial index of the main functor of each functor-argument expression of the language L is formed out of the categorial index of the index of the expression which the functor forms together with its arguments, as well as out of the subsequent indices arguments of this functor.

In the formal definition of the set S, it is required that each functor-argument constituent of the given expression should satisfy the principle (*Sc*).

Every *wfe e* of S is a *meaningful expression* of L possessing one intensional sense, one meaning, *intension* $\mu(e)$, where μ is the *operation of indicating the meaning* defined on the set S.

The *meaning* $\mu(e)$ of the *wfe e* of the set S may be intuitively understood, in accordance with the understanding of meaning of expressions by Ajdukiewicz [1], [2] and, independently, by Wittgenstein [55] as a common property of all these *wfes* of S which possess the same manner of using as that e by competent users of the language L (see [64]). It is a constituent of the knowledge $K=\mu(S)$.

Every *wfe e* of S is a *meaningful expression* of L possessing one extensional sense, one denotation, *extension* $\delta(e)$, where δ is the *operation of denoting* defined on the set S. The notion of *denoting* can, however, also be introduced as the *operation of denoting* δ_K defined on the set of constituents of knowledge K.

The *denotation* $\delta(e)$ of the meaningful expression e is defined as the set of all ontological objects (or the ontological object) of the set $O=\delta(S)$ to which refer occurrences of the expression e.

The *denotation* $\delta_K(k)$ of the constituent k of knowledge K is defined as the set of all extra linguistic, ontological objects to which k refers.

4.2. Categorial Compatibility

In logical conception of the language L, the three distinguished kinds of sense of expressions of the

language *L* must be compatible: any expression having syntactic sense, any *wfe* of *L* belonging to a syntactic category of a defined kind, has semantic, intensional sense, and extensional sense and is, simultaneously, a meaningful expression of *L* belonging to a defined intensional and, respectively, to a defined extensional semantic category. Logical sense of *wfes* of *L* is connected with compatibility of their syntactic and semantic—intensional and extensional categories. In categorial approach to language, mentioned categories of *wfes* of *L* are determined by attributing to them, like their expressions, categorial indices of the set *I*. Compatible categories have the same categorial index.

We see that categorial indices introduced by Ajdukiewicz are useful not only while establishing and examining *syntactic connection* of expressions of *L*. They serve to define categorial compatibility. They also appear, as we shall see, in the role of the tool coordinating meaningful expressions and extralinguistic objects: *intensions* and *extensions* (cf. Suszko [50-53]; Ajdukiewicz [4], Stanosz and Nowaczyk [47]).

4.2.1. Postulate of Categorial Compatibility

The postulate of categorial compatibility of syntactic and semantic categories is one of the most important conditions of the logical sense of *wfes* of the language *L*. Here is a more formal description of this postulate.

Let

S be the set of all *wfes* of *L,*

K–the set of all *intensions* of expressions of the set *S*; $K=\mu(S)$,

O–the set of all *extensions* of expressions of the set *S*; $O=\delta(S)$.

Discussed above syntactic and semantic categories of meaningful *wfes* of *L* are the following subsets of the set *S*:

(1) $Syn_\xi=\{e\in S: i_S(e)=\xi\}$, where $i_S\colon S\to I$,

(2) $Int_\xi=\{e\in S: i_K(\mu(e))=\xi\}$, where $i_K\colon K\to I$,

(3) $Eks_\xi=\{e\in S: i_O(\delta(e))=\xi\}$, where $i_O\colon O\to I$.

The syntactic (resp. *intensional, resp. extensional*) *category with the index* ξ is the set of all *wfes* of *S* that have the categorial index ξ (resp. *intensions* of which, resp. *extensions* of which have the index ξ).

The postulate of categorial compatibility defining an aspect of the logical sense of *wfes* of *L* has the following form (see [65-68]):

(P) $Syn_\xi=Int_\xi=Eks_\xi$ for any $\xi\in I$.

4.2.2. Equivalents of the Postulate of Categorial Compatibility

The formal postulate (P) does not grasp well the problem of logical sense of language expressions of *L* because it does not show relationships of the distinguished categories of *wfes* to corresponding with them extra-linguistic categories of *intensions* (constituents of knowledge) and ontological categories, in such a way that the mutual correspondence of elements of the triad:

<div align="center">Language—Knowledge—Reality</div>

and the language adequacy of syntax with bi-level semantics: intensional and extensional, have been preserved (see Diagram 2).

As it was mentioned, unambiguous determined meanings (*intensions*) and denotations (*extensions*) should be assigned to *wfes* of *L*. They belong, respectively, to suitable extra linguistic categories of objects:
categories of constituents of knowledge, meanings, *intensions* (e.g., logical notions, logical judgments, operations on them) and

ontological categories, of denotations, *extensions* (e.g., individuals, set of individuals, states of affairs, operations on them).

The categories of meanings, *intensions*, are subsets of the set K of constituents of knowledge, and ontological categories—subsets of the set O ontological objects. They are determined by categorial indices. And so, for any index $\xi \in I$:

(4) $K_\xi = \{k \in K: i_K(k) = \xi\}$,

(5) $O_\xi = \{o \in O: i_O(o) = \xi\}$.

Sematic categories [see (2) and (3)] can be defined by formulas:

(6) $Int_\xi = \{e \in S: \mu(e) \in K_\xi\}$

and

(7) $Eks_\xi = \{e \in S: \delta(e) \in O_\xi\}$

stating that: the semantic intensional (resp. extensional) category with the index ξ is the set of all *wfes* of L the meanings, *intensions* (resp. denotations, *extensions*) of which belong to the category of constituents of knowledge (resp. to the ontological category) with the index ξ.
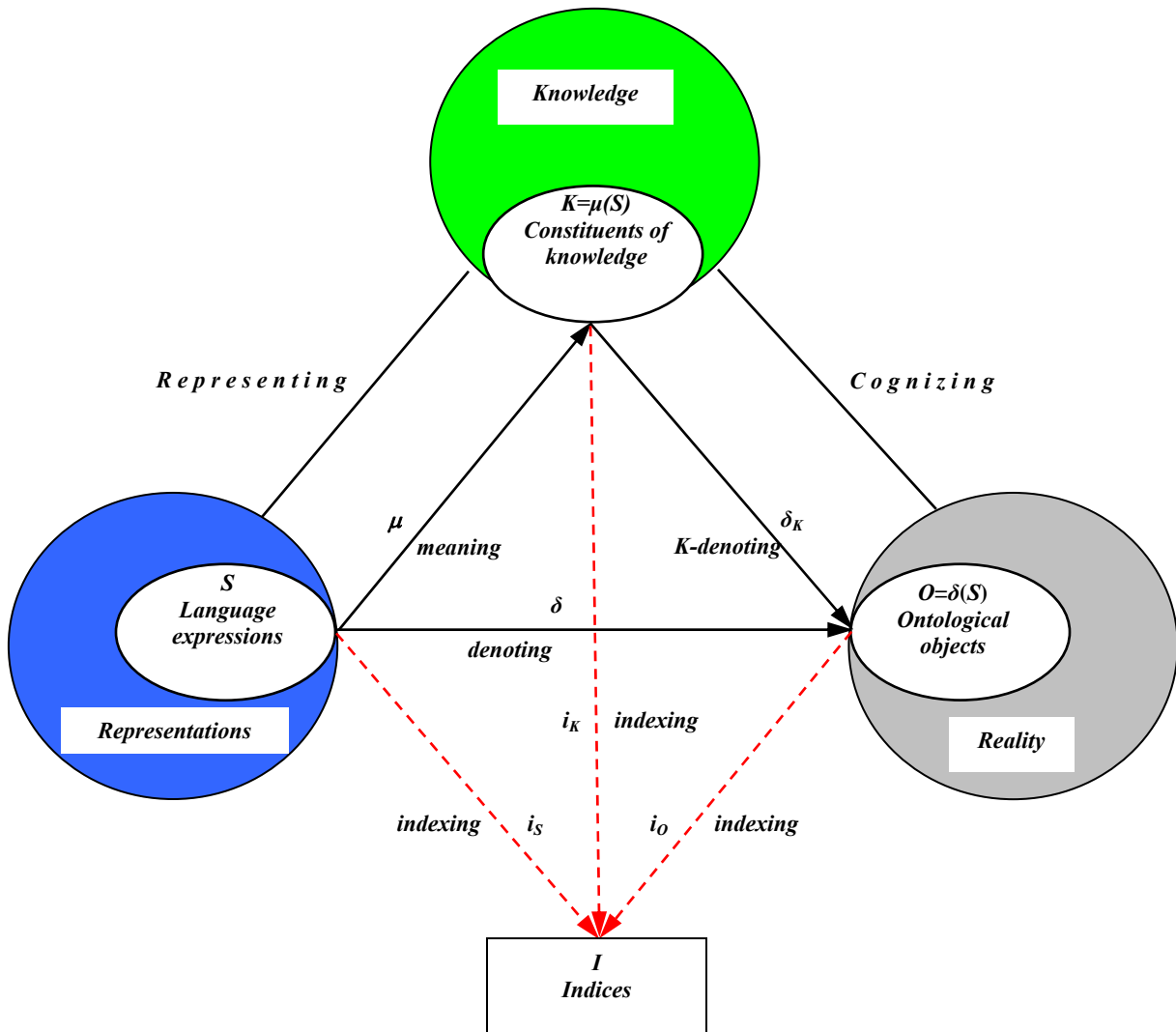

Diagram 3.

And so, for any $e \in S$ and $\xi \in I$ the postulate (P) of categorial compatibility can be replaced by the following equivalent conditions illustrated by means of Diagram 3:

(8) $e \in Syn_\xi$ if $\mu(e) \in K_\xi$ if $\delta(e) \in O_\xi$,

(9) $i_S(e) = i_K(\mu(e)) = i_O(\delta(e))$.

We see that categorial compatibility can be ensured by the identity of categorial indices: of any *wfes of L*, its meaning (*intensions*) and its denotation (*extensions*). So we see that categorial indices are also a tool of coordination *wfes* of L and corresponding to their extralinguistic objects.

4.2.3. Semantic Compatibility

From Diagram 3, we conclude that ontological objects of the set O are not only denotations of *wfes* of the set S but also denotations of corresponding to them constituents of knowledge K.

Semantic compatibility for the language L is defined by the following formula:

(10) $\delta(e) = \delta_K(\mu(e)) \in O = \delta(S) = \delta_K(K)$, for any $e \in S$,

if δ_K is the operation of constituents of knowledge.

From formula (10) immediately follows that if two expressions of L have the same meaning then they have the same denotation. It is well known that the reverse implication does not hold.

4.3. Structural Compatibility

4.3.1. On Structure of Expressions and Their Semantic Counterparts

The form of language expressions, their connectivity, well-formedness and sense, is connected with the structure of our knowledge and the structure of the cognizing fragment of reality. Its language description is composed of parts which can be separated. Some of them are independent or relatively independent and are classified as the basic language categories. They are *names* and *sentences*. Others are auxiliary, dependent constituents of language expressions, allowing to build more composed expressions from simpler ones. They are *functors*.

The categorial approach to L allows to us define the structural compatibility of its composed expressions and corresponding with them meanings and denotations. As it was mentioned, every *wfe* of L has one functor-argument structure. Functors of such expressions may be treated as partial functions defined on a proper subset of the set S and with the values in this set. The language L can be characterized then as the following partial algebra:

$$L = <S, F>,$$

where $F \subset S$, and F is the set of all functors of L.

Every composed expression e of the set S can be written in the functional-argument form:

$$e = f(e_1, e_2, ..., e_n),$$

where f is the main functor and $e_1, e_2, ..., e_n$—its subsequent arguments.

If the expression e is *wfe* (it belongs to the set S), then in accordance to the principle of syntactic connection (*Sc*) the index of its main functor f formed from the index a of e and successive indices $a_1, a_2, ..., a_n$ of successive arguments $e_1, e_2, ..., e_n$ of the functor f, can be written in the following quasi-fractional form:

$$(i_S) \quad i_S(f) = i_S(e)/i_S(e_1) \, i_S(e_2) ... \, i_S(e_n) = a/a_1 \, a_2 \, ... \, a_n.$$

The functor-function f corresponds to the function defined on meanings, respectively denotations, of arguments of this functor with subsequent indices $a_1, a_2, ..., a_n$, the value of which is the meaning, respectively the denotation, of the expression e, which the functor f forms, with the index a.

If in the language *L* we have two basic syntactic categories: names and sentences with respective indices *n* and *s*, then meanings, *intensions*—logical notions with the index *n* are assigned to names, and meanings, *intensions*—logical judgments with the index *s* are assigned to sentences. Denotations of names are usually individuals or their sets, and denotations of sentences (in situational semantics)—states of affairs, situations. They also have, respectively, indices *n* and *s*.

Example 1. Let us consider the following sentence of natural language:

(*) Peter respects Susan

with the index *s*, the main functor of which is the word "respects" of two name arguments "Peter" and "Susan" with index *n*. The expression (*) can be written in the following function-argument form:

(*') respects (Peter, Susan).

The index of the functor "respects" is *s/nn*, its meaning (with the same index) is the function which being defined on the notions of "Peter" and "Susan" with the index *n*, as meanings (*intensions*) of these names in the sentence (*') has as the value its meaning, i.e., the logical judgment with the index *s* stating that Peter respects Susan; denotation of the functor is the mapping which being defined on denotations (*denotates*) of names in (*') with index *n*, so on people Peter and Susan, has, as its value, the state of affairs: the fact that Peter respects Susan, being the denotation of the sentence (*') and having, like the sentence, the index *s*.

If somebody accepts, with accordance to the phrase-structural grammar by Chomsky that in (*) the main functor is "respects Susan" (the predicate) of one argument "Peter" then the function-argument form of (*) is as follows:

(*'') respects Susan (Peter)=(respects(Susan))(Peter)

and the index of the composed functor "respects Susan" is *s/n* and the index of the functor "respects" in it is (s/n)/n. Then the meaning and the denotation of the latter functor differ essentially from this used in (*').

As we can see in natural language, sentences may have different functor-argument structure thus different meanings and denotations. And therefore they can be treated as skeletons, schemas which represent unambiguous expressions with one functor-argument structure, one meaning and one denotation.

4.3.2. Principles of Compositionality

From the postulate (P) of categorial compatibility three principles of compositionality follow (see [60], [63-66]): one syntactic (compositionality of syntactic forms) and two semantic: compositionality of meaning and compositionality of denotation. For any composed expression $e=f(e_1, e_2,..., e_n)$ of *L* and functions $h=i_S$, μ, δ their common schema has the form:

(COM$_h$) $h(e)=h((f(e_1, e_2,..., e_n))=h(f)(h(e_1), h(e_2),...,h(e_n))$.

For $h=i_S$, we get the syntactic principle, for $h=\mu$, δ we get the semantic principles corresponding to the already known ones to Frege [23].[2]

In loose formulation, these principles state that: *The categorial index* (*the syntactic* category), resp. *the meaning*, resp. *the denotation of a well formed functor-argument expression of the language L is the value of the function of the index*, resp. *the function of the meaning*, resp. *the function of the denotation, of its main functor defined on indices*, resp. *on meanings*, resp. *on denotations subsequent arguments of this functor.*

4.3.3. Functions *h(f)*

The formulations of the principles (*COM$_h$*) define *h(f)* as functions. Indeed the index $i_S(f)$ of the functor *f* is the function:

$$i_S(f): \{(i_S(e_1)\} \times \{i_S(e_2)\} \times ... \times \{i_S(e_n))\} \rightarrow \{i_S(e)\},$$

which defined on n-tuple of indices $(i_S(e_1), i_S(e_2),..., i_S(e_n))$ has the value $i_S(e)$; hence follows the syntactic principle of compositionality ($COMi_S$).

Similarly the meaning and the denotation of the functor f, defined on meanings and, respectively, on denotations of its arguments, are functions whose values are, respectively, meanings and denotations of the expression e. However, remember that the same n-argument functor ($n{\geq}1$), e.g., "respects" in (*') of Example 1 may have different, other arguments, though its meaning, respectively denotation, are uniquely determined.

Thus, for any *wfe* $e=f(e_1, e_2,..., e_n)$ such that for indices $i_S(e)=a$, $i_S(e_l)=a_l$, where $l=1,...,n$, $\mu(f)$, is the function:

$$\mu(f): Ka_1 \times Ka_2 \times ... \times Ka_n \rightarrow Ka,$$

which for *intensions* of arguments of the functor f has the value $\mu(e)$ compatible with the principle (COM_μ), and $\delta(f)$ is the function:

$$\delta(f): Oa_1 \times Oa_2 \times ... \times Oa_n \rightarrow Oa,$$

which for denotations of arguments of the functor f has the value $\delta(e)$ compatible with the principle (COM_δ).

4.3.4. Cancelation Principles

Just as the index of the functor f in the expression $e=f(e_1, e_2,..., e_n)$ we write in a quasi-fractional form:

$$(i_S)\ i_S(f) = i_S(e)/i_S(e_1)\ i_S(e_2)... \ i_S(e_n)$$

its meaning and denotation we may write in the similar way:

$$(\mu)\ \mu(f) = \mu(e)/\mu(e_1)\ \mu(e_2)... \ \mu(e_n),$$
$$(\delta)\ \delta(f) = \delta(e)/\delta(e_1)\ \delta(e_2)... \ \delta(e_n).$$

At the established quasi-fractional records (i_S)—the index of the functor f, (μ)—of its meaning and (δ)—of its denotation, some analogues of Ajdukiewicz's rules of cancelation of fractional indices, that serve to examination of syntactic connection of complex expressions, correspond to principles of compositionality (COM_h). They follow from them and allow "to calculate" indices, meanings and denotations of functor-argument expressions of L. Their scheme, for $h=i_S, \mu, \delta$, can be written in the following way:

$$(CANh)\ h(e)/h(e_1)\ h(e_2)... \ h(e_n)(h(e_1), h(e_2),..., h(e_n)) = h(e).$$

Example 2. For the functor "respects" in the sentence

(*') respects(Peter, Susan)

The cancelation principles for $h=i_S, \mu, \delta$ have the form:

$$s/n\ n\ (n, n) = s,$$
$$\mu((*'))/\mu(Peter)\ \mu(Susan)\ (\mu(Peter), \mu(Susan)) = \mu((*')),$$
$$\delta((*'))/\delta(Peter)\ \delta(Susan)\ (\delta(Peter, \delta(Susan)) = \delta((*')),$$

while for the functor "respects Susan" in the sentence

(*") respects Susan(Peter)=(respects(Susan))(Peter)

they for $h=i_S, \mu$ are the following:

$$((s/n)/n\ (n))\ (n) = s/n\ (n) = s$$
$$\mu(respects\ Susan(Peter)) = ((\mu(respects(Susan))(\mu(Peter)) = (\mu(respects)(\mu(Susan))(\mu(Peter)) =$$
$$= (((\mu(*")/\mu(Peter))/\mu(Susan))(\mu(Susan))(\mu(Peter)) = ((\mu(*")/\mu(Peter))(\mu(Peter) = \mu(*")$$

Similarly for $h=\delta$.

4.3.5. Models of L

The principles of compositionality can be treated as some conditions of homomorphisms $h=i_S, \mu, \delta$ of the algebra of the language $\boldsymbol{L}$ into algebras its images $h(\boldsymbol{L})$, i.e.,

$$\boldsymbol{L}=<S, F> \xrightarrow{\quad h \quad} h(\boldsymbol{L})=< h(S), h(F)>,$$

where F is the set of all functor-partial functions defined on subsets of the set S with values in the set S, and $h(F)$, for $h=i_S, \mu, \delta$, is the set of functions corresponding to the functor-functions of the set F.[3]

The algebra $i_S(\boldsymbol{L})=<i_S(S), i_S(F)>$

is called the *syntactic model of the language L,* while the algebras:

$$\mu(\boldsymbol{L})=<\mu(S), \mu(F)>=< K, \mu(F)> \text{and } \delta(\boldsymbol{L})=< \delta(S), \delta(F)>=< O, \delta(F)>$$

semantic models for L; the first is called the *intensional model* for L, the other one—the *extensional model for L.*

In the process of cognition of reality, logical sense of expressions of the language L takes into account also that sentences of L were a carrier of true information about cognized fragments of reality; they should be true in the mentioned above models of L. The language as a tool for describing reality must distinguish the category of *sentences* among its syntactic categories. True sentences have the informative content and in this way allow to enrich our knowledge.

If for $h=i_S, \mu, \delta$ it is so that the sentence e of the language L is true in models $h(\boldsymbol{L})$, we may speak that our cognition by means of the sentence e is *true.*

The notions of *truthfulness* in appropriate models are introduced theoretically by means of three new primitive notions Th, satisfying for $h=i_S, \mu, \delta$ axioms:

$$\varnothing \neq Th \subseteq h(S)$$

and understood intuitively, respectively, as: the singleton consisting of the index of true sentences, the set of all true logical judgments, the set composed of the states of affairs that take place (in situational semantics) or the singleton composed of the value of truth (in Frege's semantics).

For $h=i_S, \mu, \delta$ we assume that:

The sentence e of the language L *is true in the model $h(\boldsymbol{L})$* iff $h(e) \in Th$.

4.3.6. Application of Principles of Compositionality

Now we will apply the principles of compositionality to simple expressions with universal and existential quantifiers of first-order languages.[4]

Example 3. Let us consider quantifier expressions:

$$(**)\forall x\, P(x) \text{ and } (**')\exists x\, P(x),$$

in which P is an established one-argument predicate treated as a one-argument functor-function, and quantifiers $\forall$ and $\exists$ are treated as two-argument functors-functions defined on a variable standing next to them and a sentential function with a free variable bound by the given quantifier.

The index for x is n_1, i.e., $i(x)=n_1$,

the index for P is s_1/n_1, i.e., $i(P)=s_1/n_1$, because

the index for the sentential function P(x) is s_1, i.e., $i(P(x))=s_1$,

because $i(P(x))=i(P)(i(x))=s_1/n_1(n_1)=s_1$,

The index of quantifiers $\forall$ and $\exists$ is $s/n_1 s_1$, i.e., $i(\forall)=i(\exists)=s/n_1 s_1$.

Using the principles of compositionality and cancelation we can "compute" the index of the expression (**) in its functor-argument form:

$$i\ (\forall(x, P(x))=i\ (\forall)(i(x), i(P(x)))=i(\forall)(i(x), i(P)(i(x)))=$$
$$=s/n_1s_1(n_1, s_1/n_1(n_1))=s/n_1s_1(n_1, s_1)=s.$$

In a similar way, we "calculate" the index of the expression (**')=$\exists(x, P(x))$. Thus the expressions (**) and (**') are sentences.

We will define now denotation and meaning of discussed quantifiers of the first order. We assume that

$\delta(x)=\{\delta(a):$ a is an individual name represented by x$\}=U$—the universe of individuals;

$\delta(P)$ is the function which for the argument $\delta(x)$ has the value $\delta(P(x))=\delta(P)(\delta(x))$, where

$\delta(P)(\delta(x))=\{\delta(P)(\delta(a)): \delta(a)\in U\}=$

$=\{\delta(P(x/a): P(x/a)$ is the sentence obtained from P(x) by replacing x by the name a$\}=$

$=\{$the set of states of affairs that are denotations of the sentences P(x/a)$\}=$

$=\{$the set of states of affairs that individuals $\delta(a)\in U$ have the property-function $\delta(P)\}$.

The denotation $\delta(\forall)$ (resp. $\delta(\exists)$) of the quantifier $\forall$ (resp.$\exists$) is the two-argument function, whose value $\delta((**))$ (resp. $\delta((**'))$) is assigned to arguments $\delta(x)$ and $\delta(P(x))$, because

$\delta(\forall(x,P(x)) = \delta(\forall)(\delta(x),\delta(P(x))) = \delta(**))$ and $\delta(\exists(x,P(x)) = \delta(\exists)(\delta(x),\delta(P(x))) = \delta((**'))$.

So, *The denotation $\delta(\forall)$ (resp. $\delta(\exists)$) is the function which, for arguments the universe U and the set of all states of affairs $\delta(P)(u)$, where $u\in U$, has as the value the state of affair that holds* iff *every* (*resp. some*) *individual $u\in U$ has the property-function $\delta(P)$, consisting in assigning the state of affair $\delta(P)(u)$ to u.*

In a similar way we define meanings of quantifiers $\forall$ and $\exists$ in sentences (**) and (**'). We accept then that

$\mu(x)$ is the set *C* of all logical notions that are meanings of individual names represented by the variable x,

$\mu(P(x))$ is the set *J* of all logical judgments corresponding to meanings of sentences P(*a*), where *a* is an individual name represented by x,

$\mu(P)$ is the function that assigns the set of all such logical judgments to the set *C*.

$\mu(\forall)$ (resp. $\mu(\exists)$) *is the two-argument function which being defined on the set C of logical notions and the set J of logical judgments has as the value the true logical judgment $\mu((**))$ (resp. $\mu((**'))$) iff every* (resp. *some*) *notion c of the set C has the property-function $\mu(P)$, consisting in assigning the judgment $\mu(P)(c)$ to c.*

It is obvious that quantifiers $\forall$ and $\exists$ are in logic typically ambiguous, depending on a type. In other contexts, e.g. in expressions

(***) $\forall x,y\ R(x,y)$ and (***')$\exists x,y\ R(x,y)$ or

(*v) $\forall x\ R(x,y)$ and (*v')$\exists y\ R(x,y)$

have other indices, meaning and other denotations. Their categorial index in the expressions (***) and (***') is $s/n_1n_1s_2$, where s_2 is the index of the sentential function of two individual variables, while in the expressions (*v) and (*v') they have the index: s_1/n_1s_2.

The predicate-functor R index is, of course, the index s_2/n_1n_1.

It is easy to check and "compute" that exemplary expressions are syntactically connective, so *wfes.* The first of them—(***) and (***')—are sentences, because they have the index *s*, while the others—(*v) and (*v')—are sentential functions with one free variable, because they have the index s_1.

5. Final Remarks

Logical sense of language expressions is, of course, some kind of idealization. In the logical conception of language, the sense of its expressions, both syntactic and semantic: intensional and extensional ensures their structural and semantic unambiguity and mutual syntactic and semantic compatibility, correspondence.

Common language, and often also scientific, is a living creature still developing. The degree of syntactic and semantic senses of its expressions changes can be narrower or higher depending on its skillful precision.

Structurally or semantically ambiguous expressions can always be split into expressions having unambiguous syntactic and semantic senses, and be categorially analyzed.

Expressions imprecise or vague obviously do not have a logical sense. If they are to well-play their communicative and cognitive function and truly describe defined fragments of reality, their degree of the logical sense should be increased by precision or sharping their meanings and denotations.

Only expressions with a high degree of the logical sense: syntactical and semantical (intensional and extensional) get closer to the sense and may, after a proper justification, become theorems of a given field of knowledge providing a source of knowledge and reflection of the world and intersubjective communicability about it.

Notes

1. See [3] and [4]. The notion of categorial grammar was constructed under influence of Leśniewski's theory of semantic (syntactic) categories in his systems of protothetics and ontology [36], [37], under Husserl's ideas of pure grammar [30], and under the influence of Russel's theory of logical types. The notion was shaped by Bar-Hillel [6-8] and developed by Lambek [34], [35]; Montague [39], [40]; Cresswell [19], [20]; Buszkowski [14-17]; Marciszewski [38]; Simons [44], [45], Tałasiewicz [54] and others (see also [18]).The first formalization of languages generated by the so called categorial grammar, the notion introduced and explicated by Buszkowski [15], [16], was presented in the author's book in Polish [56] and its English version [57], and then also in [63].

2. (cf. also [25], [39], [42], [43] [31-32] and [26-28], [33]).

3. The algebraic approach to syntax and semantics of language can also be found in papers of Dutch logicians, especially in those by Johan van Benthem [9-12]. However, algebraic approach presented in this paper differs substantially from that given by J. van Benthem.

4. Some general findings relating to the solution of the problem of syntactic categories of quantifiers, their denotation or/and meaning are presented in my papers [58-59] and [69].

Works Cited

Ajdukiewicz, K. "O znaczeniu wyrażeń (On meaning of expressions)." *Księga Pamiątkowa Polskiego Towarzystwa Filozoficznego we Lwowie*. Lwów, 1931. (Reprinted in: [5]).

Ajdukiewicz, K. "Sprache und Sinn." *Erkenntnis*, IV (1934): 100-38.

Ajdukiewicz, K. "Die syntaktische Konnexität." *Studia Philosophica* 1, Leopoli, 1 (1935): 1-27 (English translation: Syntactic Connection. S. McCall, ed. *Polish Logic 1920-1939*. Oxford: Clarendon Press, 1967. 202-31).

Ajdukiewicz, K. "Związki składniowe między członami zdań oznajmujących (Syntactical relations between constituents of declarative sentences)." *Studia Filozoficzne* 6.21 (1960): 73-86 (First presented in English at the International Linguistic Symposium in Erfurt, September 27-October 2, 1958).

Ajdukiewicz, K. *Język i poznanie* (*Language and Cognition*) 1 (1960). Warsaw: PWN.

Bar-Hillel, Y. "On Syntactical Categories." *Journal of Symbolic Logic* 15 (1950): 1-16 (Reprinted in Bar-Hillel [8], 19-37).

Bar-Hillel, Y. "A Quasi-Arithmetical Notation for Syntactic Description." *Language* 29 (1953): 47-58 (Reprinted in Bar-Hillel [8], 61-74).

Bar-Hillel, Y. *Language and Information. Selected Essays on Their Theory and Applications*. Reading, Mass.: Addison-Wesley Publishing Co., 1964.

van Benthem, J. "Universal Algebra and Model Theory. Two Excursions on the Border." Report ZW-7908. Groningen: Department of Mathematics of Groningen University, 1980.

van Benthem, J. "Why Is Semantics What?" J. Groenendijk, T. Janssen, and M. Stokhof, eds. *Formal Methods in the Study of Language*. Amsterdam: Mathematical Centre Tract 135, 1981. 29-49.

van Benthem, J. "The Logic of Semantics." F. Landman and F. Veltman, eds. *Varietes of Formal Semantics*. GRASS Series, 3 (1984): 55-80. Dordrecht: Foris.

van Benthem, J. *Essays in Logical Semantics*. Dordrecht: Reidel, 1986.

Bocheński, J. M. "On the Syntactical Categories." *New Scholasticism* 23 (1949): 257-80.

Buszkowski, W. "Three Theories of Categorial Grammar." In: [18], 1988. 57-84.

Buszkowski, W. *Logiczne podstawy gramatyk kategorialnych Ajdukiewicza-Lambeka* (*Logical Foundations of Ajdukiewicz's-Lambek's Categorial Grammar*). Logika i jej zastosowania (Logic and itsapplications).Warsaw: PWN, 1989.

Buszkowski, W. "O klasycznej gramatyce kategorialnej (On classical categorial grammar)." J. Pelc, ed. *Znaczenie i prawda* (*Meaning and truth*). Rozprawy semiotyczne. Warsaw: PWN, 1994. 203-20.

Buszkowski, W. "Type Logics in Grammar." V. F. Hendrics and J. Malinowski, eds. *Trends in Logic*: 50 *Years of Studia Logica*. Dordrecht: Kluwer, 2003. 321-66.

Buszkowski, W., Marciszewski W., and van Benthem J., eds. *Categorial Grammar*. Amsterdam-Philadelphia: John Benjamis Publishing Company, 1988.

Cresswell, M. J. *Logics and Languages*. London: Mathuen and Co. Ltd., 1973.

Cresswell, M. J. "Categorial Languages." *Studia Logica* 36 (1977): 257-69.

Carnap, R. *Meaning and Necessity*. Chicago: University of Chicago Press, 1947.

Frege, G. *Begriffsschrift, eine der arithmetischen nachbildete Formelsprache des reinen Denkens*. Halle, 1879 (Reprinted in: Frege [24]).

Frege, G. "Über Sinn und Bedeutung." *Zeitschrift für Philosophie und pilosophishe Kritik* 100 (1892): 25-50 (English translation: H. Feigel and W. Sellars, eds. *Readings in Philosophical Analysis.* New York: Appleton-Century-Crofts, 1949, and also in M. Beaney, ed. *The Frege Reader*. Oxford: Blackwell, 1997. 151-71).

Frege, G. *Begriffsschrift und andere Ausätze*. I. Angelelli, ed. Darmstadt-Hildesheim: Wissenschaftliche Buchgesellschaft/G. Olms, 1964.

Gamut, L.T. F. (J. van Benthem, J. Groenendijk, D. de Jongh, M. Stokhoft, and H. Vercuyl). *Logic, Language and Meaning*, vol. I: *Introduction to Logic*, Vol. II: *Intensional Logic and Logical Grammar*. Chicago-London: The University of Chicago Press, 1991.

Hodges, W. "Compositional Semantics for Language of Imperfect Information." *Logic Journal of the IGPL* 5.4 (1996): 539-63.

Hodges, W. "Compositionality Is Not the Problem." *Logic and Logical Philosophy* 6 (1998): 7-33.

Hodges, W. "Formal Features of Compositionality." *Journal of Logic, Language and Information* 10 (2001): 7-28.

Husserl, E. *Logische Untersuchungen*, Vol. I, Halle, 1900; Vol. II, Halle, 1901.

Janssen, T. M. V. "Compositionality." J. van Benthem and A. ter Muelen, eds. *Hanbook of Logic and Language*. Amsterdam-Lausanne-New York: Elsevier Science, 1996.

Janssen, T. M. V. "Frege, Contextuality and Compositionality." *Journal of Logic, Language and Information* 10 (2001): 115-36.

Kracht, M. *Interpreted Languages and Compositionality*. Dordrecht-Heildeberg-London-New York: Springer, 2011.

Lambek, J. "The Mathematics of Sentence Structure." *American Mathematical Monthly* 65 (1958): 154-70.

Lambek, J. "On the Calculus of Syntactic Types." R. Jakobson, ed. *Structure of Language and Its Mathematical Aspects. Proceedings of Symposia in Applied Mathematics* 12 (1961): 166-78. Providence, Rhode Island: AMS.

Leśniewski, S. "Grundzüge eines neuen Systems der Grundlagen der Mathematik." *Fundamenta Mathematicae* 14 (1929): 1-81.

Leśniewski, S. "Über die Grundlagen der Ontologie." *Comptes rendus des séances dela Société des Sciences et des Lettres de Varsovie*, Classe II, 23 (1930): 111-32.

Marciszewski, W. "A Chronicle of Categorial Grammar." In: [18], 1988, 7-22.

Montague, R. "Universal Grammar." *Theoria* 36 (1970): 373-98.

Montague, R. *Formal Philosophy*: *Selected Papers of Richard Montague*. Ed. and introd. R. H. Thomason. New Haven, Conn.: Yale University Press, 1974.

Partee, B. H. "Compositionality." F. Landman and F. Veltman, eds. *Varieties of Formal Semantics. Proceedings of the Fourth Amsterdam Colloqium*. Dordrecht: Foris, 1984. 281-311.

Pelletier, F. J. "Did Frege Believe Frege's Principle?" *Journal of Logic, Language and Information* 10 (2001): 87-114.

Simons, P. "Combinators and Categorial Grammar." *Notre Dame Journal of Formal Logic* 30.2 (1989): 241-61.

Simons, P. "Languages with Variable-Binding Operators: Categorial Syntax and Combinatorial Semantics." J. Jadacki and J. Paśniczek, eds. *The Lvov-Warsaw School—The New Generation*. Poznań Studies in the Philosophy of Sciences and Humanities. 89 (2006): 239-68. Amsterdam-New York: Rodopi.

Stanosz, B. and Nowaczyk, A. *Logiczne podstawy języka* (*Logical Foundations of Language*). Wrocław-Warsaw: Ossolineum, 1976.

Suszko, R. "Syntactic Structure and Semantical Reference, Part I." *Studia Logica* 8 (1958): 213-44.

Suszko, R. "Syntactic Structure and Semantical Reference, Part II." *Studia Logica* 9 (1960): 63-93.

Suszko, R. "O kategoriach syntaktycznych i denotacjach wyrażeń w językach sformalizowanych (On syntactic categories and denotation of expressions in formalized languages)." *Rozprawy logiczne* (*Logical dissertations* to the memory of Kazimierz Ajdukiewicz). Warsaw: PWN, 1964. 193-204.

Suszko, R. "Ontology in the Tractatus of L. Wittgenstein." *Notre Dame Journal of Formal Logic* 9 (1968): 7-33.

Tałasiewicz M. *Philosophy of Syntax*: *Foundational Topics*. Dordrecht: Springer, 2010.

Wittgenstein, L. *Philosophical Investigations*. Oxford: Blackwell, 1953.

Wybraniec-Skardowska, U. *Teoria języków syntaktycznie kategorialnych* (*The theory of syntactically categorial languages*). Wrocław-Warszawa: PWN, 1985.

Wybraniec-Skardowska, U. *Theory of Language Syntax. Categorial Approach*. Dordrecht-Boston-London: Kluwer, 1991.

Wybraniec-Skardowska, U. "Logical and Philosophical Ideas in Certain Approaches to Language." *Synthese* 116.2 (1998): 231-77.

Wybraniec-Skardowska, U. "On Denotations of Quantifiers." M. Omyła, ed. *Logical Ideas of Roman Suszko*. Warsaw: Faculty of Philosophy and Sociology of Warsaw University, 2001. 89-119.

Wybraniec-Skardowska, U. "Three Principles of Compositionality." *Bulletin of Symbolic Logic* 7.1 (2001): 157-8.

Wybraniec-Skardowska, U. "Meaning and Interpretation." J. Y. Beziau and A. Costa Leite, eds. *Handbook of the First World Congress and School on Universal Logic*. Unilog'05, Montreux, Switzerland, 2005, 104.

Wybraniec-Skardowska, U. "O pojęciach sensu z perspektywy logiki (On the notions of sense from the perspective of logic)." K. Trzęsicki, ed. *Ratione et Studio*. Białystok: Publishing House of Białystok University, 2005. 155-90.

Wybraniec-Skardowska, U. "On the Formalization of Classical Categorical Grammar." J. Jadacki and J. Paśniczek, eds. *The Lvov-Warsaw School—The New Generation.* Poznań Studies in the Philosophy of Sciences and Humanities, 89 (2006): 269-88. Amsterdam-New York: Rodopi.

Wybraniec-Skardowska, U. "Meaning and Interpretation." *Studia Logica* 85 (2007): Part I, 105-32; Part II, 263-76.

Wybraniec-Skardowska, U. "On Metaknowledge and Truth." D. Makinson, J. Malinowski, and H. Wanshing, eds. *Trends in Logic*: *Towards Mathematical Philosophy*. Berlin-Heidelberg: Springer, 2009. 319-43.

Wybraniec-Skardowska, U. "Three Principles of Compositionality." U. M. Żegleń, ed. *Cognitive Science and Media in Education. From Formal and Cognitive Aspects of Language to Knowledge*. Toruń: Publishing House Adam Marszałek, 2010. 28-65.

Wybraniec-Skardowska, U. "On Language Adequacy." *Studies in Logic, Grammar and Rhetoric* 40.53 (2015): 257-92.

Wybraniec-Skardowska, U. "Logic and Sense." J. Y. Beziau, S. Ural, A. Buchsbaum, I. Tasdelen, and V. Kamer, eds. *Handbook of the 5th World Congress and School on Universal Logic, UniLog 2015*. Istanbul: Istanbul University, Turkey, 2015. 303-4.

Wybraniec-Skardowska, U. "Categories of First Order Quantifiers." Book of Abstracts, 15th Congress of Logic, Methodology and Philosophy of Science, CLMPS 2015; Logic Colloquium, LC 2015. Helsinki: University of Helsinki, 2015. 674-5.