Urszula Wybraniec-Skardowska

Logic – Language – Ontology

Selected Works

Springer Nature

Preface

This monograph belongs in the domain of universal logic and therefore pertains to the broadly-understood logic which has its application in a variety of disciplines of science and knowledge. It spans these formalized aspects of classical logic, set theory and logical semiotics, which are made use of in contemporary philosophy of language and ontology. Because a task of ontology is to describe the structure of a being or reality, and because it is language that serves this purpose, language should reflect the structure of reality in compliance with its own system. Its structure, which preserves this compliance, is dealt with by the logical theory of language syntax and semantics. Accordingly, in the book, this theory is formalized in the individual chapters with the aid of means of classical formal logic and set theory. Thus, in order to understand it, it suffices to hold elementary logical knowledge and simple knowledge of issues relating to axiomatization of theories.

This book is intended for undergraduate and graduate students, as well as researchers who are interested in areas where formal logic and logic of language are applied to learn more about ontology, cognitive science, symbolic systems, information science and linguistics. Basically, the works collected in this book can be read separately, although they are for the most part connected thematically in the order they follow one another. I hope that studying the chapters of the monograph will offer the reader a solid and up-to-date view that will enable comprehending many problems which are still encountered in currently conducted research.

Warsaw, June 2021 *Urszula Wybraniec-Skardowska*

Acknowledgements

I would like to pay my debt of gratitude. First, for being given the opportunity to have my selected works published by Birkhäuser in *Studies in Universal Logic* (*SUL*) series edited by Jean-Yves Beziau. Second, I must express my gratitude to the Polish Academy of Sciences, Białystok Publishing House, Adam Marszałek Publishing House, De Gruyter and Springer for their permissions to republish these selected works in *SUL*.

It want to acknowledge the assistance that I had. I would like to express my gratitude to all the referees of my originally published works. To Zbigniew Bonikowski I own sincere thanks for his painstaking efforts to reformat the works in compliance with the style accepted by *SUL*.

Several parts of this book go back to discussions with my teacher, Jerzy Słupecki, concerning categorial syntax and then categorial semantics in scientific meetings at Tilburg University and Amsterdam University during my stays there (a grant from The Netherlands Organization for Scientific Research). Many people from these universities helped me in many different ways, rendering their works available or facilitating access to necessary literature. Here, I would like in particular to thank Johan van Benthem, Filip Buekens, Jan van Eijck, Theo Janssen and Harrie de Swart.

I received valuable comments and suggestions from my colleagues or co-workers, including Tadeusz Batóg, Andrzej Biłat, Zbigniew Bonikowski, Wojciech Buszkowski, Jaakko Hintikka, Wilfrid Hodges, Andrzej Indrzejczak, Jacek Jędrzejowski, Jacek Malinowski, Jerrold Jacob Katz, Arnold Koslow, Marek Magdziak, Witold Marciszewski, Alex Orenstein, Jerzy Perzanowski, Bartłomiej Skowron, Mieszko Tałasiewicz and Heinrich Wansing. I want to thank them all for their support.

I am also greatly indebted to my husband, Adam Skardowski, who created the best possible conditions for me to continue my research work and to work on this monograph.

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Part I
Introduction

The Dual Ontological Nature of Language Signs and the Problem of Their Mutual Relations

Abstract The subject matter of this work covers the issues or problems listed below: The problem of the ontological status of language signs and a more general philosophical problem connected with it: What is language as a system of signs, which—on the one hand—serves to: 1) represent our knowledge about the reality which is being recognized, and, on the other one to: 2) a. explore and better cognize or discover it, b. describe it in an adequate manner, and c. enable users to make an interpersonal communication? All the pragmatic functions of language require carrying out its logical-philosophical analysis and this means both regarding its syntax and semantics. Such an analysis is not possible without determining the following: How are signs perceived and what is their ontological nature in the so-called *functional approach towards logical semiotics of natural language*, founded on two ways of their usage: either as signs which we use in concrete situational or situational-language contexts or as isolated signs detached from such contexts? In the first case, they are language *tokens* (*concretes*), existing material objects perceived through the senses, with a fixed temporal-spatial location, in the other one—they are *non-concretes* and as such (as the majority or researchers in the field of philosophy and linguistics accept)—abstract objects, language *types*. The type-token distinction (differentiation between abstract and concrete) has already acquired a certain status in contemporary philosophy and is of considerable importance to metaphysics and epistemology in particular. Indeed, it is most often illustrated with reference to language signs (words, expressions) as the distinction type/token of a sign, introduced into semiotics by Ch. S. Peirce. In the semiotic analysis, and also in the linguistic one, there are used both types and tokens of signs, however, often without paying due attention to when it is said about types and when about tokens. This is related to the problems which are still considered by philosophy of language: What is the type? What is the token (specimen)? What are the mutual relations between the type and the token? Disputes concerning providing answers to these questions are related to existential issues dealing with the ontological status of these language signs and two

Paper delivered at the XIth Philosophical Convention in Lublin, September 2019. First published in *Ruch Filozoficzny* **77** (1), 7–24 (2021), in Polish.

currents within ontology of language, remaining under the influence of two fundamental concepts which have been formed in the debate: nominalism and realism. The author presents in brief different stances on the above-mentioned issues or problems, as well as argues that, from the logical point of view: 1) working out any theoretical conception of language must take into account its bi-aspectual characteristics: as a language of expression-tokens and as a language of expression-types; 2) advocating either of the standpoints: (a) types exist independent of their tokens, or (b) it is not so, can be omitted in syntactic considerations on language;3) mutual relations between sign-tokens differ from the mutual relations between sign-types, yet 4) determination of the mutual relation between sign-types and sign-tokens depends on accepting either of the standpoints: (a) or (b); 5) semantic or semantic-pragmatic concepts of language, such as: *meaning*, *denotation*, *interpretation* should be defined exclusively for types of tokens, but their definitions require certain reference to the functions which sign-tokens perform in language (words, expressions), or to relations between them; 6) The concept of an act of language communication differs from the general language communication: the first one is defined by means of sign-tokens, whereas the other one—with the use of sign-types.

Key words: sign-tokens • sign-types • dual ontological status of language signs • nominalism and realism in philosophy of language • token-token relations • type-type relations • type-token relations

1 Preliminaries

The question of the ontological status of language signs is connected with a more general philosophical problem. What is language as a system of signs, serving to represent knowledge about the reality that is being cognized, to probe this knowledge as well as to better recognize or discover the reality, to facilitate its reliable description and interpersonal communication between its users? All these pragmatic functions of language require its logical-philosophical analysis, which is not possible without determining how signs of language are conceived and what their ontological nature is like.

In a semiotic analysis and a linguistic one, there are used both types and tokens of signs, often without paying due attention when types are spoken about and when tokens are. This is in connection with the problems which are continuously discussed in philosophy of language: What is the type of a sign? What is the token (specimen) of a sign? What are mutual relations between the type and the token?

Debates held to answer these questions are related to existential issues connected with the ontological status of language signs and two trends in language ontology being under the influence of two fundamental ideas that were formed in the dispute about universals: nominalism and realism.

In the work, different standpoints on the issue of the above-raised problems and questions will be presented in brief and logical consequences resulting from opting for either of them will be analyzed.

2 The Functional Approach Towards Language as a System of Signs. Two Ways of Using Signs

In the process of cognition of reality, we acquire knowledge about it, we collect it in a system of knowledge as well as represent it in a sign system, usually the language-based one. We explore this knowledge and we process it. This leads us to a new knowledge about the reality of interest to us or to discovering it.

The effectiveness of cognition is dependent on mutual relations between the elements of the triad: language-knowledge-reality. It is achieved when language, its syntax, *adequately* reflects its semantics, therefore when language serves to adequately describe this reality,[1] but also—to secure effective interpersonal communication of its users. The above-mentioned pragmatic functions of language require philosophical and logical analyses, both of its syntax and its semantics. Such an analysis is connected with perception of signs, their ontological nature in the so-called *functional approach to language*,[2] founded on two manners of its usage: either as signs which we make use of in concrete situational or situational-language contexts, or as signs that are isolated, torn off such contexts. In the first case, they are language *concretes*, existing material objects perceptible through senses, one of a defined temporal-spatial location. In the other one—they are non-concretes, and as such (as the majority of researchers in the field of philosophy and linguistics assume)—abstract objects, language *types*.

3 Ch. S. Pierce's Distinction: Type-token of a Sign

The type-token distinction has acquired a certain status in contemporary philosophy and is significant in metaphysics and epistemology. It is most often illustrated just with reference to language signs (words, expressions) as the distinction of type-token of a sign, introduced by Ch. S. Pierce (1906 and 1908). I will quote a few fragments of his works, in which the concepts *token* (concrete, unit, specimen) and *type* appear for the first time with reference to the words:

> A common mode of estimating the amount of matter in a MS. or printed book is to count the of words. There will ordinarily be about twenty *thes* on a page, and of course they count as twenty words. In another sense of the word "word", however, there is one word "the" in the English language: it is impossible that this world should lie visibly on page or heard in

[1] See [17, 18].

[2] The concept was introduced and discussed by Jerzy Pelc in [8, 9].

> any voice, for the it is not a Single thing or Single event. It does not exist; it only determines things that do exist. Such a definitely significant form, I propose to term a *Type*. A Single event which happens once and whose identity is limited to that one happening or a Single object or thing which is in some single place at any one instant of time, such event or thing being significant only as occurring just when and where it does, such as this or that word on single line of a single page of a single copy of a book, I will venture to call a *Token*.[3]

In his letters to Lady Welby (1908), we find other relevant fragments:

> A Sign may *itself* have a "possible" Mode of Being. E.g. A hexagon inscribed in or circumscribed about a conic. It is a Sign, in that the collinearity of the intersections of opposite sides shows the curve to be a conic, if the hexagon is inscribed; but if it be circumscribed the copunctuality of its three diameters (joining opposite vertices.) Its mode of Being may be Actuality: as with any barometer. Or Necessitant: as the word "the" or any other in the dictionary. For a "possible" Sign I have no better designation than a *Tone*, though I am considering replacing this by "Mark." [—] An Actual sign I call a *Token*; a Necessitant Sign a *Type*.[4]

or

> A Sign may be an Occurrence, such as any one of a score of "thes" on a single page of single copy of an English book. I call such a Sign a *Token*.[5]

In a semiotic analysis, including also a linguistic one, there are used both types (*types*) and specimens of signs (*tokens*). However, attention is not often paid here to whether it is **types** which are meant or **tokens**. This is due to the fact that the concepts are indeterminate and mixed up.

4 Controversies Over the Ontological Status of Language Signs and Relations Between Them

Problems concerning the ontological nature of types and tokens of language sign are still the subject of discussion in philosophy of language.[6] The distinction type-token is an ontological one between the general kind of subjects and individual objects. Disputes over the ontological status of these signs concern the questions:

- What is a *type*?
- What is a *token* (specimen)?
- What are the mutual type-token relations like?

The disputes are related to the existential questions with reference to the ontological nature of these signs. They are connected with two trends in ontology of language remaining under the influence of two ideas in the dispute over universals: *nominalism* and *realism*.

[3] See [3, pp. 505, 506].

[4] See [5] and [7, p. 480].

[5] See [6].

[6] This problem area is discussed in detail by Linda Wetzel in [12].

Let us consider, for instance, the following fragment of Gertruda Stein's poem "Saint Emily":

Rose is a rose is a rose is a rose

and let us ask: *How many words are there?* The answer is ambiguous. In one sense there are three, yet in another—ten. This depends, obviously, on whether we mean word-types or word-tokens.

The difference in the meaning is connected also with the question: *Are types universals?* The answer this time depends on what universals are. We are familiar with controversies over this issue and the answers are like the following ones:

1) Yes, they are. *After all they are abstracts, but they do not have a spatial-temporal location, although they have instances; they exist if universals exist.*
2) No, they are not. They are not universals or extraordinary objects; they are in their instances (similarly as universals, they are not abstracts).[7]

Coming back to the question: *What are types?* And assuming that word-types are paradigms of types in general, it is possible to formulate briefly a few different, totally differing answers, accepted by philosophers. They are as follows:

Types are:

- *universals, abstracts, are not visible, perceptible; they are without location*,[8]
- *distributive sets of tokens*,[9] constructs,
- *kinds, species* (but not sets),[10]
- *laws* [11] *which do not exist, but are—decidedly significant forms that define things, ones that exist*,
- *norms:* [12] *since a type covers tokens and has all designable features of tokens (the same phonological structure of words and the same arrangement of letters).*

There is no clarity, either, in answering the question: What are tokens of signs? It is most often assumed that word-tokens, *tokens*, specimens are material, physical objects—**concretes** localized in time and space. They can be formed from particles of ink, ball pen ink, pixels of light, electronic dots or sections (or their lack) on a computer screen. They can also be smoke or light signals, hand gestures, sound waves, etc.

The basic difference between types of words and their tokens is marked in that the same type can possess different tokens. For instance, the type called 'sign' can possess the following different word-tokens:

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[7] It needs observing, though, that most metaphysicians regard them as a kind of universals.

[8] This stance is accepted, in particular, by Edward N. Zalta and Michael Jubien. See [20] and [2].

[9] See Willard van Quine [10].

[10] See, e.g. Nicholas Wolterstorff [13].

[11] See Charles S. Peirce [4].

[12] See Sylvian Bromberger [1].

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SIGN g *sign*

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sign

It is fairly obvious that there cannot be accordance in the answer to the following question: *What are the relations between a type and language tokens like?* The answer depends on how the *type* is conceived. Thus:

1) If the type is a set, the type-token relation is *set membership*, the *relation of belonging to the set* (belonging of sign-tokens to their type),
2) If the type is a class, this relation is *class membership*, the *relation of belonging to the class*,
3) If the type is a kind, this relation is *kind-membership, the relation of belonging to the kind*,
4) If the type is a law, this relation can be a relation of being used in compliance with the order of law.

Nonetheless, it is very often considered that this relation is:

5) *a relation of creating an 'instance'* or *an exemplification* (tokens are instances of the type; they exemplify the type)—**the instantiation view**, or
6) *a relation of representing by tokens*—**the representation view.**[13]

The last two standpoints are subject to objections raised by opposing trends. Thus:
Objection towards 5): *There are types which do not possess tokens* (e.g. infinitely long combinations of word-types).
Objection towards 6): There are instances which are not tokens of a type and they do not represent it.

(We will return to these views and objections later in this work.)

Remark 1 It is reasonable to speak of the type-token or token-type relations when types of signs exist in a sense, when they have their primary or secondary existence in linguistics and philosophy of language.

The problem of primary of secondary being of language signs in linguistics and philosophy is connected with answering the following question: *Is the linguistic type an ontologically primary or secondary expression of language in relation to linguistic token-based expression?*

This problem is therefore linked to the age-long question: *Which was first: the spirit or the matter?* When we wish to stand in support for a stance on the type-token relation, we ought to determine whether we do so:

- *in the spirit of Plato*—i.e. for the primary existence of *types* (abstract objects) and the secondary one—*tokens* (material objects), or to the contrary:
- *in the nominalistic spirit, concretistic*—for the primary existence of *tokens* (objects existing in reality) and the secondary one—*types*.

[13] In case 6), what is meant, in fact, is the token-type relation.

5 Solving the Problems Under Analysis

Accepting the Platonian version of realism, it is acknowledged that: a *type*, as an isolated, abstract being is independent on tokens (they do not form this *type*), although we can regard *tokens* as its:

- physical representations, or its
- exemplifications, concretizations or instances.

In this case we can state that the relation which connects *types* and *tokens* is either representation or concretization, or instantation, in the sense that *types* can determine their representations, exemplifications—*tokens*, which are then represented, exemplified or are instances of these types, respectively. This depends, obviously, on how we perceive this *type*.

On the other hand, while accepting the concretistic stance, it is stated that: given *tokens* (language concretes) delineate and determine the given *type* which is formed by their means, is a type of *tokens*. Then the *type is not an independent being*, but a secondarily existing object, abstracted, isolated from tokens, and the very *tokens* themselves can:

- form this *type*, construct this *type*, or
- represent this *type*.

This depends on how we conceive typology or logical abstraction.

As regards typology, *a type is an abstract creation or construct, an abstract object*, formed, distinguished, revealed or abstracted somehow intellectually, *generated out of certain objects, which have at least one property in common*. Then, a *type* is always a type of certain objects which have at least one property in common which is identical. Let us note, however, that a *type is not necessarily a type of concrete objects*, therefore not necessarily a type of *tokens*. *A type—as an abstract object—can be a type of other abstract objects*, e.g. a noun (as a part of speech) is a type of certain abstract words which—as extracted from among *token* words (inscriptions or utterances) possessing some defined, the same properties, are types themselves—types of concretes, sign-*tokens*.

Linguistics basically considers language types of concretes (sounds of speech, inscriptions): *types are simple words and sentences—abstract expressions of language*. These expressions have their own types on the meta-linguistic level. These, in turn, have their own types on the meta-metalinguistic level. Let us observe, however, that:

Independent of what types of expressions of the given language are the basis of concrete linguistic studies, all theorems of the language of linguistics (their components-words) are expression-types of their token-expressions, concretes.

5.1 The Instantiation View vs the Representation View

Let us consider objections raised by representatives of the opposing standpoints against each other with respect to the views on functioning of the *type-token relation*.

1) The objections voiced by advocates of the representation view towards those adhering to the view of forming instantiation of the given type are founded on the belief that philosophers and linguists preferring this option assume that the sole feature which each *token* of a given type is entitled to is that they have a shape or a sound and that the same shape or the same sound distinguishes them, allowing to create their type, brought out by this common property of theirs, precisely the same physical similarity.

   I am of the opinion that the misunderstanding stems from the fact that interpreters, or maybe even the very propagators of the view of creating instances themselves, do not discriminate between the name of type and this type, e.g. between the name of the type ‘color’ and the abstract being defined by this name. This abstract being, in this case, has been distinguished, generated, abstracted thanks to concrete objects due to that each of them has the same characteristic: being a colorful object which in particular can be each colorful inscription (written in different colors), or colored hearts or stars, like

   

   and every colorful thing, e.g. a concrete colorful child’s ball. This abstract being is formed or is treated as a *common property* which distinguishes in total all these colorful objects, but is not a property of these objects. The common property is an abstract creation and cannot be treated as a property of each *token* separately, cannot be their shared property. The argument against the instantiation view is not clear or full, though.

   I believe that advocates of the instantiation view probably do not mean that the given type is generated by *tokens*, abstracted by *tokens*, but they are secondary beings, physical concretizations of the existing types. It is only in this sense that *tokens are instantiations of their types* (*types* are exemplified by *tokens*). Then their standpoint is the Platonizing one. With another assumption—*concretistic* one—one can also say that *tokens* represent a *determined type* (*types* are represented by *tokens*).

   From the formal, logical point of view, a type marked by the name ‘color’ is a logical construct, a set-theory construct, an equivalence class of all these objects which are colorful. Thus, one cannot treat concrete colorful inscriptions of different time-spatial location, yet possessing some color, as instances of the type defined with the name ‘color’ through a physical similarity to the inscription found between the quotation marks.
2) Objections raised by representatives of the instantiation view against supporters of the representation approach are based—as far as I think—on the statement

that a name-type cannot be identified with the object labeled with this name. For instance, one cannot consider the type named 'horse', represented by

horse *HORSE*

to 'represent' individual horses as suggested by Zoltán Szabó[14]—a representative of the representation view and this by the comparison with iconic signs, maps, photographs, since an icon represents what it presents. The name 'horse' *designates* individual concrete horses, but it does not represent them.

6 Theoretical Approach to the Problem of Mutual Relations Between Tokens and Types of Signs

6.1 Syntax

Formalizing the language syntax, both on the level of concretes and that of types, relations between sign-tokens themselves and sign-types themselves,[15] that is, respectively:

$$E \to E \qquad \text{and} \qquad T \to T$$

a) they can be determined by grammatical rules of the given language; they are then relations of *forming signs consisting of simpler ones*, or
b) they can be determined by indiscernibility of signs from the pragmatic point of view; then, they are relations of: identifiability (broadly understood equiformity) of tokens $\simeq$ and identity of types $=$, respectively, that is:

$$e \simeq e' \quad \text{and} \quad t = t', \quad \text{for } e, e' \in E, t, t' \in T$$

whereby identifiable tokens e and e' are customarily not the same, that is:

(1) $e \simeq e' \to (e \neq e' \text{ or } e = e')$,

while the identical types t and t' are defined (represented) by the same tokens, i.e. if these types are equivalence classes of identifiable tokens, then

(2) $t = t' \to \forall e \in E\ (e \in t \leftrightarrow e \in t')$.

Relations between tokens and types, that is:

$$E \to T \qquad \text{and} \qquad T \to E$$

are defined by the ways of:
c) creating a type out of tokens of set E as well as, respectively,

[14] See [11].

[15] By E we denote the set of token-signs, briefly—tokens, and by T—the set of sign-types, briefly—types.

d) indicating tokens out of set E by a type.

In the case of way c), it denotes a relation of *constructability of a type out of tokens* (the *concretistic* version: tokens are elements of types as sets, that is classes; they are representatives of types; set-types can be then equivalence classes of identifiable tokens).

In the case of way d), the relation is *exemplification*, concretization of type (the Platonizing version: types are exemplified by tokens, they are represented by them).

Note 1 It can be proved that in syntactic considerations on language, advocating of only one of the stances: *concretistic* or *Platonizing* can be omitted.[16] Both are equivalent, which points to the advantage of the concretistic one, though.

6.2 Semantics

Word- and expression-types, as abstract beings, creations formed out of tokens, whose only mutual feature is structural similarity (spelling or phonographic), thus not necessarily physical similarity, the same shape or the same sound, can constitute the basis of syntactic studies. Such linguistic expression-types are usually devoid of a meaning, or have a number of meanings. If we wish to invest them with a semantic-pragmatic property and assign a meaning or meanings to them, with which they are used in language and the explanation of which is contained in concrete dictionaries of language, they should be treated as generated not only due to the external structural characteristic, the form of their *linguistic externals* (*tokens*), but also with respect to how these tokens are applied by language users in the given situational-linguistic context with reference to the existing reality and the studied one, e.g. linguistic.

The starting point then is the relation *use* of *using tokens of language by users of the language with reference to extralinguistic objects* called their *referents* or *correlates*, or *subject references*.

The basic semantic notions: *meaning*, *designation*, *denotation of a language sign* are defined on the *level of types*, therefore for sign-types. In the semantic-pragmatic approach, their definitions require, however, referring to semantic-pragmatic concepts of the *level of concretes*, in particular to the notion of relation *use of using tokens*, as well as the notion *Use of using their types*, defined by means of relation *use* in the following manner:

(D*Use*) User *u uses type t in the sense Use* iff he/she uses a token of type *t* in the sense *use*.

Language *tokens* (concretes) alone do not have either a meaning or designates, or denotation (they can be used only in the sense *use* with the meaning defined by their types).

[16] This statement is justified in my articles [14] and [15].

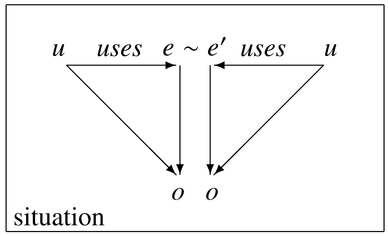

Fig. 1

Nevertheless, two different tokens e and e' can have the same subject reference o. The relation connecting tokens e and e' is then the *relation* $\sim$ *of the same manner of usage of tokens by a user of language* (see Figure 1).

The meaning of *type of language sign t* is defined as the common property of sign-types having the same manner of usage as t, whereas the relation connecting types t and t' is then the *relation* $\approx$ *of the same manner of usage of types t* and t' defined as follows:[17]

(D$\approx$) *$t \approx t'$ iff any user of language uses type t in the sense Use iff he/she uses also type t' in the sense Use as well as uses—in the sense use—a token e of type t with reference to the object o iff he uses also—in the sense use—a token e' of type t' with reference to the same object o.*

6.3 Signs in Language Communication

A communication act between sender s and receiver r of a sign involves sign-tokens. The figures below (Fig. Fig. 2a and 2b) illustrate the situation in which we come to deal with communication acts by means of one or two different sign-tokens e and e', where o denotes the object of reference of token e, while o'—the object of interpretation of sign e or e'.

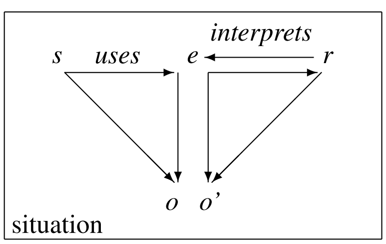

Fig. 2a

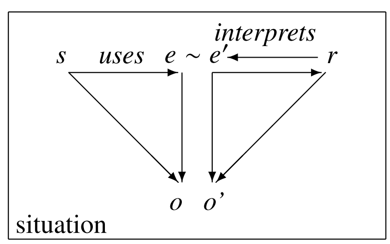

Fig. 2b

The *general notion of language communication between potential sender* S *and potential receiver* R requires a sign-type t to intermediate in it.

[17] Cf. [16].

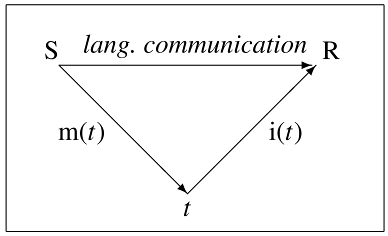

Fig. 3

This notion is illustrated in Figure 3, where m(t) means the meaning of sign-type t in it, whereas i(t)—its interpretation.

7 Conclusions

From the theoretical and logical point of view:

- working out some theoretical conception of language must take into account its bi-aspectual characteristics: as a language of expression-concretes and as one of expression-types,
- supporting one of the following standpoints:

  a) types exist independent of their tokens (the *Platonizing stance*) or,
  b) that it is not so (the *concretistic stance*),

  can be neglected in syntactic considerations on language.
- Mutual relations between sign-tokens differ from the mutual ones between sign-types, yet
- Determining the mutual relations between sign-types and sign-tokens depends on advocating one of the above-mentioned standpoints: a) or b),
- Semantic notions or semantic-pragmatic ones of language such as: *meaning and denotation*, *interpretation*, should be defined exclusively for types of tokens, but their definitions require some reference to the functions which signs perform in language (words, expressions) concretes, or to relations between them,
- The notion of an act of language communication differs from the general notion of language communication: the first is defined by means of sign-concretes, while the other—by means of sign-types.

Acknowledgements I wish to thank the Reviewers of the article for their penetrating comments and kind suggestions, which enabled me to improve on the first draft of the manuscript.

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On the Structure and Contents of the Monograph

The book is a collection of papers and aims to unify the questions of syntax and semantics of language, which are included in logic, philosophy and ontology of language. The leading motif of the presented selection of works is the differentiation between linguistic tokens (material, concrete objects) and linguistic types (ideal, abstract objects) following two philosophical trends: nominalism (concretism) and Platonizing version of realism. The opening article under the title "The Dual Ontological Nature of Language Signs and the Problem of Their Mutual Relations" provides a broad introduction into the problem area connected with this differentiation, while the logic-formal characteristics of the distinction are framed in the work entitled "On the Type-Token Relationships" (Chapter 1).

The basic part of the book deals with issues relating to syntax (Chapters 2-4) and semantics of language (Chapters 5-6), as well as pertaining to syntactic-semantic-pragmatic questions (Chapters 7-13). Throughout the book, language, categorial language, is characterized syntactically as generated by classical categorial grammar (Chapter 2) and formalized on two opposing levels: as language of expression-tokens (level of *tokens*) and language of expression-types (level of *types*).

The author's considerations contained in Chapters 2 and 4 lead to the important philosophical conclusion that in formal-logical syntactic studies on language the assumption that expression-types constitute the primary language layer while expression-tokens make the secondary one, can be neglected; thus, this speaks in favour of the opposing standpoint—the concretistic one—in the ontology of language syntax.

In the works "Meaning and Interpretations", Parts I and II (Chapters 5 and 6), it is underlined, however, that such semantic concepts as: meaning, denotation and interpretation are defined on the *types level*, yet their formal definitions require making use of notions of the *tokens level*. The semantic notions introduced in the above-mentioned articles are also used in the following works of the present selection, under the titles: "Three Principles of Compositionality" and "On Metaknowledge and Truth" (Chapters 7 and 8). They formalize two principles of compositionality that are well known in the literature on the subject, deriving from Frege, i.e. those of meaning and of denotation; they are related to the syntactic principle of compositionality which was introduced by the author. All the three principles are, at the same time, three conditions of homomorphism of categorial language algebra into three kinds of non-standard models of language (one syntactic and two semantic ones: intensional and extensional), which allows introducing three definitions of truthfulness into these models.

The next two works in the collection, entitled: "On Language Adequacy" and "What is the Sense in Logic and Philosophy of Language" (Chapters 9 and 10) concern adequacy of categorial language syntax along with its dual semantics: intensional and extensional, and categorial compatibility of any of its syntactic categories with two corresponding semantic categories: intensional and extensional, based on the compatibility the syntactic category of each language expression with the ontological category assigned to its denotatum. The well-known problem of categorial compatibility for first-order quantifiers finds its solution in the paper "Categories of First-Order Quantifiers" (Chapter 11).

In the work "Logic and Ontology of Language" (Chapter 12), being in a sense a summary of the considerations presented in the preceding chapters of the book, language is treated as an ontological being, characterized in compliance with the logical conception of language proposed by Ajdukiewicz. Application—like throughout the book—of tools of classical logic and set theory has resulted in emergence of a general formal logical theory of syntax, semantics and pragmatics of language, which takes into account duality in the understanding of linguistic expressions as tokens (concretes) and types (abstract objects). In terms that take into account a functional approach to language itself, there comes out an ontological neutrality of logic with respect to existential assumptions relating to the ontological nature of linguistic expressions and their extra-linguistic ontological counterparts.

The issues connected with applying logic while explaining the manner of using linguistic tokens and linguistic types to determine notions of language communication are raised and illustrated in the last chapter of the work, bearing the title "A Logical Conceptualization of Knowledge on the Notion of Language Communication".

Note. All the older works which were published in the past will present terminology adjusted to the whole as well as contemporarily applied set of formal-logical and set theory symbols. All the works will have an abstract and key words attached. In the version prepared for printing they are adapted to SUL Springer style and typeset in the LaTeX system by my co-worker Dr. Zbigniew Bonikowski.

Part II
Selected Works

Chapter 1
On the Type-Token Relationships

Urszula Wybraniec-Skardowska

Abstract Taking into account the dual ontological status of linguistic objects as tokens (material, concrete objects) and types (abstract objects), the paper outlines two different but equivalent, axiomatic approaches to the formalization of the theories of concrete and abstract words (the *Tk* of word tokens theory and the theory *Tp* of word types). Theories *Tk* and *Tp* are formalized on two levels. The first level of formalization of the theory *Tk* are tokens whose primary existence is assumed here, and the second level are types having derivative existence. The formalization levels for the theory *Tp* are the opposite. It can be proved that the two theories *Tk* and *Tp* are equivalent, which leads to the conclusion that in the syntactic research of language one can omit the assumption of the primary existence of linguistic types. These two different approaches to the ontology of language are fully presented in the author's work in Polish: "O dwóch podejściach do formalizacji teorii napisów" (Zeszyty Naukowe WSP w Opolu, Matematyka 27 (1990), 33–48). The ideas contained in this work are also used in the other works of this volume.

Key words: Token-type distinction • Concrete and abstract words • Axiomatic formalizations of theories of concrete and abstract words • Theory of word tokens • Theory of word types • Two ontologically opposite kinds of formalization theory of tokens and theory of types • Equivalent theories

1. The dual ontological status of linguistic objects revealed due to the distinction between "type" and "token" introduced by Ch.S. Peirce can be a base of the two-fold, both theoretical and axiomatic, approach to the language. In [1] referring to some ideas included in A.A. Markov's work [2] and in some earlier papers of the author ([4], [5] and [7]), the problem of formalization of the concrete and abstract words theories raised by J. Słupecki was solved. The construction of the theories presented in the above mentioned papers has two levels. The axiomatic theory of linguistic tokens: material, physical linguistic objects, constitutes the first one. Linguistic

This is a slightly adjusted original paper in *Bulletin of the Section of Logic of the Polish Academy of Science*, Institute of Philosophy and Sociology **15** (4), 164–171 (1986).

types, according to the literature of the subject, are defined on the other level as equivalence classes of equiform linguistic-tokens. Assuming the opposite point of view, one can accept that theory of linguistic-types: abstract objects, in which the theory it is possible to define the notion of linguistic token as well as the derivative notions should become the basis of formalization of the theory of linguistic tokens and the theory of language in general. The axioms and definitions of both theories of linguistic objects: *Tk* and *Tp* representing the other approach to the ontology of language are included in the sequel of the paper. The foundations of the theory of linguistic tokens *Tk* in which the primary assumption as to the linguistic types existence is superfluous have been referred on the basis of the monography [7]. The basis of the theory of linguistic types *Tp* which takes into account the other position has to be presented here for the first time.

2. The theories *Tk* and *Tp* are added to the theory of functional calculus with identity and to set theory. The primitive notions of the former are:

$$U, \approx, c, V,$$

i.e. respectively: the set of all linguistic-tokens, binary equiformity relation and ternary concatenation relation defined in the set U, the vocabulary of word-tokens.

The primitive notions of the theory *Tp* are:

$$\overline{U}, \cdot, V,$$

i.e. respectively: the set of all linguistic types, a binary function of concatenation of linguistic types and vocabulary of word-types.

Writing the axioms of the theory *Tk* down (resp. *Tp*) we assume that the variables

$$p, q, r, s, t, u, v \text{ (resp. } \overline{p}, \overline{q}, \overline{r}, \overline{s}, \overline{t}, \overline{u}, \overline{v}),$$

with subscripts or without them, run over the set U (resp. $\overline{U}$), while the letter X (resp. $\overline{X}$), with a subscript or without – the family 2^U (resp. $2^{\overline{U}}$).

The expression "$p \approx q$" is read: "linguistic tokens p and q are equiform", or shortly: "tokens p and q are equiform".

We read the expression "$c(p, q, r)$" as: "the linguistic token r is a concatenation of linguistic tokens p and q" and the expression "$\overline{r} = \overline{p} \cdot \overline{q}$ as: "the linguistic type $\overline{r}$ is a concatenation of linguistic types $\overline{p}$ and $\overline{q}$".

Let us note that the concatenation relation c need not be a function because it is possible to obtain many equiform tokens as concatenation of two tokens.

In the notation of some axioms of the theories *Tk* and *Tp* we shall use terms: "W" and "$\overline{W}$" which denote the set of all word-tokens and the set of all word-types, respectively. They are defined as follows:

in the theory *Tk*

$D1.$ $W = \bigcap\{X \mid V \subseteq X \wedge \forall r \forall p, q \in X(c(p, q, r) \Rightarrow r \in X)\},$

in the theory *Tp*

$\overline{D}1.$ $\overline{W} = \bigcap\{\overline{X} \mid \overline{V} \subseteq \overline{X} \wedge \forall \overline{p}, \overline{q} \in \overline{X}(\overline{p} \cdot \overline{q} \in \overline{X})\}.$

The sets W and $\overline{W}$ are the smallest sets of appropriate linguistic objects included vocabularies and closed with respect to a suitable concatenation.

The following expressions are the axioms of the theory *Tk*:

$A1.$ a) $p \approx p$,
b) $p \approx q \Rightarrow q \approx p$,
c) $p \approx q \wedge q \approx r \Rightarrow p \approx r$,
$A2.$ $\exists r\ c(p, q, r)$,
$A3.$ $c(p, q, r) \Rightarrow \neg(r \approx p) \wedge \neg(r \approx q)$,
$A4.$ $c(p, q, t) \wedge c(r, s, u) \wedge p \approx r \wedge q \approx s \Rightarrow t \approx u$,
$A5.$ $c(p, q, s) \wedge c(s, r, t) \wedge c(q, r, v) \wedge c(p, v, u) \Rightarrow t \approx u$,
$A6.$ $c(p, q, t) \wedge c(r, s, t) \Rightarrow (p \approx r \Leftrightarrow q \approx s)$,
$A7.$ $c(p, q, r) \wedge s \approx r \Rightarrow c(p, q, s)$,
$A8.$ $c(p, q, t) \wedge c(r, s, u) \wedge t \approx u \Rightarrow [p \approx r \vee \exists v\ (c(r, v, p) \vee c(p, v, r))]$,
$A9.$ $\emptyset \neq V \subseteq U$,
$A10.$ $p \in V \wedge q \approx p \Rightarrow q \in V$,
$A11.$ $c(p, q, r) \Rightarrow r \notin V$,
$A12.$ $r \in W \setminus V \Rightarrow \exists p, q \in W\ c(p, q, r)$,
$A13.$ $r \in W \wedge c(p, q, r) \Rightarrow p, q \in W$.

The following expressions are the axioms of the theory *Tp*:

$\overline{A}1.$ $\cdot : \overline{U} \times \overline{U} \to \overline{U}$ — the concatenation $\cdot$ is a binary function in the set $\overline{U}$,
$\overline{A}2.$ $\overline{p} \cdot \overline{q} \neq \overline{p} \wedge \overline{p} \cdot \overline{q} \neq \overline{q}$,
$\overline{A}3.$ $(\overline{p} \cdot \overline{q}) \cdot \overline{r} = \overline{p} \cdot (\overline{q} \cdot \overline{r})$,
$\overline{A}4.$ $\overline{p} \cdot \overline{q} = \overline{r} \cdot \overline{s} \Rightarrow (\overline{p} = \overline{r} \Leftrightarrow \overline{q} = \overline{s})$,
$\overline{A}5.$ $\overline{p} \cdot \overline{q} = \overline{r} \cdot \overline{s} \Rightarrow [\overline{p} = \overline{r} \vee \exists \overline{u}\ (\overline{p} = \overline{r} \cdot \overline{u} \vee \overline{r} = \overline{p} \cdot \overline{u})]$,
$\overline{A}6.$ $\emptyset \neq \overline{V} \subseteq \overline{U}$,
$\overline{A}7.$ $\overline{p} \cdot \overline{q} \notin \overline{V}$,
$\overline{A}8.$ $\overline{r} \in \overline{W} \setminus \overline{V} \Rightarrow \exists \overline{p}, \overline{q} \in \overline{W}\ (\overline{r} = \overline{p} \cdot \overline{q})$,
$\overline{A}9.$ $\overline{p} \cdot \overline{q} \in \overline{W} \Rightarrow \overline{p}, \overline{q} \in \overline{W}$.

The relation $\approx$ is an equivalence relation in the set U of linguistic tokens ($A1$ a–c). By $[p]$ we denote the equivalence class of the relation $\approx$ determined by p.

3. It is possible to define the notions of the theory *Tp* in the theory *Tk*—the sets of linguistic types $\overline{U}$, $\overline{V}$, $\overline{W}$, and also the function of concatenation $\cdot$. We add the following definitions to the axioms of the theory *Tk*:

$D2.$ $\overline{p} \in \overline{U} \Leftrightarrow \exists p\ (\overline{p} = [p])$,
$D3.$ $\overline{r} = \overline{p} \cdot \overline{q} \Leftrightarrow \exists p, q, r\ (\overline{p} = [p] \wedge \overline{q} = [q] \wedge \overline{r} = [r] \wedge c(p, q, r))$,
$D4.$ $\overline{p} \in \overline{V} \Leftrightarrow \exists p \in V\ (\overline{p} = [p])$,
$D5.$ $\overline{p} \in \overline{W} \Leftrightarrow \exists p \in W\ (\overline{p} = [p])$.

We define the notions of the theory *Tk* in the theory *Tp*—the sets of linguistic tokens U, V, W, and the relations of equiformity $\approx$ and concatenation c. Hence we add two axioms and definitions of the above mentioned notions to the axioms of the theory *Tp*:

$\overline{A}10.$ $\overline{p} \neq \emptyset$,
$\overline{A}11.$ $p \in \overline{q}_1 \wedge p \in \overline{q}_2 \Rightarrow \overline{q}_1 = \overline{q}_2$.

$\overline{D}2.$ $p \in U \Leftrightarrow \exists \overline{p}\ (p \in \overline{p})$,
$\overline{D}3.$ $p \approx q \Leftrightarrow \exists \overline{p}\ (p, q \in \overline{p})$,
$\overline{D}4.$ $c(p, q, r) \Leftrightarrow \exists \overline{p}, \overline{q}, \overline{r}\ (p \in \overline{p} \wedge q \in \overline{q} \wedge r \in \overline{r} \wedge \overline{r} = \overline{p} \cdot \overline{q})$,
$\overline{D}5.$ $p \in V \Leftrightarrow \exists \overline{p} \in \overline{V}\ (p \in \overline{p})$,
$\overline{D}6.$ $p \in W \Leftrightarrow \exists \overline{p} \in \overline{W}\ (p \in \overline{p})$.

4. It can be shown that the theories: *Tk* and *Tp* are equivalent. The axioms $A1$–$A13$ and definitions $D1$–$D5$ are the theorems of the theory *Tp* while the axioms $\overline{A}1$–$\overline{A}11$ and definitions $\overline{D}1$–$\overline{D}6$ are the theorems of the theory *Tk*. This justifies that

1° notions of the linguistic tokens and linguistic types, and also of the concatenation relations c and $\cdot$ are mutually definable. Let us note additionally that the theories *Tk* and *Tp* are consistent (cf. [4], [5] and [7]). It is possible, after making a suitable assumption as to the form of the set V, to reconstruct all the axioms of Tarski's metascience [3] formulated for word-types in them.

Theory *Tk* is the core of the theory of the languages considered in [6] and [7]. These theories give full axiomatic, syntactic characteristic of the languages, first on the level of tokens and then, on the level of types. Enriching theory Tp in a visible way we can give full axiomatic characteristic of the languages assuming that the basic language foundation consists of expression-types, the derivative one—of expression-tokens. Then, both approaches to the theory of languages are equivalent. It could result from above that

2° in purely syntactic theoretical researches of language philosophical aspects referring to the nature of linguistic objects may be omitted, however, the possibility of constructing a theory of languages as the theory which does not require initial assumptions as to abstract linguistic objects (theory *Tk*) shows, at the same time that

3° the primary assumption that there exist languages of expression-types is superfluous.

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Chapter 2
On the Axiomatic Systems of Syntactically-Categorial Languages

Urszula Wybraniec-Skardowska

Abstract The paper is a summary of the Polish monograph *Teorie języków syntaktycznie kategorialnych* (PWN, Warsaw–Wrocław (1985)) and its English edition: *Theory of Language Syntax. Categorial Approach* (Kluwer Academic Publishers, Dordrecht–Boston–London (1991)).

Key words: Theory of syntactic categories • Categorial grammar • Categorial indices • Syntactically categorial languages • Fundamental theorem of the theory of syntactic categories • Axiomatic systems of languages of expression tokens • Axiomatic systems of languages of expression types

1 Introduction

In the monograph four axiomatic systems of syntactically-categorial languages are presented. The first two refer to languages of expression-tokens. The others also takes into consideration languages of expression-types.

Generally, syntactically-categorial languages are languages built in accordance with principles of the theory of syntactic categories introduced by S. Leśniewski [4]; they are connected with the Ajdukiewicz's concept [1] which was a continuation of Leśniewski's idea and further developed and popularized in the research on categorial grammars, by Y. Bar-Hillel [2], [3].

To assign a suitable syntactic category to each word of the vocabulary is the main idea of syntactically-categorial approach to language. Compound expressions are built from the words of the vocabulary and then a suitable syntactic-category is assigned to each of them. A language built in this way should be decidable, which means that there should exist an algorithm for deciding about each expression of it, whether it is well-formed or is syntactically connected (in sense of Ajdukiewicz [1]).

This is a little corrected version of the original paper published in *Bulletin of the Section of Logic of the Polish Academy of Science*, Institute of Philosophy and Sociology **13** (4), 241–251 (1984).

The traditional, originating from Husserl, understanding of the syntactic category confronts some difficulties. This notion is defined by abstraction using the concept of belonging two expressions to the same syntactic category (see Ajdukiewicz [1]).

If we use the following expression:

(α) $r(p/q)s$

which we read: the expression r rises from s by the replacement of its constituent q by p; and the expressions:

(β) p and q are expressions of the same syntactic category;
(γ') r and s are expressions of the syntactic category of sentences;
(γ'') r and s are well-formed expressions;

then the schemas usually found definitions of this notion are expressions:

(I') $\alpha \Rightarrow (\beta \Leftrightarrow \gamma')$ and
(I'') $\alpha \Rightarrow (\beta \Leftrightarrow \gamma'')$.

Defining the notion of syntactic category we refer to the concept of either a sentence or a well-formed expression. The assumption, in theoretical considerations of a language, that these notions are primitive concepts seems to be groundless. An attempt of defining a concept of sentence in such a way so as to construct an algorithm of testing whether an expression is well-formed—analogous to that of Ajdukiewicz's [1], with simultaneous keeping of the definition on the scheme (I') can lead to the vicious circle. On the other hand, accepting the definition on the scheme (I'') we should agree for instance that the functors appearing in the well-formed expressions 1° and 2° (sentences and names) such as:

$$1^\circ \begin{cases} \text{John loves Helen} \\ \text{John and Helen} \end{cases} ; \qquad 2^\circ \begin{cases} 2 = 2 \\ 2 + 2 \end{cases}$$

ought to classified as belonging to the same syntactic category, although the first is a sentence-forming functor and the second is the name-forming functor.

One of the main aims of the paper is to remove the difficulties mentioned, above. So called "the fundamental theorem of the theory of syntactic categories" (briefly: *fttsc*) can be proved in each of the four presented theories of syntactically-categorial languages. Applying in its notation the expression:

(γ) r and s are expressions of the same syntactic category, its scheme has the form:
(I) $\alpha \Rightarrow (\beta \Leftrightarrow \gamma)$.

2 The Theories *TLTk* and *TETk*

The foundation of all four presented systems of syntactically-categorial languages is *TLTk* i.e., the theory of linguistic tokens and its extension, *TETk* i.e., the theory of expression-tokens. *TLTk* is based on the classical first-order functional calculus with identity and on set theory. Its primitive concepts are: the *set U of all linguistic* *tokens*, *binary equiformity relation* $\approx$ *in* U, *ternary concatenation relation* c in this set, and the *vocabulary* V . On the basis of this theory, the notion of the *set* W *of all word-tokens* is defined. It is the smallest set containing the vocabulary V and closed under concatenation relation c. *TLTk* is equivalent to the theory presented in the paper [6]. It describes the properties of any linguistic tokens (which are visually perceptible objects), regardless of how they are constructed or what symbolism is used for their notation. The axioms characterizing properties of the relations $\approx$ and c are the same as those of [6]. Together with them, there are following axioms of *TLTk* which are characterizing properties of sets U, V and W :

$A1.$ $\emptyset \neq V \subseteq U$;
$A2.$ $p \in V \wedge q \approx p \Rightarrow q \in V$;
$A3.$ $c(p, q, r) \Rightarrow r \notin V$;
$A4.$ $r \in W \setminus V \Rightarrow \exists p, q \in W\ c(p, q, r)$;
$A5.$ $r \in W \wedge c(p, q, r) \Rightarrow p, q \in W$.

Writing the axioms $A1$–$A5$ we make an agreement that variables: p, q, r, s, t, u, v, ... with subscripts or without them are representing any linguistic tokens.

In *TLTk* we define, in a natural way, by induction, a *generalized* $n+1$*-argument concatenation relation* c^n ($n \geq 2$). We read the expression: $\mathsf{c}^n(p_1, \ldots, p_n, p_0)$, in the following ways: p_0 is n-ary concatenation of tokens $p_1, \ldots, p_n$.

In syntactical analysis of categorial languages we use *categorial indices* introduced to semiotics by Ajdukiewicz [1]. By means of them we define two basic notions of *TETk*: the concept of *language-expression* and *syntactic category*. The *set* I *of all categorial indices* is defined in *TETk* by the *set* I_0 *of all basic indices*, which is a primitive concept of this theory. The set I is the smallest set containing I_0 and closed under the relation c. We postulate for sets I_0 and I that they satisfy expressions resulting from axioms $A1$–$A5$ by replacement of the, symbols "V" and "W" by the symbols "I_0" and "I", respectively. So, categorial indices are linguistic tokens. They are not words of the language, because we postulate the axiom:

$A6.$ $V \cap I_0 = \emptyset$;

from which it follows:

Theorem 1 $W \cap I = \emptyset$.

Categorial indices are "attached" to words of the language by the *relation* ι *of assigning of indices to words*, which is a new primitive notion of *TETk* characterized by the following axioms:

$A7.$ $\iota \subseteq W \times I \wedge \iota$ is a function;
$A8.$ $p \in D(\iota) \wedge q \approx p \Rightarrow q \in D(\iota) \wedge \iota(q) \approx \iota(p)$.

The third and the last primitive concept of *TETk* is a one-to-one *function* ρ *of building compound expressions*. Its left domain is the union of all finite, greater than one, Cartesian powers of the domain $D(\iota)$ of the function ι (i.e. the set of all words having categorial indices). The right domain of ρ is a subset of the set $D(\iota) \setminus V$. Namely, the following expression is an axiom of *TETk*:

*A*9. $\rho\colon \bigcup_{k=2}^{\infty} D(\iota)^k \underset{1-1}{\longrightarrow} D(\iota) \setminus V$;

Another axiom of *TETk* is:

*A*10. $p = \rho(p_0, p_1, \ldots, p_n) \Rightarrow [q \approx p \Leftrightarrow$
$\Leftrightarrow \exists q_0, q_1, \ldots, q_n (q = \rho(q_0, q_1, \ldots, q_n) \wedge \forall 0 \leq k \leq n\ (q_k \approx p_k))]$.

We read the expression being the antecedent of the implication *A*10, in the following-way: p is a compound expression-token built from $n + 1$ word-tokens: the main functor p_0 and its successive arguments $p_1, \ldots, p_n$.

The expression $\rho(p_0, p_1, \ldots, p_n)$ can be treated as an translation into the language of *TETk* of such a compound language-expression which is composed from the functor p_0 and its successive arguments $p_1, \ldots, p_n$. The word "translation" here has the meaning which is in agreement with that introduced by A. Tarski in [5]. A translation of any compound expression-tokens does not depend on the symbolism in which it was written; in particular, it does not depend on whether the notation with or without brackets was used. It depends, however, on the words it was composed from and on relations between them.

In the theory *TETk* we define: the *set E_s of all simple expression-tokens* (as the set $V \cap D(\iota)$), the *set E_c of all compound expression-tokens* (as a counter-domain of the function ρ), the *set E of all expression-tokens* (as the union of sets E_s and E_c). To the notional apparatus of this theory we also introduce the defined notion of the *constituent of a given expression-token*.

In *TETk*, the following two definitions are assumed:

Definition 1 $C_\xi = \{p \in E \mid \iota(p) \approx \xi\}, \xi \in I$.

Definition 2 $p =_C q \Leftrightarrow \exists \xi \in I\ (p, q \in C_\xi)$.

The first one defines a *syntactic category of an index ξ* as the set of all these expression-tokens whose index is equiform to ξ. The second definition defines the *categorial conformity relation*. The expression: $p =_C q$, will be used in the formulation of *fttsc*. We read it: expressions p and q belong to the same syntactic category.

Corollary 1 *The relation $=_C$ is an equivalence relation in the set E.*

3 Systems *TSCL* and *TSCω-L*

The system *TSCL* concerns simple languages of expression-tokens. In expressions of these languages there are no operators and variables bound by them. The system *TSCω-L* is a modification of the theory *TSCL* and concerns so-called ω-languages, i.e., languages of expression-tokens in which variable-bounding operators can occur. The main concept defined in these theories is a notion of a well-formed, expression-token of a language (briefly: *wfe*). In both theories the set S of all *wfe's* is defined as follows:

(*) $\quad S = \bigcup_{n=0}^{\infty} {}^{n}S,$

where ${}^{n}S$ is the *set of all wfe's of n-order* ($n \geq 0$) defined in the theory *TSCL* and *TSCω-L* separately.

Namely, in *TSCL* we have:

Definition 3 a) ${}^{0}S = E_s$,
b) $p \in {}^{k+1}S \Leftrightarrow p \in {}^{k}S \vee \exists n \geq 1 \exists p_0, p_1, \ldots, p_n \in {}^{k}S\ [p = \rho(p_0, p_1, \ldots, p_n) \wedge \wedge\, c^{n+1}(\iota(p), \iota(p_1), \ldots, \iota(p_n), \iota(p_0))]$.

A *wfe* of 0-order of a simple language is a simple expression-token of this language. A *wfe* of $k+1$-order of such language is either *wfe* of k-order or a compound expression-token of this language which is composed of *wfe's* of k-order of this language such that the index of the main functor p_0 of the expression p is a concatenation of the index of this expression and indices of successive arguments of this functor.

Let us note that in the factorial notation of Ajdukiewicz, the index of the main functor p_0 of the expression p is of the following form:

$$\frac{\iota(p)}{\iota(p_1)\iota(p_2)\ldots\iota(p_n)}.$$

In the *TSCL* system we assume a new axiom, which is a warrant of non-emptiness of the set S:

$A11.\ \ \iota(S \setminus {}^{0}S) \cap I_0 \neq \emptyset.$

In the definition of the set ${}^{n}S$ in the theory *TSCω-L* appear its primitive terms "0" and "*Vr*" designing the *set of all operators* and the *set of all variables*, respectively.

The sets satisfy the following axioms:

$A11^{\omega}.\ \ 0 \cup Vr \subseteq E_s,$
$A12^{\omega}.\ \ p \in 0 \wedge q \approx p \Rightarrow q \in 0,$
$A13^{\omega}.\ \ p \in Vr \wedge q \approx p \Rightarrow q \in Vr.$

Part b) of the definition of the set ${}^{n}S$ in the system *TSCω-L* is obtained from the part b) of the definition above by adding to the second component of the disjunction, the following clause of conjunction: $p_0 \notin 0$, and by adding the following third component to the disjunction:

$$\exists p_0, p_1, p_2, p_3\ [p = \rho(p_0, p_1, p_2) \wedge p_0 \in 0 \wedge p_1 \in Vr \wedge p_2 \in {}^{k}S \wedge p_3 (fv) p_2 \wedge \\ \wedge p_3 \approx p_1 \wedge \mathsf{c}^3(\iota(p), \iota(p_1), \iota(p_2), \iota(p_0))].$$

where "(fv)" is a defined term denoting the *relation of being a free variable in an expression-token*.

An expression which satisfies the second component of the new disjunction is called a *compound non-operator wfe of k-order of ω-language*. An expression satisfying the third component of the disjunction is called an *operator wfe of k-order* of this language. All the expressions of a finite order of the first kind constitute the set $S^{n\omega}$, those of the second kind—the set S^{ω}. Sets $S^{n\omega}$ and S^{ω} are disjoint and non-empty. We postulate that they satisfy axioms analogous to $A11$.

Remark 1 The definitions of the set S, assumed in the theories *TSCL* and *TSCω-L* gives the possibility to formulate an algorithm syntactic connection (well- formedness of expressions) which is analogous to that of given by Ajdukiewicz [1].

In both theories it is also possible to introduce the following definitions of the *set B of all basic expression-tokens* and the *set F of all functor- tokens*:

(**) a) $B = \{p \in S \mid \iota(p) \in I_0\}$,
b) $F = \{p \in S \mid \iota(p) \in I \setminus I_0\}$.

Theorem 2 *The family of equivalence classes of the relation $=_C$ in the set S is a family of non-empty and disjoint syntactic categories, whose union is equal to the set S. The set S is the union of two non-empty and disjoint sets B and F.*

To formulate *fttsc* in the theories *TSCL* and *TSCω-L* we introduce new notion of a four-argument relation $(/)$ of *replacement of a constituent of a wfe* and an auxiliary concept of relation $(/)^n$ of *replacement of a constituent of n-order of a given wfe*. The definition of relation $(/)^n$ accepted in the theory *TSCω-L* is a modification of the definition of *TSCL*. Both definitions are intuitive but rather complicated. So we omit them here, and we refer the reader to [7].

The definition of relation $(/)$ in *TSCL* and *TSCω-L* is the formula:

(***) $r(p/q)s \Leftrightarrow \exists n\; r(p/q)^n s.$

The symbolic notation of *fttsc* (see schema (I)) is:

Theorem 3 $r(p/q)s \Rightarrow (p =_C q \Leftrightarrow r =_C s).$

According to it, two expressions of a simple language (ω-language) belong to the same syntactic category if and only if replacing one of them by the other in a *wfe* of this language (ω-language) we obtain a *wfe*, which belongs to the same syntactic category.

4 Systems *DTSCL* and *DTSCω-L*

Systems *DTSCL* and *DTSCω-L* are called the dualistic theories of syntactically-categorial languages, because they permit to treat languages from two of points of view, which has a connection with a dual ontological nature of language objects. On the one hand they concern the languages of expression-tokens and the other hand, the languages of expression-types. It is because the theory *DTSCL* (*DTSCω-L*) is a definitional extension of the theory *TSCL* (*TSCω-L*), and the definitions added are the definitions of:

1° sets of abstract language objects, as for example: the set of all linguistic types, the vocabulary of word-types, the set of all word-types, the set of basic abstract indices, the set of all abstract indices, the set of all expression-types, the set of all well-formed expression-types;

2° relations $\overline{\mathsf{c}}, \overline{\mathsf{c}}^n, \overline{\iota}, \overline{\rho}, (\overline{fv})$ corresponding to relations $\mathsf{c}, \mathsf{c}^n, \iota, \rho, (fv)$, respectively, and holding between linguistic-types or word-types.

The sets listed above are defined as quotient sets of suitable sets: U, V, W, I_0, I, E, S by the equiformity relation. The elements of these sets are the appropriate equivalence classes of equiform linguistic-tokens, i.e., linguistic-types. Relations: $\overline{\mathsf{c}}$, $\overline{\mathsf{c}}^n$, $\overline{\iota}$, $\overline{\rho}$, $(\overline{fv})$ hold between equivalences classes of the relation $\approx$ if and only if the relations: c, c^n, ι, ρ, (fv) hold between representatives of the suitable classes.

There is a complete analogy between syntactical notions of languages of expression-tokens and languages of expression-types. The theorems or definitions of the theories *DTSCL* and *DTSCω-L* are all counterparts of axioms, theorems or definitions of the theories *TSCL* and *TSCω-L*, respectively. This fact has a philosophical meaning because in syntactical considerations on language it is penciled to avoid the assumption about existing of ideal objects.

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Chapter 3
The Logical Foundations of Language Syntax Ontology

Urszula Wybraniec-Skardowska

Abstract The paper formalizes the issue of logical syntax of categorial language, taking into account the dual ontological status of linguistic objects treated—on the one hand—as material, physical object-concretes (linguistic tokens) and—on the other one—as abstract ideal objects (linguistic types). At the same time, there is account taken of two opposing approaches towards formalization of language, connected with two trends in ontology of language: nominalism and Platonian version of realism. The author presents two formal theories of language syntax, which derive from opposing ontological assumptions. The first of them departs from the assumption that the primitive layer of language are expression-concretes (expression-tokens) and the secondary one—abstract expressions (expression-types), while the other assumption accepts the opposite standpoint. The author proves equivalence of the presented theories, which allows her to formulate the philosophical thesis that in formal-logical syntactic considerations on language, the assumption of the primary existence of abstract linguistic beings can be neglected.

Key words: Syntax • Categorial grammar • Dual ontological status linguistic objects • Token-types differentiation • Nominalism and realism in ontology of language • Two opposite formalizations of theories of language syntax • Two equivalent formal theories

Introduction

The question about the ontological status of language objects[1] can be brought down to one of the following two problems to be settled:

English translation and adaptation of the paper “Logiczne podstawy ontologii składni języka”, Studia Filozoficzne **6–7** (271–272), 263–284 (1988).

[1] Both in this place and further on in the paper we shall omit questions concerning acoustic languages. The logical foundations of such languages and the axiomatic theory of phonology are presented by T. Batóg [2].

1° Are inscriptions of language, including words and expressions, physical objects of determined shape, spread out in space and time?
2° Are inscriptions of language, including words and expressions, abstract objects, thus certain ideal beings?

The ontological status of inscriptions of language, which are dealt with in these two questions, is different; they belong to different ontological categories. In the logical-semiotic practice, however, we do treat them on a par with each other.

Most frequently, following the differentiation introduced by C.S. Peirce, inscriptions, words, or expressions are understood as either *concretes*, that is *tokens*, *events*—material objects perceivable through the senses, or *types*—classes of equiform tokens being abstract objects. Such a duality of conceiving a language inscription appears in the famous monograph by A. Tarski [6]. It was popularized in particular by R. Carnap in his works published in the 1940s. We adapt them in this article.

The dual ontological character of linguistic objects and making use of them in a dual manner in semiotic analysis, that is as tokens or types, points to the necessity of bi-aspectual characterizing of language in the theoretical, logical conception of it: as a language of expression-tokens and as a language of types of such expressions.

At the same time, elaboration of a given conception cannot but take into consideration the two main currents of language ontology, which are related to the two fundamental ideas that have been formed in the controversy about *universalia*: nominalism and Platonism.

Taking the nominalistic point of view, it is assumed that the basic language dimension are object-tokens, that is concretes. Abstract linguistic objects, that is types of tokens are then constructs of the secondary analysis. On the other hand, while accepting that at the foundation of studies on language lie ideal objects understood as linguistic types, while object-tokens available to cognition through the senses are secondary with reference to these objects, we accept the other ontological settlement that makes reference to Platonism.[2]

A logical conception of ontology of language syntax ought to provide the formal bases to allow both:

1° framing the theory of language in a formalized system based on the primitive concept of the object-token, in which system the abstract notions: type of tokens, words or expressions are secondary, defined constructs (the nominalistic, concretistic approach), and—in opposition—
2° framing the language theory in the formalized system based on the primitive concept of a type of linguistic objects, in which the notions of the *level of concretes*, in particular, inscription-tokens, words or expression-tokens, are defined (the Platonizing, idealistic approach).

The inspiration for considering the first framing is J. Słupecki's work (see [7]), whereas that for the idealistic approach comes from A. Tarski's monograph [6].

[2] Here, it must be clearly emphasized that in the theory of language, by Platonism it is generally understood the conviction of inalienability of senses, i.e. meanings of names and logical judgements.

Both frameworks are presented in this study (Parts II and III) as a development of the ideas sketched in [8] and [9]. Both yield the above-mentioned bi-aspectual syntactic description of language on the basis of the two different ontological stances. It needs underlining that the logic practiced in them is neutral in the controversy about *universalia*. The methodological studies discussed in Part IV lead to the conclusion that both frameworks concerning the syntactic description of language are equivalent and to a conclusion in favour of the concretistic conception of language accepted, e.g. by S. Leśniewski and being convergent with the ontological reism of T. Kotarbiński.

In my work on this article in Polish, I used a number of valuable comments offered to me by Dr Wojciech Buszkowski and Dr Jerzy Perzanowski. I would like to express my sincere thanks to them.

1 Intuitive Understanding of Categorial Language

1.1 Initial Syntactic Characteristic of Language

In order to achieve the aims set in this work, while building a determined theory of language, we shall be interested in categorial languages characterized syntactically, thus—to speak plainly—languages built in compliance with general principles of S. Leśniewski's theories of syntactic categories, in the version modified by K. Ajdukiewicz [1].

Nevertheless, it needs stressing plainly that languages built according to other principles can make formal models of language to which the basic notions considered in both frameworks presented here apply. Part of the considerations relating to both approaches may as well be applied equally effectively, e.g. to a description of languages of N. Chomsky.

Any syntactically characterized language L is defined when a set of all its sentences is determined: in a more general way—the set S of all its well-formed expressions. Defining this set is possible when out of the universal class U of all linguistic objects we are able to differentiate the vocabulary V_1 of all simple words and with the use of ternary concatenation relation c, defined in U, to generate the set W_1 out of it of all words whose subset is S.

The vocabulary V_1 can be once and for all established, closed, like in formalized languages, or open, i.e. containing potential words, like in natural languages. The concatenation relation considered on the level of concretes can be a relation of right-wise (e.g. in the European ethnic language) or left-wise (e.g. in the Hebrew language) adding on the same level a second inscription to the first one. This may be a non-linear relation of connector of inscriptions, like in hieroglyphs or in many mathematical formulae.

The simplest syntactic characteristic of language L is given by the system

$$\text{(L)} \qquad \langle U, V_1, c, W_1, S\rangle.$$

The language L characterized in the above-shown way will be a categorial language, when—while applying W. Buszkowski's nomenclature [4]—the properties of *functoriality* and *typization* will be assigned to its expressions, and additionally—*replaceability*—in traditional frameworks referring to E. Husserl's ideas.

The categorial, compliant with Ajdukiewicz's [1], language analysis which allows distinguishing the set S, takes place by means of categorial indices and concerns exclusively any complex expressions forming the subset E_1^c of the set W_1/V_1. Each expression of the set E_1^c is in this sense *functorial* so that the part (constituent) called the main functor and the parts (constituents) called its arguments can be distinguished.

The categorial indices (types) by means of which we examine the syntactic correctness of complex expressions are inscriptions of set U. They do not belong, however, to the set W_1 of words of language L. They are words of the metalanguage of the language L, serving the purpose of *typization*, that is determining syntactic categories of its expressions and examining their syntactic correctness. For the needs of giving syntactic characteristics of categorial language L, apart from the vocabulary V_1, there is a need for an auxiliary vocabulary V_2, composed of indices of basic and defined auxiliary symbols (e.g. brackets, commas), as well as a set W_2 generated from it using concatenation relation c, of all auxiliary words. The set W_2 includes the set E_2^s of all basic indices and the set E_2^c of all functoral indices, being defined concatenations formed of the indices of the set E_2^s in accordance with given rules, therefore—in sum—the set E_2 of all well-formed indices (types). In theoretical considerations, we replace the rules of forming functoral indices by one binary relation ρ_2 of forming functoral indices, while the set E_2^c is defined as its counter-domain $D_2(\rho_2)$.

Typization of given words of language L is done with the relation e of pointing to indices of the words, in other words typization relation [4]. It is from the set of words possessing indices, that is the domain $D_1(e)$ of the relation e that we distinguish the set E_1 of all expressions of categorial language L, thus both the set E_1^s of words of the vocabulary V_1 possessing indices and called a set of all simple expressions of the language L and the set E_1^c generated from that of E_1^s, of all its functorial compound expressions. The principles of formation of concatenations from words possessing indices, ones that are compound expressions of the given categorial language, are established by syntax rules of this language. In theoretical considerations, we replace these rules by one binary relation ρ_1 of forming complex expressions of the language L, whereas the very set E_1^c is defined as its counter-domain $D_2(\rho_1)$.

The set S of all well-formed expressions of categorial language L can be generated by the system

$$(\mathrm{G}_L) \qquad \langle E_1^s, E_2^s, e, r\rangle,$$

where r is the rule establishing the dependence between an index of any complex expression and an index of its main functor, as well as indices of its arguments. Freely speaking, it says that an index of the main functor of a given complex expression is a functoral index formed from an index of this expression, that is an index of the expression which the functor forms together with its arguments, and the indices of all subsequent arguments of this functor.

A functorial analysis consists in checking whether rule r holds for each constituent of the given functorial expression of the language L, which is an expression of L. If this is so, the given functorial compound expression of language L that belongs to the set S is well-formed.

The system (G_L) can be regarded as a reconstruction of classical categorial grammar, whose idea was set out in [1].[3] Such a grammar is unambiguous: a word or an expression of L has, on the *level of concretes*, with the exactitude to identifiability (equiformity), exclusively one index (type) assigned to it, thus only one syntactic category. We shall accept, with reference to a syntactically characterized categorial language, that it is unambiguously typizable [4]. This assumption is utterly natural, since finitely typizable languages, as for instance, natural languages are usually considered to be (can always be made) unambiguous through "separation" of words or expressions that a finite number of indices are assigned to. This can be accomplished by equipping them, e.g. with relevant digital or literal indices, or with diacritical marks.

Then, the notion of identifiability (equiformity in a board sense) of object-tokens—so elementary to the language L characterized on the level of concretes—can be treated most broadly by investing it with the pragmatic character—this depends on the aims that we are directed by while wanting to obtain defined results. Therefore, graphic or physical semblance of inscriptions does not have to be deciding when it comes to their equiformity or variformity.

The categorial character of language L is also connected with the logical partition $Ct(S)$ of the set S of its well-formed expressions into syntactic categories. The concept of a syntactic category is also linked to relation ($/$) of *replaceability* of expressions. In traditional frameworks, a syntactic category is a set of replaceable expressions in any sentential contexts or—more generally—in well-formed ones. Traditional definitions imply, however, difficulties: they do not exclude the possibility of getting involved in a vicious circle (see [7]). After all, their aptness can be refuted by instances.

Although we depart here from traditional definitions of the notion being discussed, it still remains in a relevant relation with replaceability. This relation is described in the paper by the so-called *fundamental theorem of the theory of syntactic categories*.

Since the syntactically characterized categorial language L will be grasped in a bi-aspectual way, we shall mark it with L^1 when we consider it on the *level of concretes* and with L^2 when it is described on the *level of types*. Also, all the syntactic notions that make the characteristics of the language L^l ($l = 1, 2$) will be marked with relevant terms corresponding to them, with attached digital super scripts 1 or 2, depending on the level of L.

[3] Categorial grammars serve to describe or discover syntactic structures of language by means of indices, in particular—to delineate the syntactic category of sentences of language. Studies on these grammars—competitive towards the well-known grammars of Chomsky—have already boasted of quite an extensive body of literature. A review of the most significant results of these studies and indicating their most important trends are given by W. Buszkowski in [5] (see also [3]).

The syntactic characteristics of the language L^l ($l = 1, 2$) is thus formulated by a system that is much more developed than (L), that is the following one

$$(\mathrm{L}^l) \qquad \langle U^l, \overset{l}{=}, c^l, V_1^l, V_2^l, W_1^l, W_2^l; e^l, \rho_1^l, \rho_2^l, E_1^{sl}, E_2^{sl}, E_1^{cl}, E_2^{cl}; S^l, Ct^l(S^l), (/^l)\rangle.$$

The symbol "$\overset{1}{=}$" means here a binary relation of identifiability (equiformity) of linguistic tokens, the symbol "$\overset{2}{=}$"—an ordinary relation of identity of linguistic types. We shall call the relation "$\overset{l}{=}$"—an equality relation in the sense l.

1.2 Preliminaries to the Theory of Categorial Languages

The theory of categorial languages is a theory of description of notions of the system L^l ($l = 1, 2$). As a result, certain conditions imposed by axioms and definitions of this theory are superimposed on the notions of the system (L^l). The notions of the system (L^l) will be described within two dual theories of the language L^l: T_k^l and T_p^l. The theory T_k^l frames the concretistic approach to language. It is developed on the *level of concretes* as theory T_k^1, and on the *level of types* as the theory T_k^2 built over the former. The theory T_p^l deals with the Platonizing approach to language and is developed first on the *level of types*—as the theory T_p^2, then on the *level of concretes*—as the theory T_p^1 built over T_p^2. The theory T_k^l comes close to those of *TSCL* and *DTSCL* built in [7]. We can find a fragment of the theory T_p^l in [9] (see also [8]).

We obtain axioms, definitions and certain distinguished theorems of the theories T_k^l and T_p^l as substitutes of certain schemata of expressions that we list below, accepting in their designations and recordings, the assumption that $l, x \in \{1, 2\}$. By writing them symbolically, we also accept some notational conventions relating to variables.

Thus, we agree to accept that the letters:

$$p^l, q^l, r^l, s^l, t^l, u^l, \ldots \;\; \text{and} \;\; p^{l_0}, p^{l_1}, p^{l_2}, \ldots, q^{l_0}, q^{l_1}, q^{l_2}, \ldots$$

are variables running over the set U^l, while the letters:

$$p_x^l, q_x^l, r_x^l, s_x^l, t_x^l, u_x^l, \ldots \;\; \text{and} \;\; p_x^{l_0}, p_x^{l_1}, p_x^{l_2}, \ldots, q_x^{l_0}, q_x^{l_1}, q_x^{l_2}, \ldots$$

are variables representing the words of the set W_x^l, the letters ξ^l, σ^l—variables representing the indexes of the set E_2^l, and the letter X^l, on the other hand, a variable running over the family of all the subsets of the set U^l.

The formal recordings of a series of schemata will be accompanied by formulations rendered in words and also by insertions relating the manner of reading of given expressions.

$\mathrm{A}^l 1$ a. $p^l \overset{l}{=} p^l$, b. $p^l \overset{l}{=} q^l \;\Rightarrow\; q^l \overset{l}{=} p^l$,
c. $p^l \overset{l}{=} q^l \;\wedge\; q^l \overset{l}{=} r^l \;\Rightarrow\; p^l \overset{l}{=} r^l$.

The relation $\overset{l}{=}$ is thus a reflexive, symmetrical and transitive relation in the set U^l.

Further, we shall read the expression $p^l \overset{l}{=} q^l$ as: the linguistic object p^l is equal in the sense l to the *linguistic object* q^l, or: the objects p^l and q^l are equal in the sense l.

We shall read the expression $c^l(p^l, q^l, r^l)$, given below, in the following way: the object r^l is a concatenation of the objects p^l and q^l.[4]

$\mathrm{A}^l 2$ $$\exists r^l \; c^l(p^l, q^l, r^l).$$

$\mathrm{A}^l 3$ $$c^l(p^l, q^l, u^l) \wedge c^l(r^l, s^l, t^l) \wedge p^l \overset{l}{=} r^l \wedge q^l \overset{l}{=} s^l \;\Rightarrow\; u^l \overset{l}{=} t^l.$$

$\mathrm{A}^l 4$ $$c^l(p^l, q^l, r^l) \wedge s^l \overset{l}{=} r^l \;\Rightarrow\; c^l(p^l, q^l, s^l).$$

Thus, for each two linguistic objects of the set U^l there exists an object of this set, being their concatenation; concatenations of two pairs of objects of the set U^l with the first and the second elements that are equal in the sense l are objects that are equal in the sense l; an object of the set U^l equal in the sense l to an object being a concatenation of two objects of the set U^l is also a concatenation of these objects.

$\mathrm{A}^l_x 5$ $$V^l_x \subseteq U^l.$$

$\mathrm{A}^l_x 6$ $$p^l \in V^l_x \;\wedge\; q^l \overset{l}{=} p^l \;\Rightarrow\; q^l \in V^l_x.$$

$\mathrm{A}^l_x 7$ $$c^l(p^l, q^l, r^l) \;\Rightarrow\; r^l \notin V^l_x.$$

Thus, the vocabulary V^l_x is a subset of the set of linguistic objects U^l, a word equal in the sense l to a word of the vocabulary V^l_x is also its word; a concatenation of two objects of the set U^l is never a word of the vocabulary V^l_x.

$\mathrm{D}^l_x 1$ $$W^l_x = \bigcap\big\{X^l \mid V^l_x \subseteq X^l \;\wedge\; \forall p^l, q^l \in X^l_x \,\forall r^l \,(c^l(p^l, q^l, r^l) \Rightarrow r^l \in X^l)\big\}.$$

$\mathrm{A}^l_x 8$ $$r^l \in W^l_x \setminus V^l_x \Rightarrow \forall p^l, q^l \in W^l_x \; c^l(p^l, q^l, r^l).$$

The set of words W^l_x is the least set of linguistic objects of the set U^l including the vocabulary V^l_x and closed under the concatenation relation c^l, whereas each word of the set W^l_x, which is not a word of the vocabulary V^l_x, is a concatenation of a certain pair of words of the set W^l_x.

$\mathrm{A}^l 9$ $$e^l \subseteq W^l_1 \times W^l_2.$$

$\mathrm{A}^l 10$ $$D_1(e^l) \cap D_2(e^l) = \emptyset.$$

[4] The notion of concatenation lies at the foundation of many formal models of language, e.g. Chomsky's grammars. Its properties are characterized in the paper by the expressions that follow.

Thus, relations e^l of indicating indices of words of the language L^l are relations whose domain is included in the set of words W_1^l of this language, the counter-domain—in the set W_2^l of its auxiliary words; the domain and the counter-domain of this relation are disjoint sets.

Then the expression $e^l(p^l, q^l)$ is read: the object q^l is an index of the word p^l of the language L^l.

A^{l}11 $$e^l(p^l, q^l) \wedge e^l(r^l, s^l) \wedge p^l \overset{l}{=} r^l \;\Rightarrow\; q^l \overset{l}{=} s^l.$$

A^{l}12 $$e^l(p^l, q^l) \wedge r^l \overset{l}{=} p^l \wedge s^l \overset{l}{=} q^l \;\Rightarrow\; e^l(r^l, s^l).$$

Thus, the indices of words of the language L^l that are equal in the sense l are equal in the sense l; an object equal in the sense l to an index of the given word of the language L^l is an index of a word equal to it in the sense l.

A$_x^l$13 $$D_1(\rho_x^l) = \bigcup_{j=2}^{\infty} D_x(e^l)^j \quad \wedge \quad D_2(\rho_x^l) \subseteq D_x(e^l) \setminus V_x^l.$$

The domain of the relation ρ_1^l of formation of complex expressions of the language L^l (respectively, the relation ρ_2^l of formation of functoral indices of this language) is a set of all finite, greater than 1, Cartesian powers of the set $D_1(e^l)$ of all the words of the language L^l possessing indices (respectively, the set $D_2(e^l)$ of all the indices of such words); the counter-domain of the relation ρ_1^l (respectively, the relation ρ_2^l) is included in the set of all the words of the language L^l possessing an index and not being simple words of this language (respectively, in a set of all the indices of words of the language L^l not belonging to the auxiliary vocabulary).

We read the expression $\rho_1^l(p_1^{l_0}, p_1^{l_1}, \ldots, p_1^{l_n}; p_1^l)$ as: the word p_1^l of the language L^l is a compound expression of this language formed of the main functor $p_1^{l_0}$ and its successive n ($n \geq 1$) arguments $p_1^{l_1}, \ldots, p_1^{l_n}$.

We read the expression $\rho_2^l(p_2^l, p_2^{l_1}, \ldots, p_2^{l_n}; p_2^{l_0})$ as: the auxiliary word $p_2^{l_0}$ of the language L^l is its functoral index formed of the index p_2^l of a word of the language L^l and successive n indices $p_2^{l_1}, \ldots, p_2^{l_n}$ of words of this language.

A$_x^l$14 $$\rho_x^l(p_x^{l_0}, p_x^{l_1}, \ldots, p_x^{l_n}; p_x^l) \;\wedge\; \rho_x^l(q_x^{l_0}, q_x^{l_1}, \ldots, q_x^{l_m}; q_x^l) \Rightarrow$$
$$\Rightarrow \left[q_x^l \overset{l}{=} p_x^l \Leftrightarrow m = n \;\wedge\; \forall 0 \leq k \leq n \, (q_x^{l_k} \overset{l}{=} p_x^{l_k})\right].$$

A$_x^l$15 $$\rho_x^l(p_x^{l_0}, p_x^{l_1}, \ldots, p_x^{l_n}; p_x^l) \;\wedge\; \forall 0 \leq k \leq n \, (q_x^{l_k} \overset{l}{=} p_x^{l_k}) \;\wedge\; q_x^l \overset{l}{=} p_x^l \Rightarrow$$
$$\Rightarrow \rho_x^l(q_x^{l_0}, q_x^{l_1}, \ldots, q_x^{l_n}; q_x^l).$$

Thus, two compound expressions of the language L^l (respectively, two functoral indices of this language) are equal in the sense l iff they are formed of the same number of equal—in the sense l—words of the language L^l in relations with each other (respectively, indices of words of this language); a word of the language L^l

(respectively, an auxiliary word of this language), equal in the sense l to the complex expression p_1^l of the language L^l (respectively, to the functorial index p_2^l of this language), is a complex expression of this language (respectively, it is its functoral index) formed of successive words (respectively, indices of words) equal in the sense l, respectively, words occurring in the same order (respectively, indices of words), of which the word p_1^l (respectively, the index p_2^l) is formed.

$\mathrm{D}_x^l 2$ a. $E_x^{sl} = V_x^l \cap D_x(e^l)$, b. $E_x^{cl} = D_2(\rho_x^l)$, c. $E_x^l = E_x^{sl} \cup E_x^{cl}$.

Thus, the set of all the simple expressions of the language L^l (the set of all the basic indices of this language) is a set of all the words of the vocabulary of this language (a set of all the words of its auxiliary vocabulary), possessing an index (being indices of words); the set of all the complex expressions of the language L^l (the set of all the functoral indices of this language) is the counter-domain of the relation of forming complex expressions (a relation of forming functoral indices); the set of all the expressions of the language L^l (the set of all the well-formed indices of this language) is the union of the set of all its simple expressions (the set of all its basic indices) and the set of all its complex expressions (the set of all the functoral indices).

We denote by ${}^nS^l$ ($n \in N \cup \{0\}$) the set of all the well-formed expressions of the order n of the language L^l.

$\mathrm{D}^l 3$ a. ${}^0S^l = E_x^{sl}$,

b. $p_1^l \in {}^{k+1}S^l \Leftrightarrow p_1^l \in {}^kS^l \vee \exists n \geq 1 \exists p_1^{l_0}, p_1^{l_1}, \ldots, p_1^{l_n} \in {}^k S^l \big[\rho_1^l(p_1^{l_0}, p_1^{l_1}, \ldots, p_1^{l_n}; p_1^l) \wedge$
$\wedge \forall 0 \leq k \leq n \forall p_2^{l_k}, p_2^l \left(e^l(p_1^{l_k}, p_2^{l_k}) \wedge e^l(p_1^l, p_2^l) \Rightarrow \rho_2^l(p_2^l, p_2^{l_1}, \ldots, p_2^{l_n}; p_2^{l_0})\right)\big]$.

c. $S^l = \bigcup_{n=0}^{\infty} {}^nS^l$.

A well-formed expression of the order 0 of the language L^l is a simple expression of this language; a well-formed expression of the order $k + 1$ of the language L^l is either a well-formed expression of the order k of this language or a complex expression of this language, formed of $n + 1$ ($n \geq 0$) of well-formed expressions of the order k of this language such that the index of the main functor of this comound expression[5] is a functorial index formed of the index of this expression and successive indices of subsequent arguments of this functor; a set of all the well-formed expressions of the language L^l is the union of all the sets of its well-formed expressions of a finite order (greater than or equal to zero).

$\mathrm{A}^l 16$ $e^l(S^l \setminus {}^0S^l) \cap E_2^{sl} \neq \emptyset$.

$\mathrm{A}^l 17$ $e^l(S^l) \subseteq E_2^l$.

[5] Compare the rule r in Part I.1.

Thus, there exists a well-formed expression of the language L^l which is not its simple expression, possessing the basic index; the indices of well-formed expressions of the language L^l are well-formed.

We denote by B^l and F^l the set of all the basic expressions of the language L^l and the set of all its functors, respectively.

D^{l}4 $$B^l = \{p_1^l \in S^l \mid \forall p_2^l \, (e^l(p_1^l, p_2^l) \Rightarrow p_2^l \in E_2^{sl})\}.$$

D^{l}5 $$F^l = \{p_1^l \in S^l \mid \forall p_2^l \, (e^l(p_1^l, p_2^l) \Rightarrow p_2^l \in E_2^{cl})\}.$$

The set of all the basic expressions (functors) of the language L^l is a set of all these well-formed expressions whose indices are basic indices (functoral).

We denote by $Ct^l_{\xi^l}$ the syntactic category with the index ξ^l; by $\overset{l}{\sim}$—the relation of syntactic categorial compatibility. We read the expression $p^l \overset{l}{\sim} q^l$: p^l and q^l belong to the same syntactic category.

D^{l}6 $$Ct^l_{\xi^l} = \{p_1^l \in E_1^l \mid \forall p_2^l \, (e^l(p_1^l, p_2^l) \Rightarrow p_2^l \overset{l}{=} \xi^l)\}.$$

D^{l}7 $$p^l, q^l \in E_1^l \Rightarrow (p_1^l \overset{l}{\sim} q^l \Leftrightarrow \exists \xi^l \; p^l, q^l \in Ct^l_{\xi^l}).$$

In accordance with the above definitions the syntactic category with the index ξ^l is a set of all these expressions of the language L^l, whose index is equal in the sense l to that of ξ^l; two expressions of the language L^l belong to the same syntactic category iff they both belong to a syntactic category with an index.

We denote by Ct^l, $Ct^l(S^l)$, $Ct^l(B^l)$, $Ct^l(F^l)$ the families of all the syntactic categories, respectively: well-formed, basic, functorial expressions of the language L^l.

D^{l}8 $$Ct^l = \{Ct^l_{\xi^l} \mid \xi^l \in e^l(E_1^l)\}.$$

D^{l}9
a. $$Ct^l(S^l) = \{Ct^l_{\xi^l} \mid \xi^l \in e^l(S^l)\},$$
b. $$Ct^l(B^l) = \{Ct^l_{\xi^l} \mid \xi^l \in e^l(B^l)\},$$
c. $$Ct^l(F^l) = \{Ct^l_{\xi^l} \mid \xi^l \in e^l(F^l)\}.$$

The family of all the syntactic categories of expressions (well-formed expressions, basic, functorial) is the family of all the syntactic categories with indices being indices of expressions (well-formed, basic, functorial).

We denote by $(/^l)^n$ the relation of replaceability of a constituent of the n-th order of the given expression. We read the expression $r^l(p^l/^l q^l)^n s^l$ as: the expression r^l is formed of the expression s^l by replacement of its constituent q^l of the n-th order by the expression p^l. We read the expression $r^l(p^l/^l q^l)s^l$ in an analogous way, omitting merely the phrase: the n-th order.

D^{l}10 a. $r^l(p^l/^l q^l)^0 s^l \Leftrightarrow s^l, r^l \in E_1^l \ \wedge \ q^l \overset{l}{=} s^l \ \wedge \ p^l \overset{l}{=} r^l$,

$$\text{b. } r^l(p^l/^l q^l)^1 s^l \Leftrightarrow$$

$$\Leftrightarrow \exists\, n \geq 1\ \exists\, s^{l_0}, s^{l_1}, \ldots, s^{l_n}, r^{l_0}, r^{l_1}, \ldots, r^{l_n}\ \big[\rho_1^l(s^{l_0}, s^{l_1}, \ldots, s^{l_n}; s^l) \wedge$$

$$\wedge\, \rho_1^l(r^{l_0}, r^{l_1}, \ldots, r^{l_n}; r^l)\ \wedge\ \exists\, 0 \geq j \geq n\ \big(q^l \overset{l}{=} s^{l_j} \wedge p^l \overset{l}{=} r^{l_j} \wedge$$

$$\wedge\, \forall\, k \neq j, 0 \leq k \leq n\ (s^{l_k} \overset{l}{=} r^{l_k})\big)\big],$$

$$\text{c. } k > 0 \Rightarrow \big[r^l(p^l/^l q^l)^{k+1} s^l \Leftrightarrow \exists\, t^l, u^l\ (r^l(u^l/^l t^l)^k s^l \wedge u^l(p^l/^l q^l)^1 t^l)\big],$$

$$\text{d. } r^l(p^l/^l q^l) s^l \Leftrightarrow \exists\, n\ r^l(p^l/^l q^l)^n s^l.$$

Thus, the expression r^l is formed of the expression s^l by replacement of its constituent q^l of the 0-th order by the expression p^l iff s^l and r^l are expressions of the language L^l such that q^l is equal in the sense l to s^l, whereas p^l is equal in the sense l to r^l; the expression r^l is formed of the expression s^l by replacement of its constituent q^l by the 1-st order expression p^l iff s^l and r^l are compound expressions of L^l formed of the same number of words of this language, equal—in the sense l—in each place with the exception of—at the most—the place where they are equal—in the sense l—to the word q^l and the word p^l replacing it, respectively; the expression r^l is formed of the expression s^l by replacement of its constituent q^l of the $k+1$-th order ($k>0$) with the expression p^l iff r^l is formed of s^l by replacement of its constituent t^l of the k-the order with some expression u^l which is formed of t^l by replacement of its constituent q^l of the 1-st order with the expression p^l; the expression r^l is formed of the expression s^l by replacement of its constituent q^l with the expression p^l iff r^l is formed of s^l by replacement of its constituent q^l of a finite order (greater than or equal to 0) with the expression p^l.

We call two expressions *dual* when one of them is noted exclusively with the use of logical constants, the terms appearing in (L^1) and variables with the upper script 1 or 1_i, where $i \in N \cup \{0\}$, the other differs from the former in this that the upper scripts substitute scripts 2 or 2_i. We also call *dual* the notions appearing in the systems (L^1) and (L^2) in the same places and the terms denoting them. Such dual expressions are, for instance, the expressions $A^1 11$ and $A^2 11$ obtained from the schema $A^l 11$, or the pairs of expressions: $A_1^1 14$ and $A_1^2 14$ as well as $A_2^1 14$ and $A_2^2 14$, obtained from the schema $A_x^l 14$.[6]

Concluding the considerations in this part of the work, let us draw attention to the fact that the formalism of theories T_k^l and T_p^l is based on set theory.

[6] By introducing further in the work, some auxiliary concepts of subsets or sub-relations of the systems (L^1) and (L^2), respectively, we broaden the notion of a pair of dual expressions, terms or concepts, in a natural way.

2 The Formal Theory T_k^l – a Concretistic Approach

The theory T_k^l is a theory of any, yet established, language L^l, whose primitive notions are the following ones of the *level of concrete*s: the set U^1 of all linguistic tokens, the binary relation $\stackrel{1}{=}$ of equiformity defined on the set U^1, the ternary relation c^1 of concatenation defined in this set, the vocabulary V_1^1 of simple word-tokens, the auxiliary vocabulary V_2^1 of auxiliary word-tokens, the binary relation e^1 of indicating indices of word-tokens and two binary relations ρ_1^1 and ρ_2^1, respectively: of forming compound expression-tokens and of forming functoral indices of word-tokens. The remaining concepts of the system (L^1), some auxiliary notions of the *level of concretes* as well as all the notions of the system (L^2) and the dual ones to the auxiliary concepts of the *level of concretes* are defined in the theory T_k^l.

2.1 The Level of Concretes; Theory T_k^1

The meaning of the primitive and derivative terms of the theory T_k^1, and therefore concepts of the set (L^1) and certain auxiliary notions of the *level of concretes* are established by axioms and definitions being, for $l = 1$ and $x \in \{1, 2\}$, substitutions of the schemata given in Sec. 1.2. They are the following: axioms $A^1$1a-c–$A^1$4, $A_1^1$5–$A_1^1$7, $A_2^1$5–$A_2^1$7, definitions $D_1^1$1 and $D_2^1$1, axioms $A_1^1$8 and $A_2^1$8, $A^1$9–$A^1$12, $A_1^1$13–$A_1^1$15 as well as $A_2^1$13–$A_2^1$15, definitions $D_1^1$2a-c and $D_2^1$2a-c, $D^1$3a-c, axioms $A^1$16 and $A^1$17 and definitions $D^1$4–$D^1$9a-c, $D^1$10a-d.

The given axioms and definitions of the theory T_k^1 come close to those accepted jointly in the theories *TLTk*, *TETk*, *TSCL* presented in monograph [7].

We will provide a series of theorems and definitions of the theory T_k^1. Using the terms expressed in words: object, word, expression, index, etc., we shall mean solely linguistic tokens.

T1 $$c^1(p^1, q^1, u^1) \wedge p^1 \stackrel{1}{=} r^1 \wedge q^1 \stackrel{1}{=} s^1 \Rightarrow c^1(r^1, s^1, u^1).$$

T2 $$c^1(p^1, q^1, u^1) \wedge r^1 \stackrel{1}{=} p^1 \wedge s^1 \stackrel{1}{=} q^1 \wedge t^1 \stackrel{1}{=} u^1 \Rightarrow c^1(r^1, s^1, t^1).$$

Thus, a token, being a concatenation of two tokens, is also a concatenation of tokens that are equiform with them; if, on the other hand, the token u^1 is a concatenation of the tokens p^1 and q^1, then the equiform token of u^1 is a concatenation of tokens equiform with p^1 and q^1.

From the definitions $D_1^1$1 and $D_2^1$2 of the sets W_1^1 of all the words and W_2^1 of all the auxiliary words there immediately follow conclusions which can jointly substitute these definitions, respectively. They are substitutions of the schemata (for $x \in \{1, 2\}$):

W_x1 a. $V_x^1 \subseteq W_x^1 \subseteq U^1$,

b. $p^1, q^1 \in W_x^1 \wedge c^1(p^1, q^1, qr^1) \Rightarrow r^1 \in W_x^1$,

c. $V_x^1 \subseteq X^1 \wedge \exists r^1 \exists p^1, q^1 \in X^1 \left(c^1(p^1, q^1, r^1) \Rightarrow r^1 \in X^1\right) \Rightarrow W_x^1 \subseteq X^1$.

The following schemata of the theorems are true:

T_x3 $r^1 \in W_x^1 \setminus V_x^1 \Leftrightarrow \exists p^1, q^1 \in W_x^1\, c^1(p^1, q^1, r^1)$,

T_x4 $r^1 \in W_x^1 \Leftrightarrow r^1 \in V_x^1 \vee \exists p^1, q^1 \in W_x^1\, c^1(p^1, q^1, r^1)$,

T_x5 $r^1 \in V_x^1 \Leftrightarrow r^1 \in W_x^1 \wedge \neg\exists p^1, q^1 \in W_x^1\, c^1(p^1, q^1.r^1)$,

The schema T_x3 follows those of A_x^18 and A_x^17 as well as W_x1b, while the schema T_x4 from those of $W_x$la and T_x3. We obtain the schema T_x5 from those of $W_x$la, A_x^17 and T_x3.

We call elements of the set $W_x^1 \setminus V_x^1$ *complex words* (when $x = 1$), respectively: *complex auxiliary words* (when $x = 2$).

According to T_x3, a complex word (a complex auxiliary word) is an object being a concatenation of two words (two auxiliary words). In compliance with T_x4, a word (an auxiliary word) is either a word of the vocabulary (a word of the auxiliary vocabulary) or a concatenation of a pair of words (auxiliary words). Following T_x5, a word of the vocabulary (the auxiliary vocabulary) is a word (an auxiliary word) not being a concatenation of any pair of words (auxiliary words).

From the schemata T_x4, A_x^16 and A_x^14 there follows the schema of theorems

T_x6 $p^1 \in W_x^1 \wedge q^1 \stackrel{1}{=} p^1 \Rightarrow q^1 \in W_x^1$.

A linguistic object equiform with a given word (an auxiliary word) is also a word (an auxiliary word).

It follows directly from the schema A_x^19 of axioms that the set $D_1(e^1)$ of all the words possessing the index is a subset of the set of all the words, while the set $D_2(e^1)$ of all the indices of words is a subset of all the auxiliary words. Thus, we have

W_x2 $D_x(e^1) \subseteq W_x^1$.

The corollaries below follow from the axioms A^111, A^112 and A^11a:

W3 $e^1(p^1, q^1) \wedge e^1(p^1, s^1) \Rightarrow q^1 \stackrel{1}{=} s^1$,

W4 a. $e^1(p^1, q^1) \wedge r^1 \stackrel{1}{=} p^1 \Rightarrow e^1(r^1, q^1)$

b. $e^1(p^1, q^1) \wedge s^1 \stackrel{1}{=} q^1 \Rightarrow e^1(p^1, s^1)$.

Thus, the index of a word is—up to equiformity—unambiguously determined; the same index corresponds to equiform words, and equiform indices correspond to the same word.

The corollaries W4a,b justify the schema that is analogous with T_x6:

T$_x$6a $$p^1 \in D_x(e^1) \ \wedge \ q^1 \stackrel{1}{=} p^1 \ \Rightarrow \ q^1 \in D_x(e^1).$$

From the schema D$^1_2$2a-c of the definition and the schema A^1_x13 of axioms we obtain the schemata of corollaries:

W$_x$5 $$\text{a. } E_x^{s1} = V_x^1, \qquad \text{b. } E_x^{c1} = D_x(e^1) \setminus V_x^1, \qquad \text{c. } E_x^1 \subseteq D_x(e^1),$$

while from the schemata W$_x$5b and W$_x$2, we get the following schemata of corollaries:

W$_x$6 $$\text{a. } E_x^{c1} = W_x^1 \setminus V_x^1, \qquad \text{b. } E_x^1 \subseteq W_x^1.$$

The sense of these corollaries is quite obvious.

From the axiom A^{1}9 and the corollaries W$_1$5c and W$_2$5c we get the theorem saying that the set of all expressions is disjoint with that of all well-formed indices, that is

T7 $$E_1^1 \cap E_2^1 = \emptyset.$$

From the schemata A^1_x15, A^1_x1a of axioms we obtain the schema of corollaries:

W$_x$7 $$\rho_x^1(p_x^{1_0}, p_x^{1_1}, \ldots, p_x^{1_n}; p_x^1) \ \wedge \ q_x^1 \stackrel{1}{=} p_x^1 \ \Rightarrow \ \rho_x^1(p_x^{1_0}, p_x^{1_1}, \ldots, p_x^{1_n}; q_x^1).$$

Tokens equiform with a compound expression (with the functoral index) formed of the main functor and its successive arguments (of the given index and successive indices) is also a complex expression (a functoral index), formed of this functor and its successive arguments (of this index and indicated successive indices).

We obtain a series of analogous theorems (cf. A^1_x6, T$_x$6, T$_x$6a), whose sense is completely clear, from the following schemata:

T$_x$8
$$\text{a. } p^1 \in E_x^{s1} \ \wedge \ q^1 \stackrel{1}{=} p^1 \ \Rightarrow \ q^1 \in E_x^{s1},$$
$$\text{b. } p^1 \in E_x^{c1} \ \wedge \ q^1 \stackrel{1}{=} p^1 \ \Rightarrow \ q^1 \in E_x^{c1},$$
$$\text{c. } p^1 \in E_x^{1} \ \wedge \ q^1 \stackrel{1}{=} p^1 \ \Rightarrow \ q^1 \in E_x^{1}.$$

The schema T$_x$8a follows immediately from the schemata D^1_x2a, A^1_x6 and T$_x$6a. The schema T$_x$8b follows directly from the schemata D^1_x2b, A^1_x13 and W$_x$7, whereas the schema T$_x$8c—from the schemata D^1_x2c and T$_x$8a,b.

We now move on to theorems relating to the set S^1 of all well-formed expressions of the language L^1.

T9
$$S^1 = \bigcap \big\{ X^1 \mid E_1^{s1} \subseteq X^1 \wedge$$
$$\wedge \forall p_1^1 \, \forall n \geq 1 \, \forall p_1^{1_0}, p_1^{1_1}, \ldots, p_1^{1_n} \in X^1 \, \big[\rho_1^1(p_1^{1_0}, p_1^{1_1}, \ldots, p_1^{1_n}; p_1^1) \wedge$$
$$\wedge \forall 0 \leq k \leq n \, \forall p_2^{1_k}, p_2^1 \, \big(e^1(p_1^{1_k}, p_2^{1_k}) \wedge e^1(p_1^1, p_2^1) \Rightarrow$$
$$\Rightarrow \rho_2^1(p_2^1, p_2^{1_1}, \ldots, p_2^{1_n}; p_2^{1_0}) \big) \Rightarrow p_1^1 \in X^1 \big] \big\}.$$

The set S^1 of all well-formed expressions is the least set of linguistic tokens including the set of all simple expressions and satisfying the condition[7] that it includes each complex expression such that the index of the main functor of this expression is the functoral index formed of the index of this expression and successive indices of successive arguments of its main functor.

We omit the proof of this theorem. In the same way we do not dwell on detailed proofs of a few next theorems which describe the properties of the set S^1; they are modelled on the proofs given in the monograph [7].

Thus, on the basis of the definitions $\mathrm{D}^1 3\mathrm{a}$, $\mathrm{D}_1^1 2\mathrm{c,b}$, $\mathrm{D}^1 3\mathrm{b}$, the axiom $\mathrm{A}_1^1 13$, corollaries $\mathrm{W}_1 5\mathrm{c}$, $\mathrm{W}_1 6\mathrm{b}$, $\mathrm{W}_1 1\mathrm{a}$ and the definition $\mathrm{D}^1 3\mathrm{c}$, the following corollaries can be justified

W8
$$\text{a. } S^1 \subseteq E_1^1 \subseteq D_1(e^1) \subseteq W_1^1 \subseteq U^1,$$
$$\text{b. } S^1 \setminus {}^0S^1 \subseteq E_1^{c1} \subseteq D_1(e^1) \setminus V_1^1 \subseteq W_1^1 \setminus V_1^1.$$

We call the set $S^1 \setminus {}^0S^1$ the *set of all compound well-formed expressions*.

Let us observe that since by virtue of the corollary W8a $S^1 \subseteq D_1(e^1)$, the image $e^1(S^1)$ of the set S^1, with respect to the relation e^1, according to the axiom $\mathrm{A}^1 17$ and the corollary $\mathrm{W}_1 5\mathrm{c}$, is included in $D_2(e^1)$, and because the axiom $\mathrm{A}^1 10$ holds, the theorem saying that a set of all well-formed expressions does not have elements which are common with a set of indices of these expressions is true, that is

T10
$$S^1 \cap e^1(S^1) = \emptyset.$$

There holds the theorem analogous with $\mathrm{T}_1 8$a-c:

T11
$$\text{a. } p^1 \in S^1 \wedge q^1 \stackrel{1}{=} p^1 \Rightarrow q^1 \in S^1,$$

whose proof (see [7]) is based on $\mathrm{D}^1 3$a-c, T8a, $\mathrm{W}_1 7$, $\mathrm{A}^1 1\mathrm{a}$, $\mathrm{T}_1 6\mathrm{a}$, $\mathrm{A}^1 11$, $\mathrm{W}_2 7$.

The direct corollaries with the axiom $\mathrm{A}^1 16$ are the following:

W9a
$$S^1 \setminus {}^0S^1 \neq \emptyset,$$

W10a
$$E_2^{s1} \neq \emptyset.$$

The corollaries below follow from them immediately (see $\mathrm{D}_2^1 2\mathrm{c}$):

[7] Compare the rule r in Sec. 1.1.

W9b $$S^1 \neq \emptyset,$$

W10b $$E_2^1 \neq \emptyset.$$

What is more, it can be easily concluded that

W9c $$^0S^1 = E_2^{s1} \neq \emptyset,$$

W10c $$E_2^{c1} \neq \emptyset.$$

From the corollaries W9a-c, W10a-c and those of W8a,b, $W_1$5a-c, $W_2$5a-c, $W_1$6a,b, $W_2$6a,b and the axiom $A_1^1 5$ or $A_2^1 5$ it follows that

T12 The sets: V_1^1, V_2^1, W_1^1, W_2^1, $W_1^1 \setminus V_1^1$, $W_2^1 \setminus V_2^1$, $D_1(e^1)$, $D_2(e^1)$, $D_1(e^1) \setminus V_1^1$, $D_2(e^1) \setminus V_2^1$, E_1^{s1}, E_2^{s1}, E_1^{c1}, E_2^{c1}, E_1^1, E_2^1, $^0S^1$, $S^1 \setminus {}^0S^1$, S^1, U^1, are non-empty.

It is easy to notice that the following are also non-empty: the set B^1 of all the basic expressions of the language L^1 and the set F^1 of all its functors. The non-emptiness of the first of them follows directly from the definition $D^1 4$, axiom $A^1 16$ and the corollary W3 as well as the theorem $T_2$8a, whereas the non-emptiness of the other—from the definition $D^2 5$, corollaries W9a and W8b, definitions $D^1 3$c,a,b and $D_2^1 2$b, the corollary W3 and the theorem $T_2$8b.

The sets B^1 and F^1, on the strength of their definitions ($D^1 4$ and $D^1 5$), the axiom $A^1 17$, the definition $D_2^1 2$c and the corollaries $W_2$5a,b, are disjoint. It is also obvious that since S^1 is a set of words that possess indices (Corollary W8a), while indices of such words are elements of the set E_2^1 (Axiom $A^1 17$), then by virtue of the definitions $D_2^1 2$c, $D^1 4$ and $D^1 5$: $S^1 = B^1 \cup F^1$. Thus, the following theorem holds:

T13 $$S^1 = B^1 \cup F^1 \ \wedge \ B^1 \neq \emptyset \ \wedge \ F^1 \neq \emptyset \ \wedge \ B^1 \cap F^1 = \emptyset.$$

We can prove easily, too, that

W11 $$\text{a. } (S^1 \setminus {}^0S^1) \cap B^1 \neq \emptyset, \qquad \text{b. } {}^0S^1 \cap F^1 \neq \emptyset.$$

Thus, among the well-formed compound expressions of the language L^1 we will always find a basic expression, while among the simple expressions of this language there will always be a functor.

From the definitions $D^1 4$ and $D^1 5$, axiom $A^1 15$, theorem T11 and the corollary W4a there follow theorems which are analogous with T11:

T14 $$\text{a. } p^1 \in B^1 \ \wedge \ q^1 \overset{1}{=} p^1 \ \Rightarrow \ q^1 \in B^1,$$
$$\text{b. } p^1 \in F^1 \ \wedge \ q^1 \overset{1}{=} p^1 \ \Rightarrow \ q^1 \in F^1.$$

We give the theorems of the theory T_k^1 connected with the notion of syntactic category.

From the definitions $D^1 7$ and $D^1 6$, respectively: the relation $\overset{1}{\sim}$ of syntactic categorial compatibility and the syntactic category with the indicated index of the word, on the basis of the axioms $A^1$1b,c,a and the corollary W3 we obtain the corollary:

W12 a. $p_1^1, q_1^1 \in E_1^1 \;\Rightarrow\; \left[p_1^1 \overset{1}{\sim} q_1^1 \;\Leftrightarrow\; \left(e^1(p_1^1, p_2^1) \;\wedge\; e^1(q_1^1, q_2^1) \;\Rightarrow\; p_2^1 \overset{1}{=} q_2^1\right)\right].$

From the definitions $D^1$7, $D^1$6, $D^1$4 and $D^1$5, on the basis of the corollary W8a we obtain the following theorem:

T15 Relation $\overset{1}{\sim}$ is an equiformity relation in the sets E_1^1, S^1, B^1, F^1.

The sets E_1^1, S^1, B^1, F^1 are non-empty. The relation $\overset{1}{\sim}$ establishes thus a logical partition of these sets into non-empty and disjoint equivalence classes, jointly yielding a relevant set. Let us observe further that on the strength of the definition $D^1$6 and the corollary W3, if p^1 belongs to the set E_1^1 (respectively: S^1, B^1, F^1) and $e^1(p^1, \sigma^1)$, then $p^1 \in Ct_\delta^1$ and if $\xi^1 \overset{1}{\sim} \delta^1$, then $Ct_\xi^1 = Ct_\delta^1$, and because the index of an expression (an element of the sets S^1, B^1, F^1) up to equiformity is—by virtue of the corollary W3—determined unambiguously, on the basis of the definitions $D^1$7 and $D^1$6, we can easily conclude that the equivalence class with a representative in the set E_1^1 (S^1, B^1, F^1) is the syntactic category with the index determined by the suitable representative of this class. In this way, we have justified the following theorem:

T16 The family Ct^1 ($Ct^1(S^1)$, $Ct^1(B^1)$, $Ct^1(F^1)$) of all syntactic categories of expressions (well-formed, basic, functor expressions) of the language L^1 is equal to the quotient family $E_1^1/\overset{1}{\sim}$ ($S^1/\overset{1}{\sim}$, $B^1/\overset{1}{\sim}$, $F^1/\overset{1}{\sim}$); it is a logical partition of the set E_1^1 (S^1, B^1, F^1) determined by the relation $\overset{1}{\sim}$.

Furthermore, on the basis of the theorems T16 and T13 we obtain the theorem:

T16a The set S^1 is the union of the union of all the basic syntactic categories of the language L^1 and the union of all the functorial categories of this language, which is disjoint with it.

Let us note that from the definition $D^1$10a-c, theorem $T_2$8c, axioms $A^1$1b,c and $A^1$15, as well as the definition $D^1$10d there follows the theorem given below:

T17 $r^1(p^1/^1q^1)s^1 \;\wedge\; r_1^1 \overset{1}{=} r^1 \;\wedge\; p_1^1 \overset{1}{=} p^1 \;\wedge\; q_1^1 \overset{1}{=} q^1 \;\wedge\; s_1^1 \overset{1}{=} s^1 \;\Rightarrow\; r_1^1(p_1^1/^1q_1^1)s_1^1.$

In order to prove the *fundamental theorem of the theory of syntactic categories*, which was mentioned in Sec. 1.1, concerning relations between the notions of syntactic category of replaceability of expressions, it is necessary to justify the following two lemmas:

$$r^1, s^1 \in S^1 \;\wedge\; r^1(p^1/^1q^1)^n s^1 \;\wedge\; r^1 \overset{1}{\sim} s^1 \;\Rightarrow\; p^1 \overset{1}{\sim} q^1,$$
$$r^1, s^1 \in S^1 \;\wedge\; r^1(p^1/^1q^1)^n s^1 \;\wedge\; p^1 \overset{1}{\sim} q^1 \;\Rightarrow\; r^1 \overset{1}{\sim} s^1.$$

The proofs of the lemmas are inductive. If $n = 0$, then the truthfulness of the lemmas is justified by $D^1$10a, T11, W8a, W4a, $A^1$1b and W12. The proofs of the lemmas for $n = 1$ are more difficult (see [7]). We do make use in them of $D^1 10b$, W8a, $A^1_1 13$, $T_1 6a$, $D^1 3$, $D^1 2b$, $A^1_1 14$, $A^1 1b,c$, W12, W4a, W3 and $A^1_2 14$. The proofs of the lemmas based on the inductive assumption follow immediately from $D^1 10c$ and from their truthfulness for $n = 1$.

We obtain the *fundamental theorem of the theory of syntactic categories* directly from these lemmas on the basis of the definition $D^1 10d$:

$$\text{T18} \qquad r^1, s^1 \in S^1 \ \wedge \ r^1(p^1/{}^1q^1)s^1 \Rightarrow \ (p^1 \overset{1}{\sim} q^1 \ \Leftrightarrow \ r^1 \overset{1}{\sim} s^1).$$

Two expressions of the language L^1 belong to the same syntactic category iff replacing either of them by the other one in a well-formed expression of the language L^1 and having obtained from it a well-formed expression of this language, we find that it belongs to the same syntactic category as the expression itself.

2.2 The Level of Types; Theory T^2_k

The theory T^2_k is built over the theory T^1_k by enriching it with definitions of notions of the language L^2, thus a language considered on the *level of types*. All the concepts of the system (L^2) and some auxiliary notions of the *level of types* are thus derivative constructs, defined by means of concepts of the *level of concretes*.

In the definitions of the notions of the *level of types* we shall make use of equivalence classes of the relation $\overset{1}{=}$ of equiformity, which in accordance with the axioms $A^1 1a$-c is an equivalence relation in the set U^1. We will denote the equivalence class of this relation with the representative p^1 by $[p^1]$.

If we were to accept that Z^1 represents the terms

$$(\text{Z}^1) \qquad U^1, V^1_1, V^1_2, W^1_1, W^1_2, D_1(e^1), D_2(e^1), E^{s1}_1, E^{s1}_2, E^{c1}_1, E^{c1}_2, E^1_1, E^1_2, {}^nS^1, S^1, B^1, F^1,$$

then the terms that are dual to those of the system (Z^1), denoting sets of the system (L^2) or distinguished sub-sets of these sets, are replaced by the variable Z^2 and defined by means of the following schema:

$$\text{DZ}^2 \qquad p^2 \in Z^2 \ \Leftrightarrow \ \exists\, p^1 \in Z^1 \ (p^2 = [p^1]).$$

The set Z^2 of object-types is thus a quotient family of equivalence classes determined by the equiformity relation and tokens of the set Z^1.

In the theory T^2_k, the notion which is dual to the set $Ct^1_{\xi^1}$ corresponds to the syntactic category of types of expressions with the index ξ^2 of a word-type, therefore the set $Ct^2_{\xi^2}$, defined by the formula

$$\text{DCt}^2_{\xi^2} \qquad p^2 \in Ct^2_{\xi^2} \ \Leftrightarrow \ \exists\, \xi^1 \in D_2(e^1)\ \exists\, p^1 \in Ct^1_{\xi^1} \ (\xi^2 = [\xi^1] \ \wedge \ p^2 = [p^1]).$$

An expression-type belongs to the syntactic category with the index ξ^2 of a word-type iff it is an equivalence class of the equiformity relation determined by some expression-token belonging to the syntactic category with some index of a word-token, being a representative of the equivalence class which determined the index ξ^2.

If we were to accept that R^1 represents the terms of the system:

$$(\mathrm{R}^1) \qquad c^1, e^1, \rho_1^1, \rho_2^1, (/^1)^n, (/^1),$$

then the dual terms denoting respective relations of the system (L^2) or some of their sub-relations will be replaced by the variable R^2 and defined by means of the schema:

$$(\mathrm{R}^2) \qquad R^2(p^{2_0}, p^{2_1}, \ldots, p^{2_n}) \;\Leftrightarrow\; \exists\, p^{1_0}, p^{1_1}, \ldots, p^{1_n} \,\big(p^{2_0} = [p^{1_0}] \wedge \wedge\, p^{2_1} = [p^{1_1}] \;\wedge\; \ldots \;\wedge\; p^{2_n} = [p^{1_n}] \;\wedge\; R^1(p^{1_0}, p^{1_1}, \ldots, p^{1_n})\big).$$ [8]

The relation R^2 holds between the object-types iff the types are equivalence classes of equiform object-tokens, between the representatives of which relation R^1 holds.

The other notions of the theory T_k^1, i.e. the relation $\overset{1}{\sim}$, families of syntactic categories, respectively: Ct^1, $Ct^1(S^1)$, $Ct^1(B^1)$, $Ct^1(F^1)$, in the theory T_k^2 correspond to dual notions defined by dual definitions of these concepts, thus: $\mathrm{D}^2 7$, $\mathrm{D}^2 8$, $\mathrm{D}^2 9$a-c, respectively.

Similarly as in [7], on the basis of the axiom and definition of the theory T_k^2, as well as the definition with the form of DZ^2, $\mathrm{DCt}^2_{\xi^2}$, DR^1 we prove that

Theorem I(T_k^2). The theorems of the theory T_k^2, and therefore T_k^1, are the expressions: $\mathrm{A}^2 2$, $\mathrm{A}^2 3$, $\mathrm{A}_1^2 5$–$\mathrm{A}_1^2 7$ and $\mathrm{A}_2^2 5$–$\mathrm{A}_2^2 7$, $\mathrm{DA}_1^2 1$ and $\mathrm{D}_2^2 1$, $\mathrm{A}_1^2 8$ and $\mathrm{A}_2^2 8$, $\mathrm{A}^2 9$–$\mathrm{A}^2 11$, $\mathrm{A}_1^2 13$, $\mathrm{A}_1^2 14$ and $\mathrm{A}_2^2 13$, $\mathrm{A}_2^2 14$, $\mathrm{D}_1^2 2$a-c and D_2^2a-c, $\mathrm{D}^2 3$a-c, $\mathrm{A}^2 16$ and $\mathrm{A}^2 17$, $\mathrm{D}^2 4$–$\mathrm{D}^2 6$ as well as $\mathrm{D}^2 10$a-d.

In view of the fact that $\overset{2}{=}$ is an identity relation and the formulas $\mathrm{A}^2 1$a-c, $\mathrm{A}^2 4$, $\mathrm{A}^2 12$, $\mathrm{A}^2 15$ hold in an evident way, and because the definitions $\mathrm{D}^1 7$, $\mathrm{D}^1 8$, $\mathrm{D}^1 9$a-c of the theory T_k^1 correspond to definitions which are dual to them, we can state:

Corollary I(T_k^2). Each dual counterpart of the axiom and definition of the theory T_k^1 is a theorem or a definition of the theory T_k^2.

Hence, there follows further:

Corollary II(T_k^2). Each dual counterpart of the thesis (acknowledged sentence) of the theory T_k^1 is a thesis (acknowledged sentence) of the theory T_k^2.

In the theory T_k^2, we can formulate several theorems which in *sensu stricto* do not have dual counterparts in the theory T_k^1. Thus, for instance, from the theorems $\mathrm{A}^2 2$ and $\mathrm{A}^2 3$ it follows that the concatenation relation c^2 is a binary operation in the set of linguistic types U^2. Since the theorem of the theory T_k^2 is $\mathrm{A}^2 11$, the function is the relation e^2. Functions, and injections to that, are also the relations ρ_1^2 and ρ_2^2; 1-1 is secured by the theorems $A_1^2 14$ and $A_2^2 14$.

Using the fact that e^2 is a function and replacing—as it is usually done—the expression $e^2(p_1^2, p_2^2)$ by the expression $p_2^2 = e^2(p_1^2)$, where $e^2(p_1^2)$ is a value of the function e^2 for the argument p_1^2, we can conclude that in the theory T_k^2, the theorem $D^2 3b$ is equivalent to the expression

$D^2$3b'
$$p_1^2 \in {}^{k+1}S^2 \iff p_1^2 \in {}^kS^2 \vee$$
$$\vee\, \exists\, n \geq 1\ \exists\, p_1^{2_0}, p_1^{2_1}, \ldots, p_1^{2_n} \in {}^kS^2\ \left(\rho_1^l(p_1^{2_0}, p_1^{2_1}, \ldots, p_1^{2_n}; p_1^2) \wedge\right.$$
$$\left.\wedge\, \rho_2^2\big(e^2(p_1^2), e^2(p_1^{2_1}), \ldots, e^2(p_1^{2_n}); e^2(p_1^{2_0})\big)\right).$$

In the proof of equivalence of these expressions, moreover, the fact that ${}^kS \subseteq D_1(e^2)$ is made use of, as well as that $D_1(\rho_1^2) \subseteq D_1(e^2)$ (Theorem $A_1^2 13$).

Therefore, because e^2 is a function and $S^2 \subseteq D_1(e^2)$ and $E_1^2 \subseteq D_1(e^2)$ (the theorem dual to W8a), we prove that the following expressions are theorems of the theory T_k^2 and as suitably equivalent can replace the theorems $D^2 4$–$D^2 6$ of this theory:

(i) $$B^2 = \left\{p^2 \in S^2 \mid e^2(p^2) \in E_2^{s2}\right\},$$

(ii) $$F^2 = \left\{p^2 \in S^2 \mid e^2(p^2) \in E_2^{c2}\right\},$$

(iii) $$Ct_{\xi^2}^2 = \left\{p^2 \in E_1^2 \mid e^2(p^2) = \xi^2\right\}.$$

The manner of reading the theorems $D^2 3b$, (i), (ii), (iii) does not differ from the way of reading the expressions $D^2 3b$, $D^2 4$–$D^2 6$, respectively.

3 The Formal Theory T_p^l – the Platonizing Approach

It is assumed in the theory T_p^l that types which are linguistic creations understood as a classes of tokens being thus abstract objects, have an independent and objective being, and are primary with reference to linguistic tokens.

The primitive concepts of the theory T_p^l are the following notions of the system (L^2): the set U^2 of all linguistic types, the vocabulary V_1^2 of simple word-types, the auxiliary vocabulary V_2^2 of auxiliary word-types, the ternary concatenation relation c^2, being a binary operation in the set U^2, the binary relation e^2 of indicating indices of word-types, being a function and the binary relations ρ_1^2 and ρ_2^2, respectively: formation of compound functorial expression-types and formation of functoral indices of word-types; both of these relations are functions.

The remaining notions of the system (L^2), with the exception of the identity relation $\stackrel{2}{=}$ are defined in the theory T_p^l. There are also some auxiliary notions of the *level of types* defined in it, as well as all the notions of the *level of concretes*, that is the notions of the system of (L^1).

3.1 The Level of Types; Theory T_p^2

In the theory T_p^2, the meaning of all the primitive terms of the theory T_p^l is established, as well as all the other terms of the system (L^2) with the exception of the term $\overset{2}{=}$. The meaning of the these terms is established by axioms and definitions which are either substitutions of the schemata given in Sec. 1.2 at $l=2$ and $x=1,2$, or they are equivalent to expressions formed in this way.

They are the following: axioms $A^2 2$, $A^2 3$, $A_1^2 5$–$A_1^2 7$, $A_2^2 5$–$A_2^2 7$, definitions $D_1^2 1$ and $D_2^2 1$, axioms $A_1^2 8$, $A_2^2 8$, $A^2 9$–$A^2 11$, $A_1^2 13$, $A_1^2 14$ and $A_2^2 13$, $A_2^2 14$, definitions $D_1^2 2$a-c, $D_2^2 2$a-c, definitions $D^2 3$a,b' (see Sec. 2.2) and $D^2 3$c, axioms $A^2 16$ and $A^2 17$, definitions (i), (ii), (iii) (see Sec. 2.2) as well as $D^2 7$–$D^2 10$a-d.

The final comments given in Sec. 2.2 lead to the statement that in the theory T_p^2, the theorems which say that the relations c^2, e^2, ρ_1^2 and ρ_2^2 are functions are true. The fact that e^2 is a function permits—instead of the inductive definitions $D^2 3$a,b of the set ${}^n S^2$—to accept the equivalent definition $D^2 3$a,b', and instead of the definitions of the notions B^2, F^2, $C^2_{\xi^2}$, that is the expressions $D^2 4$, $D^2 5$ and $D^2 6$, to accept equivalent expressions (i), (ii) and (iii), respectively.

Regarding the theory T_p^2, the dual (to $A^1 1$a-c, $A^1 4$, $A^1 12$, $A^1 15$, respectively) expressions: $A^2 1$a-c, $A^2 4$, $A^2 12$ and $A^2 15$ are obviously true.

Thus, selecting the axioms and theorems of the theory T_p^2 in the above-shown way, we can obtain:

Theorem I(T_p^2). Each thesis (acknowledged sentence) of the theory T_p^2 is either a dual counterpart of a thesis (acknowledged sentence) of the theory T_k^1 or can be translated into a dual counterpart of such a thesis (acknowledged sentence). Moreover, each dual counterpart of a thesis (acknowledged sentence) of the theory T_k^1 is a thesis (acknowledged sentence) of the theory T_p^2.

From the theoretical, syntactic point of view, framing a theory of language as the theory T_k^1 of the language of expression-tokens or as the theory T_p^2 of the language of expression-types remains therefore without an influence on the corpus of theorems relating to the categorial languages discussed here.

3.2 The Level of Concretes; Theory T_p^1

The theory T_p^1 is a definitional extension of the theory T_p^{2+} which is formed of the theory T_p^2 by enriching it with the following two axioms reflecting intuitions that we relate to linguistic types:

A^{+}1 $\quad p^2 \neq \emptyset$,

A^{+}2 $\quad p^1 \in p^2 \ \wedge\ p^1 \in q^2 \ \Rightarrow\ p^2 \overset{2}{=} q^2$.

According to the axiom A^{+1}, linguistic types (see conventions in Sec. 1.2) are non-empty sets; on the strength of the axiom A^{+2}, types are equal when they possess a common element.

Elements of linguistic types are tokens. This follows from the following definition of the set U^1, belonging to the theory $T_p^(1)$:

DU^1 $$p^1 \in U^1 \ \Leftrightarrow\ \exists\, p^2 \in U^2\ (p^1 \in p^2).$$

The definitions of the theory T_p^1 are definitions of all the notions of the *level of concretes*, that is all the notions of the system (L^1) and some auxiliary notions of this level.

The definition DU^1 falls under the general schema of the definitions of terms of the system (Z^1) (see Sec. 2.2) accepted in the theory T_p^1 and denoting the subsets of the set U^1:

DZ^1 $$p^1 \in Z^1 \ \Leftrightarrow\ \exists\, p^2 \in Z^2\ (p^1 \in p^2).$$

The variable Z^2 represents here terms which are dual to those of the system (Z^1).

The set Z^1 is thus a set of such linguistic tokens that are elements of some type of the set Z^2 which is dual to the set Z^1.

In the theory T_p^1, the notion $Ct^1_{\xi^1}$ of the syntactic category of expression-tokens with the index ξ^1 is defined by the formula

DCt$^1_{\xi^1}$ $$p^1 \in Ct^1_{\xi^1} \ \Leftrightarrow\ \exists\, \xi^2 \in D_2(e^2)\, \exists\, p^2 \in Ct^2_{\xi^2}\ (\xi^1 \in \xi^2 \ \wedge\ p^1 \in p^2).$$

The primary notion of the relation $\overset{1}{=}$ of equiformity, which is fundamental for the theory T_k^1, is defined in the theory T_p^1 by the formula:

D$\overset{1}{=}$ $$p^1 \overset{1}{=} q^1 \ \Leftrightarrow\ \exists\, p^2\ (p^1, q^1 \in p^2).$$

Two linguistic tokens are then equiform iff they belong to some (the same) type.

The definition D$\overset{1}{=}$ can be included in the following general schema defining other relations of the *level of concretes*, that is the relation R^1 of the system (R^1) (see Part II.2). Relations R^1 are defined by means of the dual relations R^2. Thus, we have:

DR^1 $$R^1(p^{1_0}, p^{1_1}, \ldots, p^{1_n}) \ \Leftrightarrow\ \exists\, p^{2_0}, p^{2_1}, \ldots, p^{2_n} \big(p^{1_0} \in p^{2_0} \ \wedge\ p^{1_1} \in p^{2_1} \ \wedge \\ \wedge\ \ldots\ \wedge\ p^{1_n} \in p^{2_n} \ \wedge\ R^2(p^{2_0}, p^{2_1}, \ldots, p^{2_n})\big).$$

According to DR^1, the relation R^1 between linguistic tokens holds iff between the appropriate linguistic types to which the tokens belong there holds the relation R^2.

The remaining notions of the level of concretes, i.e. the relation and the families: Ct^1, $Ct^1(S^1)$, $Ct^1(B^1)$, $Ct^1(F^1)$, are defined in the theory T_p^1 like in the theory T_k^1, that is by definitions $\mathrm{D}^1 7$, $\mathrm{D}^1 8$, $\mathrm{D}^1 9$a-c, respectively.

On the basis of the axioms and definitions of the theory T_p^{2+} and the definitions DZ^1, $\mathrm{DCt}^1_{\xi^1}$, $\mathrm{D}\overset{1}{=}$, DR^1 of the theory T_p^1, we prove that the theorems of the theory T_p^1 are all the axioms and definitions of the theory T_k^1, with the exception of the definitions $\mathrm{D}^1 7$, $\mathrm{D}^1 8$, $\mathrm{D}^1 9$a-c, which are the same in both theories.

That the theorems of the theory T_p^1 are axioms $\mathrm{A}^1 1$a-c–$\mathrm{A}^1 4$, $\mathrm{A}^1_1 5$–$\mathrm{A}^1_1 7$ of the theory T_k^1, the definition $\mathrm{D}^1_1 1$ and axiom $\mathrm{A}^1_1 8$ of this theory, was specifically proved in [9]. The theorems of the theory T_p^1 are certainly counterparts of the expressions $\mathrm{A}^1_1 5$–$\mathrm{A}^1_1 7$, $\mathrm{D}^1_1 1$ and $\mathrm{A}^1_1 8$ concerning auxiliary words, that is $\mathrm{A}^1_2 5$–$\mathrm{A}^1_2 7$, $\mathrm{D}^1_2 1$ and $\mathrm{A}^1_2 8$. The proofs of the remaining axioms and definitions of the theory T_k^1, different from $\mathrm{D}^1 7$, $\mathrm{D}^1 8$ and $\mathrm{D}^1 9$a-c, do not pose any major difficulty. In the proofs of expressions written with the use of the equiformity relation $\overset{1}{=}$ it is convenient to make use of the following corollary of the definition $\mathrm{D}\overset{1}{=}$ and the axiom $\mathrm{A}^+ 2$: $p^1 \in p^2 \wedge q^1 \overset{1}{=} p^1 \Rightarrow q^1 \in p^2$, according to which a token equiform with one belonging to the given linguistic type also belongs to it.

Therefore, the following is true:

Theorem II(T_p^1). The theorems or definitions of the theory T_p^1 and therefore also of the theory T_p^l are all the axioms and definitions of the theory T_k^1.

From the theorem given above there follows:

Corollary I(T_p^1). Each thesis (acknowledged sentence) of the theory T_k^1 is a thesis (acknowledged sentence) of the theory T_p^1, and therefore also of the theory T_p^l.

4 Metalogical and Philosophical Consequences

The two dualistic approaches to bi-aspectual formalization of language, presented in Sections 1 and 2, suggest the question concerning the mutual relationships between the theories ${}^{l}_{k}$ and T_p^l as well as the notions described by them.

In compliance with the theorem II(T_p^1), in the theory T_p^l, it is possible to ground the axiom system and all the definitions of the theory T_k^1. Thus, all the notions of the system (L^1), in a more general sense: all the notions of the *level of concretes*, are definable in the theory T_p^l.

And contrariwise: since, according to the theorem I(T_k^2) and the final considerations in Sec. 2.2. dealing with equivalence of the expressions $\mathrm{D}^2 3$b and $\mathrm{D}^2 3$b', $\mathrm{D}^2 4$ and (i), $\mathrm{D}^2 5$ and (ii) as well as $\mathrm{D}^2 6$ and (iii), the axiom system and all the definitions of the theory T_p^2 can be embedded in the theory T_k^l, therefore, all the notions of the *level of types* are definable in the theory T_k^l.

Finally, we can formulate the following conclusions:

(1) The system of notions of the *level of concretes* (the language L^1) and the system of notions of the *level of types* (the language L^2) are syntactically mutually definable.

The mutual definability of notions, which are mentioned in (1), is possible thanks to the fact that in the theory T^l_p on the *level of concretes* (in the theory T^1_p) definitions of sets and relations of this level are accepted, that is DZ^1, $DCt^1_{\xi^1}$, $D\overset{1}{=}$, DR^1, while in the theory T^l_k on the *level of types* (in the theory T^2_k) the definitions DZ^2, $DCt^2_{\xi^2}$, DR^2 of sets and relations on the *level of types* are introduced.

It can be proved that the definitions DZ^2, $DCt^2_{\xi^2}$, DR^2 of the theory T^2_k are theorems of the theory T^l_p (the theory T^1_p), whereas the definitions DZ^1, $DCt^1_{\xi^1}$, $D\overset{1}{=}$, DR^1 of the theory T^1_p are theorems of the theory T^l_k (the theory T^2_k). The last fact is based, among others, on that in the theory T^1_k the following theorems are true:

$$q^1 \in Z^1 \ \wedge \ p^1 \overset{1}{=} q^1 \ \Rightarrow p^1 \in Z^1$$

(see the conventions on the variables in Sec. 1.2, as well as the axioms $A^1_1 6$ and $A^2_2 6$, theorems $T_1 6$ and $T_2 6$, theorems $T_1 6a$ and $T_2 6a$, $T_1 8a$-c, $T_2 8a$-c, T11 and T14a,b) and also

$$R^1(q^{1_0}, q^{1_1}, \ldots, q^{1_n}) \ \wedge \ \forall 0 \leq k \leq n \, (p^{1_k} \overset{1}{=} q^{1_k}) \ \Rightarrow \ R^1(p^{1_0}, p^{1_1}, \ldots, p^{1_n})$$

(see the theorem T2, axioms $A^1 12$, $A^1_1 15$ and $A^1_2 15$, the theorem T17).

Moreover, we can note that the theorems of the theory T^2_k (T^l_k) are both—added to the theory T^2_p (the theory T^{2+}_p)—axioms: A^+1 and A^+2. They follow directly from the agreement on the variables p^2 and q^2, definition DU^2 and the properties of equivalence classes.

The considerations in this part of the paper lead thus not only to the statement that on the basis of the theories T^l_k and T^l_p the definitions of the sets and relations of both levels are equivalent, but also to concluding that both theories are equivalent. The theorems or definitions of the theory T^l_p are all the axioms and definitions of the theory T^1_k (the theorem $II(T^1_p)$), and moreover—all the definitions of the theory T^2_k, whereas the theorems or definitions of the theory T^l_k are all the axioms and definitions of the theory T^2_p (the theorem $I(T^2_k)$, theorems $D^2 3b$', (i)–(iii) of the theory T^2_k, axioms A^+1 and A^+2 of the theory T^{2+}_p, as well as all the definitions of the theory T^1_p.

Thus, we have, in consequence:

(2) The theories T^l_k and T^l_p are equivalent.

The theories T^l_k and T^l_p—from the point of view of ontology of language—represent two different approaches towards formalization of language syntax. Therefore, we can conclude, at the same time, the following:

(3) Two dual approaches to language syntax represented by the theories T^l_k and T^l_p are equivalent.

Furthermore, there exists a complete analogy between syntactic notions of the languages L^1 and L^2. According to the corollary II(T_k^2), each property possessed by tokens (the theory T_k^1) can be translated into a property belonging to types as equivalence classes of equiform objects (the theory T_k^2). And on the contrary: in compliance with the theorem I(T_p^2) and the corollary I(T_p^1), each property of types (the theory T_p^2) can be translated into one that its tokens have (the theory T_p^1). Formalizing, however, syntactically the language in the form of the theory T_k^l, we not only do not deplete the resource of theorems connected with the syntax of categorial languages, but we also are able to bypass this without postulating the existence of ideal beings which are linguistic types, word-types, expression-types, understood as classes of tokens.

Thus, the argumentation presented speaks in favor of the following philosophical conclusion:

(4) In syntactic considerations of a language, it is possible to omit assumptions of the existence of ideal linguistic creations conceived as a class of equiform tokens.

In syntactic considerations of a language, it suffices, therefore, to assume only the existence of determined objects of the *level of concretes*, and to treat languages of expression-types, abstract expressions, as substitute convenient forms of languages of expression-tokens, expression-concretes.

The studies presented in this paper and their consequences concern simple categorial languages, languages of expressions (see Sec. 1.1) not including operators which bind variables. All formal-logical and philosophical inquiries can be generalized, though, in such a way as should concern also languages of expressions including operators binding variables.

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Chapter 4
On the Eliminatibility of Ideal Linguistic Entities

Urszula Wybraniec-Skardowska

To the memory of Jerzy Słupecki

Abstract With reference to Polish logico-philosophical tradition two formal theories of language syntax have been sketched and then compared with each other. The first theory is based on the assumption that the basic linguistic stratum is constituted by object-tokens (concrete objects perceived through the senses) and that the types of such objects (ideal objects) are derivative constructs. The other is founded on an opposite philosophical orientation. The two theories are equivalent. The main conclusion is that in syntactic researches it is redundant to postulate the existence of abstract linguistic entities. Earlier, in a slightly different form, the idea was presented in [27] and signalled in [26] and [25].

Key words: Dual nature of linguistic objects • Differentiation type-token • Syntax • Two opposite approaches to formalizations of language syntax • Two formal theories of language syntax • Eliminatibility of existence of abstract linguistic objects

Idealization, and so also abstraction, has become an indispensable procedure nowadays widely made use of in sciences, the science of language included. While its product are ideal entities, derivative in relation to physical objects, idealization may lead to useful fiction that facilitates considering physical objects. Still, one should also allow for another, specific idealization, e.g., mathematical or logical ascertainment. Many mathematicians and logicians are familiar with the belief that the truths of mathematical and logical theories, their axioms and theorems, are not material recordings but geometrical products, abstract objects whose representations are concrete, material recordings, that is—physical objects.

Nothing then, I believe, hinders accepting the fact that in the theory of language there exist both material linguistic objects, taking the shape of inscriptions or sounds of speech, as well as abstract linguistic entities. Such is after all, though often unconsciously managed, semiotic praxis.

From the point of view of philosophy, however, it is not indifferent whether these linguistic objects of double ontological nature are ascribed an independent existence

The paper first published in Studia Logica **48** (4), 587–615 (1989).

or not and if not—to which of them the primitive existence is ascribed, and to which the derivative one. The philosophical assumptions influence also the choice of this and no other formal concept of language. Assuming, for instance, that the simplest linguistic objects are geometrical products possessing the primitive existence we may, similarly as Euclidean geometry does, postulate their existence by accepting the appropriate axioms.

Let us note that Alfred Tarski in his famous work [21] devoted to the problem of truth, while expounding the axioms of metascience, postulates that language expressions are abstract entities, intuitively understood as classes of equiform concrete inscriptions.

We will slightly modify and develop the idea of Tarski, as related to metalogic, in order to sketch in Sec. 5 (cf. [25–27]) a formal theory **T2** of language syntax, the theory deriving from certain abstract objects, namely types of inscriptions which function as primitive entities. Material inscriptions (concretes) will be defined in it. This theory will be compared with the theory **T1** which is built in Sec. 4 and presents an opposite approach (cf. [23], [24]). The effect of the comparison of the two theories (Secs. 6 and 7) explains the title of the work—a reflection of some views of Jerzy Słupecki.

At the end of his life Jerzy Słupecki inclined towards Leśniewski's nominalism; in the question of the nature of linguistic objects he accepted Kotarbiński's assumptions of ontological reism. As far as I know, Słupecki was the first to attempt to formalize certain linguistic aspects referring to concrete and abstract words. He initiated first some research in this direction [7] with reference to the theory of algorithms of A. A. Markov [16], and then inspired researches on language carried out by the author of the present work, which were crowned with a monograph [24]. A common idea of these studies was a concretizing approach to language, i.e., postulating the existence of inscriptions and words as concretes and ascribing derivative existence to the types of inscriptions or words treated as certain abstract products—trough linguistic abstraction.

I would like to believe that the present text successfully draws out from dimness and develops certain ideas worked out by my teacher, and in this way—by the linguistic concretization—calls him from the non-existence into derivative existence — now only intentional.

1 Non-uniform Semiotic Characterization of Language

Certainly, one of the turning points in the twentieth-century linguistics was *Cours de linguistique générale* by Ferdinand de Saussure—a work published posthumously in 1916. It includes a postulate of scientific description of language as *la langue*—the system of wholeness of elements, signs bound by certain relations and performing certain functions, the system which is, at the same time, the mechanism serving as a tool of the communicative act between people. The postulate is by all means up-to-date. It requires a wide scientific characterization of language, taking into consideration the famous tripartition of semiotics advocated by C. Morris [17], which divides that discipline into syntax, semantics and pragmatics.

It does not mean, however, that language, in the theory of language, is not characterized in a narrower sense—exclusively syntactically, as, for example, in the epochal work by Noam Chomsky—*Syntactic Structures* (1957), or at most, with the semantic component added only.

A uniform semiotic characterization of language is made difficult because of the interpretative concept of language as a product built of words. In the above-mentioned division of semiotics the concrete and abstract linguistic entities occupy different, though equivalent places. The abstract expressions perform theoretical role. In pragmatics they serve the purpose of explaining the process of communication between people; in semiotics, by their means, such basic terms as denotation, truth, or meaning are explained; in syntactic studies they help to formulate grammatical rules. Linguistic description on the pragmatic level, which concerns the functionality of language (see e.g. [3] or [20]), is connected with the use of expressions in context, and consequently, without doubt, with linguistic concretes. Also an analysis of syntactic correctness of a given expressions and, in reference to it—making use of, for example, K. Ajdukiewicz's algorithm [1] (as a system of psycho-physical activities) demands the use of linguistic concretes.

Thus, language is a construct of a double nature: it consists of *tokens* (concretes) and *types* (ideal objects). The differentiation *types-tokens* made by C. S. Pierce (see [19]) and propagated through works by R. Carnap and Y. Bar-Hillel (see e.g. [11], [2], [3]) has been adopted for good in logic and semiotics. *Types* are generally understood here as classes of equiform (or equisounding) *tokens*. Yet it is not always so. As Witold Marciszewski[1] rightly observed they may be understood as concretes, e.g. some undetermined equiform inscriptions with data defined by means of D. Hilbert's eta-operator of indefinite description.

2 Preliminary Conventions Concerning Language

For the purpose of the present work it seems indispensable and useful to establish certain unification of language and, consequently—some conventions. Thus:

1. Language will be characterized exclusively syntactically;
2. Language analysis will not concern spoken language;[2]
3. Language will be considered in two aspects: as the language of *tokens* (*token level*) and as the language of *types* (*type level*);
4. *Tokens* will be understood as empirical objects perceived by sight; *types*—as sets of *tokens* established by equiformity relation, ie. as some abstract products;
5. *Tokens* may, yet need not, be inscriptions on paper, table, sign-board, stone, etc. They may be some configurations of stars or colourful objects, smoke signals,

[1] The observation was included in the review of [24].

[2] A formal concept of such a language is presented by T. Batóg in [5].

or light illuminations, or the so-called "live pictures" during entertainments and shows, and so on;

6. Equiformity of *tokens* is determined by the pragmatic aim. We will assume that equiformity is an equivalence relation;
7. The syntactic characterization of language will consider an approach referring to the theory of syntactic categories of S. Leśniewski [14] in the version modified by K. Ajdukiewicz [1] (cf. also M. J. Cresswell [12], [13] and A. Nowaczyk [18]). The idea of such an approach is to generate *concatenations* from a vocabulary of a given language which would be its *functorial expressions* (ie. composed of the main functor and its arguments) and to assess which of them are well formed. The assessment is made with the help of *categorial indices* (types) ordered one' by one (on the *token level* with the exactitude to equiformity) to every expression of a given language and precisely delimiting, at the same time, syntactic category of every language expression. It consists in checking if for every constituent of a given functorial expression the rule which expresses the *superior principle of the theory of syntactic categories* holds: the index of the main functor of a compound expression is determined by the index of this expression and indices of the arguments of its main functor. The language thus characterized is called *categorial language* (cf. [12], [13], [18], [23], [24], [25];
8. A complete categorial characterization of language will include the division of the set of all well-formed expressions into syntactic categories;
9. The syntactic characterization of language will allow us to conceive it as a language generated by a *classical categorial grammar*, the idea going back to K. Ajdukiewicz [1][3], and also as a *typed functorial language* whose precise algebraic description has been proposed by W. Buszkowski [9];
10. In the present work language will be characterized in a formal way by the axiomatic method (cf. [14] and [17]), within two contrastive theories: **T1** and **T2** which assume set-theoretical formalism.

3 Dual Theories Concepts and Expressions

Theories **T1** (Sec. 4) and **T2** (Sec. 5) grasp the dual ontological approach to the syntax of language. They are presented at two levels as dualistic theories. Now **T1** provides formal foundations of categorial languages by adopting the *nominalistic* (*concretistic*) standpoint in the philosophy of language and assumes that *tokens*, and hence concrete objects, form the fundamental level of language, while *types*

[3] The term "categorial grammar" was introduced by Y. Bar-Hillel et al. in [4]. A historical survey of categorial grammars as well as the basic terms referring to them is given by W. Marciszewski in [15]. Categorial grammars are formal grammars developed in parallel to N. Chomsky's generative grammars. A significant share in the development of their mathematical foundations has been contributed by W. Buszkowski, who has also been popularizing the grammars in his works (see [8–10]). A contemporary formulation of categorial semantics has been developed by J. van Benthem in [22].

are constructs obtained in a derived analysis. The formalization of that theory is accordingly carried out first at the *token level* and yields the theory **T1**(tk), and then expanded at the *type level* it yields the dual theory **T1**(tp). Now **T2** represents the opposite, *Platonic*, philosophical orientation in the syntax of language as it assumes that the study of language is based on *types*, and hence ideal objects, while *tokens* as their concrete representations are the subject matter of derived analysis. Hence that theory is constructed first as **T2**(tp) which describes objects at the *type level*, and then expanded as the dual theory T2(tk) which describes objects at the token level.

Dual theories describe syntactic concepts which belong to the two different levels mentioned above. Hence the theories **T1**(tk) and **T2**(tp) as well as **T1**(tp) and **T2**(tk) are *dual*, too.

The syntactic concepts at the *token level* include sets and relations which enable us to formally describe an arbitrary but fixed categorial language $\mathscr{L}$ as a language of expression-tokens. They are (1) sets of tokens which belong to the following system (S):

U	– the set of all tokens, that is the universe of $\mathscr{L}$,
V^1	– the vocabulary of $\mathscr{L}$,
V^2	– the auxiliary vocabulary of $\mathscr{L}$,
W^1	– the set of all words of $\mathscr{L}$,
$W^1 \backslash V^1$	– the set of all compound words of $\mathscr{L}$,
W^2	– the set of all auxiliary words of $\mathscr{L}$,
$W^2 \backslash V^2$	– the set of all compound auxiliary words of $\mathscr{L}$,
$D_1(\mathfrak{i})$	– the domain of the relation $\mathfrak{i}$ of indication of the indices of word-tokens,
$D_1(\mathfrak{i}) \backslash V^1$	– the set of all those compound words of $\mathscr{L}$ which have an index,
$D_2(\mathfrak{i})$	– the counterdomain of $\mathfrak{i}$,
$D_2(\mathfrak{i}) \backslash V^2$	– the set of all those compound auxiliary words of $\mathscr{L}$ which are indices of words,
E^1_s	– the set of all simple expressions of $\mathscr{L}$,
E^1_f	– the set of all functorial expressions of $\mathscr{L}$,
E^1	– the set of all expressions of $\mathscr{L}$,
E^2_s	– the set of all basic index-tokens,
E^2_f	– the set of all functoral index-tokens,
E^2	– the set of all well-formed index-tokens,
E_n	– the set of all well-formed expressions of the n-th order of $\mathscr{L}$,
E	– the set of all well-formed expressions of $\mathscr{L}$,
$E \backslash E_0$	– the set of all well-formed compound expressions of $\mathscr{L}$,
B	– the set of all basic expressions of $\mathscr{L}$,
F	– the set of all functors of $\mathscr{L}$,
Ct_t	– a syntactic category with the index t,
$Ct(E^1)$	– the family of all syntactic categories of the expressions $\mathscr{L}$,
$Ct(E)$	– the family of all syntactic categories of the expressions of E,
$Ct(B)$	– the family of all basic syntactic categories of $\mathscr{L}$,
$Ct(F)$	– the family of all functoral syntactic categories of $\mathscr{L}$,

and (2) the relations holding among the tokens from the universe U and belonging to the following system (R):

$\approx$ – the equiformity relation among tokens,
$\mathfrak{c}$ – the relation of the concatenation of tokens,
$\mathfrak{i}$ – the relation of the indication of the indices of word-tokens,
$\mathfrak{r}_1$ – the relation of the formation of functorial expressions of $\mathscr{L}$
$\mathfrak{r}_2$ – the relation of the formation of functoral indices of word-tokens,
$(/)^n$ – the relation of replacement of n-th order constituents of expression-tokens,
$(/)$ – the relation of replacement of expression-tokens,
$\tilde{c}$ – the relation of the categorial agreement among expression-tokens.

The concepts from the systems (S) and (R) describe the theories **T1**(tk) and **T2**(tk).

The syntactic concepts at the *type level* include those sets and relations (functions) which make it possible to describe an arbitrary but fixed categorial language $\underline{\mathscr{L}}$ as a language of expression-types. They are (1) concepts from the system (**S**) which is obtained from (S) by the replacement of its successive concepts by the appropriate sets of types belonging to the universe of $\underline{\mathscr{L}}$ or by families of such sets, and (2) concepts from the system (**R**), which is obtained from (R) by the replacement of the relation of equiformity $\approx$ by the relation of identity $=$ and the replacement of the remaining relations by the successive appropriate relations holding among the types of the universe of $\underline{\mathscr{L}}$.

The concepts which occupy the same place in the order in (S) (resp. (R)) and (**S**) (resp. (**R**)) are termed *dual*. The terms which denote dual concepts are also called *dual*. The terms which are dual relative to one another are distinguished only by the use, in the case of the terms from (**S**), of bold type without any change in the shape of the type used in the terms from (S), and in the case of terms from (**R**) of the single underline without any change in the shape of the type used in the terms from (R).

Let the letters (v) x, y, z, t, resp. (V) X, Y, Z, T, with or without subscripts and/or superscripts, range over set U tokens, resp. types, from the universe U, resp. $\boldsymbol{U}$. The letters (v) resp. (V), with superscripts k, where $k = 1, 2$, are reserved for words from W^k, resp. $\boldsymbol{W}^k$. It is also assumed that the letter A (resp. $\boldsymbol{A}$) stands for subsets from the universe U of $\mathscr{L}$ (resp. $\boldsymbol{U}$ of $\underline{\mathscr{L}}$).

Two expressions are called *dual* if one of them is recorded solely with the use of logical constants, specific terms occurring in (**S**) and/or (**R**), letters from (v) and/or the letter A, and brackets, while the other differs from the former by having the specific terms of the former replaced by dual terms (printed in bold type), lower-case letters by analogous capital letters from (V), and the letter A by the letter $\boldsymbol{A}$. An expression dual to α is denoted by $\mathrm{d}(\alpha)$.

4 Theory T1

The theory **T1** has as its primitive terms: U, $\approx$, $\mathfrak{c}$, V^1, V^2, $\mathfrak{i}$, $\mathfrak{r}_1$, $\mathfrak{r}_2$. They are at the same time the primitive terms of the fragment **T1**(tk) of **T1**. Those terms which denote the remaining concepts at the *token level* and also all those terms which denote concepts at the *type level* are defined in **T1**.

T1 refers to the theory of categorial languages presented in [24].

4.1 Formalization of T1 at the Token Level; Theory T1(tk)

4.1.1 Axioms and Definitions of T1(tk)

Now **T1**(tk) is an axiomatic theory of the language $\mathscr{L}$ characterized by all primitive and derived concepts at the *token level*. The formulation of axioms and definitions will be preceded by suitable remarks in most cases pertaining to the intuitive interpretation of the concepts which categorially describe the language $\mathscr{L}$.

The *universe* U of $\mathscr{L}$ is the set of all tokens, in which we distinguish certain subsets which enable us to define that language.

The *relation of equiformity* $\approx$ is a binary relation in U. Two tokens between which that relation holds are called *equiform*. The equiformity of tokens is determined by pragmatic aspects, acts in which they are used, and not by physical similarity. For instance, two inscriptions printed in different type but consisting successively of the same letters of alphabet may be equiform, whereas two nouns or two adjectives, printed in the same type, may be not equiform if one of them occurs in a sentence with an adjunct or is itself an adjunct, while the other does not or is not.[4]

We adopt the following axiom characterizing equiformity:

A1a. $x \approx x$,
b. $x \approx y \Rightarrow y \approx y$,
c. $x \approx y \wedge y \approx z \Rightarrow x \approx z$.

The *relation of concatenation* $\mathfrak{c}$ is a ternary relation in U. Any token z which is in the relation $\mathfrak{c}$ with the tokens x and y, i.e., satisfies the expression $\mathfrak{c}(x, y, z)$, is called the *concatenation* of x and y. In the European ethnic languages, any inscription z obtained from an inscription equiform with x by the writing on the right of the latter, immediately after it and at the same level, of an inscription equiform with y, is a concatenation of the inscriptions x and y. In a similar way, but by writing the second inscription on the left of the first, we obtain a concatenation, e.g., in Hebrew or Arabic languages. Concatenations are not always obtained by a linear connection of two tokens, which can be seen in the case of hieroglyphs and mathematical formulas.

[4] If one should use a simile here, it is like having two crystal flower-vases of the same shape and cut when one is empty and the other is full of beautiful red roses, or like comparing the shape of the figure of a beautiful actress posing in exactly the same posture and background in two photos, in one of which she appears clothed, while in the other—naked.

Two equiform tokens may be concatenations of the same two tokens, which shows that the relation of concatenation $\mathfrak{c}$ is not the function. The concept, of concatenation is at the basis of many formal models of language, especially the formal languages in Chomsky's sense. The concept is described in detail in [24–26].

We adopt the following axioms which describe the fundamental properties of concatenation:

A2. $\exists z\, \mathfrak{c}(x, y, z)$,
A3. $\mathfrak{c}(x, y, z) \wedge \mathfrak{c}(x', y', t) \wedge x \approx x' \wedge y \approx y' \Rightarrow z \approx t$,
A4. $\mathfrak{c}(x, y, z) \wedge t \approx z \Rightarrow \mathfrak{c}(x, y, t)$.

Thus for every two tokens in $\mathscr{L}$ there is a token in U which is their concatenation; concatenations of two pairs of tokens in $\mathscr{L}$ with first and second elements pairwise equiform yield equiform tokens; a token which is equiform with the concatenation of two tokens is also their concatenation. □

The *vocabulary* V^1 of $\mathscr{L}$ is a set of simple word-tokens of that language. It is fixed once and for all if $\mathscr{L}$ is a formalized language, or is open and includes potential words if $\mathscr{L}$ is, for instance, a natural language. It is used to generate, by means of the relation of concatenation, the *set* W^1 of *all words* of $\mathscr{L}$. It has as its subset the *set* E of *all* its *well-formed expressions* (briefly: wfe), which determines the language $\mathscr{L}$. Hence the simplest syntactic characterization of $\mathscr{L}$ is given by the system:

($\mathscr{L}$) $\langle U, \mathfrak{c}, V^1; E\rangle$.

The categorial characterization of $\mathscr{L}$, which makes it possible to distinguish the set E, is done by the use of *categorial indices* assigned to the appropriate words of $\mathscr{L}$. They are tokens from U, hut are not in the set W^1 of the words of $\mathscr{L}$, but are words in the metalanguage of that language. They are the so-called *auxiliary words* of $\mathscr{L}$ and are in the set W^2 of all such words. W^2 is generated from the *auxiliary vocabulary* V^2 of $\mathscr{L}$ by means of the relation $\mathfrak{c}$. V^2 consists of *basic indices* and auxiliary symbols, such as brackets, commas, fraction lines, etc.

It is assumed concerning the vocabularies V^k $(k = 1, 2)$ that they satisfy the following axioms:

A^{k}5. $V^k \subseteq U$,
A^{k}6. $x \in V^k \wedge t \approx x \Rightarrow t \in V^k$,
A^{k}7. $\mathfrak{c}(x, y, z) \Rightarrow z \notin V^k$.

Thus, for $k = 1, 2$, V^k is a set of tokens; a token which is equiform with a word from V^k is also such a word; no concatenation of any pair of tokens is a word in V^k. □

The meaning of the terms W^k $(k = 1, 2)$ is fixed by the following definitions and axioms:

D^{k}1. $W^k = \bigcap\big\{A\colon V^k \subseteq\ \wedge \forall x, y \in A\, \forall z\, \big(\mathfrak{c}(x, y, z) \Rightarrow z \in A\big)\big\}$,
A^{k}8. $t \in W^k \backslash V^k \ \Rightarrow\ \exists x, y \in W^k\, \mathfrak{c}(x, y, t)$.

The set of words W^k $(k = 1, 2)$ is thus the smallest set of tokens containing the vocabulary V^k and closed under concatenation, while every compound word (resp. auxiliary compound word) is the concatenation of a pair of words from W^1 (resp. W^2).

Categorial indices are assigned to the appropriate words of $\mathscr{L}$ by the binary *relation* $\mathfrak{i}$ *of indication of indices of words*, that is—to use Buszkowski's terminology—by the *typization* of words.

The relation $\mathfrak{i}$ is described by the four axioms given below. In the recording of the last two axioms we use the expression of the form $\mathfrak{i}(x, y)$, which we read thus: y is the index of the word x of $\mathscr{L}$.

A9. $\mathfrak{i} \subseteq W^1 \times W^2$,
A10. $D_1(\mathfrak{i}) \cap D_2(\mathfrak{i}) = \emptyset$,
A11. $\mathfrak{i}(x^1, x^2) \wedge \mathfrak{i}(y^1, y^2) \wedge x^1 \approx y^1 \Rightarrow x^2 \approx y^2$,
A12. $\mathfrak{i}(x^1, x^2) \wedge z \approx x^1 \wedge t \approx x^2 \Rightarrow \mathfrak{i}(z, t)$.

Typization is to be used in the analysis of the syntactic correctness of the *expressions* of $\mathscr{L}$. They are in the set E^1 and can be either simple expressions from E^1_s, distinguished from the vocabulary V^1 and, of course, the set $D_1(\mathfrak{i})$, or compound expressions, i.e., *functorial expressions* from E^1_f, distinguished from the set $D_1(\mathfrak{i})\backslash V^1$. The principles of the construction of functorial expressions are, self-evidently, determined by the syntactic rules of $\mathscr{L}$. In theoretical considerations we shall replace them by a single binary *relation* $\mathfrak{r}_1$ *of the formation of functorial expressions* of $\mathscr{L}$. If we assume that

($\mathfrak{r}_1$) $\quad \mathfrak{r}_1(x^1_0, x^1_1, \ldots, x^1_n; x^1)$,

is an expression in the theory **T1**(tk), which we read: x^1 is a functorial expression consisting of the main functor x^1_0 and its successive arguments $x^1_1, \ldots, x^1_n$ ($n \geq 1$), then x^1 in ($\mathfrak{r}_1$) may be treated as a substitute of any expression of $\mathscr{L}$ which is formed of the main functor x^1_0 and its successive arguments $x^1_1, \ldots, x^1_n$, regardless of the way in which that expression in the form of the appropriate concatenation occurs in $\mathscr{L}$. Hence the same expression of the language of **T1**(tk), having the form ($\mathfrak{r}_1$), may replace expressions of $\mathscr{L}$ constructed according to various rules, for instance sentential and nominal expressions of natural language, provided that those expressions are formed of the same number of words of which one is a functor and the remaining ones are its arguments (the position occupied in the concatenation by a sentence-forming functor may obviously differ from the place which in another concatenation is occupied by a name-forming functor). The same expression of the language of **T1**(tk) of the form ($\mathfrak{r}_1$) may replace different but synonymous expressions in various languages, for instance languages of the sentential calculus. Note that the following expressions:

$$p \Leftrightarrow (q \Rightarrow r); \qquad \phi\big(p, \diamond(q, r)\big); \qquad EpCqr,$$

recorded respectively in three notations: the one which is used in the present paper, Leśniewski's notation, and Łukasiewicz's parenthesis-free notation, are expressions taken from the various languages of the sentential calculus but each of them consists of the functor of equivalence and its the same arguments.

The *categorial indices* by means of which the typization of the words of $\mathscr{L}$ is carried out are, as we know, auxiliary words in that language. They are in the set E^2 and are classed into *basic* ones (which are in the set E_s^2) and *functoral ones* (which are in the set E_f^2). The latter are formed of the basic ones in accordance with definite rules, which in theoretical considerations are replaced by the single binary *relation* $\mathfrak{r}_2$ *of the formation of functoral indices*. If we assume that

$(\mathfrak{r}_2)\quad \mathfrak{r}_2(x^2, x_1^2, \ldots, x_n^2; x_0^2),$

is an expression in **T1**(tk) which we read thus: x_0^2 is a functoral index formed of the index x^2 and, successively, the indices $x_1^2, \ldots, x_n^2$, then x_0^2 may be treated as a substitute for any functoral index determined by the index x^2 and the successive indices $x_1^2, \ldots, x_n^2$ $(n \geq 1)$, regardless of the rules of concatenation of indices provided for $\mathscr{L}$. If, for instance, $V^2 = \{s, n, /, '\}$ and concatenation is to consist in right-sided linear juxtaposition, then x_0^2 equally well corresponds, for instance, to the index s/nn of a sentence-forming functor of two name arguments, and the index $s/nn//s/nn$, s/nn of a functor-forming functor which forms such a functor and also has such functors as its two arguments.

The relation $\mathfrak{r}_k$ $(k = 1, 2)$ is formally described by the following axioms:

A^{k}13. $D_1(\mathfrak{r}_k) = \bigcup_{n=2}^{\infty} D_k(\mathfrak{i})^n \ \wedge\ D_2(\mathfrak{r}_k) \subseteq D_k(\mathfrak{i}) \backslash V^k,$

A^{k}14. $\mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \wedge \mathfrak{r}_k(y_0^k, y_1^k, \ldots, y_n^k; y^k) \Rightarrow$
$\Rightarrow [y^k \approx x^k \ \Leftrightarrow\ m = n \ \wedge\ \forall 0 \leq j \leq n\, (y_j^k \approx x_j^k)],$

A^{k}15. $\mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \ \wedge\ \forall 0 \leq j \leq n\, (y_j^k \approx x_j^k) \ \wedge\ y^k \approx x^k \Rightarrow$
$\Rightarrow \mathfrak{r}_k(y_0^k, y_1^k, \ldots, y_n^k; y^k).$

Thus, the relation of the formation of functorial expressions (functoral indices) of $\mathscr{L}$ has as its domain the set of all finite Cartesian powers (greater than 1) of the set $D_1(\mathfrak{i})$ (the set $D_2(\mathfrak{i})$) of all those words of $\mathscr{L}$ which have indices (all indices of such words) and the counterdomain of $\mathfrak{r}_1$ $(\mathfrak{r}_2)$ is included in the set of all compound words of $\mathscr{L}$ which have an index (compound auxiliary words of $\mathscr{L}$ which are indices of words); two functorial expressions (functoral indices) of $\mathscr{L}$ are equiform if and only if they are formed of the same number of pairwise equiform words (indices of words) of $\mathscr{L}$; a word (an auxiliary word) of $\mathscr{L}$ which is equiform with a functorial expression x^1 (functoral index x^2) of that language is a functorial expression (functoral index) formed of successive words (indices of words) which are pairwise equiform with the words (indices of words) occurring in the same order, of which the word x^1 (index x^2) is formed. □

The *set* E_s^1 (E_s^2) *of all simple expressions* (*all basic indices*) of $\mathscr{L}$ is defined as the set of all words of the vocabulary (auxiliary vocabulary) of that language which have an index (are indices of words). The *set* E_f^1 (E_f^2) *of all functorial expressions* (*functoral indices*) of $\mathscr{L}$ is defined as the counterdomain of the relation of the formation of functorial expression (functoral indices). The *set* E^1 (E^2) *of all expressions* (*all well-formed indices*) of $\mathscr{L}$ is defined as the sum of the sets E_s^1 and E_f^1 $(E_s^2$ and $E_f^2)$. Hence the following definitions $(k = 1, 2)$ oblige in **T1**(tk):

D^{k}2a. $E_s^k = V^k \cap D_k(\mathfrak{i}),$

b. $E_f^k = D_2(\mathfrak{r}_k)$,
c. $E^k = E_s^k \cup E_f^k$.

The concept of the *set E of all well-formed expressions* (wfe), which is fundamental for the categorial language $\mathscr{L}$ is defined by reference to the set E_n, of all such expressions of the *n*-th order; E_n is defined by induction:

D3a. $E_0 = E_s^1$,
b. $x^1 \in E_{k+1} \iff x^1 \in E_k \vee \exists n \geq 1\, \exists x_0^1, x_1^1, \ldots, x_n^1 \in E_k \big[\mathfrak{r}_1(x_0^1, x_1^1, \ldots, x_n^1; x^1) \wedge$
$\wedge\ \forall 0 \leq j \leq n\, \forall x_j^2, x_j \, \big(\mathfrak{i}(x_j^1, x_j^2) \wedge \mathfrak{i}(x^1, x^2) \implies \mathfrak{r}_2(x^2, x_1^2, , \ldots, x_n^2; x_0^2)\big)\big]$.

Thus a wfe of the 0 order in $\mathscr{L}$ is any simple expression of that language. A wfe of the $k+1$-th order in $\mathscr{L}$ is either a wfe of the k-th order or a functorial expression formed of wfes of the k-th order: the main functor and its arguments such that any index of the main functor of that expression satisfies the rule which expresses the *superior principle of the theory of syntactic categories* (cf. Sec. 2), briefly **sptsc**, formulated as follows:

(sptsc) *The index of the main functor of a functorial expression is a functoral index formed of the index of that expression and the successive indices of the successive arguments of that functor.*

The set E is defined as the sum of all wfes of a finite order ≥ 0. Hence

D3c. $E = \bigcup_{n=0}^{\infty} E_n$.

We also assume that

A16. $\overrightarrow{\mathfrak{i}}(E \setminus E_0) \cap E_s^2 \neq \emptyset$,[5]
A17. $\overrightarrow{\mathfrak{i}}(E) \subseteq E^2$,

which is to say that there is at least one compound wfe of $\mathscr{L}$ which has a basic index and that the indices of wfes of $\mathscr{L}$ are well-formed.

We show below that A16 guarantees the non-emptiness of U so that there is at least one token.

Note that the set E of all wfes of the categorial language $\mathscr{L}$ can be generated (cf. Sec. 2, p. 9) by the system

$$\mathscr{G}_{\mathscr{L}} = \langle U, \mathfrak{c}, \mathfrak{r}_1, \mathfrak{r}_2, E_s^1, E_s^2, \mathfrak{i}\rangle$$

which may be treated as a reconstruction of the *classical categorial grammar*, whose idea going back to [1] (cf. [8, 9]). That grammar is said to be *rigid* [8]: every word or expression of $\mathscr{L}$ has one categorial index assigned to it (up to equiformity). That follows from the axioms Ala and A11.

A more precise categorial characterization of the language $\mathscr{L}$ described by **T1**(tk) is thus given by the pair

$$\mathscr{L}(\mathscr{G}_{\mathscr{L}}) = \langle \mathscr{G}_{\mathscr{L}}; E\rangle.$$

[5] The expression $\overrightarrow{\mathfrak{i}}(A)$ represents the image of set A with respect to the relation $\mathfrak{i}$.

The categorial analysis and the estimation of the syntactic correctness of given expression of $\mathscr{L}$ refers solely to its functorial expressions and consists in finding whether the rule **sptsc** is satisfied for every constituent of such an expression. The functorial expressions of $\mathscr{L}$ are compound expressions formed of its *basics expressions* and auxiliary expressions, that is *functors*. □

The *set B of all basic expressions* of $\mathscr{L}$ is defined as the set of well-formed expressions with basic indices, and the *set F of all functors* of $\mathscr{L}$ is defined as the set well-formed expressions of $\mathscr{L}$ with functoral indices. The formal definitions of those sets are as follows:

D4. $B = \{x^1 \in E : \forall x^2 \left(\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \in E^2_s\right)\}$,
D5. $F = \{x^1 \in E : \forall x^2 \left(\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \in E^2_f\right)\}$.

The singling out of the sets B and F from the set E does not give the complete syntactic categorial description of $\mathscr{L}$, which consists in the possibility of carrying out a logical partition of E into *syntactic categories* (see Sec. 2, p. 8).

The traditional definitions of syntactic category link it—in accordance with the ideas advanced by E. Husserl—to the set of expressions *replaceable* in any sentential contexts, or, more generally, in any well-formed ones (see [14], [21], [1], [6]). Such definitions not only do not eliminate the risk of a vicious circle (cf. [24]), but also have other undesirable consequences.[6] In any case, when carrying out a categorial analysis of a given expression it is most convenient to define its syntactic category by making use of the index of that expression, and to include in one and the same syntactic category any two expressions which have equiform indices, that is such which are *categorially in agreement*.

Let then the *syntactic category with the index t* correspond to the set Ct_t of all those expressions of $\mathscr{L}$ whose index is equiform with t. In symbols:

D6. $Ct_t = \{x^1 \in E^1 : \forall x^2 \left(\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \approx t\right)\}$, where $t \in \overrightarrow{\mathfrak{i}}(E^1)$.

Further, let any two expressions x and y be categorially in agreement if they bear to one another the relation $\tilde{c}$ defined by the formula:

D7. $x, y \in E^1 \;\Rightarrow\; \left[x\tilde{c}y \Leftrightarrow \exists\, t\ (x, y \in Ct_t)\right]$.

Note that $\tilde{c}$ determines the logical partition of E^1 into syntactie categories, and hence also the expected logical partition of E and, consequently, of B and F (see Theorem 8 below). Those partitions are, correspondingly, families of sets Ct(S), where S $\in \{E^1, E, B, F\}$, called *families of all syntactic categories of the expressions belonging to the set* S. The definitions of those families are obtained correspondingly from the schema:

[6] There exist expressions included into the same syntactic category of, for example, names, that are not replaceable in any sentence or well-formed expression. For instance, the noun “man”, personal pronoun “he”, or cardinal numeral “8” are names. Hence, by replacing the noun by the pronoun or numeral in the well-formed expressions: “a noble man”, “John is a noble man”, we obtain meaningless expressions. On the other hand, the expressions: “8 = 8” and “8–8” are also well-formed, though the latter emerges by replacing the sentence-forming functor “=” by the name-forming functor “–”—ie., by a functor of another syntactic category.

D8(S). $Ct(\mathrm{S}) = \{Ct_t : t \in \overrightarrow{\mathfrak{i}}(\mathrm{S})\}$, for $\mathrm{S} \in \{E^1, E, B, F\}$. □

By adopting definition D7 we deviate from the traditional definitions of the concept of syntactic category, But we still associate that concept with the concept of replaceability of expressions, important in the theory of syntactic categories. This will be reflected in the *fundamental theorem of the theory of syntactic categories* (Theorem 9 below).

The *relation of replaceability* $(/)$ is a four-argument relation in E^1. Its definition is based on the auxiliary concept of the *relation* $(/)^n$ *of the replaceability of a constituent* of the n-th order. The latter will be defined by induction. In the recording of the definitions of both relations we shall make use of the expression $y(t/z)^n x$, which we read: an expression y of $\mathscr{L}$ is obtained from an expression x of that language by the replacement of its z constituent of the n-th order by an expression t. The formulation $y(t/z)x$ we read analogically with the omission of the element: of the n-th order.

D9a. $y(t/z)^0 x \;\Leftrightarrow\; x, y \in E^1 \wedge z \approx x \wedge t \approx y.$

b. $y(t/z)^1 x \;\Leftrightarrow\; \exists n{\geq}1\, \exists x_0, x_1, \ldots, x_n\, \exists y_0, y_1, \ldots, y_n$
$\big[\mathfrak{r}_1(x_0, x_1, \ldots, x_n; x) \;\wedge\; \mathfrak{r}_1(y_0, y_1, \ldots, y_n; y) \;\wedge$
$\wedge\; \exists\, 0{\leq}j{\leq}n\, \big(z \approx x_j \;\wedge\; t \approx y_j \;\wedge\; \forall\, k{\neq}j \wedge 0{\leq}k{\leq}n\; (x_k \approx y_k)\big)\big].$

c. $y(t/z)^{k+1} x \;\Leftrightarrow\; \exists x_1, x_2\, \big(y(x_2/x_1)^k x \;\wedge\; x_2(t/z)^1 x_1\big)$, for $k > 0$.

d. $y(t/z)x \;\Leftrightarrow\; \exists n{\geq}0\, (y(t/z)^n x).$ □

4.1.2 Major Theorems of the Theory T1(tk)

As has been mentioned previously, Axiom A16 leads us in particular to the conclusion that tokens exists. This is so because it guarantees the non-emptiness of $E \backslash E_0$, and hence, by D3c, a, the non—emptiness of some set E_n $(n > O)$. Since by D3a, b, c, D[1]2a, b, c, and A[1]13 we have the following lemma:

(1) $E_n \subseteq E \subseteq E^1 \subseteq D_1(\mathfrak{i})$, for all $n \geq 0$,

the non-emptiness of some of the sets E_n $(n > 0)$ implies 1) the non-emptiness of the sets E, E^1, $D_1(\mathfrak{i})$, and since by A9 and D[1]1 the following inclusion holds:

(2) $D_1(\mathfrak{i}) \subseteq W^1 \subseteq U$

it also implies 2) the non-emptiness of the sets W^1 and U. This means that there exist not only tokens in general, but, in particular, word-tokens, word-tokens which have an index, expression-tokens, well-formed expression-tokens.

One can formulate a more general theorem which guarantees the non-emptiness of sets at the *token level*, that is sets in the system (S).

To do so note first that since some E_n is non-empty, by D3b, a every set E_n is non-empty (including $E_0 = E^1_s$). By D[1]2a the same applies to the vocabulary V^1. Since $E \backslash E_0 \neq \emptyset$ and since, by D3a, b, c, D[1]2b, A[1]13 and (2) we have

(3) $E \backslash E_0 \subseteq E^1_f \subseteq D_1(\mathfrak{i}) \backslash V^1 \subseteq W^1 \backslash V^1$,

we find that the sets E^1_f, $D_1(\mathfrak{i})\backslash V^1$, $W^1\backslash V^1$ are non-empty. Note further that Axiom A16 also guarantees the non-emptiness of E^2_s, and the fact that $E\backslash E_0$ is non-empty guarantees the non-emptiness of E^2_f by Definitions D3c, b, (3) and Definition D^{2}2b, Thus D^{2}2c yields immediately the non-emptiness of E^2, D^{2}2a—the non-emptiness of V^2 and $D_2(\mathfrak{i})$; D^{2}2b and A^{2}13 yield the non-emptiness of $D_2(\mathfrak{i})\backslash V^2$. It yields the non-emptiness of $W^2\backslash V^2$ (A9) and W^2.

The fact that B is non-empty follows from its definition (Def. D4), A16, the fact that by A11 the index of a word is determined unambiguously up to equiformity, and from the theorem stating that a word index which is equiform with a basic index is basic, too (see formula (∗) of Theorem 2 below). The fact that F is non-empty follows from D5, the non-emptiness of $E\backslash E_0$, (3), D3c, a, b and D^{2}2b, A11, and the theorem stating that a index of word which is equiform with a functoral index is functoral, too (see formula (∗) of Theorem 2 below). It can also easily be seen that if $t \in \overrightarrow{\mathfrak{i}}(E^1)$, then the category Ct_t is non-empty by A11 and A1a and D6. By D8(S), the families Ct(S) are non-empty, too, if $\mathrm{S} \in \{E^1, E, B, F\}$, because $\mathrm{S} \neq \emptyset$ and $\overrightarrow{\mathfrak{i}}(\mathrm{S}) \neq \emptyset$ in view of (1) and the correctness of the inclusions $B \subseteq E$ and $F \subseteq E$.

The foregoing leads us to the following theorem:

Theorem 1 *All sets in the system* (S) *are non-empty.* □

The next two theorems describe important properties of the relation of equiformity.

Theorem 2 *A token which is equiform with a token from any set* S *of the system* (S) *is also an element of* S*, i.e., for any set of tokens of* (S) *the following formula holds:*
(∗) $x \in \mathrm{S} \ \wedge\ t \approx x \ \Rightarrow\ t \in \mathrm{S}$.

Proof If $\mathrm{S} = U$, then (∗) follows directly from the convention concerning the variables x and y (Sec. 3). If $\mathrm{S} = V^k$ $(k = 1, 2)$, the truth of (∗) is based on Axioms A^{k}6, To substantiate (∗) when $\mathrm{S} = W^k\backslash V^k$ and $\mathrm{S} = W^k$ note first that the concept of concatenation bears important relations to certain sets in (S). Since it follows immediately from the definitions of the sets W^k for $k = 1, 2$ (Def. D^{k}1) that

(4) $V^k \subseteq W^k \subseteq U$,
(5_k) $x, y \in W^k \ \wedge\ \mathfrak{c}(x, y, z) \ \Rightarrow\ z \in W^k$,

the Lemmas (5_k) and Axioms A^{k}7 and A^{k}8 $(k = 1, 2)$ yield the relationships:

(6_k) $z \in W^k\backslash V^k \ \Leftrightarrow\ \exists x, y{\in}W^k\ \mathfrak{c}(x, y, z)$,

and the formulas (4) and (6_k), the relationships:

(7_k) $z \in W^k \ \Leftrightarrow\ z \in V^k \ \vee\ \exists x, y{\in}W^k\ \mathfrak{c}(x, y, z)$.

Thus the truth of (∗) for $\mathrm{S} = W^k\backslash V^k$ follows from (6_k) and A4, and for $\mathrm{S} = W^k$, from (7_k) and A^{k}6 and A4. Note further that the same index corresponds to equiform words while equiform indices correspond to the same word, so that

(8) $\mathfrak{i}(x, y) \ \wedge\ t \approx x \ \Rightarrow \mathfrak{i}(t, y)$,
(9) $\mathfrak{i}(x, y) \ \wedge\ t \approx x \ \Rightarrow \mathfrak{i}(x, t)$.

This follows from A1a and A12. The properties (8) and (9) substantiate the correctness of $(*)$ for $\mathrm{S} = D_k(\mathfrak{i})$ $(k = 1, 2)$, and hence, by A1b and A^k6, its correctness for $\mathrm{S} = D_k(\mathfrak{i})\backslash V^k$.

The implication $(*)$ holds for $\mathrm{S} = E_s^k$ $(k = 1, 2)$, which follows directly from $\mathrm{D}^k2\mathrm{a}$ and from the fact that it holds for $\mathrm{S} = \mathrm{D}_k(\mathfrak{i})$ and $\mathrm{S} = V^k$; the fact that it holds for E_f^k $(k = 1, 2)$ follows from $\mathrm{D}^k2\mathrm{b}$ and the following conclusions from A1a and A^k15:

$$(10_k)\quad \mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \ \wedge\ y^k \approx x^k \ \Rightarrow\ \mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; y^k).$$

In view of the above $(*)$ holds for the sets $\mathrm{S} = E^k$ $(k = 1, 2)$ on the strength of their definitions (Def. $\mathrm{D}^k2\mathrm{c}$).

The substantiation of the fact that the implication $(*)$ is valid for $\mathrm{S} = E$ is based on D3c and the lemma

$$(11)\quad x \in E_n \ \wedge\ t \approx x \ \Rightarrow\ t \in E_n, \quad \text{for all } n \geq 0.$$

whose proof by induction is in turn based on the statement that the formula holds for $E_0 = E_s^1$ (D3a) and at the inductive assumption, which is to say that its truth is assumed for $n = k$, on the statement that it holds for $n = k + 1$ on the basis of D3b, (10_k), (8) and A1b.

Now $(*)$ is also self-evidently correct for $\mathrm{S} = E\backslash E_0$, which follows from D3c and (11). To complete the proof of Theorem 2 one has to prove $(*)$ for $\mathrm{S} = B$, $\mathrm{S} = F$, and $\mathrm{S} = Ct_t$. Since $(*)$ is valid for $\mathrm{S} = E$, the justification of the first two cases is based directly on Definition D4, resp. D5, Lemma (8), and A1b. The truth of $(*)$ for $\mathrm{S} = Ct_t$ follows from D6, the fact that it is true for $\mathrm{S} = E^1$, and Lemma (8). □

Theorem 3 *The following implication holds for every relation* R *in the system* (R)*:*
$$(**)\quad R(x_0, x_1, \ldots, x_n) \ \wedge\ \forall\, 0 \leq l \leq n\ (y_l \approx x_l) \ \Rightarrow\ R(y_0, y_1, \ldots, y_n), \quad \textit{for } n \geq 1.$$

Proof If R equals the equiformity relation $\approx$, then $(**)$ follows immediately from Axioms A1b, c. If $\mathrm{R} = \mathfrak{c}$, it follows from A4 and the lemma

$$(12)\quad \mathfrak{c}(x_0, x_1, x_2) \ \wedge\ y_0 \approx x_0 \ \wedge\ y_1 \approx x_1 \ \Rightarrow\ \mathfrak{c}(y_0, y_1, x_2).$$

The proof of (12) is based on A2 applied to the tokens y_0 and y_1, A3 and A4. If $\mathrm{R} = \mathfrak{i}$, then $(**)$ is a substitution of A12, and if $\mathrm{R} = \mathfrak{r}_k$, $(k = 1, 2)$, then it is a substitution of A^k15.[7] If $\mathrm{R} = (/)^n$, then the proof of that formula is by induction: for $n = 0$ it follows immediately from D9a, a substitution instance of $(*)$ (we set E^1 for S), A1b and A1c; for $n = 1$ it follows immediately from Definition D9b, the correctness of $(**)$ for $\mathrm{R} = \mathfrak{r}_1$, and A1a, c; by assuming the truth of that formula for $n = k$ we arrive at stating its truth for $n = k + 1$ on the basis of D9c, A1a, and the fact that it is true for $n = 1$. If $\mathrm{R} = (/)$, then it follows from the fact that it is true for $\mathrm{R} = (/)^n$ by D9d. Finally, $\mathrm{R} = \tilde{c}$, then $(**)$ follows from D7 and D6, the truth of $(*)$ for $\mathrm{S} = E^1$, the Lemma (8) and A1b. □

The successive theorems illustrate certain properties of E.

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[7] Relations $\mathfrak{r}_k$ $(k = 1, 2)$ are $n + 1$-argument relations, $n \geq 0$.

Theorem 4 *The set E of all well-formed expression of $\mathscr{L}$ is the least set of tokens from the universe U of that language containing the set of all its simple expressions and satisfying the condition that it contains every functorial expression of $\mathscr{L}$ that satisfies the rule* **sptsc**.

An analogous theorem is given in [24] together with its proof. The proof of Theorem 4 is modelled on the latter. It is omitted here.

As has been mentioned in Sec. 4.1.1, categorial indices are not words of $\mathscr{L}$, but are words of the metalanguage of that language, namely auxiliary words. This applies in particular to the indices of wfes of $\mathscr{L}$. That fact follows from the relationships:

$$E \subseteq D_1(\mathfrak{i}) \subseteq W^1, \quad \overrightarrow{\mathfrak{i}}(E) \subseteq D_2(\mathfrak{i}) \subseteq W^2.$$

The first of them is a direct consequence of Lemmas (l), (2), while the second follows from Al7, $D^2$2c, a, b, $A^2$13 and A9. In view of A10 we can accordingly state that the following theorem holds:

Theorem 5 *The set of all well-formed expressions of $\mathscr{L}$ is disjoint from the set of the indices of those expressions, ie.*

$$E \cap \overrightarrow{\mathfrak{i}}(E) = \emptyset.$$

Theorem 6 *The set E of all well-formed expressions of $\mathscr{L}$ is the sum of two non-empty and disjoint sets: the set B of all basic expressions of $\mathscr{L}$ and the set F of all its functors. In symbols:*

$$E = B \cup F \ \wedge \ B \neq \emptyset \ \wedge \ F \neq \emptyset \ \wedge \ B \cap F \neq \emptyset.$$

Proof Since by Lemmas (1) and (2) E is the set of those words which have indices, and by A17 the indices of such words are elements of E^2, $E = B \cup F$ by $D^2$2c, D4, and D5. By Theorem 1 B and F are non-empty sets. They are also disjoint, which follows from their definitions, A17, $D^2$2c, a, b, and $A^2$13, because the indices of their expressions belong, respectively, to the disjoint sets E^2_s and E^2_f. □

Note in this connection that a theorem analogous to Theorem 5 holds for the set E^2 of all well-formed indices because for $k = 1, 2$ the following formula is valid:

(13) $E^k = E^k_s \cup E^k_f \ \wedge \ E^k_s \neq \emptyset \ \wedge \ E^k_f \neq \emptyset \ \wedge \ E^k_s \cap E^k_f \neq \emptyset.$

The sets B and F, which form a partition of the set E, have, correspondingly, common elements with the disjoint sets of expressions $E \backslash E_0$ and E_0. The fact that there is a basic expression of $\mathscr{L}$ which is a compound well-formed expression follows from A16, D4, A1a and A11, and the formula $(*)$ for $\mathrm{S} = B$. On the other hand, as we know, $E \backslash E_0$, is a non-empty set, and by D3c, b, a there is a functorial expression of $\mathscr{L}$ and there is also such its main functor belonging to $E_0 \subseteq E$ that its arbitrary index is, by $D^2$2b, a functoral index. The functor is, therefore, by D5, also an element of F. We accordingly have

Theorem 7

a. $(E \backslash E_0) \cap B \neq \emptyset$,
b. $E_0 \cap F \neq \emptyset$.

In accordance with the convention 8 in Sec. 2, the categorial character of $\mathscr{L}$ should reflect a more detailed logical partition of E than Theorem 6 indicates, namely a logical partition of that set into syntactic categories. Formally this is so in fact, because the more general theorem holds:

Theorem 8 *If* $\mathrm{S} \in \{E^1, E, B, F\}$*, then*

(i) $\mathrm{S} = \bigcup Ct(\mathrm{S})$ – S *is the sum of all syntactic categories of the expressions of* S
(ii) $\forall\, Ct_t \in Ct(\mathrm{S})\ (Ct_t \neq \emptyset)$ – *which are non-empty*
(iii) $\forall\, Ct_t, Ct_{t'} \in Ct(\mathrm{S})\ (Ct_t \neq Ct_{t'} \ \Rightarrow Ct_t \cap Ct_{t'} \neq \emptyset)$ – *and pairwise disjoint.*

Proof Let $\mathrm{S} \in \{E^1, E, B, F\}$. By Lemma (1), Definitions D4, D5, and D6, and axioms A1a and A11, an arbitrary token x from S belongs to some syntactic category with an index t (such that $\mathfrak{i}(x,t)$. The relation $\tilde{c}$ is thus reflexive on S. It follows directly from D7 that it is symmetric in that set. It is also transitive in that set, for if $x, y \in Ct_{t_1}$ and $y, z \in Ct_{t_2}$, then by D6 the index of the expression y is equiform with both t_1 and t_2, whence it follows that $t_1 \approx t_2$ and then $Ct_{t_1} = Ct_{t_2}$ and *eo ipso* $x, z \in Ct_{t_1}$. Since S is non-empty (Theorem 1), the equivalence relation $\tilde{c}$ determines the logical partition $\mathrm{S}/_{\tilde{c}}$ of S into non-empty and pairwise disjoint equivalence classes relative to $\tilde{c}$. Note further that an equivalence class relative to $\tilde{c}$ is a syntactic category whose index is the index of the expression which is a representative of that class, ie.,

(14) $x \in \mathrm{S} \ \wedge\ \mathfrak{i}(x,t) \ \Rightarrow\ [x]_{\tilde{c}} = Ct_t, \quad$ for $\mathrm{S} \in \{E^1, E, B, F\}$.

In fact, if $y \in [x]_{\tilde{c}}$, then by D7 $x, y \in Ct_{t_1}$, and since $\mathfrak{i}(x,t)$ by applying D6 we obtain $t \approx t_{t_1}$ and $Ct_t = Ct_{t_1}$, and then $y \in Ct_t$. And conversely: note that $x \in Ct_t$ because it follows by assumption and from A1a and A11 that $x \in E^1$ and if $\mathfrak{i}(x, x^2)$, then $x^2 \approx t$ for any x^2. Thus, if $y \in Ct_t$, then $x\tilde{c}y$ and $y \in [x]_{\tilde{c}}$.

Thus (14) is true, and since the index of a word-token is determined unambiguously by up to equiformity (Axiom A11) while syntactic categories with equiform indices are identical, by D8(S) the quotient family $\mathrm{S}/_{\tilde{c}}$ is equal to the family $Ct(\mathrm{S})$ of all syntactic categories of expressions in S. This proves formulas (i)–(iii). □

Finally, we proceed to formulate the aforementioned (Sec. 4.1.1) *fundamental theorem of the theory of syntactic categories*:

Theorem 9 (fttsc) *Two expressions of $\mathscr{L}$ belong to the same syntactic category if and only if on replacing one by the other in a well-formed expression of $\mathscr{L}$ and obtaining from it a well-formed expression of that language we find that it belongs to the same syntactic category as the former. In symbols:*

$$x, y \in E \ \wedge\ y(t/z)x \ \Rightarrow\ (t\tilde{c}z \ \Leftrightarrow\ y\tilde{c}x)$$

Proof The proof of this theorem is based on the following two lemmas:

(15) $x, y \in E \ \wedge\ y(t/z)^n x \ \wedge\ t\tilde{c}z \Rightarrow y\tilde{c}x,$ for $n \geq 0$,
(16) $x, y \in E \ \wedge\ y(t/z)^n x \ \wedge\ y\tilde{c}x \Rightarrow t\tilde{c}z,$ for $n \geq 0$,

The proofs of these lemmas are carried out by induction. When $n = 0$ their truth is substantiated by reference to D9a, D7, D6, Lemma (1), and A1b, A12, and A11. The proofs for $n = 1$ are more difficult. In this case we shall prove only (15) and leave the proof of (16) to the Reader (see [24]). In the case under consideration it follows from the assumption of (15) and from D9b, D7, and D6 that

(a) $x, y \in E$
(b) $\mathfrak{r}_1(x_0, x_1, \ldots, x_n; x) \wedge \mathfrak{r}(y_0, y_1, \ldots, y_n; y)$
(c) $z \approx x_{j_1} \wedge t \approx y_{j_1},$ for $0 \leq j_1 \leq n$
(d) $\forall\, k \neq j_1 \wedge 0{\leq}k{\leq}n\ (x_k \approx y_k)$

and

(e) $z, t \in E^1$
(f) $\forall\, z^2\, \big(\mathfrak{i}(z, z^2) \ \Rightarrow\ z^2 \approx t_1\big) \ \wedge\ \forall\, t^2\, \big(\mathfrak{i}(t, t^2) \ \Rightarrow\ t^2 \approx t_1\big).$

It follows from (a) and Lemma (1) that

(g) $x, y \in E^1$.

To prove that $x\tilde{c}y$, which is to say that x and y are elements in the same syntactie category we assume, on the basis of (g) and Lemma (1), that x^2 and y^2 are their respective index-tokens, ie.,

(h) $\mathfrak{i}(x, x^2) \wedge \mathfrak{i}(y, y^2)$.

We shall now demonstrate that $x^2 \approx y^2$. Note that since, in accordance with (a) and (b), x, y are compound well-formed expressions, their respective elements $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ are also well-formed expressions by D3c, b and as such have their indices (Lemma (1)). Let therefore

(i) $\forall\, 0{\leq}k{\leq}n\ \big(\mathfrak{i}(x_k, x_k^2) \wedge \mathfrak{i}(y_k, y_k^2)\big).$

Now (c), (8), and (i) yield:

(j) $\mathfrak{i}(z, x_{j_1}^2) \wedge \mathfrak{i}(t, y_{j_1}^2)$.

In view of (f) it follows from (j) and A1b, c that $x_{j_1}^2 \approx y_{j_1}^2$, and in view of (d) it follows from (i) and A11 that, for every $k \neq j_1$ and $0{\leq}k{\leq}n$, $x_k^2 \approx y_k^2$. Hence, for every $0{\leq}k{\leq}n$, $x_k^2 \approx y_k^2$, and in particular

(k) $x_0^2 \approx y_0^2$.

The well-formed expressions x, y are of the form (b) and as such must satisfy the rule which expresses **sptsc**. On the basis of assumptions (i) and (h) we take that rule into consideration (see D3b) in the following formula:

(l) $\mathfrak{r}_2(x^2, x_1^2, \ldots, x_n^2; x_0^2) \wedge \mathfrak{r}_2(y^2, y_1^2, \ldots, y_n^2; y_0^2).$

Now $x^2 \approx y^2$ follows from A14 and the formulas (l) and (k).

By assuming now that $\mathfrak{i}(x,t)$ we would obtain, by (h) and A1a and A11, $t \approx x^2$. This allows us to state, by (g) and D6, that $x \in Ct_{x^2}$. Likewise we demonstrate that $y \in Ct_{y^2}$, and since $x^2 \approx y^2$, $Ct_{x^2} = Ct_{y^2}$. Now it follows that x and y belong to the same syntactic category, which allows us to state, by D7, that $x\tilde{c}y$ $(y\tilde{c}x)$.

The proofs of Lemmas (15) and (16) follow immediately, by inductive assumption, from D9c and the fact that they are true for $n = 1$.

Theorem 8 is a direct consequence of these lemmas and D9c. □

4.2 Formalization of T1 at the Type Level; Theory T1(tp)

The formalization of the theory **T1** at the *type level* consists (see Sec. 3) in the expansion of the theory **T1**(tk) in the form of its dual theory **T1**(tp), which describes all the concepts at that level, that is the concepts of the systems (**S**) and (**R**). The theory **T1**(tp) allows us to describe any fixed categorial language $\underline{\mathscr{L}}$ as a language of expression-types. All concepts at the *type level* are derived constructs defined by means concepts at the *token level*. Every set **S** of types, which is an element of the system (**S**), except for the set $\boldsymbol{Ct_T}$, is defined as follows by means of the dual set S of tokens:

(D**S**) $X \in \mathbf{S} \;\Leftrightarrow\; \exists x{\in}\mathrm{S}\,\big(X = [x]\big)$, i.e. $\mathbf{S} = \mathrm{S}/{\approx}$.

In the above schema, and also further in the text, we use the symbol $[x]$ for the equivalence class represented by x and determined by the equiformity relation.

The syntactic category with the index type T, that is the set $\boldsymbol{Ct_T}$, is the family of all equivalence classes of equiform tokens belonging to the syntactic category with an index-token which is a representative of the equivalence class that determines the index T. In symbols:

D$\boldsymbol{Ct_T}$. $\boldsymbol{Ct_T} = \big\{X \in \boldsymbol{E^1} \colon \exists\, x \in Ct_t \;(X = [x] \wedge T = [t])\big\}$.

The remaining concepts of (**S**), that is the families **Ct**(**S**) of all syntactic categories of expression-types from **S**, where $\mathbf{S} \in \{\boldsymbol{E^1}, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{F}\}$, are defined by definitions which are dual to Definition D8(S). Hence

D$\boldsymbol{Ct}$(**S**). $\boldsymbol{Ct}(\mathbf{S}) = \big\{\boldsymbol{Ct_T} \colon T \in \mathfrak{i}(\mathbf{S})\big\}$, for $\mathbf{S} \in \{\boldsymbol{E^1}, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{F}\}$.

The relation $\underline{\tilde{c}}$ of the categorial agreement of expression-types is defined by a definition dual to D7, namely

D$\boldsymbol{\tilde{c}}$. $X, Y \in \boldsymbol{E^1} \;\Rightarrow\; \big[X\underline{\tilde{c}}Y \;\Leftrightarrow\; \exists\, T\; (X, Y \in \boldsymbol{Ct_T})\big]$.

Each of the remaining relations **R** from (**R**) is defined by its dual relation R from (R) in the following way:

D**R**. $\mathbf{R}(X_0, X_1, \ldots, X_n) \;\Leftrightarrow\; \exists\, x_0, x_1, \ldots, x_n\, \big(X_0 = [x_0] \;\wedge\; X_1 = [x_1] \;\wedge\; \ldots$
$\ldots \;\wedge\; X_n = [x_n] \;\wedge\; \mathrm{R}(x_0, x_1, \ldots, x_n)\big)$, where $n \geq 1$.

Thus a relation **R** holds between types if and only if they are such equivalence classes of equiform tokens that a dual relation R holds between their representatives.

In view of the axioms and definitions of the theory **T1**(tk) and the definitions of the concepts of the systems (**S**) and (**R**) of the theory **T1**(tp) we can substantiate the following.

Fact 1 *Every expression dual to a thesis of the theory* **T1**(tk) *is a thesis of the theory* **T1**(tp).

Fact 1 is substantiated directly by the observation that the following holds:

Fact 1a *Every expression dual to an axiom or definition of the theory* **T1**(tk) *is a theorem or definition of the theory* **T1**(tp).

By a *thesis* of a theory we mean in this paper its axioms, definitions and derived theorems.

Now Fact 1a follows from the fact that 1) the expressions d(A1a), d(A1b), d(A1c), d(A4), d(A12), d(A15), are theorems in the theory **T1**(tp); 2) in the theory **T1**(tp) Definitions d(D7), d(D8(S)) hold for S $\in \{E^1, E, B, F\}$; 3) the following expressions are theorems in **T1**(tp): d(A2), d(A3); d(A^{k}5), d(A^{k}6), d(A^{k}7), d(D^{k}1), d(A^{k}8), for $k = 1, 2$; d(A9), d(A10), d(A11); d(A^{k}13), d(A^{k}14), d(D^{k}2a), d(D^{k}2b), d(D^{k}2c) for $k = 1, 2$; d(D3a), d(D3b), d(D3c), d(A16), d(A17), d(D4), d(D5), d(D6), d(D9a), d(D9b), d(D9e), d(D9d).

The proofs of theorems given under 3) are fairly similar to the proofs of the corresponding theorems given in [24]. By way of example we shall give proofs of d(A^{k}8), $k = 1, 2$ and for d(D6).

d(A^{k}8). $T \in \boldsymbol{W}^1 \backslash \boldsymbol{V}^1 \;\Rightarrow\; \exists\, X, Y \in \boldsymbol{W}^k\ \underline{\mathfrak{c}}(X, Y, Z)$.

Proof Let $T \in \boldsymbol{W}^1 \backslash \boldsymbol{V}^1$ $(k = 1, 2)$. It follows from D$\boldsymbol{W}^k$ that $T = [t_1]$ and $t_1 \in W^k$, and from D$\boldsymbol{V}^k$, that for any $x \in V^k$, $T \neq [x]$. Hence $t_1 \in W^k \backslash V^k$, and by Axioms A^{k}8 $(k = 1, 2)$ we have that $\mathfrak{c}(x_1, y_1, z_1)$ and $x_1, x_2 \in W^k$. Note that in accordance with D$\boldsymbol{W}^k$ we have: $[x_1], [x_2] \in \boldsymbol{W}^k$, and in accordance with D$\underline{\mathfrak{c}}$ we have: $\underline{\mathfrak{c}}([x_1], [x_2], T)$. The truth of the consequent of the implication which is being proved follows immediately therefrom. □

d(D6). $\boldsymbol{Ct}_T = \left\{X^1 \in \boldsymbol{E}^1 : \forall\, X^2\ \left(\underline{\mathfrak{i}}(X^1, X^2) \;\Rightarrow\; X^2 = T\right)\right\}$.

Proof Let $X^1 \in \boldsymbol{Ct}_T$. Then by D$\boldsymbol{Ct}_T$ we have $X^1 \in \boldsymbol{E}^1$, $X^1 = [x^1]$, $x^1 \in Ct_{t_1}$, and $T = [t_1]$. Assume additionally that $\underline{\mathfrak{i}}(X^1, X^2)$. Then by D$\underline{\mathfrak{i}}$ we have: $X^1 = [x']$, $X^2 = [x^2]$, and $\underline{\mathfrak{i}}(x', x^2)$. Since $x^1 \approx x'$, it follows from (8) and D6 that $x^2 \approx t_1$. Hence $T = X^2$. Thus the inclusion $\subseteq$ holds. To prove the converse inclusion we assume that $X^1 \in \boldsymbol{E}^1$ and that for any X^2 if $\underline{\mathfrak{i}}(X^1, X^2)$, then $X^2 = T$. We want to show that $X^1 \in \boldsymbol{Ct}_T$. By D$\boldsymbol{E}^1$ we have that there is an $x^1 \in E^1$ such that $X^1 = [x^1]$, and since Lemma (1) holds, there is a t_1 such that $\mathfrak{i}(x^1, t_1)$, and in view of D$\underline{\mathfrak{i}}$ we can state that $\underline{\mathfrak{i}}(X^1, [x_1])$. It follows from the assumption that $T = [t_1]$. Hence, in order to state that $X^1 \in \boldsymbol{Ct}_T$ (by applying D$\boldsymbol{Ct}_T$) it suffices to state that $x^1 \in Ct_{t_1}$. That is in fact so in view of D6, because $x^1 \in E^1$, and if $\mathfrak{i}(x^1, x^2)$, then $x^2 \approx t_1$ by A1a and A11. □

In the theory **T1**(tp) we can formulate several theorems which are equivalent to expressions that are dual analogues of theses of the theory **T1**(tk) but are not such theses themselves. Thus we have

Theorem 10 *If* $\mathbf{R} \in \{\underline{\mathfrak{c}}, \underline{\mathfrak{i}}, \underline{\mathfrak{r}}_1, \underline{\mathfrak{r}}_2\}$*, then* $\mathbf{R}$ *is a function. The functions* $\underline{\mathfrak{r}}_1$ *and* $\underline{\mathfrak{r}}_2$ *are* 1–1 *functions.*

Proof The fact that the concatenation relation $\underline{\mathfrak{c}}$ is a function follows from d(A2) and d(A3). Since in **T1**(tp) the theorems $D_2(\underline{\mathfrak{i}}) \neq \emptyset$ and d(A11) hold, the relation $\underline{\mathfrak{i}}$ is a function. Inasmuch as $D_2(\underline{\mathfrak{r}}_k) \neq \emptyset$ for $k = 1, 2$, d($\mathrm{A}^1$14), and d($\mathrm{A}^2$14), we immediately conclude that the relations $\underline{\mathfrak{r}}_1$ and $\underline{\mathfrak{r}}_2$ are 1–1 functions. □

Writing $X^2 = \underline{\mathfrak{i}}(X^1)$ instead of $\underline{\mathfrak{i}}(X^1, X^2)$ and $X = \underline{\mathfrak{r}}_k(X_0, X_1, \ldots, X_n)$, for a fixed $k = 1, 2$, for $\underline{\mathfrak{r}}_k(X_0, X_1, \ldots, X_n; X)$ we can record two facts:

Fact 2 *The theorem* d(D3b) *of the theory* **T1**(tp) *is, on the basis of that theory, equivalent to the expression:*

(v) $X \in \boldsymbol{E}_{k+1} \;\Leftrightarrow\; X \in \boldsymbol{E}_k \;\vee\; \exists n{\geq}1\; X_0, X_1, \ldots, X_n \in \boldsymbol{E}_k$
$\big[X = \underline{\mathfrak{r}}_1(X_0, X_1, \ldots, X_n) \;\wedge\; \underline{\mathfrak{i}}(X_0) = \underline{\mathfrak{r}}_2\big(\underline{\mathfrak{i}}(X), \underline{\mathfrak{i}}(X_1), \ldots, \underline{\mathfrak{i}}(X_n)\big)\big].$

Fact 3 *The theorems* d(D4)*,* d(D5)*, and* d(D6) *of the theory* **T1**(tp) *are, on the basis of that theory, equivalent respectively to the following expressions:*

(i) $\boldsymbol{B} = \big\{X \in \boldsymbol{E} : \underline{\mathfrak{i}}(X) \in \boldsymbol{E}^2_s\big\},$
(ii) $\boldsymbol{F} = \big\{X \in \boldsymbol{E} : \underline{\mathfrak{i}}(X) \in \boldsymbol{E}^2_f\big\},$
(iii) $\boldsymbol{Ct}_T = \big\{X \in \boldsymbol{E}^1 : \underline{\mathfrak{i}}(X) \in T\big\}.$

In the proof of the equivalence of d(D3b) and (v) we avail ourselves of the lemma which is dual to Lemma (1) $\big(\boldsymbol{E}_k \subseteq \boldsymbol{D}_1(\underline{\mathfrak{i}})\big)$ and the theorem d(A13) $\big(\boldsymbol{D}_2(\underline{\mathfrak{r}}_2) \subseteq \boldsymbol{D}_1(\underline{\mathfrak{i}})\big)$. In the proof of Fact 3 we avail ourselves of the lemma which is dual to Lemma (1): $\boldsymbol{E} \subseteq \boldsymbol{E}^1 \subseteq \boldsymbol{D}_1(\underline{\mathfrak{i}})$. □

5 Theory T2

The theory **T2** has as its primitive terms the following symbols: $\boldsymbol{U}$, $\underline{\mathfrak{c}}$, $\boldsymbol{V}^1$, $\boldsymbol{V}^2$, $\underline{\mathfrak{i}}$, $\underline{\mathfrak{r}}_1$, $\underline{\mathfrak{r}}_2$. They are at the same time the primitive terms of its fragment **T2**(tp). The terms which denote the remaining concepts at the *type level* and also all terms denoting concepts at the *token level* are defined in **T2**.

5.1 Formalization of T2 at the Type Level; Theory T2(tp)

The theory **T2**(tp) is an axiomatic theory which describes the language $\underline{\mathscr{L}}$ characterized categorially as a language of expression-types.

The axioms and definitions of **T2**(tp) are either dual analogues of the axioms and definitions of **T1**(tk) or expressions equivalent to the latter. They are listed here. They are: Axioms d(A2), d(A3); Axioms d(A^k5), d(A^k6), d(A^k7) for $k = 1, 2$; Definitions d(D^k1) for $k = 1, 2$; Axioms d(A^k8) for $k = 1, 2$; Axioms d(A9), d(A10), d(A11); Axioms d(A^k13), d(A^k14) for $k = 1, 2$; Definitions d(D^k2a), d(D^k2b), d(D^k2c), for $k = 1, 2$; Definitions d(D3a), (v), (see Sec. 4.2) and d(D3c); Axioms d(A16), d(A17); Definitions (i)–ii) (see Sec. 4.2) and d(D7), d(D8(S)), d(D9a), d(D9b), d(D9c), d(D9d).

On the adoption of these axioms and definitions we can prove that the relations $\underline{\mathfrak{c}}$, $\underline{\mathfrak{i}}$, $\underline{\mathfrak{r}}_1$ and $\underline{\mathfrak{r}}_2$ are functions (see Theorem 10 in Sec. 4.2). The concatenation of two types yields one type, a word-type has one corresponding index-type, etc. This justifies the recording of d(D3a), (v), and (i)–(iii). These definitions are, respectively, equivalent to the expressions which are dual to D3a, b, D4–D6.

Note that the following expressions are theorems in **T2**(tp): d(A1a), d(A1b), d(A1c), d(A4), d(A12), d(A15). Hence by accepting axioms and definitions of **T2**(tp) in the way described above we can state, on the one hand,

Fact 4 *Every expression dual to a thesis in* **T1**(tk) *is a thesis in* **T2**(tp)*, i.e.,*

$$\textit{If } \mathbf{T1}(\mathrm{tk}) \vdash \alpha \textit{ then } \mathbf{T2}(\mathrm{tp}) \vdash \mathrm{d}(\alpha),$$

and on the other,

Fact 5 *Every thesis in* **T2**(tp) *is either an expression dual to a thesis in* **T1**(tk) *or an expression equivalent to a dual analogue* (*an expression which is translatable into a dual analogue*) *of a thesis in the latter theory, that is*

$$\textit{If } \mathbf{T2}(\mathrm{tp}) \vdash \underline{\alpha} \textit{ then } \underline{\alpha} = \mathrm{d}(\alpha) \textit{ and } \mathbf{T1}(\mathrm{tk}) \vdash \alpha,$$

or there is a $\underline{\beta}$ *such that* ($\mathbf{T2}(\mathrm{tp}) \vdash \underline{\alpha}$ *if and only if* $\mathbf{T2}(\mathrm{tp}) \vdash \underline{\beta}$) *and* $\underline{\beta} = \mathrm{d}(\beta)$ *and* $\mathbf{T1}(\mathrm{tk}) \vdash \beta$.

Facts 4 and 5 reveal the close connection between the theory **T1**(tk) and its dual theory **T2**(tp). From the formal point of view, if we consider only the syntactic single-level characterization of language, there is thus no essential difference between the two ontologically opposed methods of describing language by dual theories **T1**(tk) and **T2**(tp).

5.2 Formalization of T2 at the Token Level; Theory T2(tk)

We join to **T2**(tp) two additional axioms which render certain intuitions which we associate with the concept of *type* as a non-empty class of equiform *tokens*:

A^t1. $X \neq \emptyset$ – a type is non-empty set,

A^t2. $x \in X \wedge x \in Y \Rightarrow X = Y$ – two types are equal if they have an element in common.

The formalization of **T2** at the *token level* requires a definitional expansion of the theory **T2**(tp), namely the theory $\mathbf{T2}^t$(tp) by its *enrichment* with definitions of all concepts at the *token level*, defined by the appropriate dual concepts from the *type level*. The theory **T2**(tk), dual to **T2**(tp), is a fragment of **T2** which includes those definitions and describes a categorial language $\mathscr{L}$ as a language of expression-tokens.

The definitions of all sets of tokens from the system (S), except for the set Ct_t, have in **T2**(tk) the following schema:

DS. $x \in \mathrm{S} \;\Leftrightarrow\; \exists\, X \in \mathbf{S}\, (x \in X).$

The set $\mathrm{S} \neq Ct_t$ of tokens is a set of those tokens which are elements (concrete representatives) of some type that belongs to the dual set **S**.

Since the universe $\boldsymbol{U}$ of $\underline{\mathscr{L}}$ is non-empty (Fact 1), in agreement with Axiom $\mathrm{A}^t 1$ and Definition DU elements of a type are tokens of U. The types are thus really sets of tokens.

The concept Ct_t of syntactic category with an index t is defined thus:

DCt_t. $Ct_t = \big\{x \in E_1 : \exists\, T\, \exists\, X \in \boldsymbol{Ct}_T\; (x \in X \wedge t \in T)\big\}.$

The remaining elements of (S), that is the family Ct(S), where $\mathrm{S} \in \{E^1, E, F, R\}$, are defined as in **T1**(tk), and hence by definitions of the form D8(S).

The definitions of all relations in (R), except for $\tilde{\mathfrak{c}}$, have the following form:

DR. $\mathrm{R}(x_0, x_1, \ldots, x_n) \;\Leftrightarrow\; \exists\, X_0, X_1, \ldots, X_n\; \big(x_0 \in X_0 \wedge x_1 \in X_1 \wedge \ldots x_n \in X_n \wedge$
$\wedge\, \mathbf{R}(X_0, X_1, \ldots, X_n)\big),$ where $n \geq 1$.

The relation $\tilde{\mathfrak{c}}$ is defined identically as in **T1**(tk).

Note that the definition D$\approx$ of $\approx$ can be recorded in a simpler way:

D$\approx$. $x \approx y \;\Leftrightarrow\; \exists\, X\; (x, y \in X).$

Thus by assuming in **T2** that types are primitive entities, while tokens are derived constructs as elements of types (Definition DU), we are in a position to formally show that in accordance with our intuition any type is a set of equiform tokens. Hence in particular Definition D$\boldsymbol{U}$ in **T1**(tp) is a theorem in **T2**.

We shall discuss in greater detail that fragment **T2**(tk) of **T2** which is developed on the basis of $\mathbf{T2}^t$(tp). Owing to the definitions which are valid in that fragment, namely definitions of the concept at the *token level*, it can be used to describe a categorial language $\mathscr{L}$ in a manner analogous to how it is done in **T1**(tk). This is so because we have to do with the following

Fact 6a *Every axiom and every definition in* **T1**(tk) *is a theorem or a definition in* **T2**(tk)*,*

and hence with

Fact 6 *Every thesis in* **T1**(tk) *is a thesis* (*theorem or definition*) *in* **T2**(tk)*.*

The complete substantiation of Fact 6a is rather labour-consuming. The detailed substantiation of the fact that Axioms A1a, b, c – A4 and $A^1 5$—$A^1 8$ and Definition $D^1 1$ are theorems in **T2**(tk) is to be found in [26]. Those theorems pertain to the tokens from the universe U or its subsets V^1 and W^1. The substantiation of the fact that the analogues of the expressions $A^1 5$–$A^1 8$ and $D^1 1$ pertaining to the auxiliary words in the sets V^2 and W^2, that is the expressions $A^2 5$–$A^2 8$ and $D^2 1$, are theorems in **T2**(tk), is analogous. The proofs of the fact that the remaining axioms and definitions in **T1**(tk) other than D7 and D8(S), where $S \in \{E^1, E, B, F\}$ (assumed also in **T2**(tk)) are theorems in **T2**(tk) present no major problems. To show the functioning of the definitions and axioms given in this section we shall prove by way of example the expressions $A^k 15$ ($k = 1, 2$) and the simple inclusion yielding D4.

$A^k 15$. $\mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \ \wedge\ \forall 0{\le}j{\le}n\ (y_j^k \approx x_j^k) \ \wedge\ y^k \approx x^k \ \Rightarrow$
$\Rightarrow\ \mathfrak{r}_k(y_0^k, y_1^k, \ldots, y_n^k; y^k).$

Proof It follows from the first assumption of $A^k 15$ and from $D\mathfrak{r}_k$ ($k = 1, 2$) that there are types $X_0^k, X_1^k, \ldots, X_n^k$ such that, for any $0 \le 1 \le n$, $x_l^k \in X_l^k$, $x^k \in X^k$, and $\boldsymbol{R}(X_0^k, X_1^k, \ldots, X_n^k; X^k)$. Since the following lemma

L1. $x \in X \ \wedge\ y \approx x \ \Rightarrow\ y \in X.$

is true in view of $A^t 2$ and D≈, it follows from the remaining assumptions of $A^k 15$ that for any $0 \le 1 \le n$, $y_l^k \in X_l^k$ and $y^k \in X^k$. By availing ourselves again of $D\mathfrak{r}_k$ we obtain the thesis of $A^k 15$. □

D4(⊆). $B \subseteq \left\{x^1 \in E \colon \forall x^2 \left(\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \in E_s^2\right)\right\}.$

Proof Let $x^1 \in B$. By DB there is a type $X^1 \in B$ such that $x^1 \in X^1$. Then by Definition (i) $X^1 \in \boldsymbol{E}$ and $\underline{\mathfrak{i}}(X^1) \in \boldsymbol{E}_s^2$. By applying D$E$ we have $x^1 \in E$. Let us assume for the purpose of the proof that $\mathfrak{i}(x^1, x^2)$. Then there are Y^1, Y^2 such that $x^1 \in Y^1$, $x^2 \in Y^2$, and $\underline{\mathfrak{i}}(Y^1, Y^2)$, i.e., $Y^2 = \underline{\mathfrak{i}}(Y^1)$ (Definition D$\mathfrak{i}$). Hence, by Axiom $A^t 2$, $Y^1 = X^1$ which is to say that $x^2 \in \underline{\mathfrak{i}}(X^1)$, and by DE_s^2 we have $x^2 \in E_s^2$. This proves that the inclusion under consideration is true. □

6 The Equivalence of the Theories T1 and T2

The two various formalizations of the theory of the syntax of language, presented by the theories **T1** and **T2** treated in their two aspects, make us above all reflect on whether both theories equally well describe the language syntactically or whether they differ in the sets of their theses, ie. whether **T1 = T2**.

As it is know, two axiomatic theories that do not differ from one another by the sets of their theses are equivalent, and to demonstrate that it suffices to show that every axiom and every definition in one theory is a thesis in the other theory, and conversely, every axiom and every definition in the latter is a thesis in the former.

Let us accordingly make a formal comparison of **T1** and **T2**.

Note first that all concepts at the *token level* are definable in **T2**(tk) in terms of concepts from the *type level* (definitions with the schemata DS, DCt_t, DR) or are such as in **T1**(tk) (Definitions D7 and D8(S)) and, what is more, they can be *characterized* as in **T1**(tk) (Fact 6): every axiom and every definition in **T1**(tk) is a thesis in **T2**(tk) (Fact 6a), Note also that Definitions D$\tilde{\boldsymbol{c}}$ and D$\boldsymbol{Ct}$(**S**) in **T1**(tp) are such as in **T2**(**T2**(tp)). It can be demonstrated that the remaining definitions in **T1**(tp), that is definitions with the schema D**S**, where **S** is a set from the system (**S**) other than $\boldsymbol{Ct}_T$, Definition D$\boldsymbol{Ct}_T$, and definitions with the schema D**R**, where **R** is a relation in the system (**R**) other than $\underline{\tilde{\boldsymbol{c}}}$, are theorems in **T2**(**T2**(tk)). For Definition D$\boldsymbol{U}$ that fact was mentioned already in Sec. 5.2. We shall now prove the correctness of that statement only for expressions of the form D**S**.

DS. $X \in \mathbf{S} \Leftrightarrow \exists x \in S\,(X = [x])$.

Proof Let $X \in \mathbf{S}$. Since by Axiom A^t1 some $x_1 \in X$, by Definition DS of S, $x_1 \in \mathrm{S}$. Note that $X = [x_1]$, because if $y \in X$, then in view of $x_1 \in X$ Definition D$\approx$ implies that $y \approx x_1$ and hence $y \in [x_1]$, and if $y \in [x_1]$, then $y \approx x_1$, and, by Lemma L1, $y \in X$. Thus the simple implication in D**S** is true. To prove the converse implication note that if $x_1 \in \mathrm{S}$ and $X = [x_1]$, then by Definition DS $x_1 \in X_1$ and $X_1 \in \mathbf{S}$. Now since $x_1 \in X$, it follows from Axiom A^t2 that $X_1 = X$ and $X \in \mathbf{S}$. □

The foregoing considerations lead us to the conclusion that the theory **T1** can be grounded in the theory **T2**.

We shall prove that the converse also holds. Note first that the axioms and definitions adopted in **T2**(tp) either are dual analogues of the axioms and definitions of **T1**(tk) or are equivalent to the dual analogues of definitions of that theory (the expressions d(3a), (v), (i)–iii)). As such they are, in agreement with Fact 1a, theorems or definitions in **T1** (**T1**(tp)). Thus all concepts at the *type level* can be *characterized* in **T1**(tp) in the same way as in **T2**(tp). This is possible owing to the fact that all concepts at that level are in **T1**(tp) definable in terms of concepts from the *token level* (definitions with the schemata D**S**, D$\boldsymbol{Ct}_T$, D**R**) or are the same as in **T2**(tp) (Definitions D$\boldsymbol{Ct}$(**S**), D$\underline{\tilde{\boldsymbol{c}}}$). Both axioms of $\mathbf{T2}^t$(tp) joined to **T2**(tp) are also theorems in **T1**(**T1**(tp)). This follows directly from the convention that X, Y are variables which represent types, Definition D$\boldsymbol{U}$, and the properties of equivalence classes. Further all the definitions of concepts from the *token level* adopted in **T2**(**T2**(tk)) are theorems or definitions in **T1**(tp). Definitions D$\tilde{c}$ and DCt(S) are the same in both theories, and the expressions with the schemata DS, DCt_t, and DR are provable in **T1**(tp). In their proofs in fact use is made of Theorems 2 and 3 (the formulas ($*$) and ($**$)). In this way every axiom and every definition of **T2** is a thesis in **T1**. Thus **T2** can be grounded in **T1**.

As a result of the above we may state

Fact 7 *The theories* **T1** *and* **T2** *are equivalent.* □

7 Final Conclusions and Remarks

From the point of view of the philosophy of language the theories **T1** and **T2** represent, respectively, two dual approaches to the syntax of language, the *nominalistic* (*concretistic*) and the *Platonic*. In the light of Fact 7 we may accordingly state that

(I) *The two ontologically opposed approaches to the syntax of language represented by the theories* **T1** *and* **T2**, *are equivalent*.

The biaspectual formalizations of **T1** and **T2** at two different levels, that of *tokens* and that of *types*, as presented above, show clearly the analogies between the properties of the objects belonging to those two different levels. Dual expressions describe the analogous properties of dual concepts. The said analogies can be grasped in two ways. On the one hand, they can be perceived separately within both **T1** and **T2**. It suffices to compare any thesis of **T1**(tk), which describes the properties of concepts at the *token level* with the dual thesis of the dual theory **T1**(tp), which describes the properties of concepts at the *type level* (Fact 1), and to compare any thesis of **T2**(tp), which describes the properties of concepts at the *type level*, with either the dual thesis of the dual theory **T2**(tk) or its translation into the dual thesis of that theory—the theory which describes the properties of concepts at the *token level* (Fact 4, 5, and 6). On the other hand, we find these analogies when we compare the theories **T1** and **T2**, strictly speaking when we compare the theses of **T1**(tk) with the theses of the dual theory **T2**(tp), and conversely. This is so because, in accordance with Fact 4, every property which is an attribute of an object at the *token level* is also an attribute of the dual object at the *type level*, while in accordance with Fact 5, every property which is an attribute of an object at the *type level* either is an attribute of the dual objects at the *token level* or can be transformed into such a property.

The above observations will be recorded as the following conclusion:

(II) *There is a formal mutual analogy between dual syntactic concepts at the token level and the type level.*

In view of the equivalence of the theories **T1** and **T2** it follows from the comments made above that whether elements of language are concrete or abstract entities is of no importance in theoretical enquiries concerned with the syntax of language. Hence note that

(III) *In purely theoretical syntactic considerations the philosophical aspects pertaining to the double ontological nature linguistic objects may be disregarded.*

But the conclusion (I) and (II) given above speak in favour of Słupecki ideology concerning the nature of linguistic objects. The possibility of constructing a theory of the syntax of language as the theory **T1**, which represents the *concretistic* approach and does not require, for the description of the basic syntactic concepts, the assumption of the existence of ideal objects (that is *types* understood as sets of equiform *tokens*) leads us in fact to the following essential conclusion of the present paper:

(IV) *In the syntactic analyses of language one may eliminate the assumption on the existence of ideal linguistic entities interpreted as classes of equiform tokens.*

* * *

Some final remarks. The studies presented in this paper cover only the categorial description of languages which do not include operators that bind variables. These studies can be generalized so as to cover such languages as well (see [24]). Further. This paper presents only a most essential fragment of syntactic problems. It discusses those syntactic concepts which are used for a general description of a language constructed in the spirit of Leśniewski and Ajdukiewicz. But it seems that the formulation of the fundamental philosophical conclusion present in this paper (Conclusion (IV)) can be affected neither by the expansion of the conceptual apparatus used and of the scope of syntactic problems, nor by the construction of the theoretical foundations of the. syntax of language which would consider other formal models, such as Chomsky's transformational-generative models. The analyses pertaining to the two dual ontological approaches to the syntax of language can probably be easily adjusted to the construction of other theory of the syntax of language, in particular the theories of formal languages in Chomsky's spirit.

* * *

Acknowledgements I wish first of all to thank W. Buszkowski for his penetrating comments and suggestions which helped me to give the final form to this paper and its version [27]. I have also availed myself of the comments made by the reviewers of my earlier papers, namely T. Batóg, W. Marciszewski, J. Perzanowski, and O. A. Wojtasiewicz. I wish to express my gratitude to all of them. I would like to particularly thank the editor of the volume—J. Zygmunt, for his special concern for the proper drafting of the text.

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Chapter 5
Meaning and Interpretation. Part I

Urszula Wybraniec-Skardowska

Abstract The paper is an attempt at a logical explication of some crucial notions of current general semantics and pragmatics. A general, axiomatic, formal-logical theory of meaning and interpretation is outlined in this paper. In the theory, according to the *token-type* distinction of Peirce, language is formalised on two levels: first as a language of *token*-objects (understood as material, empirical, enduring through time-and space objects) and then—as a language of *type*-objects (understood as abstract objects, as classes of tokens). The basic concepts of the theory, i.e. the notions: *meaning*, *denotation* and *interpretation* of well-formed expressions (*wfes*) of the language are formalised on the *type*-level, by utilising some semantic-pragmatic primitive notions introduced on the *token*-level. The paper is divided into two parts. In Part I a theory of meaning and denotation is proposed, and in Part II - its expansion to the theory of meaning and interpretation is presented. The meaning, respectively the interpretation, of a *wfe*, is defined as an equivalence class of the *relation possessing same manner of use of types*, respectively, the *relation possessing same manner of interpreting of types* (cf. Ajdukiewicz [2], Wittgenstein [49]). The concept of denotation is defined by means of the relation of referring which holds between *wfe-types* and objects of reality described by the given language.

Key words: *Token-type* distinction • *Token*-syntax • *Type*-syntax • Meaning • Referring • Denotation • Synonymy • Unambiguity • Ambiguity

Introduction

The words ‘meaning’ and ‘interpretation’ possess many different meanings, the terms being, (at the same time), applied in different sciences. Despite the fact that they are included into the key terms of the logical theory of language and philosophy of language, they do not have a fixed meaning in these disciplines. It happens only

First published in Studia Logica **85** (1), part 1, 105–132 (2007).

too often that the notion of ‘interpretation’ is identified with the one of ‘meaning’, that is with an indication of extension (denotation) of language expressions. Then the difference between the notions of ‘meaning’, ‘denotation’ and ‘interpretation’ becomes blurred. Differentiation of the terms, providing their definitions and mutual relations is the basic aim of the present work.

1 Preliminaries

1.1 The Problem of the Meaning of ‘Meaning’

The word ‘meaning’, lacking precision as regards its meaning, requires logical explication. Searching for its precision has been and still is the goal of numerous attempts undertaken in the literature pertaining to logic and philosophy. As we may conclude, the question concerning the meaning of ‘meaning’ is still of principal importance (see Putnam [45]), and in particular the one with reference to the concept of ‘concept’, here—to the logic-oriented concept of the term ‘meaning of expression’ (see e.g. Marciszewski [34, 35]).

Answering the following question:

WHAT IS THE MEANING OF ‘MEANING’?

is a task for the theory of meaning. And this is not an easy task as none of the known theories of meaning has come to be commonly accepted.

One or another conceptualization of the knowledge on the notion of ‘meaning’ and related notions, such as referring, denotation, interpretation must be constituted through determined philosophical settlements referring to the nature of meaning, as well as through settlements related to a selection of primitive notions of interest to us here, since at the foundation of any definition of the notion of ‘meaning’ and related notions, there must always be found some primitive notions that allow for their logical explication.

There exist different philosophical conceptions concerning the nature of meaning and various theories of meaning, an extensive review and discussion of which can be found in *Dictionaries* edited by Robert Audi [5] and by Witold Marciszewski [34, 35].

There are philosophers-logicians, like W. Quine and N. Goodman, who claim that the notion of meaning can not be defined at all. Other philosophers supply various hypotheses as to the nature of meaning, especially whether meaning is:

a) *an extra-linguistic creation* included into
  - a1) the domain of objective and real being (the connotation-related meaning introduced by J. S. Mill), or
  - a2) the psychic sphere (meaning as an idea, a thought associated with a form of sign—in the associationism originating from J. Locke), or still

- a3) a sphere of objective-ideal objects (meaning either as an ideal object—in E. Husserl's philosophical version [28], or as an intentional object—in the framework proposed by R. Ingarden [29], or as an abstract object—in the logic-semantic version by G. Frege [25], later developed by A. Church [16]),

b) *a creation of the very language itself* (meaning as a property of expressions of the same intensional structure or the same intension—in the concept of meaning by R. Carnap [14]),

c) *a creation determined through the way of using linguistic expressions* (meaning as a way of using an expression—in the concept of meaning deriving from L. Wittgenstein [49] and independently from K. Ajdukiewicz [1, 2]),

d) *a creation determined through certain conditions* being either:
- d1) conditions which allow recognizing truthfulness of a sentence, under which it can be verified, certified as acceptable (verificationism of R. Carnap [12, 13, 15], or
- d2) conditions of truthfulness of a sentence (the truth-conditional conception of meaning by D. Davidson [19, 20] drawing in the spirit of Tarski's theory of truth), or still
- d3) conditions of assertability of sentences (the theory of meaning by M. Dummett [21–24]).

The traditional conceptions which place the meaning of expressions in the psychic sphere are of historical importance nowadays. Others are being developed or modified and have their followers and opponents. None of the conceptions is commonly acceptable, or—as M. Dummett puts it [23]—is a satisfying theory of meaning. Building such a theory—in the opinion of the researcher—is one of the most urgent tasks of contemporary analytical philosophy. However, the very theory of meaning itself proposed by Michael Dummett has already been severely criticized (see e.g. Gunson [27]), especially by followers of the leading truth-conditional theory of meaning advocated by Donald Davidson.

It must be underlined that since the time of Gotlob Frege [25] the notion of meaning *Sinn* (English *intension*) has been differentiated from the one of denotation *Bedeutung* (English *extension*). The differentiation intension-extension was introduced by Rudolf Carnap [14]. The notion of meaning as intension in opposition to the one of extension was for the first time used formally by Richard Montague [37]. This notion is considered in numerous works by the latter author and followers of his ideas. The notion of intension remains thus in a close relation with the one of interpretation of a linguistic expression: it consists in attributing a suitable meaning to it, usually through placing it in a context (index) and referring it to certain reality (a possible world) through pointing its extension in this context (index).

It must also be stressed that the notion of meaning in logical semantics is the so-called *cognitive*, *informative* meaning. We deal with it while performing a logical analysis of texts. Questions connected with emotive or expressive meanings of expressions are discussed mainly in ethics.

Completing this brief review of different philosophical and logical conceptions of meaning, it can be observed that none of them is a general theory in the sense given below.

1.2 What is a General Theory of Meaning and Interpretation?

1. A general theory of meaning and interpretation is a theoretical conception which allows to grasp, in an overall way, the nature of linguistic expressions and what their meaning is, and also what their interpretation is.
2. It does not depend on the specific character of any language (whether it is a formalized or natural language), shape or structure of its expressions, their syntactic categorization (whether these are names, sentences or functors, etc.).
3. It is based on a universal theory of syntax, in which it is determined what a well-formed linguistic expression is, not taking into account its specific internal structure.
4. It is a part of general semantics and general pragmatics. As such, it does not pay attention to who the user of the language is, what his philosophical views are on the nature of the world (worlds, respectively) to which the language refers, and in particular whether this world is represented (whether these worlds are represented, respectively) by situations.
5. Thus, it takes no notice of what ontological beings the objects considered by the language are and what their structure is, to what ontological category these objects belong. Therefore, an object considered by the language may be anything that one can talk about, think about, etc, by means of linguistic expressions.
   Let us observe that in order to provide a theoretical construction of general notions of meaning and interpretation we will not need, in any particular way, the logical notion of meaning dependent on the context, or taking into account what a situation is (the central notion of situational semantics), or what its set-theoretical representation is.
6. A general theory of meaning and interpretation attempts to answer the question:

   «What is meaning and interpretation at all? »

   but not the question:

   «What is meaning or interpretation in its dependence on the context? »

   Similar remarks refer to the notion of denotation (extension). A general theory of meaning and interpretation does not make use of the well-known and accepted results from the founders of the so-called theory of contextual use of expressions (see Montague [37–41]; Scott [46], Cresswell [17] and others; cf. also Tokarz [48]).
7. Its task is to explain and explicate the general notions of meaning and interpretation and also to characterize principal relations between these and related notions, such as reference and denotation, interpretational denotation.

8. In compliance with the above-discussed differentiation made by Frege [25] and the distinction of intension-extension (see Carnap [14]) there are introduced in it differentiations between the meaning (*Sens*, *intension*) and denotation (*Bedeutung*, denotation, *extension*), as well as one between interpretation and interpretational denotation.
9. The notions of the theory include the one of synonymy, unambiguity and ambiguity. The theory offers certain criteria of unambiguity and ambiguity of linguistic expressions, establishes some relations between the notions of meaning and interpretation, and also between these notions and the ones of linguistic communication.

Working out a general theory of meaning and interpretation, which would satisfy the requirements mentioned above, encounters considerable difficulty due to the divergent tasks which are posed for it to fulfill. The difficulty seems also to arise from already developed habits.

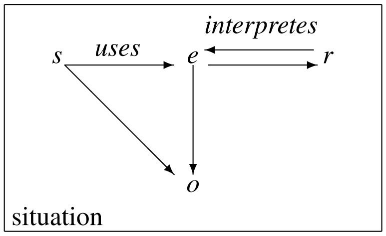
1.3 The Aim and Assumptions of the Work

The aim of the present work is to outline the foundations of a certain general axiomatic formal-logical theory of meaning and interpretation, as a semantic-pragmatic theory. Although this theory will concern meaning and interpretation of expressions of any language, it will take into consideration, to a certain degree, the following two aspects:

1. cognitive-communicative function of natural language, according to its genesis, and also the so-called
2. *functional approach* to logical analysis of this language.

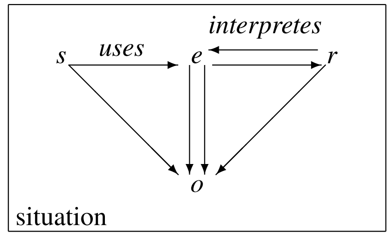
The work makes use of the assumption that the primitive linguistic beings are material creations, e.g. given sounds, written signs, physical objects placed somehow in time and space, concrete objects which have some referents attributed to them, and which are called *tokens*. Everything points to the fact that explaining the process of the formation of language, according to its genesis and cognitive-communicative function, assumes that the tokens are primitive beings of natural language applied in communication acts between their sender and receiver. In these acts the sender s calls, uses a token e of a sign with reference to a broadly conceived object o, while the receiver r interprets it, in compliance with or in discordance with the sender's intention, as this or another object o' (see Diagram 1a); if in compliance—there follows understanding (see Diagram 1b); if in discordance—there follows misunderstanding (see Diagram 2a); it may happen that the receiver will not be able to interpret the sign, which results in incomprehension (see Diagram 2b).

Communicating



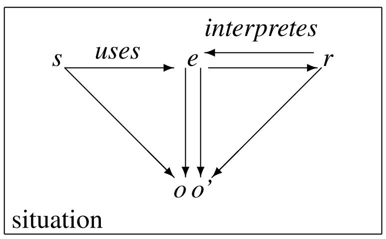
Diag. 1a

Understanding



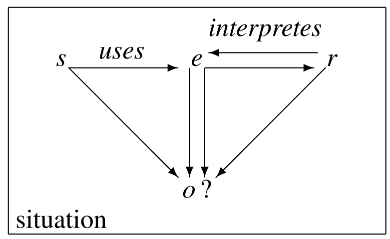
Diag. 1b

Misunderstanding



Diag. 2a

Incomprehension



Diag. 2b

Pertaining to aspect 1), the paper assumes that signs-examples (*tokens*) are primitive linguistic beings. According to the well-known *token-type* distinction made by Peirce [42], we differentiate *token*-signs from *type*-signs, which are abstract linguistic objects, and whose physical representations are just *tokens*. In the proposed theory, the basic semantic-pragmatic notions, therefore notions of meaning and interpretation, are defined by means of expression-*types*, still their definitions use such primitive notions of the theory as *using* and *interpreting* of expression-*tokens*.

Taking into account aspect 2) in the proposed conception, that is the *functional approach* to natural language analysis, this is marked by taking into consideration the *manner of use* expressions (see Pelc [43, 44]). Following Pelc, we distinguish two understandings of this statement: in the first of them, the manner of *use* takes place only in given conditionings, in determined language-situational contexts and concerns only expression-*tokens*; in the other—the manner of *Use* (*usage*) characterizes the meaning of the expression. This manner is somehow built into this meaning. In this case an expression can be treated as isolated, static, torn from the context, e.g. as an entry in a dictionary. It is then an expression-*type*, a class of its concrete occurrences, a distributive class of expression-*tokens used* either to represent a given object, or in concrete acts of communicating in given linguistic-situational contexts, with reference to only one, widely conceived object or more than one object, yet of the same kind. Different, repetitive *tokens* of an expression-*type Used* in an unambiguous way are thus *used* with reference to the same object, or with reference to different but similar objects of the same kind. For example, two single *tokens* of the word-*type* ‘ball-point pen’, having a fixed meaning (the manner of *Use*) in English, can be *used* in a similar linguistic-situational context either with reference to the same ball-point pen, e.g. the one I am holding in one of my hands, or with reference to two different ball-point pens, e.g. in a situation of teaching a child the meaning of the word 'ball-point pen' through an ostensive definition. When x utters twice the *tokens* of the sentence-*type*: 'This is a ball-point pen' pointing to, respectively, the first and then to the other ball-point pen in succession, then the sentence-*tokens* uttered by x refer to two different states of affairs: one in which the first object indicated by x at the moment t1 is called a ball-point pen and the other in which the object pointed to by x at time t2 is also called a ball-point pen. We can regard both states of affairs as references to objects of sentence-*tokens* successively uttered by x, each of which belongs to another sub-type (sub-class) of *tokens* of the same sentence-*type* 'This is a ball-point pen'. Let us observe that the above-mentioned sentence-*type* includes the indexical word 'this' of a changeable manner of *Use* and as such can be treated as a sentence isolated from the linguistic-situational context, a sentence-*type* without a fixed manner of *Use*, thus without fixed meaning, whereas each of the mentioned sub-types (sub-classes) of *tokens* of the whole sentence-*type* used to acquaint the child with the name of the indicated (usually a few times) concrete ball-point pen has a fixed manner of *Use* determined just by the *use* of its *tokens* with reference to the state of affairs of the same kind—pointing and calling the same ball-point pen by its name.

The presented exemplification did not aim at examining the ways of use of linguistic expressions at all, but was meant to underline that there exist two different such ways and in order to settle what the meaning of an expression is it is justifiable to refer to the other the manner of *Use* which concerns only expression-*types*, yet which employs the manner of *use* of the first kind.

The relation of *using*, concerning all the relations of physical object-based reference of expression-*tokens* made by users of language, will be a primitive notion of the theory proposed here. This relation is a set of all such physical relations. The relation *Using* is, on the other hand, a relation defined by means of the relation of *using* and applied by users of language for expression-*types*. The difference between these relations is explained by the fact that two persons can *Use* the same expression-*type* by means of its two different *tokens*, that is by *using* its two different *tokens*.

By taking into account, in the proposed theoretical conception of meaning and interpretation, the manner of *Use* of expressions (expression-*types*) by users of language, we are referring to the traditional considerations which date back to Aristotle and the Middle Ages, and which are now included into the logical pragmatics. In this way, out of different options concerning the nature of meaning we are choosing option c), in Section 1.1, connected with the ideas advocated by Ludwig Wittgenstein [49] and Kazimierz Ajdukiewicz [1, 2], that is the one connected with understanding of the meaning of expression-*type* as a manner of its *Use*.

The paper explicates the notions of meaning and interpretation on the basis of a formal theory $\boldsymbol{T}$ of language syntax and its expansion by semantic and pragmatic components. In the theory $\boldsymbol{T}$, according to the *token-type* distinction of Peirce, language is formalised on two levels: first as a language of *token*-objects and then—as a language of *type*-objects. The most important syntactic notion of a *well-formed expression* (a *wfe*) is defined separately on the *token*-level and on the *type*-level. Some foundations of the two-level *theory* ***T*** *of language syntax* are outlined in Section 2. A formalization of ***T*** is given in the author's books [50, 53]. Two level *token-type* formalisation of syntax allows us to outline a new semantic-pragmatic theory of meaning. In Section 3 the theory ***T*** will be expanded by semantic and pragmatic notions connected with meaning, referring and denotation. A new, general theory ***TM*** of meaning is proposed and outlined in this Section. In Part II of this paper (in Section 4) the theory ***TM*** is also expanded to the theory ***TMI*** of meaning and interpretation.

All theories presented or outlined in the paper assume set-theoretical formalization.

2 Syntax for Language; the Theory *T*

2.1 Two Kinds of Syntax: a *token*-syntax and a *type*-syntax

In the theory ***T*** a language L will be formalized dually, as a creation of a double ontological nature: both as a language of *tokens* (at the *token*-level) and a language of *types* (at the *type*-level), according to the *token-type* distinction by Peirce. *Tokens* are intuitively understood as concrete, material, empirical, enduring through time-and-space objects, which are perceived by sight and are usually inscriptions, but do not have to be inscriptions. They can be on a paper, a notice-board, a blackboard, a stone, etc.; they may be configurations of such things as jigsaw-puzzle pieces, leaves, stones, stars, or smoke signals, or illuminated advertisements, and so on. *Types* are understood as sets (classes) of *tokens* bearing an *identifiability* relation to each other, i.e. types are ideal, abstract beings. The relation of *identifiability of tokens* is determined by pragmatic factors and not physical similarity, and it is understood very broadly. For instance, two inscriptions printed in different type but consisting successively of the same letters of alphabet may be identifiable, e.g. the words

LOGIC – written by means of capital letters,

logic – printed in *italic* or

logic – printed in bold type

can be regarded as identifiable words.

We will assume that the identifiability of *tokens* is an equivalence relation.

The *expressions* of the language L are some *concatenations*. *Concatenations* will be obtained by means of a ternary *relation of concatenation*. They on the *token*-level may be, but do not have to be, sequences of two *tokens*. Intuitively, a concatenation of two written *tokens* p and q, for example in an European language (or respectively a Semitic one), is a written *token* r that is made up by adding to a *token* p^*, identifiable with p, on the right side (respectively on the left side) the written *token* q^*, identifiable with q. For example, in Latin, the concatenations of the following word-*tokens*:

LOGICA

S
t
u
d
i
a

the second and the first, is the name-*token* of the title of a journal:

`Studia Logica`

and any name-*token* identifiable with it, in particular the *token* in a vertical line:

STUDIA
LOGICA

or any *token* on the cover of any copy of the journal 'Studia Logica'. So, the relation of concatenation defined on *tokens* is not a set-theoretical function and the relation of identifiability is not a relation of physical similarity. Moreover, a concatenation of the first name-*token* in the central line and the same first name-*token* is for instance the token:

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The twofold ontological character of linguistic objects understood as *tokens* (material objects) and *types* (abstract objects) should be emphasized in the formalization of the theory of language syntax $\boldsymbol{T}$ on two levels. We can choose as the first level of formalization of the theory $\boldsymbol{T}$ the *token*-level and describe *token*-syntax, and as the second one – the *type* level, on which we can describe *type*-syntax.

2.2 Some Basis of the Theory *T*

Let L be a given language. For formalizing the *token*-syntax of L we assume that primitive linguistic objects are material, physical linguistic entities, i.e. *tokens*. The simplest syntactic characterization of any language L on the *token*-level gives the following six-tuple:

(L) $\langle U_L, \sim, c, V, W; S\rangle$, where

U_L is the *linguistic universe of* L, i.e. the set of all *tokens* of language L;
$\sim$ – the binary *relation of identifiability* defined on U_L;
V – the *vocabulary of all simple word-tokens of* L;
c – the ternary *relation of concatenation* defined on U_L to generate from V words of L;
W – the *set of all word-tokens of* L;
S – the *set of all well-formed expression-tokens* of L which is a subset of W.

If, for instance, L is the English language, then the universe U_L includes all such sign-*tokens* as: Latin letters, punctuation signs, brackets, and all their concatenations; V is the set of all simple word-*tokens* of English, together with auxiliary sign-*tokens* used to create expressions of English; W is the set of all strings (concatenations) composed from tokens of V, e.g. to W belong the tokens written after the colon:

`of` it *on* **and** AN if book a book Eve book a it a a book
Eve *Eve* **book**:book OF IT
a - book a a book lies `a book of lies` *a book* **lies** on
A BOOK LIES **on a** table
it is a book lies a book - lies a book on a table . . .
a ***book lies*** on a table a pen **lies** on a table

and so on. The set S consists of tokens of V, e.g.

Ann *Eve* book on a and

and some composed words of W, e.g.

a book a book and a pen Eve lies on a table
A book lies on a table and a pen lies on a table.

The notions: U_L, $\sim$, c, V are primitive notions of the theory $\boldsymbol{T}$. They are characterized axiomatically in a theory of word-*tokens* (cf. Wybraniec-Skardowska [50, 53]) which is the basis of the theory $\boldsymbol{T}$. Some axioms for the relation $\sim$ of identifiability state that it is an equivalence relation in the linguistic universe U_L of all *tokens* of L. Axioms for concatenation do not have to be modeled on Tarski's axioms [47] of his theory of strings (a theory of concatenations) or on any axioms for formal grammar because our concatenations are defined on *tokens* (not on *types*) and do not have to be linear and associative. The notions W and S are derived, defined notions of $\boldsymbol{T}$. The set W of all words of L is defined as a smallest set including the vocabulary V and closed under the relation of concatenation c.

For the definition of the *set S of all well formed expression-tokens* (for short: *wfes*) of L we can introduce, on the basis of the theory $\boldsymbol{T}$, a system of notions that can be regarded as a reconstruction of a *categorial grammar* for L, generating the set S.

The notion of categorial grammar originated from Ajdukiewicz [3, 4] and was constructed under the influence of Leśniewski's theory of semantic (syntactic) categories in his protothetics and ontology systems [32, 33], under Husserl's ideas of pure grammar [28], and under the influence of Russell's theory of logical types. The notion was shaped by Bar-Hillel [6–8] and developed by Lambek [30, 31], Montague [40, 41], Cresswell [17, 18], Buszkowski [9, 11] and others (see also Marciszewski [36]). A categorial two-level characterization of language L as the language generated by the so-called *classical categorial grammar*, the notion introduced and explicated by Buszkowski [9, 11], is given by Wybraniec-Skardowska [51–53].

In Wybraniec-Skardowska's approach, the formal-logical characterization of *token*-syntax of L on the basis of $\boldsymbol{T}$ requires the consideration of an ordered system much more complex than (L). Then, the theory $\boldsymbol{T}$ formalizes the basic principles of Ajdukiewicz's-Leśniewski's theory of semantic (syntactic) categories. For the definition of the set S, only such concatenations of the set W of all words of L are taken into consideration that are *functor-argument expressions* of L, i.e. expressions that are compounded from a main part (a *functor*) and complementary parts (*arguments of that functor*). It aims to assess which of them have syntactic sense, i.e. are *well formed* of S, using *categorial index assignment* and the *principle of syntactic connection*.

On the *type*-level the categorial language will be described by means of the notions of the following system ($\underline{L}$) of sets of *types* and relations defined on the *types*, which is dual to the system (L):

($\underline{L}$) $\langle \underline{U}_L, =, \underline{c}, \underline{V}, \underline{W}; \underline{S} \rangle$, where

$\underline{U}_L$ is the set of all *types* of language L;
$\underline{V}$ – the *vocabulary of all simple word-types of* L;
$\underline{c}$ – the ternary *relation of concatenation* defined on *types* $\underline{U}_L$;
$\underline{W}$ – the *set of all word-types of* L;
$\underline{S}$ – the *set of all well-formed expression-types* of L which is a subset of $\underline{W}$.

At the *type*-level the theory $\boldsymbol{T}$ is formalized as an extended theory described above. All notions of the system ($\underline{L}$) in such an extended theory are derived constructs defined by means of the dual notions from the *token*-level. Any set $\underline{Set}$ of *types*, from ($\underline{L}$), is obviously defined as the quotient family of the set *Set*, from the *token*-level, i.e.

$$\underline{Set} = Set/\sim .$$

A linguistic *type* $\underline{p}$, belonging to the universe $\underline{U}_L$ of types of language L or to its subsets, is of course an equivalence class $[p]_\sim$ of *tokens* of U_L or *tokens* of its suitable subsets, with respect to the relation $\sim$ of *identifiability*, i.e.

$$\text{if } \underline{p} \in \underline{Set} \text{ then } \exists\, p \in Set \left(\underline{p} = [p]_\sim = \{q \in Set\colon q \sim p\}\right).$$

For example, in English to the set $\underline{U}_L$ belong:

- the set of all tokens identifiable with the token: **book**
- the set all tokens identifiable with the token: a book *lies on*
- and the set of all tokens identifiable with the token:

A *book lies* on a *table*.

The first of them belongs to the set $\underline{V}$, all the sets belong to the set $\underline{W}$, but only the first and the third ones belong to the set $\underline{S}$.

Let us observe that the relation concatenation $\underline{c}$ on types is defined by means of the relation concatenation c on *tokens*. Then it is the two-place function on *types*.

The two-level, logical explication of the notion of a *wfe* (the notions of the sets S and $\underline{S}$) allows us to introduce, on the basis of $\boldsymbol{T}$, basic concepts connected with the semantic and pragmatic notions considered in this paper.

In further parts of this paper we will not use *wfe-types* as elements of the set $\underline{S}$ but as some subtypes of *wfe-types* of this set. By *wfe-types* of L we will understand all elements of the set $\underline{S}^* \supset \underline{S}$ defined as follows:

$$\underline{S}^* = \{\underline{e} \subseteq \underline{p}\colon \underline{e} \neq \emptyset \wedge \underline{p} \in \underline{S}\}.$$

Let us observe that

$$\forall \underline{e} \in \underline{S}^{*} \; (\emptyset \neq \underline{e} \subseteq S).$$

So, each *wfe-type* of $\underline{S}^{*}$ is a nonempty set of identifiable *wfe-tokens* of S.

Let us observe that in English to the set $\underline{S}^{*}$ belong, in particular, all subsets of the set of all tokens identifiable with the token:

a book lies on a table,

e.g. the singleton {a book lies on a table} and the three-element set {a book lies on a table, *a book lies on a table*, a book lies on a table}.

Let us note that we do not assume anything about syntactic categorization of the set $\underline{S}^{*}$. Syntactic categories of $\underline{S}^{*}$ can be (but do not have to be) the category of sentences, the category of terms, and different categories of functors.

3 The General Theory of Meaning: the Theory *TM*

3.1 Meaning

The formalization of some semantic or semantic-pragmatic issues requires the enriching of the conceptual apparatus of the theory ***T***. Semantic-pragmatic issues which we here consider are connected with the meaning of linguistic expressions and, more exactly, with Carnap's dualism in the meaning of expressions, i.e. his *intension-extension* distinction (see Carnap [14]) and also with Frege's *Sinn* (*meaning*) and *Bedeutung* (*denotation*) distinction (see Frege [25]). The formal conception of meaning presented here also has some connection with the understanding of the meaning of expressions as a manner of their use. Such an approach refers to the conception of meaning originating from Wittgenstein [49] and, independently, from Ajdukiewicz [1, 2].

New primitive notions of the theory of meaning—the theory ***TM***—as an expansion of the theory ***T*** of syntax for a language L, are the non-empty sets: *the set* User *of all users of a given language* L, the *set* Ont *of all extra-linguistic objects described by* L, and the two-place *operation* use *of using the well-formed expression-tokens*.

The set User of users of language L can be composed not only of the current, but also the former or future users of this language. We do not assume anything about the nature of the objects of the set Ont. They can be not only material objects, but for instance, fictional or abstract creations described by language L as well. The objects under consideration are cognizable in particular, that is they are what we recognize (what we recognized, will recognize) in communication acts. We do not assume anything about the ontological categorization of the set Ont, either. The following may be (but do not have to be) ontological categories: category of individuals, category of a set of individuals (satisfying a certain property), various categories of set-theoretical relations and functions, category of situations (states of affairs), etc. We understand the operation use also in a very broad sense: as an operation of producing, calling, using, exposing or interpreting *wfe-tokens* in order to refer them to corresponding objects of *Ont*. We can also call the operation *use*: *a function of object reference of wfe-tokens by users of language L*. It can be understood as a set of all physical activities of users of language *L* being applied currently, applied in the past or possible to be applied in future with the aim to refer concrete *wfe-tokens* to objects of the set *Ont* in relevant situations.

These primitive notions of the theory ***TM*** satisfy only the following axioms:

Axiom (sets: *User and Ont*) $\mathit{User} \neq \emptyset \wedge \mathit{Ont} \neq \emptyset$.

Axiom (*use*) *use is a partial function of*

$$\mathit{User} \times S \to \mathit{Ont},$$
$$\mathrm{Dom}_1(\mathit{use}) = \mathit{User} \text{ and } \mathrm{Dom}_2(\mathit{use}) \subset S.$$

The expression: $\mathit{use}(u,e) = o$, where $u \in \mathit{User}$, $e \in S$ and $o \in \mathit{Ont}$ is read: *u uses* (*makes or exposes*) *the wfe-token e to refer to the object o*. This object *o* is called the *referent of the wfe-token e* assigned by its user *u*.

From the second axiom it follows that every user of *uses* at least one *wfe-token* of *L* to refer to an object. Not every *wfe-token* must have a referent.

Definition 1 (possessing of reference)

$$e \text{ has object reference iff } e \in S \ \wedge\ \exists u \in \mathit{User}\ \exists o \in \mathit{Ont}\ \big(\mathit{use}(u,e) = o\big).$$

In accordance with Definition 1: an object reference has only such *wfe-token* which is used by some user of *L* to refer to an extra-linguistic object.

Definition 2 (possessing the same manner of *use*)

$$e \approx e' \text{ iff } \exists o \in \mathit{Ont}\ \big[\exists u \in \mathit{User}\ \big(\mathit{use}(u,e) = o\big)\ \wedge\ \exists u \in \mathit{User}\ \big(\mathit{use}(u,e') = o\big)\big],$$

i.e. *e* and *e'* have the same object reference.

Axiom (*Use*) $\emptyset \neq \mathit{Use} \subseteq \mathit{User} \times \underline{S}^*$,

where the relation *Use* is defined as follows:

Definition 3 (*Using* types)

$$u\ \mathit{Use}\ \underline{e} \text{ iff } \exists e \in \underline{e}\ \exists o \in \mathit{Ont}\ \big(\mathit{use}(u,e) = o\big).$$

In accordance with this definition a *user u Uses the wfe-type* $\underline{e}$ if and only if the user *u uses a wfe-token* of $\underline{e}$ to refer to some referent.

Immediately from the above axioms and Definition 1, it follows that

Corollary 1

a. $\forall u \in \mathit{User}\ \exists e \in S\ \exists o \in \mathit{Ont}\ (\mathit{use}(u,e) = o)$,

b. $\forall u \in \mathit{User}\ \exists \underline{e} \in \underline{S}^*\ (u\ \mathit{Use}\ \underline{e})$,

c. $\neg u\ \mathit{Use}\ \underline{e}$ *iff* $\forall e \in \underline{e}\ \forall o \in \mathit{Ont}\big(\neg \mathit{use}(u,e) = o\big)$.

d. *If e has object reference, then* $\exists u \in \mathit{User}\ \exists \underline{e} \in \underline{S}^*(e \in \underline{e}\ \wedge\ u\ \mathit{Use}\ \underline{e})$.

The next definition is the definition of the relation ≅ possessing same manner of *Use* of the *wfe-types*:

Definition 4 (possessing the same manner of *Use* of types)

$$\underline{e} \cong \underline{e}' \text{ iff } \forall u \in \mathit{User}\,\big[(u\ \mathit{Use}\ \underline{e} \Leftrightarrow u\ \mathit{Use}\ \underline{e}') \wedge \\ \wedge \forall e \in \underline{e}\,\forall o \in \mathit{Ont}\,\big((\mathit{use}(u, e) = o) \Rightarrow \exists e' \in \underline{e}'\,(\mathit{use}(u,e') = o)\big) \wedge \\ \wedge \forall e' \in \underline{e}'\,\forall o \in \mathit{Ont}\,\big((\mathit{use}(u, e') = o \Rightarrow \exists e \in \underline{e}\,(\mathit{use}(u, e) = o))\big].$$

Definition 4 states that *two wfe-types* $\underline{e}$, $\underline{e}'$ *have the same manner of Use in L* if and only if any user of the given language *Uses* one of them if and only if he/she *Uses* the second of them, and for every *token* of $\underline{e}$ *used* by him to refer to any object o there exists a *token* of $\underline{e}'$ *used* by him to refer to the same object o and, conversely, for every *token* of $\underline{e}'$ *used* by him to refer to any object o there exists a *token* of $\underline{e}$ *used* by him to refer to the same object o.

Let us observe that it is difficult to speak about *the same manner of Use* of two expression-*types* in *L* when one of them is *Used* by some user while the other is not (whoever the user is). Thus, if two expressions have *the same manner of Use* then they are either both *Used* by any user of *L* or they are not *Used* by the user at all (whoever the user is). In the latter case the expressions do not have any manner of *Use* (that is they have *the same manner of Use*).

We can give a simpler, equivalent definition to the above because of the following theorem:

Theorem 1

$$\underline{e} \cong \underline{e}' \text{ iff } \forall u \in \mathit{User}\,\big[(u\ \mathit{Use}\ \underline{e} \Leftrightarrow u\ \mathit{Use}\ \underline{e}') \wedge \\ \wedge \forall o \in \mathit{Ont}\big(\exists e \in \underline{e}\,(\mathit{use}(u, e) = o) \Leftrightarrow \exists e' \in \underline{e}'(\mathit{use}(u, e') = o)\big)\big].$$

Let us note that the terms: ‘teenager’ and ‘adolescent’ have *the same manner of Use*, because if any user *Uses* neither of them, all the conditions of Theorem 1 are satisfied, and if any user *Uses* both of them, he/she *uses* a token of the term ‘teenager’ to any referent who is a young man/woman iff he/she *uses* some token of the term ‘adolescent’ to the same young man/woman.

Theorem 1 states that *two wfe-types* $\underline{e}$, $\underline{e}$’ *have the same manner of Use* if and only if any user of the given language *Uses* one of them if and only if he/she also *Uses* the second of them, and every object is a referent of some *token* of the type $\underline{e}$ (*used* by the user) if and only if it is a referent of some *token* of the second *type* $\underline{e}$’ (*used* by the user).

Proof We make use of the following law of the predicate calculus:

(∗) *If v is not a free variable in formula F then*

$$\forall v\,(E \Rightarrow F) \Leftrightarrow (\exists v\,E \Rightarrow F).$$

(1) $$\underline{e} \cong \underline{e}' \text{ iff } \forall u \in \mathit{User}\big[(u\ \mathit{Use}\ \underline{e} \Leftrightarrow u\ \mathit{Use}\ \underline{e}') \wedge \\ \wedge \forall e \in \underline{e}\,\forall o \in \mathit{Ont}\,\big((\mathit{use}(u, e) = o) \Rightarrow \exists e' \in \underline{e}'(\mathit{use}(u,e') = o)\big) \wedge \\ \wedge \forall e' \in \underline{e}'\,\forall o \in \mathit{Ont}\,\big((\mathit{use}(u, e') = o) \Rightarrow \exists e \in \underline{e}\,(\mathit{use}(u, e) = o)\big)\big].$$

From (∗) we have

(2) $$\forall e \in \underline{e}\,\forall o \in \mathit{Ont}\,\big((\mathit{use}(u, e) = o) \Rightarrow \exists e' \in \underline{e}'\,(\mathit{use}(u,e') = o)\big) \text{ iff} \\ \text{iff } \forall o \in \mathit{Ont}\,\big(\exists e \in \underline{e}\,(\mathit{use}(u, e) = o) \Rightarrow \exists e' \in \underline{e}'\,(\mathit{use}(u,e') = o)\big).$$

and

(3) $\forall e' \in \underline{e}' \, \forall o \in \mathit{Ont} \left((\mathsf{use}(u, e') = o) \Rightarrow \exists e \in \underline{e} \left(\mathsf{use}(u,e) = o\right)\right)$ iff
iff $\forall o \in \mathit{Ont} \left(\exists e' \in \underline{e}' \left(\mathsf{use}(u, e') = o\right) \Rightarrow \exists e \in \underline{e} \left(\mathsf{use}(u,e) = o\right)\right)$.

From formulas (2), (3) and (1) we obtain our Theorem 1. □

A relationship between two different relations of possessing the same manner of use yields

Theorem 2
$\exists u \in \mathsf{User}\,(u \; \mathsf{Use} \; \underline{e}) \wedge \underline{e} \cong \underline{e}' \Rightarrow \exists e \in \underline{e} \, \exists e' \in \underline{e}'(e \approx e')$.

If two *Used* *wfe-types* $\underline{e}$ and $\underline{e}'$ have the same manner of *Use* (in the second sense) than there exist tokens e and e' of $\underline{e}$ and $\underline{e}'$, respectively, that have also the same manner of *use* (in the first sense).

It is easy to see that using Theorem 1 we get:

Theorem 3
The relation $\cong$ *is an equivalence relation in the set* $\underline{S}^*$ *of well-formed expression-types of the language L.*

The basic notion of the presented theory—the notion of meaning, is defined as follows:

Definition 5 (meaning)

$$\mu(\underline{e}) = [\underline{e}]_{\cong}.$$

The *meaning* (*intension*) $\mu(\underline{e})$ *of the wfe-type* $\underline{e}$ is the equivalence class of the relation $\cong$ possessing same manner of *Use* of types determined by type $\underline{e}$. It can intuitively be understood as a common property of all *wfe-types* possessing the same manner of *Use* as the expression $\underline{e}$, called *the manner of Use of the expression* $\underline{e}$.

Definition 5 gives us the definition of the *operation* μ *of meaning*, which is the map:

$$\mu : \underline{S}^* \to 2^{\underline{S}^*}$$

Two *wfe-types* $\underline{e}$ and $\underline{e}$' *are synonymous* (*have the same meaning*, *are intensionally agreeable*) if and only if their meanings are equal:

Definition 6 (synonymous)

$\underline{e}$ and $\underline{e}'$ are *synonymous* iff $\mu(\underline{e}) = \mu(\underline{e}')$.

So, expressions that have the same meaning have the same manner of *Using* them, i.e. we have:

Corollary 2

a. $\mu(\underline{e}) = \mu(\underline{e}')$ *iff* $\underline{e} \cong \underline{e}'$,
b. *Meaning of any wfe-type* $\underline{e}$ *is the equivalence class of all expressions synonymous with the expression* $\underline{e}$.

For example, the predicate-names 'teenager' and 'adolescent' are synonymous in English. Also, the sentences show below have the same meaning:

'Robert met a teenager' and 'Robert met an adolescent'.

When any *wfe-type* $\underline{e}$ is regarded as a set of all identifiable *wfe-tokens*, i.e. $\underline{e} \in \underline{S}$, then it can have more than one meaning determined by subsets of *tokens* of this set. The same remark concerns many *types* of the set $\underline{S}^*$. Then they do not have an established meaning.

For example: The term-*type* 'key' regarded as the set of all tokens identifiable with the token *key* have a different meaning than the proper subset 'key^1' $\subset$ 'key' composed only from such tokens identifiable with the given above that refer only to musical keys.

Let us introduce the definitions of properties of: *possessing of a meaning* and *possessing of an established meaning*.

Definition 7 (possessing of meaning)

a. $\underline{e}$ *has a meaning* iff $\exists\, \underline{e}' \subseteq \underline{e}\, \exists\, M \subseteq \underline{S}^* \big(M = \mu(\underline{e}')\big)$,
i.e. there exists a set of well-formed expression-*types* which is the meaning of a subtype of the *wfe-type* $\underline{e}$,

b. $\underline{e}$ *has an established meaning* iff $\forall\, \underline{e}' \subseteq e\ \big(\mu(\underline{e}') = \mu(\underline{e})\big)$,
i.e. $\neg\exists\, \underline{e}' \subseteq \underline{e}\ \big(\underline{e}' \neq e \ \wedge\ \mu(\underline{e}') \neq \mu(\underline{e})\big)$,
i.e. no proper sub-*type* of $\underline{e}$ has the meaning different from $\underline{e}$.

From Definition 7 we get the following corollaries:

Corollary 3

a. *Every wfe-type has at least one meaning,*
b. *If $\underline{e}$ has an established meaning then $\underline{e}$ has one meaning,*
c. *If $\underline{e}$ has not an established meaning then $\underline{e}$ has more than one meaning.*

The following theorem establishes some relationships between introduced notions:

Theorem 4

a. *$\underline{e}$ has not an established meaning iff*

$$\exists\, \underline{e}_1 \subseteq \underline{e}, \underline{e}_2 \subseteq \underline{e}\ \big(\underline{e}_1 \neq \underline{e}_2 \ \wedge\ \mu(\underline{e}_1) \neq \mu(\underline{e}_2)\big),$$

b. *If $\underline{e}$ has an established meaning then*

$$\forall\, e_1, e_2 \in \underline{e}\ (e_1, e_2 \textit{ have object reference} \Rightarrow e_1 \approx e_2),$$

c. *If $\exists\, e_1, e_2 \in \underline{e}\big(\neg(e_1 \approx e_2)\big)$ then $\underline{e}$ has not an established meaning,*
d. *If $\underline{e}$ has not an established meaning then*

$$\exists\, u \in \mathsf{User}\, \exists\, e_1, e_2 \in \underline{e}\, \forall\, o \in \mathsf{Ont}\, \neg\big(\mathsf{use}(u, e_1) = o = \mathsf{use}(u, e_2)\big),$$

i.e. *If an expression-type has not an established meaning then there exists a user of language L who does not* use *at least two its tokens to refer to the same object.*

In accordance with Theorem 4, part b, users use an expression-*type*, e.g. the word ‘a problem’ in an established meaning, when they say (by means of its *tokens*) only about the same object, here about the same problem. Otherwise, according to part c. of the theorem, the word, here the word ‘a problem’, has not an established meaning. The notion not possessing an established meaning is different from the one of *ambiguity*. The notion of *ambiguity* requires introducing the notion of referring and denotation.

3.2 Denotation

The relation *Ref* is a binary relation between *wfe-types* and objects of reality considered by the given language *L*. Formally this is a binary relation defined as a subset of the Cartesian product of the set $\underline{S}^*$ of well-formed expression-*types* and the set Ont of all objects described by the language *L*, i.e.

$$Ref \subseteq \underline{S}^* \times \mathsf{Ont}\,,$$

and its definition is:

Definition 8 (referring)

$$\underline{e}\ Ref\ o \text{ iff } \exists\, u \in \mathsf{User}\ \exists\, e \in \underline{e}\ \big(\mathsf{use}(u, e) = o\big).$$

The *wfe-type* $\underline{e}$ *refers to the object o* iff there exist a user of the language *L* using some token of the expression $\underline{e}$ to refer to the object *o*. Thus the *wfe-type* $\underline{e}$ does not refer to the object *o* iff no user of language *L* uses any token of the expression $\underline{e}$ to refer to the object *o*.

For example, the term ‘book’ refers to a book on my desk but does not refer to any computer.

Every object to which $\underline{e}$ refers is called a *denotatum* of the expression-*type* $\underline{e}$. The set of all denotata of $\underline{e}$ is denoted by $\delta(\underline{e})$ and called the *denotation* (*extension*) *of the expression-type* $\underline{e}$. Thus

Definition 9 (denotation)

$$\delta(\underline{e}) = \{o \in \mathsf{Ont}\colon \underline{e}\ Ref\ o\}.$$

The operation δ:

$$\delta\colon \underline{S}^* \to 2^{\mathsf{Ont}},$$

is called the *operation of denotation*.

Let us observe that from Definitions 9, 8 and 1 it follows that the denotation of the expression-*type* $\underline{e}$ is a nonempty set of all its denotata if and only if a user of language Uses the expression $\underline{e}$. So, we can formulate

Theorem 5

a. $\exists\, u \in \mathsf{User}\ (u\ \mathsf{Use}\ \underline{e})$ *iff* $\delta(\underline{e}) \neq \emptyset$;

b. $\delta(\underline{e}) = \emptyset$ *iff* $\forall\, u \in \mathsf{User}\ (\neg u\ \mathsf{Use}\ \underline{e})$;

c. *If* $\forall u \in \mathsf{User}\ (u\ \mathsf{Use}\ \underline{e})$ *then* $\delta(\underline{e}) \neq \emptyset$.

This theorem explains a certain misunderstanding connected with the so-called 'empty names' as names which have empty denotation. It is said, for example, that the name-*type* 'dwarf' ('brownie', 'genie', 'leprechaun') is empty as it does not have designators. In accordance with Theorem 5b this name has empty denotation if no user of a language in which this name functions, applies any of its occurrences (*token*) to any ontological object, or in other words, there is no user of such a language, who would refer by any occurrence (*token*) of the word 'dwarf' to anything. If, then, all users of a language assume that objects described by this language, objects of the set Ont, can not be fictional beings, this name is treated as empty since it does not have denotata (it does not have designators either, that is objects of reference existing in reality). When somebody uses the name 'dwarf' and does not follow the nominalistic view, and includes to the set Ont fictional beings, here fairy-tale dwarfs, then the denotation of the name 'dwarf' is a non-empty set to this person and the name functions in this person's language as a general non-empty term (there are a great number of dwarfs in the world of the fable). So, in this approach to semantics, the denotata of the so-called 'empty names', that are not the so-called contradictory names, can be fictional or intentional non-physical objects.

We accept the convention that if $\delta(\underline{e}) = \{o\}$ and $\underline{e}$ is a proper name, then its denotation equals o. So, for non-nominalists denotations of any proper names are individual objects, i.e. their denotata, while the denotations of predicate-names are sets of all their denotata.

For example, the denotation of the definite description 'the highest mountain on Earth' is the singleton with the element Mount Everest, the denotation of the proper name 'Ann' is Ann, the denotation of the empty proper name 'Zeus' is the mythological Zeus, while the denotation of the predicate-name 'book' is the set of all books and the denotation of the empty fairytale predicate-name 'dwarf' is the set of all fictional dwarfs. For nominalists this set is the empty set.

Let us note here that we do not refer to so-called contradictory names or internally contradictory expressions, for example 'the square circle'.

Let us also observe that denotata of declarative sentences do not have to be truth values—as Frege assumed—but can be states of affairs described by these sentences.

Applying Definition 8 and Definition 9 we obtain:

Theorem 6

$$\forall u {\in} \mathsf{User}\ \forall e, e' {\in} \underline{e}\ \forall o, o' {\in} \mathsf{Ont}\ \left[\left(\mathsf{use}(u, e) = o \wedge \mathsf{use}(u, e') = o'\right) \Rightarrow o, o' {\in} \delta(\underline{e})\right].$$

In accordance with Theorem 6 if two expression-*tokens* of the *wfe-type* have two referents, respectively, then these objects belong to the denotation of the expression-*type*. More exactly: for any user u, for any couple of tokens e, e' of the *wfe-type* $\underline{e}$ and for any objects o, o' if the user u uses the expression e to refer to the object o and the expression e' to refer to the object o' then referents o, o' are equal (e.g. if $\underline{e}$ is a proper name) or they are different but belong to the denotation of the expression $\underline{e}$ (e.g. if $\underline{e}$ is a general term).

Theorem 7

If $\underline{e}'$ *is a subtype of the wfe-type* $\underline{e}$ *then* $\delta(\underline{e}') \subseteq \delta(\underline{e})$.

Theorem 7 immediately follows from Definition 9 and Definition 8.

By means of the notion of denotation we can introduce the notions of *unambiguity* and *ambiguity* of *wfe-types*.

Definition 10 (unambiguity and ambiguity)

a. $\underline{e}$ *is unambiguous* iff $\neg\exists\, \underline{e}' \subseteq \underline{e}\ \big(\delta(\underline{e}\backslash\underline{e}') \neq \emptyset \ \wedge\ \delta(\underline{e}') \cap \delta(\underline{e}\backslash\underline{e}') = \emptyset\big)$,

b. $\underline{e}$ *is ambiguous* iff $\underline{e}$ *is not unambiguous*,
i.e. $\exists\, \underline{e}' \subseteq \underline{e}\ (\delta(\underline{e}\backslash\underline{e}') \neq \emptyset \ \wedge\ \delta(\underline{e}') \cap \delta(\underline{e}\backslash\underline{e}') = \emptyset)$,
i.e. $\exists\, \underline{e}' \subseteq \underline{e}\ \big[\forall\, o \in \mathsf{Ont}\ \big(\underline{e}'\ \mathsf{Ref}\ o \ \Rightarrow\ \neg(\underline{e}\backslash\underline{e}')\ \mathsf{Ref}\ o\big) \ \wedge\ \delta(\underline{e}\backslash\underline{e}') \neq \emptyset\big]$.

So, the *wfe-type* $\underline{e}$ is ambiguous iff there exists at least one such sub-*type* of $\underline{e}$, whose denotation does not have any common *denotata* with the non-empty denotation of the difference of the expression $\underline{e}$ and this sub-*type*.

A known ambiguous word is the word ‘key’, because it includes a sub-*type*, let us say ‘key’^{2}, which refers only to keys for moving the bolt of a lock, and does not refer to any other keys belonging to the denotation of the difference of this total *type* ‘key’ and its sub-*type* ‘key’^{2} consisting of such keys as musical tone or style, the operating part of a typewriter, piano, flute, etc, in key-board instruments, and any other keys referred to by the word being the difference (of the word ‘*key*’ and its sub-*type* ’*key*’^{2}).

It is obvious that ambiguity of expression type possessing only a narrower and wider (global) meaning, secures that the denotation of the one having the narrower meaning and the difference of the denotation of the one having the wider meaning and denotation of the one having the narrower meaning are disjoint sets. Therefore, e.g. the term ‘*logic*’ as a term for some science, is ambiguous, because in a narrower sense, as a proper subset of this term, for which we can use the word ‘*logic*’^{1} it refers only to formal logic, and in the wider (global) meaning, as a total *type*, it refers not only to formal logic but also to logical theory of language (logical semiotics). A suitable criterion of ambiguity in this case is established by the following theorem:

Theorem 8
$\exists\, \underline{e}' \subseteq \underline{e}\ \big[\delta(\underline{e}\backslash\underline{e}') \neq \emptyset \ \wedge\ \big(\delta(\underline{e}\backslash\underline{e}') = \delta(\underline{e})\backslash\delta(\underline{e}')\big)\big]$ *then* $\underline{e}$ *is ambiguous.*

The next two theorems yield the necessary conditions for ambiguity of expression-*types*:

Theorem 9
If $\underline{e}$ *is ambiguous then* $\exists\, \underline{e}' \subseteq \underline{e}\ \big(\delta(\underline{e}') \neq \delta(\underline{e})\big)$.

So, if a *wfe-type* e is ambiguous, then it possesses a sub-*type* with different denotation.

The converse of Theorem 9 does not hold. For example, if $\underline{e}$ is the term-*type* ‘car’ and $\underline{e}'$ is its one-*token* subset $\{car\}$ which refers (only) to my car, then $\delta(\underline{e}') \neq \delta(\underline{e})$ but $\underline{e}$ is not ambiguous (at least it would be very difficult to find such subset of $\underline{e}$ that would be satisfying the Definition 10b).

Theorem 10
If $\underline{e}$ *is ambiguous then* $\underline{e}$ *has not an established meaning.*

In a justification of this theorem we use a theorem showing some relation between the notions: meaning and denotation. It will be given later.

Let us note that introducing the notions of *unambiguity* and *ambiguity* we do not need to refer to (as it might seem) the notion of meaning, although—as we will show later on—both notions are connected with each other.

Let us also note that all the properties of expression-*types*, connected with meaning, can be transferred in a straightforward manner to properties for their expression-*tokens*. One can say, in this way, that an inscription on the blackboard in my office, *identifiable* with the inscription

Jack has forgotten a key

is ambiguous as a *token* of the ambiguous sentence-*type* 'Jack has forgotten a key' which, as a type, is ambiguous due to the use in it of the ambiguous word-*type* 'key'.

When two expression-*types* have the same denotation we say that they are extensionally equivalent:

Definition 11 (extensional equivalence)

$\underline{e}$ and $\underline{e}$' *are extensionally equivalent* iff $\delta(\underline{e}) = \delta(\underline{e}')$.

Instead of saying that two expressions are extensionally equivalent, we can also say that they are *extensionally agreeable*.

Extensionally agreeable expressions can not be intensionally agreeable. There is a difference between the introduced notions: meaning and denotation.

3.3 Meaning and Denotation

The principal relationship between the concepts meaning and denotation gives us the following theorem:

Theorem 11

a. $\mu(\underline{e}) = \mu(\underline{e}') \Rightarrow \delta(\underline{e}) = \delta(\underline{e}')$,
b. $\delta(\underline{e}) \neq \delta(\underline{e}') \Rightarrow \mu(\underline{e}) \neq \mu(\underline{e}')$.

Part a. of the above theorem, on the basis of Definitions 6 and 11, states that:
If two expressions are synonymous then they are extensionally equivalent.

In another formulation Theorem 11a states that: *If two expressions have the same intension then they have the same extension,* i.e. *if two expressions are intensionally agreeable then they are extensionally agreeable.*

Proof (of part a.) Let us, for a *reductio ad absurdum*, assume that

(1) $$\mu(\underline{e}) = \mu(\underline{e}')$$

and that

(2) $$\delta(\underline{e}) \neq \delta(\underline{e}').$$

From (1) and Corollary 2a we have

(3) $$\underline{e} \cong \underline{e}',$$

thus from Definition 4 we get

(4) $$\forall u{\in}\mathit{User}\ \forall e{\in}\underline{e}\ \forall o{\in}\mathit{Ont}\ \left[\mathit{use}(u,e) = o\ \Rightarrow\ \exists e'{\in}\underline{e}'\ \left(\mathit{use}(u,e') = o\right)\right],$$

and

(5) $$\forall u{\in}\mathit{User}\ \forall e'{\in}\underline{e}'\ \forall o'{\in}\mathit{Ont}\ \left[\mathit{use}(u,e') = o'\ \Rightarrow\ \exists e{\in}\underline{e}\ \left(\mathit{use}(u,e) = o'\right)\right],$$

Furthermore, because of the assumption (2), there exists some object o_1 such that

(6) $$o_1 \in \mathit{Ont}$$

and for which holds one of the following two cases:

(i) $$o_1 \in \delta(\underline{e})\ \wedge\ o_1 \notin \delta(\underline{e}')$$

or

(ii) $$o_1 \in \delta(\underline{e}')\ \wedge\ o_1 \notin \delta(\underline{e}).$$

We have to show that they lead to contradiction. Let us consider case (i). Then by Definition 9 we have

(i1) $$\underline{e}\ \mathit{Ref}\ o_1$$

and

(i2) $$\neg\, \underline{e}'\ \mathit{Ref}\ o_1.$$

From (i1) and Definition 8 we get

(i3) $$\exists u \in \mathit{User}\, \exists e \in \underline{e}\ (\mathit{use}(u,e) = o_1);$$

however, from (i2) and Definition 8 it follows that

(i4) $$\forall u \in \mathit{User}\, \forall e' \in \underline{e}'\ \neg\left(\mathit{use}(u,e') = o_1\right).$$

From (i3) we obtain

(i5) $$u_1 \in \mathit{User}\ \wedge\ e \in \underline{e}\ \wedge\ \mathit{use}(u_1,e) = o_1.$$

Applying formula (4) to the formulas (i5) and (6), we can state that there exists an expression-*token* $e_1' \in \underline{e}'$ such that (i6) $$\mathit{use}(u_1, e_1') = o_1.$$

In view of (i5) $u_1 \in \mathit{User}$, and because $e_1' \in \underline{e}'$ from (i4) we get the formula

(i7) $$\neg\, \mathit{use}(u_1, e_1') = o_1,$$

which is in contradiction to formula (i6).

The proof that case (ii) leads also to contradiction is completely analogous and based on the formula (5). So, our theorem to be proved is valid. □

It is well-known that the converse implication to that given in Theorem 10a does not hold: two expressions can have the same denotation but they need not be synonymous, e.g., the expressions 'the Morning Star' and 'the Evening Star' (see Frege [25]), or the sentences (cf. Gamut [26, p. 7]):

(i) John is looking for the current Commander-in-Chief of the U.S. Armed Forces
(ii) John is looking for the current President of the United States of America.

Taking into account Theorem 11b we get:

Theorem 12

a. $\exists\, o \in \mathit{Ont}\left[(\underline{e}_1\ \mathit{Ref}\ o \wedge \neg\, \underline{e}_2\ \mathit{Ref}\ o) \vee (\underline{e}_2\ \mathit{Ref}\ o \wedge \neg\, \underline{e}_1\ \mathit{Ref}\ o)\right] \Rightarrow \mu(\underline{e}_1) \neq \mu(\underline{e}_2),$
b. *Any wfe-type has not an established meaning, if it possesses two sub-types such that some object is a denotatum only one of them.*

For example, the term '*key*' has not an established meaning because it includes at least two sub-*types* whose *denotatum* is not the same key. The term '*key*' is besides an ambiguous term.

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Chapter 6
Meaning and Interpretation. Part II

Urszula Wybraniec-Skardowska

Abstract The paper enriches the conceptual apparatus of the theory of meaning and denotation that was presented in Part I (Section 3). This part concentrates on the notion of *interpretation*, which is defined as an equivalence class of the *relation possessing the same manner of interpreting types*. In this part, some relations between meaning and interpretation, as well as one between denotation an *interpretational denotation* are established. In the theory of meaning and interpretation, the notion of *language communication* has been formally introduced and some conditions of correctness of communication have been formulated.

Key words: Interpretation • Interpretational referring • Interpretational denotation • Meaning • Language communication

4 The Theory of Meaning and Interpretation: the Theory *TMI*

4.1 Interpretation and Language Communication

The definitions of meaning, denotation and related notions given in Section 3 (Part I), define semantic-pragmatic concepts of the formal theory of meaning ***TM***. They are based on the notions of the set User of all users of the language L, the set Ont of all objects considered by users of the language and the operation use of using expression-*tokens* of L.

Because a formal theory of language should explain, at least to a certain theoretical degree, the phenomenon of language communication among people, its conceptual apparatus has to refer to the notion of *interpretation* of language expressions and to empirical acts of communication. Let us notice that the notion of *interpretation* has not to be connected solely with sign-systems of communication; in semantics it plays a crucial, specific role.

First published in Studia Logica **85** (2), 261–274 (2007).

In formal considerations the notion of *interpretation* will be defined on *wfe-types* by means of the notion int of *interpreting token* in a way completely analogous to the one given in Section 3 for the notion of meaning μ.

The notion int of *interpreting tokens*, like the notion use of using *tokens*, is a new primitive concept of our extended theory which will be denoted by ***TMI***. The notion int is the two-place operation corresponding to the *operation* use of using *tokens*, and will be regarded as its restriction:

Axiom (interpreting) *int is a partial function of the function use, i.e.*

$$\emptyset \neq \mathsf{int} \subset \mathsf{use}$$
$$\textit{and } \mathrm{Dom}_2(\mathsf{int}) \subset \mathrm{Dom}_2(\mathsf{use}) \subset S.$$

The expression $\mathsf{int}(u, e) = o$, for $u \in \mathsf{User}$, $e \in S$ and $o \in \mathsf{Ont}$ is read: *the user u interprets (understands) the wfe-token e as a sign-token of the object o*. This object o is called the *interpretandum of the wfe-token e*.

An easy consequence of the above axiom is

Corollary 4

a. $\forall u \in \mathrm{Dom}_1(\mathsf{int}) \forall e \in \mathrm{Dom}_2(\mathsf{int}) \big(\mathsf{int}(u, e) = \mathsf{use}(u, e)\big)$,

b. $\exists u \in \mathsf{User} \exists e \in S \exists o \in \mathsf{Ont} \big(\mathsf{use}(u, e) = o = \mathsf{int}(u, e)\big)$,

c. $\mathrm{Dom}_1(\mathsf{int}) \subseteq \mathrm{Dom}_1(\mathsf{use}) = \mathsf{User}$.

The notion of interpreting *tokens* appears when we speak about communication by means of *tokens*. The user of the given language can participate in an act of communication as a *sender* of an expression (sign)-*token*, i.e. as a person using the expression (sign)-*token* to refer to an object, or as a *receiver* of the expression (sign)-*token interpreting* this expression as referring to an object.

We say that *s* and *r participate in an act of communication by means of expression-token e* (symbolically: $acome_e(s, r)$) if and only if $s, r \in \mathsf{User}$, $e \in S$ and there exist objects $o, o' \in \mathsf{Ont}$ such that the user *s*—the *sender* of the expression *e—uses e* to refer to the object *o* and the user *r*—the *receiver* of the expression *e—interprets e* as a sign-*token* of the object *o*'. If the referent *o* and the interpretandum *o*' are equal, then in the act of communication between *s* and *r* holds *understanding* (see Diagram 1b (Part I)); if they are different, then between *s* and *r* holds *misunderstanding* (see Diagram 2a (Part I)). So, we have

Definition 12 (participating in an act of communication)
$acome_e(s, r)$ iff $s, r \in \mathsf{User} \wedge e \in S \wedge \exists o, o' \in \mathsf{Ont}\big(\mathsf{use}(s, e) = o \wedge \mathsf{int}(r, e) = o'\big)$.

Definition 12a (understanding)
$und_e(s, r)$ iff $s, r \in \mathsf{User} \wedge e \in S \wedge \exists o \in \mathsf{Ont}\ \big(\mathsf{use}(s, e) = o = \mathsf{int}(r, e)\big)$.

Definition 12b (misunderstanding)
$misund_e(s, r)$ iff $s, r \in \mathsf{User} \wedge e \in S \wedge \exists o, o' \in \mathsf{Ont}\big(\mathsf{use}(s, e) = o \neq o' = \mathsf{int}(r, e)\big)$.

Abbreviations '$und_e(s, r)$' and '$misund_e(s, r)$' are here used for the expressions: 'Between *s* and *r* in an act of communication by means of the *wfe-token e* exists understanding' and, respectively, 'Between s and r in an act of communication by means of the expression-*token* e exists misunderstanding'.

It may happen that an attempt at an act of communication ends in failure because of *non-understanding* between the sender and the interpreter of the *wfe-token* e, if the user uses the expression e to refer to a referent but the receiver is not able to interpret this expression. So, symbolically

Definition 12b (non-understanding)

$$non\text{-}und_e(s,r) \text{ iff } s,r \in \mathsf{User} \ \wedge \ e \in S \ \wedge \\ \wedge\, \exists\, o \in \mathsf{Ont}\,\big(\mathsf{use}(s,e) = o \ \wedge \ \forall\, o\text{'} \in \mathsf{Ont}\,\big(\neg\, \mathsf{int}(r,e) = o\text{'}\big)\big).$$

From Corollary 4b and Definition 12a follows that:

Corollary 5 $\exists\, u \in \mathsf{User}\, \exists\, e \in S\, \big(acom_e(u,u) \ \wedge \ und_e(u,u)\big)$.

According to the corollary there exists at least one user of the language L who takes part in an act of communication by means of an *expression-token* as a sender and as a receiver, and in the act holds understanding.

Acts of communication can be carried out by means of two different expression-*tokens* of the same *wfe-type*, if a sender **uses** a *token* and a receiver interprets another *token* but a *token* of the same expression-*type*; such a situation is, for instance, in communication by means of e-mail or the Internet.

Communication by means of expression-*tokens* has to be distinguished from communication $C_{\underline{e}}$ in the given community *User* by means of a *wfe-type* $\underline{e}$. Then communication $C_{\underline{e}}$ is a value of an *operation communication* C defined on the *type* $\underline{e}$. The operation communication C is a function:

$$C : \underline{S}^* \rightarrow 2^{\mathsf{User} \times S \times \mathsf{User}}$$

which every *wfe-type* $\underline{e}$ of $\underline{S}^*$ maps to the relation $C_{\underline{e}} \subseteq \mathsf{User} \times S \times \mathsf{User}$ consisting of all ordered triples, such that the first component (sender) **uses** a *wfe-token* of $\underline{e}$ and the third component (receiver) *interprets* a *token* of $\underline{e}$. So the formal definition of the *operation C of communication by means of types* is the following:

Definition 13 (communication by means of types)

$$C_{\underline{e}} = \big\{\langle s,e,r\rangle :\ s,r \in \mathsf{User} \ \wedge \ e \in \underline{e} \ \wedge \ \exists\, o \in \mathsf{Ont}\,\big(\mathsf{use}(s,e) = o\big) \ \wedge \\ \wedge\, \exists\, e\text{'} \in \underline{e}\, \exists\, o\text{'} \in \mathsf{Ont}\,\big(\mathsf{int}(r,e\text{'}) = o\text{'}\big)\big\}.$$

It follows from the above Definitions 12 and 13, and Corollary 5 that communication $C_{\underline{e}}$ by means of the *wfe-type* $\underline{e}$ includes all acts of communications by means of expression-*tokens* of the *type* $\underline{e}$ and that there exist a *wfe-type* $\underline{e}$ such that communication $C_{\underline{e}}$ by means of the *type* $\underline{e}$ is nonempty set. So,

Corollary 6

a. $\big\{\langle s,e,r\rangle :\ s,r \in \mathsf{User} \ \wedge \ e \in \underline{e} \ \wedge \ acom_e(s,r)\big\} \subseteq C_{\underline{e}}$,

b. $\exists\, \underline{e} \in S(C_{\underline{e}} \neq \emptyset)$.

Moreover,

Corollary 7 $\langle s,e,r\rangle \in C_{\underline{e}} \ \Rightarrow \ s \ \mathsf{Use}\ \underline{e} \ \wedge \ r \ \mathsf{Int}\ \underline{e}$.

The term ‘*Int*’ in Corollary 7 denotes the relation of *interpreting types*, and its definition is dual to Definition 3:

Definition 3ⁱ (communication by means of types)

$$u\ \mathit{Int}\ \underline{e} \text{ iff } \exists\, e \in \underline{e}\ \exists\, o \in \mathit{Ont}\ \big(\mathit{int}(u,e) = o\big).$$

The expression: $u\ \mathit{Int}\ \underline{e}$, is read: the user u interprets the expression-*type* $\underline{e}$. The above Definition 3^i functions if we accept the following axiom:

Axioms (domains of *Int*)

a. $\mathrm{Dom}_1(\mathit{Int}) \subseteq \mathrm{Dom}_1(\mathit{int}) \subseteq \mathit{User}$,

b. $\underline{e} \in \mathrm{Dom}_2(\mathit{Int}) \;\Rightarrow\; \forall\, e \in \underline{e}\ \big(e \in \mathrm{Dom}_2(\mathit{int})\big)$.

Of course, the relation *Int* is a nonempty binary relation and a sub-relation of the relation *Use*. It follows from Corollary 6b, Corollary 7, Axiom(interpreting), Definition 3 and Definition 3^i that

Corollary 8

a. $\mathit{Int} \neq \emptyset$,

b. $\mathit{Int} \subseteq \mathit{Use}$.

Effective, successful communication in community *User* by means of the expression-*type* $\underline{e}$ is based on the agreed meaning $\mu(\underline{e})$ of the expression-*type* $\underline{e}$ used by users who are senders of *tokens* of $\underline{e}$ in acts of communication, and based on the correlation $\mu(\underline{e})$ with the *interpretation* $\iota(\underline{e})$ of the expression-*type* $\underline{e}$ interpreted by users who are receivers of these *tokens* in the acts. A disagreement in the meaning and the interpretation of the expression-*type* leads to *misunderstanding* in communication, and ignorance of interpretation of the expression-*type* leads to *non-understanding*.

The definition of *interpretation* of the expression-*type* $\underline{e}$ is dual to the definition of meaning of $\underline{e}$. The *interpretation* $\iota(\underline{e})$ of the expression-*type* $\underline{e}$ is the equivalence class of all expressions *possessing the same manner of interpreting* (*understanding*) them, and can intuitively be understood as a common property of all expression-*types* possessing the same manner of interpretation (understanding) as the expression *type* $\underline{e}$. The property can be called the *manner of interpreting of the expression* $\underline{e}$. So,

Definition 5ⁱ (interpretation)

$$\iota(\underline{e}) = [\underline{e}]_{\cong_i},$$

where $\cong_i$ denotes the *relation of possessing the same manner of interpretation of expression-types*.

The definition of the relation $\cong_i$ is dual to Definition 4, and arises from the latter by the replacement of the expressions *use* by *int* and *Use* by *Int*.

Definition 4ⁱ (possessing the same manner of interpretation of types)

$$\begin{aligned}\underline{e} \cong_i \underline{e}\text{'} \text{ iff } \forall\, u \in \mathit{User}\ \big[&(u\ \mathit{Int}\ \underline{e} \;\Leftrightarrow\; u\mathit{Int}\ \underline{e}\text{'}) \wedge\\ &\wedge \forall\, e \in \underline{e}\ \forall\, o \in \mathit{Ont}\ \big(\mathit{int}(u,e) = o \;\Rightarrow\; \exists\, e\text{'} \in \underline{e}\text{'}\ \big(\mathit{int}(u,e\text{'}) = o\big)\big) \wedge\\ &\wedge \forall\, e\text{'} \in \underline{e}\text{'}\ \forall\, o\text{'} \in \mathit{Ont}\ \big(\mathit{int}(u,e\text{'}) = o\text{'} \;\Rightarrow\; \exists\, e \in \underline{e}\ \big(\mathit{int}(u,e) = o\text{'}\big)\big)\big].\end{aligned}$$

The relation $\cong_i$ is given if its arguments belong to the $\mathrm{Dom}_2(\mathit{Int})$. We accept the following axiom:

Axiom (domains of $\cong_\mathbf{i}$) $\cong_i \subseteq (\mathrm{Dom}_2(\mathit{Int}) \times \mathrm{Dom}_2(\mathit{Int})) \cap \cong$.

from which, on the basis of Definitions 4^i and 5^i it follows that

Corollary 9
a. $\cong_i \subseteq \cong$,
b. $\mathrm{Dom}(\iota) \subseteq \mathrm{Dom}_2(\mathit{Int})$.

Because the relation $\cong_i$ is reflexive, and $\mathrm{Dom}_2(\mathit{Int})$ is a nonempty set (Corollary 8a), we have the following conclusion:

Corollary 10 $\cong_i \neq \emptyset \ \wedge \ \mathrm{Dom}(\iota) \neq \emptyset$.

from which it follows that the operation ι on types is well defined.

It is obvious that the Definitions 8^i, 9^i of the notions: Ref^i of the *relation of interpretational referring* and the *operation* σ^i *of interpretational denotation* are dual to the Definitions 8 and 9, respectively.

Definition $8^\mathbf{i}$ (interpretational referring)

$$\underline{e}\ Ref^i\, o \text{ iff } \exists\, u \in \mathit{User}\, \exists\, e \in \underline{e}\ \big(\mathit{int}(u, e) = o\big).$$

Definition $9^\mathbf{i}$ (interpretational denotation)

$$\delta^i(\underline{e}) = \{o \in \mathit{Ont} \colon \underline{e}\ Ref^i\, o\}.$$

Of course, all theorems in the theory ***TM*** remain valid if we replace the notions:

$$(*) \qquad \mathit{use}, \mathit{Use}, \cong, \mu, Ref, \delta,$$

by their dual counterparts:

$$(**) \qquad \mathit{int}, \mathit{Int}, \cong_i, \iota, Ref^i, \delta^i$$

in the theory ***TMI***.

4.2 Meaning and Interpretation

The close relationship described here between semantic notions connected with the concept of meaning and the concept of interpretation causes these concepts to be often identified. It also suggests that all notions related to the notion of meaning are very often formulated by means of the notion of interpretation or related notions. However, the notions of the system $(*)$ and those of the system $(**)$ are not the same. Here we accept the postulate that in communication acts the sender, in order to send the message, applies the function *use* connected with the object reference of a *wfe-token*, whereas the receiver, in order to receive the message, applies another function—the function *int* which is a restriction of the former one. The pair ⟨*user*, *token*⟩, which has an object reference, may have no corresponding *interpretandum*, when—for instance—this *token* can not be received or was used with the intention of being interpreted not by the sender, but someone else. The fact is, however, that each pair that has an *interpretandum* also has the same referent (see Corollary 4a).

It follows from the discussion in Section 4.1 that each relation or function of the system (∗∗) is a sub-relation its counterpart in the system (∗). Thus, the meaning $\mu(\underline{e})$ of a *wfe-type* $\underline{e}$ and the interpretation $\iota(\underline{e})$ of the *type* $\underline{e}$ may differ. In that case the communication $C_{\underline{e}}$ by means of the expression-*type* $\underline{e}$ is not correct. We can justify it formally.

We will supply a few theorems of the theory ***TMI*** which are not dual counterparts of theorems of the theory ***TM***.

First, let us note that from Corollary 9a, Definitions 5^i and 5, Axiom(interpreting of types), Definitions 9, 8 and dual Definitions 9^i, 8^i follows that

Theorem 13 *If $\underline{e}$ is wfe-type that has determined interpretation, then*
a. $\iota(\underline{e}) \subseteq \mu(\underline{e})$,
b. $\delta^i(\underline{e}) \subseteq \delta(\underline{e})$.

The next theorems provide us some sufficient conditions for equality of meaning and interpretation, and ordinary denotation and interpretational denotation.

Theorem 14 *Let $\underline{e}$ be wfe-type that has determined interpretation.*
If $\forall\, u \in \mathsf{User}\, (u\ Use\ \underline{e} \Leftrightarrow u\ \mathsf{Int}\ \underline{e})$ *then*
a. $\mu(\underline{e}) = \iota(\underline{e})$,
b. $\delta(\underline{e}) = \delta^i(\underline{e})$.

If every user of language L uses the wfe-type $\underline{e}$ if and only if he/she interprets the expression then a. meaning and interpretation of the wfe-type $\underline{e}$ are equal, b. denotation and interpretational denotation of the expression $\underline{e}$ are equal.

Proof (of part a.) Let us assume that

(i) $$\forall\, u \in \mathsf{User}\, (u\ Use\ \underline{e} \Leftrightarrow u\ \mathsf{Int}\ \underline{e})$$

On the basis Theorem 13a we have $\iota(\underline{e}) \subseteq \mu(\underline{e})$. It is sufficient to prove only the reverse inclusion. For this purpose let us suppose that

(1) $$\underline{e}\text{'} \in \mu(\underline{e}).$$

Apllying Definition 5 and (1) we state that

(2) $$\underline{e} \cong \underline{e}\text{'}$$

and from Theorem 2 and (2) we get:

(3) $$\forall\, u \in \mathsf{User}\, \big[(uUse\ \underline{e} \Leftrightarrow u\ Use\ \underline{e}\text{'}) \wedge \\ \wedge \forall\, o \in \mathsf{Ont}\, \big(\exists\, e \in \underline{e}\ \big(\mathsf{use}(u, e) = o\big) \Leftrightarrow \exists\, e\text{'} \in \underline{e}\text{'}\ \big(\mathsf{use}(u, \underline{e}\text{'}) = o\big)\big)\big].$$

We want to justify the dual counterpart of the formula (3).
Let

$$\text{(3.1)} \qquad u \in \mathit{User}.$$

Taking into account our assumption (i) and (3) we get:

$$\text{(3.2)} \qquad u\ \mathit{Int}\ \underline{e} \ \Leftrightarrow\ u\ \mathit{Int}\ \underline{e}\text{'}$$

Let now

$$\text{(3.2.1)} \qquad o \in \mathit{Ont}.$$

Then, on the basis of (3.1), Definition 3, (i), Axioms (domains of *Int*) parts b, a, Corollary 4a we obtain the following sequences of relationships:

$$\begin{aligned}
&\exists\, e \in \underline{e}\ \big(\mathit{use}(u,e) = o\big) \ \Leftrightarrow\ e_1 \in \underline{e} \ \wedge\ \mathit{use}(u,e_1) = o \ \wedge\ u\ \mathit{Use}\ \underline{e} \ \Leftrightarrow \\
&\quad \Leftrightarrow e_1 \in \underline{e} \ \wedge\ \mathit{use}(u,e_1) = o \ \wedge\ u\ \mathit{Int}\ \underline{e} \ \Rightarrow \\
&\quad \Rightarrow \underline{e} \in \mathrm{Dom}_2(\mathit{Int}) \ \wedge\ e_1 \in \underline{e} \ \wedge\ \mathit{use}(u,e_1) = o \ \wedge\ u \in \mathrm{Dom}_1(\mathit{Int}) \ \Rightarrow \\
&\quad \Rightarrow e_1 \in \mathrm{Dom}_2(\mathit{int}) \ \wedge\ u \in \mathrm{Dom}_1(\mathit{int}) \ \wedge\ \mathit{use}(u,e_1) = o \ \wedge\ e_1 \in \underline{e} \ \Rightarrow \\
&\quad \Rightarrow e_1 \in \underline{e} \ \wedge\ \mathit{int}(u,e_1) = o \ \Rightarrow\ \exists\, e \in \underline{e}\ \big(\mathit{int}(u,e) = o\big).
\end{aligned}$$

Since $\mathit{int} \subset \mathit{use}$ (Axiom (interpreting)) we can state the following equivalence:

$$\text{(3.2.2)} \qquad \exists\, e \in \underline{e}\ \big(\mathit{use}(u,e) = o\big) \ \Leftrightarrow\ \exists\, e \in \underline{e}\ \big(\mathit{int}(u,e) = o\big).$$

Similarly we get

$$\text{(3.2.3)} \qquad \exists\, e\text{'} \in \underline{e}\text{'}\ \big(\mathit{use}(u,e\text{'}) = o\big) \ \Leftrightarrow\ \exists\, e\text{'} \in \underline{e}\text{'}\ \big(\mathit{int}(u,e\text{'}) = o\big).$$

From (3), (3.3.1), (3.2.2) and (3.2.3) we have:

$$\text{(3.2.4)} \qquad \exists\, e \in \underline{e}\ \big(\mathit{int}(u,e) = o\big) \ \Leftrightarrow\ \exists\, e\text{'} \in \underline{e}\text{'}\ \big(\mathit{int}(u,e\text{'}) = o\big).$$

(3.2.1) implies (3.2.4). Thus

$$\text{(3.3)} \qquad \forall\, o \in \mathit{Ont}\ \big(\exists\, e \in \underline{e}\ \big(\mathit{int}(u,e) = o\big) \ \Leftrightarrow\ \exists\, e\text{'} \in \underline{e}\text{'}\ \big(\mathit{int}(u,e\text{'}) = o\big)\big).$$

(3.1) implies (3.2) and (3.3). Thus

$$\begin{aligned}
\text{(4)} \quad &\forall\, u \in \mathit{User}\ \big[(u\ \mathit{Int}\ \underline{e} \Leftrightarrow u\ \mathit{Int}\ \underline{e}\text{'}) \ \wedge \\
&\quad \wedge\ \forall\, o \in \mathit{Ont}\ \big(\exists\, e \in \underline{e}\ \big(\mathit{int}(u,e) = o\big) \ \Leftrightarrow\ \exists\, e\text{'} \in \underline{e}\text{'}\ \big(\mathit{int}(u,e\text{'}) = o\big)\big)\big].
\end{aligned}$$

Applying Theorem 2^i and (4) we obtain: $\underline{e} \cong_i \underline{e}$', and according to Definition 5^i we have:

$$\text{(5)} \qquad \underline{e}\text{'} \in \iota(\underline{e}).$$

From (1) we have (5) and our inclusion: $\mu(\underline{e}) \subseteq \iota(\underline{e})$. This proves part a. of Theorem 14. □

Proof (of part b.) Let us assume that

$$(1) \qquad \forall u \in \mathit{User}\ (u\ \mathit{Use}\ \underline{e} \Leftrightarrow u\ \mathit{Int}\ \underline{e})$$

Applying Definitions 9, 8 and 3, formula (1), Axioms (domains of *Int*) a, b, Corollary 4a, Axiom (interpreting), Definitions 3^i, 8^i and 9^i we get the following sequences of equivalences:

$$\begin{aligned}(2)\ \ & o \in \delta(\underline{e}) \ \Leftrightarrow\ \underline{e}\ \mathit{Ref}\ o \ \wedge\ o \in \mathit{Ont} \ \Leftrightarrow \\ & \Leftrightarrow o \in \mathit{Ont} \ \wedge\ \exists u \in \mathit{User}\, \exists e \in \underline{e}\ \big(\mathit{use}(u, e) = o\big) \ \Leftrightarrow \\ & \Leftrightarrow o \in \mathit{Ont} \ \wedge\ u_1 \in \mathit{User} \ \wedge\ u_1\ \mathit{Use}\ \underline{e} \ \wedge\ \exists e \in \underline{e}\ \big(\mathit{use}(u_1, e) = o\big) \ \Leftrightarrow \\ & \Leftrightarrow o \in \mathit{Ont} \ \wedge\ u_1 \in \mathit{User} \ \wedge\ u_1\ \mathit{Int}\ \underline{e} \ \wedge\ e_1 \in \underline{e} \ \wedge\ \mathit{use}(u_1, e_1) = o \ \Leftrightarrow \\ & \Leftrightarrow o \in \mathit{Ont} \ \wedge\ u_1 \in \mathit{User} \ \wedge\ e_1 \in \underline{e} \ \wedge\ \mathit{int}(u_1, e_1) = o \ \Leftrightarrow \\ & \Leftrightarrow o \in \mathit{Ont} \ \wedge\ \exists u \in \mathit{User}\, \exists e \in \underline{e}\ \big(\mathit{int}(u_1, e) = o\big) \ \Leftrightarrow \\ & \Leftrightarrow o \in \mathit{Ont} \ \wedge\ \underline{e}\ \mathit{Ref}^i\ o \ \Leftrightarrow\ o \in \delta^i(\underline{e}).\end{aligned}$$

From (2) we have

$$(3) \qquad \forall o \in \mathit{Ont}\ \big(o \in \delta(\underline{e}) \ \Leftrightarrow\ o \in \delta^i(\underline{e})\big).$$

Hence by (3) we get: $\delta(\underline{e}) = \delta^i(\underline{e})$. □

The next theorem follows easily from Theorem 14a, b and the following lemma:

Lemma $\mathrm{Dom}_1(\mathit{Int}) = \mathrm{Dom}_1(\mathit{Use}) \ \wedge\ \forall u \in \mathrm{Dom}_1(\mathit{Int})\ (u\ \mathit{Use}\ \underline{e} \ \Leftrightarrow\ u\ \mathit{Int}\ \underline{e}) \ \Rightarrow$
$\Rightarrow\ \forall u \in \mathit{User}\ (u\ \mathit{Use}\ \underline{e} \ \Leftrightarrow\ u\ \mathit{Int}\ \underline{e})$.

Theorem 15 *Let* $\underline{e}$ *be wfe-type that has determined interpretation. If* $\mathrm{Dom}_1(\mathit{Int}) = \mathrm{Dom}_1(\mathit{Use})$ *then*
a. $\mu(\underline{e}) = \iota(\underline{e})$,
b. $\delta(\underline{e}) = \delta^i(\underline{e})$.

If $\underline{e}$ *is wfe-type that has determined interpretation, and all users Using types are users interpreting these types, then a. the meaning and the interpretation of the expression-type* $\underline{e}$ *are equal, and b. the ordinal denotation and the interpretational denotation of* $\underline{e}$ *are also equal.*

Proof Let us suppose that $\underline{e}$ be *wfe-type*, that has determined interpretation. Then $\underline{e}$ has determined meaning and interpretation. Let us assume that

$$(1) \qquad \mathrm{Dom}_1(\mathit{Int}) = \mathrm{Dom}_1(\mathit{Use}).$$

Since, by our assumption, $\underline{e}$ has determined interpretation $\iota(\underline{e})$, we have

$$(2) \qquad \underline{e} \in \mathrm{Dom}(\iota)$$

and from Corollary 8b and Axioms (domains of *Int*) part b. we obtain:

(3) $$\forall\, e \in \underline{e}\ \big(e \in \mathrm{Dom}_2(\mathit{int})\big).$$

Corollary 4a states that

(4) $$\forall\, u \in \mathrm{Dom}_1(\mathit{int})\ \forall\, e \in \mathrm{Dom}_2(\mathit{int})\ \big(\mathit{int}(u, e) = \mathit{use}(u_1, e)\big).$$

Thus, from (3), (4) and Axioms (domains of *Int*) part a. we have

(5) $$\forall\, u \in \mathrm{Dom}_1(\mathit{int})\ \forall\, e \in \underline{e}\ \big(\mathit{use}(u, e) = \mathit{int}(u_1, e)\big).$$

$\mathrm{Dom}_1(\mathit{Int}) \subseteq \mathit{User}$. Applying formula (5) to Definition 3 and the dual Definition 3^i we can state that

(6) If $u \in \mathrm{Dom}_1(\mathit{Int})$ then
$u\ \mathit{Use}\ \underline{e}\ \Leftrightarrow\ \exists\, e \in \underline{e}\ \exists o \in \mathit{Ont}\ \big(\mathit{use}(u, e) = o\big)\ \Leftrightarrow$
$\exists\, e \in \underline{e}\ \exists o \in \mathit{Ont}\ \big(\mathit{int}(u, e) = o\big)\ \Leftrightarrow\ u\ \mathit{Int}\ \underline{e}.$

In view of (6) we have

(7) $$\forall\, u \in \mathrm{Dom}_1(\mathit{Int})\ (u\ \mathit{Use}\ \underline{e}\ \Leftrightarrow\ u\ \mathit{Int}\ \underline{e}).$$

Applying our assumption (1) and formula (7) to the above Lemma we obtain:

(8) $$\forall\, u \in \mathit{User}\ (u\ \mathit{Use}\ \underline{e}\ \Leftrightarrow\ u\ \mathit{Int}\ \underline{e}).$$

From (8) and Theorem 14a, b we obtain:

a. $\mu(\underline{e}) = \iota(\underline{e})$ and b. $\delta(\underline{e}) = \delta^i(\underline{e})$.

This concludes the proof. □

By means of the notions of 'meaning' and 'interpretation' we can define the notions of a correct communication and non-correct communication:

Definition 14 (correct communication)
If $\underline{e}$ has n $(n \geq 1)$ meanings determined by its subtypes $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$, then $C_{\underline{e}}$ is a *correct communication* iff $\forall\, k = 1, \ldots, n$ ($\underline{e}_k$ has a determined interpretation and $\iota(\underline{e}_k) = \mu(\underline{e}_k)$).

Now we shall give some sufficient or/and necessary conditions for the correctness or non-correctness of communications by means of types.

Corollary 11

a. *If $\underline{e}$ has n $(n \geq 1)$ meanings determined by its subtypes $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$, then $C_{\underline{e}}$ is not a correct communication iff $\exists\, k = 1, \ldots, n$ ($\underline{e}_k$ has not a determined interpretation or $\iota(\underline{e}_k) \neq \mu(\underline{e}_k)$).*

b. *If $\underline{e}$ is has an established meaning and $\underline{e}$ has a determined interpretation, then $C_{\underline{e}}$ is a correct communication iff $\iota(\underline{e}) = \mu(\underline{e})$.*

c. *If $\underline{e}$ has established meaning and $\underline{e}$ has not a determined interpretation, then $C_{\underline{e}}$ is not a correct communication.*
d. *If $\underline{e}$ has an established meaning, $\underline{e}$ has a determined interpretation and $\iota(\underline{e}) \neq \mu(\underline{e})$, then $C_{\underline{e}}$ is not a correct communication.*

5 Final Remarks

The present work provides a logical explication of certain crucial notions of contemporary syntax, semantics and pragmatics, and is the result of a process of conceptualization over many years of knowledge on the syntactic sense, meaning and interpretation of linguistic expressions.

The theory of meaning and interpretation outlined in the present work differs from other theories referring to meaning and interpretation of linguistic expressions in that that it is the most general, axiomatic one and based on the two-level theory of language syntax. As such it takes into account the dual—*token* and *type*—ontological character of linguistic entities. The basic notions of the theory: meaning, referring, denotation and interpretation, interpretational referring, interpretational denotation were introduced on the second, *type*-level of formalization and required earlier formalization of some notions introduced on the first level, the *token*-level. Moreover, the principal notions can be explicated because our theory has both levels. However, even the *type*-level is essentially based on tokens, which means our theory does not have to be Platonistic, but can be nominalistic.

The subject matter of this paper does not consider the many issues, both syntactic and semantic, connected the with the problems of compositionality and substitutivity of expressions without changing their syntactic (categorial form) and semantic (meaning, denotation) sense. These important issues are the subject matter of separate considerations, which can be found in the author's papers [54–56]. The considerations are based on categorial approach to language and functor-argument structure of its expressions.

Acknowledgements The basis of the theory outlined in the two parts of this paper was prepared during my stay as a visiting research scholar at The Graduate School & University Center of The City University of New York; spring 2002. I thank CUNY for the chance to participate in the Exchange Visitor Program No. P1 3705, that gave me these research opportunities. I owe my cordial thanks to Eddie Kopiecki from CUNY who was the first reader of my paper.

I express my gratitude to all Reviewers or Debators who have suggested changes or corrections that influenced the final form of this paper. I would also like to thank Dr. Edward Bryniarski for his stimulating comments.

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Chapter 7
Three Principles of Compositionality

Urszula Wybraniec-Skardowska

Abstract This paper is a theoretical reflection on syntax and two kinds of semantics: (i) *intensional* (*conceptual*)—semantics comprising of the relationship between language and cognition (*conceptual reality*) and (ii) *extensional* (*denotational*)—semantics describing the relationships between language and reality (*ontological reality)* to which the language refers. The paper outlines a formal theory of compositionality and presents some methatheoretical results that can be regarded as a sample of the metatheory of syntax, meaning and interpretation (denotation). The theory of compositionality assumes as the basic axioms three formulas which formalise three different ways of understanding the so-called Principle of Compositionality, taking into account one syntactic and two semantic, both *intensional* (conceptual) and *extensional* (denotational), aspects. The formalisation of the syntactic principle of compositionality is connected with the rule of cancellation of categorial indices used by Ajdukiewicz [1] in his algorithm of checking the syntactic connection of language expressions. The formalisation of the two remaining principles takes into consideration Carnap's dualism: *intension-extension* in the meaning of linguistic expressions. All three axioms as proposed in this paper have the same scheme and define suitable homomorphisms of a partial algebra of language into three partial algebras determined by 1) syntactic categorial indices (types), 2) *intensional* (conceptual) semantics and 3) *extensional* (denotational) semantics. Fundamental theorems of the theory as given refer to some replacement principles and truth value principles. The theory is presented as a modification and development of the author's axiomatic Theory of Language Syntax and its expansion on semantic components ([32], [34], [35], [36], [37], [39], [40], [41], [42]).

First published in: Żegleń, U. M. (ed.) Cognitive Science and Media in Education. From Formal and Cognitive Aspects of Language to Knowledge, pp. 28–65. Publishing House Adam Marszałek, Toruń (2010). The first version of this paper was prepared in the Spring of 2000 during my stay as a visiting professor at Tilburg University and as a guest of the University of Amsterdam in the Netherlands. Its principal content was presented at Logic Colloqium 2000 in La Sorbone, Paris, July 2000 and its abstract was published in [37]. The present paper is a complete and developed text of the earlier version. Some synthetic results of the paper are included in my work [42].

Key words: Syntax • Intensjonal semantics • Extensional semantics • Meaning • Denotation • Axiomatization • Syntactic and two semantic principles od compositionality • Partial algebras • Replacement principles • Truth values principles

1 Introduction

The Principle of Compositionality has been intensively discussed and studied in recent literature.[1] The principle is commonly regarded as a condition imposed on semantics for languages; i.e. as the Principle of Compositionality of Meaning that some semantics satisfy and some do not. The colloquial formulation of the Principle is imprecise and allows for different interpretations. The most often given formulation of it is as follows (cf. Partee [25]; Partee et al. [26]; Hodges [14], [16]; Jansen [17], [18]):

(C) *The meaning of a complex expression of language is a function of the meanings of its constituent parts and syntactic rules by means of which it is formed.*

The formulation (C) uses imprecise or undefined syntactic terms such as: *function of*, *expression* (*well-formed expression*), *constituent*, *part of expression*, *syntactic rule*, and a single semantic term—*meaning*.

Let us observe that if the Principle of Compositionality is to have both cognitive and theoretical value all the above given terms have to have given explicit sense in a formal theory (cf. Hodges [15],[16]).

In this paper I also start from another observation. The framework of compositionality only as a semantic principle contains an essential gap. It is not normally assumed that compositionality is a property of syntax and, moreover, there is not a full view of compositional semantics, because the framework does not take into account a commonly known Carnap's dualism: intension-extension in the meaning of linguistic expressions.

The principle (C) relates the meaning of a complex expression with its syntactic form. The syntactic form of the complex expression has to be a function of the syntactic forms of the constituents of the expression, and in this connection it must be based on a syntactic principle of compositionality. This observation will be a starting point of our theoretical considerations. So, a preliminary principle of compositionality should be the syntactic *Principle* (*PCS*) *of Compositionality of Syntactic Forms*, which can loosely be formulated as follows:

(*PCS*) *A correct syntactic form of a complex expression is dependent on and is a function of the syntactic forms of the expression constituents from which it was formed, by means of syntactic rules of language.*

[1] According to Wilfrid Hodges [15] the words 'compositional' and 'compositionality' were introduced into semantic literature through the paper of Katz and Fodor [19, p. 503]; and cf. Katz [20, p. 152]. A comprehensive treatment of the history of the notion of compositionality can be found in the papers of Janssen [17] and Hodges [15] (cf. also Pelletier [27]).

The paper allows us to distinguish different versions of compositionality and considers compositionality as a condition on syntax (compositional syntax) for categorial languages, and as two conditions on two-level (*intensional* and *denotational*) semantics (compositional semantics) for such languages. These three conditions are formulated as the three Principles of Compositionality.

The paper is not so much a discussion on these three principles, but a formal theoretical conception which, first of all, tries to answer the following questions:

- How by means of formally-logical tools to give precise basic assumptions for compositional syntax and two-level compositional semantics for a broad class of languages—categorial languages (i.e. languages generated by categorial grammars),
- how on the basis of these assumptions (axioms and definitions) to derive fundamental theorems concerning the properties of some syntactic and semantic notions.

The principal aim of this paper is a presentation of certain foundations of the formal theory ***TC***, called the *theory of compositionality*. It is a combination of a theory of syntax and its extension to a theory of semantics, applying the axiomatic-deductive method, and to a large degree, the formalization method.

In formalizing theories of syntax and semantics we replace the imprecise and semantically fuzzy natural language by interpreted formal language. This language contains logical, syntactic and semantic terms whose sense is specified and explained by the theory in the inter-subjectively available manner by means of a systematized set of axioms, definitions and derived theorems. The main benefit from the axiomatization and formalization of theories of syntax and semantics consists of making order out of their notions, assumptions and theorems and making their foundations, using accepted principles and claims independently of often fallible intuitions.[2] The formalization gives clarity as to primitive notions and assumed postulates (axioms), strengthens the definability and validity of accepted syntactic/semantic theorems, provides the inter-subjective verification of proofs and eliminates a *non sequitur* error and a vicious circle. Following the construction and the making of a synthesis of results of the formalizations of theories of syntax and semantics, it is possible to find the explicit deductive structure of the metatheory of syntax & semantics.

It is obvious that the character of the paper excludes presenting the complete formalization of the theory ***TC***, in particular giving its complete axiomatization. However, it should be stressed that the formalized Three Principles of Compositionality are main axioms of ***TC*** from which the most important theorems of the theory ***TC*** are derived.[3] The form of these axioms and theorems allows us to point out their schemes as a part of the deductive metatheory of compositional syntax & semantics.

[2] As it is well known, using solely an intuitive comprehension of terms of a theory can be fallible, can also slow down a development of the theory and even lead to contradiction.

[3] It is obvious that if someone represents the view that compositionality might be false, he/she may be focused on a different theory—a theory of non-compositionality, accepting negations of the principles of compositionality as axioms; the case is quite similar to the one offered by Euclidean geometry and non-Euclidean geometry by Łobaczewski.

The theory ***TC*** is presented in general terms in Section 2. In Section 3 it is outlined formally. The main points of the results of my paper are explained in Section 4.

2 The Theory *TC* – a Non-formal Characterisation

2.1 Ideas

The idea behind the formalization of the three principles of compositionality on the basis of the theory ***TC***, arises not only from the need of a precision of terms, from which they are formulated, but also from intellectually-cognitive needs. The theory ***TC*** mirrors relationships defined by the triad (see Wybraniec-Skardowska [41], [42]):

language – cognition – reality.

The three principles of compositionality formalized on the basis of ***TC*** correspond to the three elements of this triad. They are some criteria on the compatibility of these elements.

The theory ***TC*** gives a new view concerning theories of syntax and semantics. This view is based on the assumption that reliability of cognition of reality by means of language is given only by an agreement of syntactic and two kind (*intensional* and *extensional*) semantic knowledge, which corresponds to three levels of knowledge about components of the triad. This knowledge can be called the *knowledge referring to three realities*:

(i) *language reality*, in which results of cognitive activities such as concepts and propositions are expressed,
(ii) *conceptual reality*, in which products of cognition of ordinary reality such as concepts and propositions are considered and
(iii) *ontological reality*, which contains objects of cognition, among other denotations of language expressions.

The language reality will be described by a theory of syntax and the conceptual and ontological realities—by its expansion to a theory of semantics. These theories of syntax and semantics create the theory ***TC***.

2.2 Main Assumptions of the Theory of Syntax

A theoretical description of syntax (compositional syntax) is partially based on a modification of the author's previous results referring to the Theory of Language Syntax ([32], [33], [34]). A fragment of the theory, considered here as a basis of ***TC***, can be regarded as a formal, axiomatic theory of categorial grammar built in the spirit of the theories of Leśniewski ([21], [22]), Ajdukiewicz ([1], [2]) and Bar-Hillel

([3], [4], [5]). The task of the theory of syntax is to provide an exact definition of the notion of a *well-formed expression* (a *wfe*) of any language L without taking into account its specific structure or symbolism, the notation of its expressions, its calligraphic system, and so on. We denote the *set of all wfes of the language* L by the symbol 'S'. The set S is the basic notion of the theory. It determines the *language reality* related to L.

Wfes of the set S are defined by means of *categorial indices* (*types*). Categorial indices are auxiliary words rated among words of the metalanguage of the language L; they point out for syntactic forms, syntactic categories of expressions of L. The set *IND* of all categorial indices determines a *metalanguage reality* by means of which we describe the language reality S. The set *IND* will be called the *indexation reality* related to the reality S. Every expression of S has one categorial index (type) from *IND*. Categorial indices of *IND* are determined by the *function* ι *of the indication of indices* to some words of language; the function ι is a primitive notion of the theory of syntax. The set *IND* is the counter-domain of the function ι.

The essence of the approach proposed here is that any complex expression of L has always a *functor-argument structure*: it is compounded from a main part, called the *main functor*, and complementing parts, called its *arguments*. Functor-argument expressions are defined as elements of the counter-domain of the *relation* r *of the formation of the functor-argument expressions* of L; the relation r is a primitive notion of the theory and replaces every rule of the formation of complex expressions of L. An expression of L is a *wfe* if it is either a possessing categorial index word of the vocabulary of the language, or it is a functor-argument expression such that it and every complex *constituent* of this expression satisfies the following *principle* (*SC*) *of syntactic connection*:

(*SC*) *The index of the main functor of a functor-argument expression is the complex (functoral) index formed from the index of that expression and the successive indices of the successive arguments of this functor.*

Indices of *wfes* can be *basic* (simple auxiliary words) or *functoral* (complex). We postulate that s is a basic index. It determines *wfes* of L that are *sentences* of the language. Complex, functoral indices serve to assign *wfes* of L which are *functors*, i.e. main functors of some functor-argument *wfes* of L. Functoral indices are formed from other indices by means of the *relation* r^{ι} *of the formation of indices* which is a primitive notion of theory and which replaces any rule of the formation of categorial indices. The index of the main functor of any expression can be written down as a quasi-fraction in which its numerator is the index of the expression, that the functor forms together with its arguments, while its denominator includes successive indices of arguments of that functor. It can be formally proved that every functor is a set-theoretical function defined on successive arguments of the functor, i.e. on some expressions of L. So, every functor-argument expression can be noted in the form of a function-argument expression.

Example 1 It is usually accepted for natural languages that names, apart from sentences, are their basic expressions. Functors of natural languages create a branched hierarchy.

Let the functor-argument form of the English sentence phrase:

$$(*) \qquad \textit{John loves Mary}$$

be the following expression with function-argument notation:

$$(1) \qquad \textit{loves}(\textit{John}, \textit{Mary})$$

and with the index s of sentences. Then the main functor in $(*)$ is the two-argument predicate '*loves*'. It has the functoral index s/nn, where n is the basic index of names, so also the index of proper names '*John*' and '*Mary*'. This functoral index is the index of the sentence-forming functor of any two-name arguments. The expression (1) contains only constituents of the 1^{st}-order: '*John*', '*Mary*' and '*loves*'. It satisfies the principle (*SC*) and is a well-formed functor-argument sentence of English.

Now let us consider the sentence phrase $(*)$ in accordance with its commonly used grammatical parse: N + VP. Then the functor-argument form of the sentence $(*)$ will be noted as the following function-argument expression with the index s:

$$(2) \qquad \textit{loves Mary}(\textit{John}).$$

The main functor '*loves Mary*' in (2) has one name-argument '*John*' and it has the functoral index s/n, which is the index of one-argument predicate. The functor '*loves Mary*', like the name '*John*', is a constituent of (2) of the 1^{st}-order. But it is a complex expression and its functor-argument form can be described by the following function-argument expression:

$$(3) \qquad \textit{loves}(\textit{Mary})$$

in which the main functor '*loves*' and the name '*Mary*' are constituents of the expression (2) of the 2^{nd}-order. The functor '*loves*' has the functoral index $(s/n)/n$. It is easy to see that the expression

$$(4) \qquad \textit{loves}(\textit{Mary})(\textit{John})$$

is a well-formed functor-argument sentence of English because it and its proper complex constituent (3) satisfy the principle (*SC*).

It is possible to consider another way of assigning indices to functors in (2) referring to Montague's Grammar [23].

The phrase '*constituent of an expression*' used in *Example 1* is a derived term of the theory. Another syntactic term—'*replaceability of expressions*' is one of the most important derived terms of the discussed theory of syntax. It occurs in formulations of all fundamental theorems of the theory ***TC*** that are consequences of the principles of compositionality.

From the theory of syntax we obtain the theory of compositional syntax accepting axiomatically the formalized principle (*PCS*) stating that: *the index of any functor-argument* (*complex*) *wfe of L is functionally determined by the index of its main functor and indices of successive arguments of this functor*. This principle corresponds to the following *rule* (*rc*) *of cancellation of quasi-fractional indices* used by Ajdukiewicz [1] in his algorithm of checking of the syntactic connection of language expressions:

$$(rc) \qquad a/a_1 a_2 \dots a_n(a_1, a_2, \dots, a_n) \rightarrow a.$$

The rule (rc) serves to check the syntactic connections of expressions of L.

Example 2 Let us observe that the rule (rc) functions for expressions (1), (2), (3) and (4) given in our *Example 1*. Using a variant of the principle (*PCS*) of compositionality of syntactic forms and the rule (rc) we "calculate" indices of expressions (1) and (4) in the following way:
$\iota(1) = \iota(loves)(\iota(Mary), \iota(John)) = s/nn(n, n) = s.$
$\iota(4) = \iota(loves(Mary))(\iota(John)) = \iota(loves)(\iota(Mary))(\iota(John)) = (s/n)/n(n)(n) =$
$= s/n(n) = s.$

2.3 Main Assumptions of the Theory of Semantics and Some Metatheoretical Issues

The theory of semantics (compositional semantics) is an expansion of the theory of syntax discussed in Section 2.2. The idea of the formalization of compositional semantics is based on a new theoretically semantic approach to language. The idea consists in accepting the following three referential relationships of *wfes* of the *language reality S* to three *realities* to which *wfes* of *S* refer: one syntactic—*metalinguistic* relationship, and two semantic: *conceptual* (*intensional*) and *denotational* (*extensional*)[4] relationships. These relationships are described in ***TC***, respectively, by three operations on *wfes* of *L*: the syntactic operation—the *function ι of the indication of indices*, and two new, semantic operations: the *meaning* (*intensional*) *operation μ* and the *denotation* (*extensional*) *operation δ*. Corresponding to *wfes* of language three *realities*, including values of these operations, are called: *indexation reality IND*, *conceptual reality CON* and *ontological reality ONT*, respectively.

The reality *IND* was defined earlier. The notions of realities *CON* and *ONT* are primitive notions of the theory of semantics. *Meanings* of *wfes*, as the values of the operation μ, belong to the reality *CON*. *Denotations* of *wfes*, as the values of the operation δ, belong to the reality *ONT*.

The notion of the *meaning* of an expression is distinguished from the notion of the *denotation* of the expression. This gives much greater flexibility in talking about meaning. For example, we can speak about meaning change without denotation change. We understand the *meaning* of a *wfe* as the logical meaning[5] that is here close to the so-called *intension*, which in Carnap's dualism: *intension-extension* in the meaning of linguistic expressions is the first factor; the second factor—*extension*, corresponds to the notion of the *denotation* (see Carnap [7], [8]; cf. also Frege [9].

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[4] Some assumptions on *denotational semantics* were presented in my earlier papers ([35], [36], [38], [40]. See also Buszkowski [6] and Wybraniec-Skardowska and Rogalski [43].

[5] This notion is different from Grice's *psychological understanding of meaning* (see Grice [12]).

The difference leads us to a formal description of the two-level semantics: *intensional* (*conceptual*) and *extensional* (*denotational*).[6]

Let us note that the conceptual reality *CON* contains all logical propositions (called *conceptual states of affairs*); they are the meanings of sentences of language. And, if the language includes names, the reality *CON* contains all logical concepts—as meanings of these names. The reality *CON* also includes all functions corresponding to functors of *L*, e.g. functions defined on conceptual states of affairs, logical concepts, and so on. The ontological reality *ONT* contains all denotations of sentences; they are understood here as states of affairs described by these sentences (not as the truth-values as Frege assumed). To *ONT* can also belong individuals—as denotations of proper names, sets of individuals—as denotations of predicate names, and different functions corresponding to suitable functors of *L*: functions defined on states of affairs, individuals, sets of individuals, and so on.

The notion of the meaning operation μ like the notion of the function ι of the indication of indices is a primitive notion of the theory ***TC***. We start from the assumption that the meaning of a *wfe* determines its denotation. Thus we also accept the assumption that the denotation operation δ is the notion defined by means of the meaning operation μ. The operation δ is defined as the composition of the operation μ and the *operation* δ_c *of conceptual denotation*, which is a new primitive notion of ***TC***. The operation δ_c assigns object of reality *ONT* to the meanings of *wfes* of the reality *CON*, i.e. to some objects of the reality *CON*.

From the definition of the denotation operation δ immediately follows that two *wfes* have the same denotations if they have the same meanings. The reverse implication is not valid. The meanings of two expressions can be changed without changing their denotations.

The distinction between understanding of meaning and denotation can be explained by the following example:

Example 3 Let us consider two sentences (cf. Gamut [10, p. 7]):

(i) John is looking for the current Commander- in-Chief of the U.S. Armed Forces.
(ii) John is looking for the current President of the United States of America.

The definite descriptions in (i) and (ii) have the same denotation, which is the person of Joe Biden, but they have not the same meaning. The meaning of the description in (i) is the logical concept of the current Commander-in-Chief of the U.S. Armed Forces, and the meaning of the description in (ii) is the logical concept of the current President of the United˙ States of America. Conceptual denotations of these two logical concepts are also the same; they determine the person of Joe Biden. So, the conceptual denotations of meanings of the definite descriptions in (i) and (ii) are compatible to ordinary denotations of these descriptions. Similarly, the sentences (i) and (ii) have the same denotation (refer to the same state of affairs looking for the person of Joe Biden by John), but they do not have the same meaning (the meaning of the sentence (i) is a conceptual state of affairs looking for the current

[6] We can accept that the zero-level of semantics is the *indexation level*, in which we consider the relationships between *wfes* and their indices, thus their syntactic forms/syntactic categories.

Commander-in-Chief of the U.S. by John, and the meaning of the sentence (ii) is a conceptual states of affairs looking for the current President of the United States of America by John). Conceptual denotations of the meanings of sentences (i) and (ii) and the denotations of these sentences are compatible.

There are some relationships between operations ι, μ and δ. In theoretical considerations, we assume for δ and ι the axiom stating that if denotations of two *wfes* are the same, then their categorial indices (syntactic categories, forms) are also the same. The reverse implication is not valid, e.g. two sentences can have different denotations. From the axiom it follows that if two *wfes* have the same meaning then they also have the same index (syntactic category, forms).

Three different operations ι, μ and δ determine three different principles of compositionality: one syntactic, corresponding to (*PCS*), mentioned in Section 2.2 and formulated by means of ι, and two semantic, corresponding to (*C*) and formulated by means of the operations μ and δ, respectively. We formalise them as three axioms (*PCS*$_1$), (*PCM*$_1$) and (*PCD*$_1$) by means of three primitive terms of ***TC***: r^{ι}, r^{μ}, r^{δ}, denoting *relations of the formation of indices, meanings* and *denotation*, respectively. These relations are one-to one functions correlated with the relation *r* of the formation of the functor-argument expressions of *L*. In theoretical considerations they replace any rules of operating on indices, meanings and denotations, respectively (see Diagram).

The relation r^{ι} of the formation of indices of syntactic forms of L	$\approx$	*Any syntactic rule operating on indices corresponding to expressions of L*
$\uparrow$		$\uparrow$
The relation r of the formation of complex expressions of L	$\approx$	*Any syntactic rule operating on expressions of L*
$\downarrow$		$\downarrow$
The relation r^{μ} of the formation of meaning of complex wfes of L	$\approx$	*Any semantic rule operating on meanings of wfes of L*
$\downarrow$		$\downarrow$
The relation r^{δ} of the formation of denotations of complex wfes of L	$\approx$	*Any semantic rule operating on denotations of wfes of L*

Diagram

The three axioms of compositionality (*PCS*$_1$), (*PCM*$_1$) and (*PCD*$_1$) ensure an adequacy of syntax and two-level (*intensional* and *extensional*) semantics. They have the same metatheoretical and formal scheme that can be described in the following way:

Scheme of axioms (compositionality)

The correlate of any functor-argument wfe of L is the value of the correlated function defined on the tuple of correlates of the main functor of the wfe and successive arguments of the functor.

The *correlate of a wfe* should be interpreted in the following three different ways: as its categorial index, as its meaning or as its denotation. The correlated function is one of the relations r^{ι}, r^{μ}, r^{δ} corresponding to the relation (function) r.

From the principle (*PCD_1*) of compositionality of denotation follows another semantic principle (*PCDc*) which can be called the *principle of compositionality of conceptual denotation* and which states that:

(*PCDc*) *The conceptual denotation of the meaning of any functor-argument wfe of L is the value of a function defined on the tuple of the conceptual denotations of meanings of the main functor of the wfe and successive arguments of the functor.*

The principle (*PCD_1*) can be replaced by the principle (*PCDc*).

The functor-argument structure of complex expressions of *L* allows us to give three variants of the principles of compositionality (*PCS_2*), (*PCM_2*) and (*PCD_2*) corresponding to the principles (*PCS_1*), (*PCM_1*) and (*PCD_1*), respectively. They take into account some ideas of Frege [9], and in a metatheoretical formulation, putting them together, we can say:

Metatheorem (compositionality)
The correlate of the main functor of any functor-argument wfe is a function defined on the tuple of successive correlates of successive arguments of this functor, and the correlate of the functor-argument wfe is the value of the function.

Let us note that the principles (*PCS_2*), (*PCM_2*) and (*PCD_2*) serve to "calculate" suitable correlates of any functor-argument *wfes* by means of suitable correlates of constituents of the *wfes* (cf. *Example 2*).

This new look at the principle of compositionality can be justified by the fact that on the basis of ***TC*** we can prove that the correlates of functors, like functors themselves, are set-theoretical functions, partial operations, defined on tuples of successive correlates of arguments of these functors. This fact allows us to treat the language reality *S* and corresponding to it $\iota-$, $\mu-$ and $\delta-$ images of *S*, i.e. $\iota(S)$—a fragment of the indexation reality *IND*, $\mu(S)$—a fragment of the conceptual reality *CON* and $\delta(S)$—a fragment of the ontological reality *ONT* as some algebraic structures (some partial algebras), and the suitable variants of compositionality as the requirements of homomorphisms between them determined by the operations ι, μ and δ, respectively.[7] In this way on the level of the metatheory of ***TC*** it is possible to show the unity of the formation of *wfes* of the language reality *S* and the formation of their correlates in the conceptual reality *CON* and in the ontological reality *ONT*.

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[7] Ideas about the algebraisation of language can already be found in Leibniz's papers. We can also find the algebraic approach to issues connected with syntax, semantics and compositionality in Montague's 'Universal Grammar' [23] and in the papers of van Benthem ([29], [30], [31], Janssen [17], Hendriks [13]. The difference between their approaches and the approach which we shall present here lies in the fact that carriers of the so-called *syntactic* and *semantic algebras* discussed in this paper include functors or, respectively, their suitable correlates, i.e. their $\iota-$ or some other semantic-function images (see *Fact 4*). Simple functors and their suitable $\iota-$, $\mu-$ or $\delta-$ images are partial operations of these algebras. They are set-theoretical functions determining these operations.

In the theory ***TC*** the notions of a model of the language L and truthfulness are derived concepts. Models of L are non-standard models. They are mentioned algebraic structures (partial algebras) determined by the fragments $\iota(S)$, $\mu(S)$ and $\delta(S)$ of the realities *IND*, *CON* and *ONT*, respectively. If we denote by $\boldsymbol{L}$ the language algebraic structure determined by the reality S then we can consider three kinds of models of L correlated with $\boldsymbol{L}$: one *syntactic* $\iota(\boldsymbol{L})$ (which is the partial algebra determined by $\iota(S)$) and two *semantic—intensional* $\mu(\boldsymbol{L})$ (which is the partial algebra determined by $\mu(S)$) and *extensional* $\delta(\boldsymbol{L})$ (which is the partial algebra determined by $\delta(S)$).

For three models $\iota(\boldsymbol{L})$, $\mu(\boldsymbol{L})$ and $\delta(\boldsymbol{L})$ of the language L we define three notions of truthfulness. For this purpose we introduce to ***TC*** three new primitive notions:

- a non-empty subset $T\iota$ of *IND* consisting only the symbol 'T' as the index of any true sentences,
- a nonempty subset $T\mu$ of *CON* consisting all meanings of sentences of L that are true logical propositions, i.e. true conceptual states of affairs and
- a nonempty subset $T\delta$ of *ONT* consisting all denotations of sentences of L that are states of affairs that obtain.

All three definitions of a true sentence in one of the models $\iota(\boldsymbol{L})$, $\mu(\boldsymbol{L})$ and $\delta(\boldsymbol{L})$ of L are analogous and are substitutions of the following definition scheme:

Scheme of definitions (truthfulness) *For* $h = \iota, \mu$ *and* δ

The sentence e is true in the model $h(\boldsymbol{L})$ *if and only if the h-correlate of e belongs to* Th.

The definitions of a true sentence correspond to the **truth value principles** (cf. Hodges [14, p. 540]) according which: *the correlate of a sentence* (i.e. its index, meaning or denotation, respectively) *determines whether or not it is true in a suitable model.*

We accept the next two axioms: (1) if a sentence has the index T of true sentences then it is a true sentence in the intensional model $\mu(\boldsymbol{L})$, and (2) if a sentence is true in the intensional model $\mu(\boldsymbol{L})$ then it is true in the extensional model $\delta(\boldsymbol{L})$, i.e. if the conceptual state of affairs, which is a meaning of a sentence, is true, then the state of affairs, which is the denotation of the sentence, obtains.

Let us note that the reverse implication to the one in the axiom (2) cannot be true.

Example 4 Let us consider the given in *Example 3* sentence (i) as a functor-argument *wfe* with the main functor 'is looking for' and its two arguments 'John' and 'the current Commander-in-Chief of the U.S. Armed Forces'. Using the semantic principles of compositionality (PCD_2) and (PCM_2) we state that denotation of (i) can be the fact that John is looking for a concrete person—Joe Biden, but the meaning of (i), i.e. the conceptual state that John is looking for the current Commander of the U.S. Armed Forces cannot be true.

From the above-given axioms (1) and (2) it follows that if a sentence of L has the index T then it is true both in the intensional model and in the extensional model of L, i.e. the conceptual state of affairs, which is the meaning of this sentence, is true and the state of affairs, which is the denotation of this sentence, obtains.

Let us observe that it is not valid in ***TC*** that denotations of all true sentences in any of three understanding of truthfulness are equal; they are not the value of truth as Frege assumed. Of course it is valid in ***TC*** that indices of all true sentences in the syntactic model are equal to the symbol 'T' indicating true sentences of L.

The most important theorems of ***TC*** which follow from the principles of compositionality (PCS_2), (PCM_2) and (PCD_2) use the syntactic notion of the *relation of replacement of a constituent of a given wfe of L*. In referring to the truth value principles we can justify three theorems of ***TC*** which we obtain from the following metatheorem:

Metatheorem (referring to the truth value principles)
Replacing in any sentence of L its constituent by a wfe that has the same correlate never alters the truth value of the replaced sentence in the suitable model (corresponding to the correlates of the constituent and the replacing wfe).

The above metatheorem of ***TC*** follows from the principal one:

Metatheorem (replacement principles)
Two expression have the same correlate if and only if by the replacement of one of them by the other in any wfe we obtain a wfe which has the same correlate as the wfe from which it was derived.

In literature we only know the immediate conclusions of the theorem (cf. Frege [9], Gamut [10, p. 11]):

Corollaries (replacement principles)
If two wfes have the same meaning (resp. denotation) then the replacement of one of them by another in a third wfe does not change the meaning (resp. denotation) of a wfe that arises from this third expression.

From *Theorems* (replacement principles), *Theorems* (referring to the truth value principles), axioms on relationships between operations ι, μ and δ and between three kinds of truthfulness we get some other corollaries pertaining to the replacement and the truth value principles. They will be given in Section 3.

Let us also note that in order to deduce some stronger theorems referring to the truth value principles we have to enrich ***TC*** with some new axioms (cf. Gerhard [11, p .280], Jansen [17, p. 463]). These axioms and stronger theorems will be given formally in the next section. On the level of the metatheory of ***TC*** they shall be have corresponding schemes.

3 The Theory *TC* – a Formal Approach

3.1 Some Formal Foundations of the Theory of Syntax

Any syntactic characterised language L is fixed if the set S of all well-formed expressions (briefly: *wfes*) of L is determined. Formalisation of a theory of syntax as a part of the theory ***TC***, and the set S, first requires providing a general characterisation of the language L as the following systems of notions:

$$(L) \qquad \langle U_L, V_1, c, W_1, E; S\rangle$$

compounded from: the set U_L—the linguistic *universe* of L, the *vocabulary* V_1 *of all simple words* of L, the ternary *relation* c *of concatenation* defined on U_L, the *set* W_1 *of all word*s of L, the *set* E *of all expressions* of L, the *set* S *of all wfe*s of L.

The first three notions of (L) are primitive notions of the theory of syntax and the remaining are derived ones. The set W_1 of all words is defined as the smallest set containing the vocabulary V_1 and closed under the relation c of concatenation; the concatenation relation is here a two-argument set-theoretical function satisfying some axioms (cf. Wybraniec-Skardowska [34]). The set E of all expressions will be defined as a subset of the set W_1. Because not every expression of L has to be a *wfe*, the set S will be defined as a subset of the set E.

The set S will be regarded as generated by the so-called classical categorial grammar (the notion shaped by Bar-Hillel [3], [4], [5]) which reconstruction, on the basis of the theory, is the following system of notions of the theory:

$$CG_L = \langle E_1^s, E_2^s, \iota, r, r^\iota, (SC), S\rangle, \quad \text{where}$$

- E_1^s – is the *vocabulary* of simple words possessing indices, i.e. *simple expressions of L*,
- E_2^s – the *auxiliary vocabulary* of simple indices, i.e. *basic indices for L*,
- ι – the function of the *indication of categorial indices* (*types*) *to some words of L*,
- r – the relation of the *formation of complex, functor-argument expressions of L*,
- r^ι – the relation of the *formation of indices for expressions of L*,
- (SC) – the *principle of syntactic connection*,
- S – the *set of all well-formed expressions of L.*

Let us characterise the notions of CG_L. They use the notion of a *categorial index*. Categorial indices (types) are distinguished from the *set* W_2 of all auxiliary words of L. The set W_2 is defined like the set W_1. It is the smallest set containing the *vocabulary* V_2 *of all simple auxiliary words of* L—a new primitive notion of the theory—and closed under the relation c of concatenation. We postulate that the vocabularies V_1 and V_2 are disjoint and non-empty subsets of the universe U_L. We also postulate that no word of the vocabulary V_1 or V_2 is a concatenation of two words, while every word that is not a simple word is a concatenation of two words. So:

Axioms (vocabularies)

a. $\emptyset \neq V_1 \subseteq U_L$ *and* $\emptyset \neq V_2 \subseteq U_L$,
b. $V_1 \cap V_2 = \emptyset$,
c. *If* $x \in V_i$ *then* $\neg \exists\, y, z \in W_i\; c(y, z, x)$, *for* $i = 1, 2$,
d. *If* $x \in W_i \backslash V_i$ *then* $\exists\, y, z \in W_i\; c(y, z, x)$, *for* $i = 1, 2$.

Categorial indices are determined by the *relation* ι *of indication of indices*. It is a new primitive notion of the theory, satisfying the following axiom:

Axiom (indication of indices)

$$\iota \subseteq W_1 \times W_2 \text{ and } \iota \text{ is a function,}$$

which says that ι is a partial function from $W_1 \times W_2$. The expression $\iota(x)$ we read: the index of the word x. The domain $D_1(\iota)$ of ι is the set of all words possessing indices and the counter-domain $D_2(\iota)$ of ι is the set of all auxiliary words which are indices of words.

Now we can introduce the following definitions:

Definition (set of all categorial indices)

$$IND = D_2(\iota),$$

Definition (vocabulary of simple expressions)

$$E_1^s = V_1 \cap D_1(\iota),$$

Definition (vocabulary of basic indices)

$$E_2^s = V_2 \cap IND.$$

The *set* E *of all expressions of* L is defined as the sum of the vocabulary E_1^s of simple expressions and the set of all functor-argument expressions of L obtained from simpler ones by using the relation r. The relations r of the formation of functor-argument expressions and r^ι of the formation of indices are further primitive notions of the theory, satisfying the following axioms:

Axiom (formation of functor-argument, complex expressions)

$$r\colon \bigcup_{k=2}^{\infty} D_1(\iota)^k \xrightarrow[1-1]{} D_1(\iota) \backslash V_1,$$

Axiom (formation of indices)

$$r^\iota\colon \bigcup_{k=2}^{\infty} IND^k \xrightarrow[1-1]{} IND.$$

So, we postulate that relations r and r^ι are one-to-one functions defined on finite tuples of words possessing indices or, on finite tuples of indices of words, respectively, and their values are complex words possessing indices or, indices of words, respectively.

The fact that the relation r holds for the $n+1$-tuple $(f, e_1, e_2, \dots, e_n)$ of words of $D_1(\iota)$ possessing categorial indices and the word e can be written down as:

$$r(f, e_1, e_2, \dots, e_n; e) \quad (n > 0).$$

and then the word e belongs to the counter-domain $D_2(r)$ of r $(D_2(r) \subseteq D_1(\iota)\backslash V_1 \subseteq W_1\backslash V_1)$ and e is called the *functor-argument expression* or the *complex expression formed from the main functor* f *and its consecutive arguments* $e_1, e_2, \dots, e_n$.

Definition (set of all expressions)

$$E = E_1^s \cup D_2(r),$$

where the counter-domain $D_2(r)$ of r is the set of all functor-argument expressions of *L*.

The fact that the relation r^ι holds for the $n+1$-tuple $(a, a_1, a_2, \dots, a_n)$ indices of words and the word index b is written down in this way:

$$r^\iota(a, a_1, a_2, \dots, a_n; b) \quad (n > 0).$$

Then the index b belongs to the counter-domain $D_2(r^\iota)$ of r^ι and b can be called the *index formed from the index* a *and indices* $a_1, a_2, \dots, a_n$. The index b can be a basic index from E_2^s or a *complex index* from $IND\backslash E_2^s \subseteq W_2\backslash V_2$.

We accept the following axiom stating that the index of the main functor of any functor-argument expression is a complex index, i.e. a concatenation of some auxiliary words:

Axiom (index of the main functor)

If f *is the main functor of an expression of* $E\backslash E_1^s = D_2(r)$*, then* $\iota(f) \in IND\backslash E_2^s$.

The functor-argument expression e can be written down as a function-argument expression according to the following convention:

Definitional convention

(*fe*) $$e = f(e_1, e_2, \dots, e_n) \underset{df}{=} r(f, e_1, e_2, \dots, e_n; e).$$

The function-argument writing of complex expressions of *L* is justified because the main functor of any functor-argument expression is a set-theoretical function defined on tuples of successive arguments of this functor. We can state this because r is a one-to-one function and the convention (*fe*) provides us with:

Fact 1

If $e = f(e_1, e_2, \dots, e_n)$ *and* $e' = f'(e_1', e_2', \dots, e_n')$*, then*

$$e = e' \text{ iff } f = f' \text{ and } e_i = e_i' \text{ for any } i = 1, 2, \dots, n$$

from which follows:

Fact 2

For any words $f, e_1, e_2, \ldots, e_n, e_1', e_2', \ldots, e_n' \in D_1(\iota)$ *then*
if $e_i = e_i'$ *for any* $i = 1, 2, \ldots, n$*, then* $f(e_1, e_2, \ldots, e_n) = f(e_1', e_2', \ldots, e_n')$.

Moreover, if $e = f(e_1, e_2, \ldots, e_n) \in E \backslash E_1^s$ and $\iota(f) = b$, $\iota(e_k) = a_k$ for $k = 1, 2, \ldots, n$, and $\iota(e) = a$ then we accept the following convention:

Definitional convention

(*fi*) $$b = a/a_1 a_2 \ldots a_n \underset{df}{=} r^\iota(a, a_1, a_2, \ldots, a_n; b).$$

If $b = a/a_1 a_2 \ldots a_n$ is the index of the main functor f of the above expression e, we call b the *complex or functoral index formed from the index a of e* and *successive indices* $a_1, a_2, \ldots, a_n$ *of the successive arguments of this functor*, and we say that b *is the index of any functor forming expressions with the index a of n arguments with successive indices* $a_1, a_2, \ldots, a_n$.

For defining the notion of the set S as a subset of the set E we use the principle (*SC*) of syntactic connection whose verbal formulation was given in Section 2. We say that:

Definition (to satisfy the principle (*SC*))

The expression $e = f(e_1, e_2, \ldots, e_n) \in E_1$ satisfies the principle (*SC*) if and only if $\iota(f) = a/a_1 a_2 \ldots a_n$ where $a = \iota(e)$ and $a_k = \iota(e_k)$ for each $k = 1, 2, \ldots, n$.

The set S *of all wfes of L* can be defined by means of an inductive definition of the set S^n ($n \geq 0$) of all *wfes of the n-th order*. Then the set S is the sum of all sets of *wfes* of a finite order (greater than or equal to 0).

Definition (sets of *wfes*)

a. $S^0 = E_1^s$,
b. $e \in S^{k+1}$ iff $e \in S^k$ or
$\exists\, n > 0\, \exists\, f, e_1, e_2, \ldots, e_n \in S^k \big(e = f(e_1, e_2, \ldots, e_n)$ and e satisfies $(SC)\big)$,
c. $S = \bigcup_{n=0}^{\infty} S^n$.

The set of all *wfes* of the 0-order is the set of all simple expression of L, and to the set of all *wfes* of the $k+1$-order ($k \geq 0$) belong such expressions which are *wfes* of k-order or are functor-arguments expression of L, formed from such expressions and satisfy the principle (*SC*) of syntactic connection.

Using the above given definitions we can prove the following theorem:

Theorem (set of *wfes*)

The set S is the smallest set including the set E_1^s *of all simple expressions and containing every functor-argument expression such that it and all its proper functor-argument constituents satisfy the principle* (*SC*). [8]

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[8] It can be observed that the definition of the set S of all *wfes* allows us to describe an algorithm of examination of syntactic correctness of functor-argument expressions of language L by means of categorial indices (see Wybraniec-Skardowska [32], [34],[39]).

The notion of a *proper constituent of a functor-argument expression e* is defined by means of the notion of a *constituent of e of some finite order*.

Definitions (constituent) Let $e \in E \backslash E_1^s$.
a. A *constituent of e of the* 0*-order* is the expression e.
b. A *constituent of e of the 1st-order* is either the main functor of e or an argument of this functor.
c. A *constituent of e of the* $k+1$*-order* is a constituent of the 1st-order of some constituent of e of the k-order ($k > 0$).
d. A *proper constituent of e* is a constituent of e of some finite order $n > 0$.

It is easy to see that:

Fact 3
a. *If* $e \in S$ *then every proper constituent of e belongs to S.*
b. $S \subseteq E \subseteq D_1(\iota) \subseteq W_1$,
c. $\iota(S) \subseteq IND$.

From the parts b. and c. of *Fact 3* it follows that every *wfe* of S possesses one categorial index: basic or complex, functoral.

From the vocabulary E_2^s of basic indices we distinguish the so-called *main index s*. The index s determines the *set Sen of all sentences* of L as a subset of the set S.

Axiom (main index)

$$s \in E_2^s.$$

Definition (set of sentences)

$$Sen = \{e \in S : \iota(e) = s\}.$$

We postulate that there is at least one functor-argument *wfe* of L that is a sentence:

Axiom (existence of complex sentences)

$$(S \backslash E_1^s) \cap Sen \neq \emptyset.$$

Complex, functoral indices are assigned to *functors of L*. The set F *of all functors of L* is denoted by 'F' and the set of all simple functors of L is denoted by 'F_0'.

Definition (sets of functors)
a. $f \in F$ iff $\exists n > 0 \, \exists e_1, e_2, \ldots, e_n, e \in S \, \big(e = (f(e_1, e_2, \ldots, e_n)\big)$.
b. $F_0 = F \cap E_1^s$.

The set F_0 is called the *set of all simple functors* of L.

Let us note that from the above given definition it follows that the so-called main functors of functor-argument *wfes* are really functors of L. According to *Axiom* (index of the main functor) we state that functors are some *wfes* with complex, functoral indices:

Fact 4
If $f \in F$, *then* $f \in S$ *and* $\iota(f) \in IND \backslash E_2^s$.

Let us also note that functors, in particular simple functors, can be regarded as set-theoretical functions defined on some *wfes* of the set S with values in S. Hence,

Fact 5
Functors, in particular simple functors, are partial operations on the set S.

Fact 5 follows from *Fact 2*, *Fact 3b* and *Definition* (sets of functors).
From our assumptions the non-emptiness of distinguished subsets of S follows:

Fact 6
a. $E_1^s \neq \emptyset$, $S \backslash E_1^2 \neq \emptyset$, $Sen \neq \emptyset$, $F \neq \emptyset$, $F_0 \neq \emptyset$ *and*
b. $Sen \cap F = \emptyset$,
c. $Sen, F, F_0 \subseteq S$.

So, we can distinguish in the set S two non-empty and disjoint sets: the set *Sen* of all sentences of L and the set F of all functors of L.

Compositional character of syntax of the language L is defined by the following axiom of the theory ***TC***:

Axiom (compositionality of syntactic forms)
(PCS_I) *If* $e = f(e_1, e_2, \ldots, e_n) \in S$, *then* $r^{\iota}\big(\iota(f), \iota(e_1), \iota(e_2), \ldots, \iota(e_n); \iota(e)\big)$.

The axiom (PCS_I) corresponds to the Principle (*PCS*) of Compositionality of Syntactic Forms, whose verbal formulation was given in Section 2. The syntactic theorems following from this axiom will be given together with fundamental semantic ones. They use another important syntactic notion—the notion of the *replaceability of expressions* whose definition will be given before formulations of these theorems.

3.2 Formal Foundations of the Theory of Semantics and Some Metatheoretical Issues

A theoretical description of semantics of the language L requires the enriching of the conceptual apparatus of the syntactic part of the theory ***TC*** on semantic notions at two levels: *intensional* (*conceptual*) and *extensional* (*denotational*).

Referential relationships between *wfes* of L, i.e. *wfes* of the so-called *language reality S related to L*, and the three realities: *IND*, *CON* and *ONT*, to which they refer, was described generally in Section 2.3. The syntactic, so-called *metalinguistic* relationship between *wfes* of the set S and the so-called *indexation reality IND* was formally characterised in Section 3.1 by means of the operation ι of the indication of categorial indices to *wfes* of L. The two semantic (*conceptual* and *denotational*) relationships between *wfes* of L and realities: the *conceptual reality CON* and the *ontological reality ONT* will be formally characterised by means of the *intensional*, *meaning operation* μ and the *extensional*, *denotational operation* δ, respectively.

The notions: *CON*, *ONT* and μ, are primitive, semantic notions of the theory ***TC***; the notion δ will be regarded as a derived notion defined by means of the *operation* δ_c *of conceptual denotation* which is also included in primitive notions of ***TC***. We axiomatically assume that the realities *CON* and *ONT* are non-empty sets, that the operation μ maps the set *S* into the set *CON*, and the operation δ_c maps the set $\mu(S)$ of all meanings of *wfes* of *S* into the set *ONT*. So,

Axiom (non-emptiness of realities)

$$CON \neq \emptyset \text{ and } ONT \neq \emptyset,$$

Axiom (meaning operation)

$$\mu : S \rightarrow CON,$$

Axiom (operation of conceptual denotation)

$$\delta_c : \mu(S) \rightarrow ONT,$$

Definition (denotation operation)

$$\delta : \mu \circ \delta_c, \text{ i.e. } \delta(e) = \delta_c(\mu(e)) \text{ for every } e \in S.$$

So, the denotation operation δ is the composition of the meaning operation μ and the operation δ_c of conceptual denotation; it is obvious that δ maps the set *S* into the set *ONT*:

Fact 7

$$\delta : S \rightarrow ONT.$$

Moreover, the following relationship between μ and δ holds:

Fact 8

If $\mu(e) = \mu(e')$, *then* $\delta(e) = \delta(e')$ *for any* $e, e' \in S$.

Fact 8 states that: *If two wfes have the same meaning* (*intension*) *then they have the same denotation* (*extension*). It was shown in *Example 3* that the reverse implication cannot be true.

Let us note some relationships between the semantic operations μ and δ, and the syntactic operation ι.

Axiom (relationship between δ and ι)

If $\delta(e) = \delta(e')$ *then* $\iota(e) = \iota(e')$ *for any* $e, e' \in S$.

This axiom and *Fact 8* provides us with:

Fact 9

If $\mu(e) = \mu(e')$ *then* $\iota(e) = \iota(e')$ *for any* $e, e' \in S$.

So, if two *wfes of L* have the same meaning or denotation then they have the same categorial indices (syntactic categories, forms). The reverse implication is not valid, e.g. two sentences (with the index *s*) can have different denotations, thus different meanings.

Compositional character of two-level (*intensional* and *extensional*) semantics of L is defined by two axioms (PCM_1) and (PCD_1) of *compositionality of meaning* and *compositionality of denotation*, respectively. They correspond to the semantic principle (C) and are analogous to the axiom (PCS_1) given in Section 3.1. They are rewritten by means of two primitive terms of ***TC***: the term 'r^{μ}', denoting the *relation of the formation of meanings of wfes of L* and the term 'r^{δ}', denoting the *relation of the formation of denotations of wfes of L*. The relations r^{μ} and r^{δ} are analogous to the syntactic relations r and r^{ι}. So, we postulate that the relations r^{μ} and r^{δ} are one-to-one functions defined on finite tuples of meanings, respectively denotations of *wfes* of L, and their values are meanings, respectively denotations of *wfes*. So, we accept the following two axioms analogous to axioms for the relations r and r^{ι}:

Axiom (formation of meanings)

$$r^{\mu}\colon \bigcup_{k=2}^{\infty} \mu(S)^k \underset{1-1}{\longrightarrow} \mu(S),$$

Axiom (formation of denotations)

$$r^{\delta}\colon \bigcup_{k=2}^{\infty} \delta(S)^k \underset{1-1}{\longrightarrow} \delta(S).$$

The fact that the relations r^{μ} and r^{δ} hold for their arguments is written down in an analogous way as in the case of relations r and r^{ι}.

The two semantic axioms of compositionality (PCM_1) and (PCD_1) are the following:

Axiom (compositionality of meaning)

(PCM_1) *If* $e = f(e_1, e_2, \ldots, e_n) \in S$, *then* $r^{\mu}\big(\mu(f), \mu(e_1), \mu(e_2), \ldots, \mu(e_n); \mu(e)\big)$,

Axiom (compositionality of denotation)

(PCD_1) *If* $e = f(e_1, e_2, \ldots, e_n) \in S$, *then* $r^{\delta}\big(\delta(f), \delta(e_1), \delta(e_2), \ldots, \delta(e_n); \delta(e)\big)$.

From the axiom (PCD_1) and Definition (denotation operation) follows the condition (PCD_{1c}) which can replace that axiom and which can be called the *principle of compositionality of conceptual denotation* (see Section 2.3, the principle (PCD_c)):

Fact 10 (compositionality of conceptual denotation)

(PCD_{1c}) *If* $e = f(e_1, e_2, \ldots, e_n) \in S$, *then*

$$r^{\delta}\big(\delta_c(\mu(f)), \delta_c(\mu(e_1)), \delta_c(\mu(e_2)), \ldots, \delta_c(\mu(e_n)); \delta_c(\mu(e))\big).$$

Let us note that on the level of the metatheory of ***TC*** the three axioms of compositionality: (PCS_1), (PCM_1) and (PCD_1) have one scheme (a verbal description of this scheme was given in Section 2.3):

Scheme of axioms (compositionality) For $h = \iota, \mu, \delta$

If $e = f(e_1, e_2, \ldots, e_n) \in S$, *then* $r^h\big(h(f), h(e_1), h(e_2), \ldots, h(e_n); h(e)\big)$.

Because relations r^ι, r^μ and r^δ are one-to-one functions, from the *Scheme of axioms* (compositionality) we can obtain the following counterparts of *Fact 1*:

Fact 11 *For* $h = \iota, \mu, \delta$

If $e = f(e_1, e_2, \ldots, e_n)$, $e' = f(e_1', e_2', \ldots, e_n') \in S$, *then*

$$h(e) = h(e') \text{ iff } h(f) = h(f') \text{ and } h(e_i) = h(e_i') \text{ for any } i = 1, 2, \ldots, n.$$

Accepting the following conventions (*hfe*) analogous to the convention (*fe*):

Definitional conventions For $h = \iota, \mu, \delta$ and for any $e, f, e_1, e_2, \ldots, e_n \in S$

(*hfe*)
$$h(e) = h(f)(h(e_1), h(e_2), \ldots, h(e_n)) \underset{df}{=}$$
$$\underset{df}{=} r^h(h(f), h(e_1), h(e_2), \ldots, h(e_n); h(e)),$$

we obtain the four following variants of the principles of compositionality corresponding to axioms (PCS_1), (PCM_1) and (PCD_1) and the condition (PCD_{1c}) of *Fact 10*:

Theorem (variants of compositionality)

(PCS_2) *If* $f(e_1, e_2, \ldots, e_n) \in S$, *then*
$$\iota\big(f(e_1, e_2, \ldots, e_n)\big) = \iota(f)\big(\iota(e_1), \iota(e_2), \ldots, \iota(e_n)\big),$$

(PCM_2) *If* $f(e_1, e_2, \ldots, e_n) \in S$, *then*
$$\mu\big(f(e_1, e_2, \ldots, e_n)\big) = \mu(f)\big(\mu(e_1), \mu(e_2), \ldots, \mu(e_n)\big),$$

(PCD_2) *If* $f(e_1, e_2, \ldots, e_n) \in S$, *then*
$$\delta\big(f(e_1, e_2, \ldots, e_n)\big) = \delta(f)\big(\delta(e_1), \delta(e_2), \ldots, \delta(e_n)\big),$$

(PCD_{2c}) *If* $f(e_1, e_2, \ldots, e_n) \in S$, *then* $\delta_c\big(\mu\big(f(e_1, e_2, \ldots, e_n)\big)\big) =$
$$= \delta_c\big(\mu(f)\big)\big(\delta_c\big(\mu(e_1)\big), \delta_c\big(\mu(e_2)\big), \ldots, \delta_c\big(\mu(e_n)\big)\big).$$

On the metatheoretical level the variants of the principles of compositionality (PCS_2), (PCM_2) and (PCD_2) have the same scheme:

Scheme (variants of compositionality) For $h = \iota, \mu, \delta$

(C_h) *If* $f(e_1, e_2, \ldots, e_n) \in S$, *then*
$$h\big(f(e_1, e_2, \ldots, e_n)\big) = h(f)\big(h(e_1), h(e_2), \ldots, h(e_n)\big),$$

which points out for an adequacy of compositional syntax and two-level (*intensional* and *extensional*) compositional semantics.

From the scheme (C_h) and *Fact 11* we get a scheme of counterparts of *Fact 2*:

Fact 12 *For* $h = \iota, \mu, \delta$ *and for any* $f(e_1, e_2, \ldots, e_n), f(e_1', e_2', \ldots, e_n') \in S$

If $h(e_i) = h(e_i')$ *for any* $i = 1, 2, \ldots, n$, *then*

$$h(f)\big(h(e_1), h(e_2), \ldots, h(e_n)\big) = h(f)\big(h(e_1'), h(e_2'), \ldots, h(e_n')\big).$$[9]

From *Fact 12* it follows that (cf. *Metatheorem* (compositionality) in Section 2.3): *Categorial indices, meanings and denotations, respectively, of the main functors of complex wfes of L, like these functors, are set-theoretical functions defined on tuples of categorial indices, meanings and denotations, respectively, of consecutive arguments of these functors*. From (C_h) it follows that: *Values of these functions are indices, meanings and denotations of wfes of L, respectively*. In this way we obtain the following fact (cf. *Fact 5*):

Fact 13

a. *Indices of functors are partial operations on the* ι*-image* $\iota(S)$ *of the set S,*
b. *Meanings of functors are partial operations on the* μ*-image* $\mu(S)$ *of the set S,*
c. *Denotations of functors are partial operations on the* δ*-image* $\delta(S)$ *of the set S.*

Let us observe that the principles (PCS_2), (PCM_2) and (PCD_2) serve to 'calculate' categorial indices, meanings, denotations, respectively, of *wfes* of the language *L*. In particular, from the principle (PCS_2) and in accordance with the principle (*SC*) of syntactic connection, for $e = f(e_1, e_2, \ldots, e_n) \in S$ and $\iota(e) = a$, $\iota(f) = b$, $\iota(e_i) = a_i (i = 1, 2, \ldots, n)$, we obtain, on the basis of our theory, the following reconstruction of the mentioned in Section 2.2 rule (*rc*) of cancellation of indices used by Ajdukiewicz (1935):

(*rc*') $$a = a/a_1a_2 \ldots a_n(a_1, a_2, \ldots, a_n).$$[10]

From *Fact 5* and *Fact 13* it follows that the variants (PCS_2), (PCM_2), (PCD_{2c}) and (PCD_2) of the principles of compositionality can be regarded as the requirements of homomorphisms between some algebras corresponding to *L*.[11]

Metatheorem (compositionality and homomorphisms)

The variants (PCS_2), (PCM_2), (PCD_{2c}) and (PCD_2) of the principles of compositionality are, respectively, conditions of homomorphisms of the following partial algebras:

$$\begin{aligned}
\boldsymbol{L} = \langle S, F_0; E_1^s\rangle \quad &and \quad \iota(\boldsymbol{L}) = \langle \iota(S), \iota(F_0); \iota(E_1^s)\rangle,\\
\boldsymbol{L} = \langle S, F_0; E_1^s\rangle \quad &and \quad \mu(\boldsymbol{L}) = \langle \mu(S), \mu(F_0); \mu(E_1^s)\rangle,\\
\mu(\boldsymbol{L}) = \langle \mu(S), \mu(F_0); \mu(E_1^s)\rangle \quad &and \quad \delta(\boldsymbol{L}) = \langle \delta(S), \delta(F_0); \delta(E_1^s)\rangle,\\
\boldsymbol{L} = \langle S, F_0; E_1^s\rangle \quad &and \quad \delta(\boldsymbol{L}) = \langle \delta(S), \delta(F_0); \delta(E_1^s)\rangle.
\end{aligned}$$

The algebras $\boldsymbol{L}$ and i $\iota(\boldsymbol{L})$ can be called *syntactic algebras* connected with *L*, while the algebras $\mu(\boldsymbol{L})$ and $\delta(\boldsymbol{L})$—*semantic algebras* corresponding to the syntactic algebra $\boldsymbol{L}$.

[9] *Fact 11* and *Fact 12* belong to the metatheory of ***TC***. They are schemas of three facts formulated separately for $h = \iota$, for $h = \mu$ and for $h = \delta$.

[10] See *Example 2*.

[11] See footnote 7.

Of course, in the algebra $\boldsymbol{L}$ all simple functors of the set F_0 are treated as partial operations on its carrier S of all *wfes* of L, and the set E_1^s of all simple expressions of L is the set of generators of $\boldsymbol{L}$. In every algebra $h(\boldsymbol{L})$, for $h = \iota, \mu, \delta$, the h-image $h(F_0)$ of the set F_0 consists of partial operations on the h-image $h(S)$ of the set S, and the h-image $h(E_1^s)$ of the set E_1^s is the set of generators. So, we have

Scheme (homomorphisms of algebras) For $h = \iota, \mu, \delta$

$$\boldsymbol{L} = \langle S, F_0; E_1^s \rangle \xrightarrow[h]{\text{hom}} h(\boldsymbol{L}) = \langle h(S), h(F_0); h(E_1^s) \rangle.$$

Let us note that described homomorphisms determined by the principles of compositionality (PCS_2), (PCM_2) and (PCD_2) can be regarded as homomorphisms between algebraic structures (partial algebras) of realities: *S* and *IND*, *S* and *CON*, and *S* and *ONT*, respectively.

Let us introduce to the theory ***TC*** three notions of a *model of L* and related to them three notions of *truthfulness* of any sentence of L.

Definitions (models)

a. The syntactic model of L is the syntactic algebra $\iota(\boldsymbol{L})$.

b. The intensional model of L is the semantic algebra $\mu(\boldsymbol{L})$.

c. The extensional model of L is the semantic algebra $\delta(\boldsymbol{L})$.

The models $\mu(\boldsymbol{L})$ and $\delta(\boldsymbol{L})$ are called *semantic models of L.*

The notions of the *truthfulness* of any sentence of L are defined by means of three new primitive notions of ***TC*** (see Section 2.3): the distinguished index T (understood as the index of all true sentences of L), the set T_μ (understood as the set of all such meanings of sentences of L that are true logical propositions, i.e. true conceptual states of affairs) and the set T_δ (understood as the set of all such denotations of sentences of L that are states of affairs that obtain).

Axioms (for T, T_μ and T_δ)

a. $T \in IND$,

b. $\emptyset \neq T_\mu \subseteq \mu(Sen)$,

c. $\emptyset \neq T_\delta \subseteq \delta(Sen)$.

Definitions (truthfulness) For any $e \in Sen$

(T_ι) e is a *true sentence in the model* $\iota(\boldsymbol{L})$ iff $\iota(e) \in T_\iota = \{T\}\}$,

(T_μ) e is a *true sentence in the model* $\mu(\boldsymbol{L})$ iff $\mu(e) \in T_\mu$,

(T_δ) e is a *true sentence in the model* $\delta(\boldsymbol{L})$ iff $\delta(e) \in T_\delta$.

All three definitions of a true sentence of L are analogous and correspond to the formulated in Section 2.3 **truth value principles**. They are represented by the same scheme (see Section 2.3):

Scheme of definitions (truthfulness) For $h = \iota, \mu, \delta$ and for any $e \in Sen$

e is true in the model $h(\boldsymbol{L})$ *if and only if* $h(e) \in T_h$.

Some relationships between the above-given notions of a true sentence are described by the following axioms:

Axioms (relations between threefold truthfulness) *For any* $e \in Sen$

(1) *If e is a true sentence in the model* $\iota(\boldsymbol{L})$*, then e is a true sentence in the model* $\mu(\boldsymbol{L})$.

(2) *If e is a true sentence in the model* $\mu(\boldsymbol{L})$*, then e is a true sentence in the model* $\delta(\boldsymbol{L})$.

The axiom (2) states that: *if the conceptual state of things, which is the meaning of a sentence, is true, then the state of things, which is the denotation of the sentence, obtains*. *Example 4*, given in Section 2.3, shows that the reverse implication to axiom (2) is not valid.

It is easy to see that *Axioms* (1) and (2) provide us to the following fact:

Fact 14

If $e \in Sen$ *and* $\iota(e) = s = T$ *(i.e. e is a true sentence), then e is a true sentence both in the intensional model* $\mu(\boldsymbol{L})$ *and in the extensional model* $\delta(\boldsymbol{L})$.

3.3 Fundamental Theorems of TC

In the previous Section 3.2 we concentrated on the formalization of the three principles of compositionality and their variants. In this section we shall formulate some important theorems of the presented theory ***TC***, which are some consequences of accepted (in ***TC***) axioms of compositionality and their variants. These theorems are connected with the truth value principle and some *replacement principles* (cf. Gamut 1991, p.11). Their formulations require the introduction of the syntactic notion of the *relation* (/) *of the replacement of a constituent of a given wfe of L*. This notion is introduced by means of the notion of the *relation* ($/^n$) *of the replacement of a constituent of the n-order* ($n \geq 0$). In recording the definitions of these notions we shall use the expressions $e' = e(p/p')$ and $e' = e(p/^n p')$ which we read: the expression e' is obtained from the expression e by the replacement of its constituent p, respectively, its constituent p of the n-th order, by the expression p'. The definition of the notion ($/^n$) is inductive. The notion of replacement is one of the most important notions of the syntactic part of ***TC***.

Definition 1 (replacement) Let $e, e' \in S$. Then

a. $e' = e(p/^0 p')$ iff $p = e$ and $p' = e'$,

b. $e' = e(p/^1 p')$ iff e and e' are some functor-argument expressions of the set $S \backslash E_1^s$ with the same number of arguments of their main functors and differ from one another only by the same syntactic position when in e occurs the constituent p and in e' occurs the constituent p'.

c. $e' = e(p/^{k+1}p')$ iff there are such q, q' that $e' = e(q/^k q')$ and $q' = q(p/^1 p')$
d. $e' = e(p/p')$ iff for some $n \geq 0$, $e' = e(p/^n p')$

By means of *Fact 11* we can easily obtain the one *fundamental syntactic replacement theorem* and two *fundamental semantic replacement theorems* which are the suitable substitutions of the following metatheorem of ***TC*** (see Section 2.3):

Metatheorem (replacement principles) *For* $h = \iota, \mu, \delta$
If $e, e' \in S$ *and* $e' = e(p/p')$*, then* $h(p) = h(p')$ *iff* $h(e) = h(e')$*,*

from which it follows that: *Two expressions have the same categorial index* (*the same syntactic category*), *the same meanings, the same denotation, respectively, if and only if by the replacement of one of them by the other in any wfe of L we obtain a wfe of L which has the same categorial index* (*the same syntactic category*), *the same meaning, the same denotation, respectively, as the expression from which it was derived.*[12]

The above given *Metatheorem* (replacement principles) is the immediate conclusion of the two following lemmas:

Lemma 1 *For any* $h = \iota, \mu, \delta$
If $e, e' \in S$, $e' = e(p/^n p')$ *and* $h(p) = h(p')$*, then* $h(e) = h(e')$.

Lemma 2 *For any* $h = \iota, \mu, \delta$
If $e, e' \in S$, $e' = e(p/^n p')$ *and* $h(e) = h(e')$*, then* $h(p) = h(p')$.

Proof (Lemma 1) Let h be one from the functions ι, μ, δ. Let us assume that $e, e' \in S$, $e' = e(p/^n p')$ and that $h(p) = h(p')$. We shall inductively prove that $h(e) = h(e')$.

a. If $n = 0$, then from part a. of *Definition* (replacement) it follows that $p = e$ and $p' = e'$, hence $h(e) = h(e')$ (because ι, μ and δ are functions).
b. If $n = 1$, then from part b. of *Definition* (replacement) it follows that for *wfes* e and e' there exist expressions $f, e_1, e_2, \ldots, e_n, f', e_1', e_2', \ldots, e_n'$ such that

$$e = f(e_1, e_2, \ldots, e_n) \text{ and } e' = f'(e_1', e_2', \ldots, e_n')$$

and either

b1. $p = f$, $p' = f'$ and for any $k = 1, 2, \ldots, n$, $e_k = e_k'$,

or

b2. $f = f'$ and for some j $(1 \leq j \leq n)$ $p = e_j$, $p' = e_j'$, and $e_k = e_k'$ for each $k \neq j$, $1 \leq k \leq n$.

[12] Let us note that *Theorem* (replacement principle) for $h = \iota$ was proved earlier (see Wybraniec-Skardowska [32], [34], without using the syntactic principle of compositionality, as the so-called *fundamental theorem of the theory of syntactic categories*.

Since $h(p) = h(p')$, in both cases: b1 and b2, we get: $h(f) = h(f')$ and $h(e_i) = h(e_i')$ for any $i = 1, 2, \ldots, n$. Hence by *Fact 11* $h(e) = h(e')$.
It remains to prove that if *Lemma 1* holds for the natural number k then it also holds for the natural number $n = k + 1$.

c. Let Lemma 1 be true for n = k. We shall prove that it is valid for n = k+1.
Let us assume that $e' = e(p/^{k+1}p')$. By part c. of *Definition* (replacement) we can infer that there are expressions q, q' such that $e' = e(q/^k q')$ and $q' = q(p/^1 p')$.

Because the lemma is valid for $n = 1$, we can state that $h(q) = h(q')$ and by the inductive assumption we get: $h(e) = h(e')$.

This concludes the inductive proof of *Lemma 1*. □

Proof (Lemma 2) Let h be one from the functions ι, μ, δ. Let $e, e' \in S$, $e' = e(p/^n p')$ and that $h(e) = h(e')$. We shall inductively prove that $h(p) = h(p')$.

a. If $n = 0$, then from *Definition* (replacement) we get $e = p$ and $e' = p'$, hence $h(p) = h(p')$.
b. If $n = 1$, then on the basis of part b. of *Definition* (replacement) there exist expressions $f, e_1, e_2, \ldots, e_n, f', e_1', e_2', \ldots, e_n'$ such that

$$e = f(e_1, e_2, \ldots, e_n) \text{ and } e' = f'(e_1', e_2', \ldots, e_n')$$

and p, p' are standing on the same position constituents of e and e', respectively, of the 1st-order. Because in accordance with the assumption that $h(e) = h(e')$ by *Fact 11* it follows that $h(f) = h(f')$ and $h(e_i) = h(e_i')$ for any $i = 1, 2, \ldots, n$, so $h(p) = h(p')$.
c. Let *Lemma 2* be true for $n = k$. Let us assume that $e' = e(p/^{k+1}p')$. By *Definition* (replacement) it follows that $e' = e(q/^k q')$ and $q' = q(p/^1 p')$. Because the lemma is valid for $n = k$, $h(q) = h(q')$, and because we proved the lemma for $n = 1$, we obtain $h(p) = h(p')$.

This concludes the inductive proof of *Lemma 2*. □

The following immediate conclusions of *Metatheorem* (replacement principles) are known in literature theorems. They were formulated as *Corollaries* (replacement principles) in Section 2.3:

Corollaries (replacement principles)

a. *If* $e, e' \in S$ *and* $e' = e(p/p')$ *and* $\mu(p) = \mu(p')$, *then* $\mu(e) = \mu(e')$,
b. *If* $e, e' \in S$ *and* $e' = e(p/p')$ *and* $\delta(p) = \delta(p')$, *then* $\delta(e) = \delta(e')$.

Using *Metatheorem* (replacement principles) for $h = \mu, \delta$, *Fact 8*, *Fact 9* and *Axiom* (relationship between δ and ι), on the level of metatheory of ***TC***, we can also get:

Fact 15

a. *If* $e, e' \in S$ *and* $e' = e(p/p')$ *and* $\mu(p) = \mu(p')$, *then* $\delta(e) = \delta(e')$ *and* $\iota(e) = \iota(e')$,

b. *If* $e, e' \in S$ *and* $e' = e(p/p')$ *and* $\mu(e) = \mu(e')$, *then* $\delta(p) = \delta(p')$ *and* $\iota(p) = \iota(p')$,

c. *If* $e, e' \in S$ *and* $e' = e(p/p')$ *and* $\delta(e) = \delta(e')$, *then* $\iota(p) = \iota(p')$,

d. *If* $e, e' \in S$ *and* $e' = e(p/p')$ *and* $\delta(p) = \delta(p')$, *then* $\iota(e) = \iota(e')$.

In connection with the truth value principle, *Metaheorem* (replacement principles) and *Scheme of definitions* (truthfulness) easily yield (see Section 2.3):

Metatheorem (referring to truth value principles) *For* $h = \iota, \mu, \delta$
If $e, e' \in Sen$ *and* $e' = e(p/p')$ *and* $h(p) = h(p')$, *then*
e is true in $h(\boldsymbol{L})$ *iff* e' *is true in* $h(\boldsymbol{L})$,

from which we obtain three important theorems of ***TC***. They together state that:
Replacing in any sentence its constituent by an expression which has the same index, the same meaning, the same denotation, respectively, never alters the truth value of the replaced sentence in the given syntactic, intensional, extensional, respectively, model.

It follows from *Metatheorem 1* for $h = \mu$, and *Axiom* (2) that:

Fact 16

If $e, e' \in Sen$ *and* $e' = e(p/p')$ *and* $\mu(p) = \mu(p')$, *then*
if e is true in $\mu(\boldsymbol{L})$, *then* e' *is true in* $\delta(\boldsymbol{L})$.

So: *Replacing in any true sentence in the intensional model its constituent by an expression that has the same meaning, we get a sentence which is true in the extensional model.*

From *Metatheorem 1* for $h = \mu$ and $h = \delta$, and for $e = p$ and $e' = p'$ we have:

Fact 17

a. *If* $e, e' \in Sen$ *and* $\mu(e) = \mu(e')$, *then* (*e is true in* $\mu(\boldsymbol{L})$ *iff* e' *is true in* $\mu(\boldsymbol{L})$).

b. *If* $e, e' \in Sen$ *and* $\delta(e) = \delta(e')$, *then* (*e is true in* $\delta(\boldsymbol{L})$ *iff* e' *is true in* $\delta(\boldsymbol{L})$).

So: *If two sentences have the same meaning, respectively, denotation then they have the same truth value in the intensional, respectively, extensional model.*

The recognition of the following

Metatheorem (referring to truth value principles) *For* $h = \iota, \mu, \delta$
If $e, e' \in Sen$ *and* $e' = e(p/p')$, *then*
$h(p) = h(p')$ *iff* (*e is true in* $h(\boldsymbol{L})$ *iff* e' *is true in* $h(\boldsymbol{L})$),

requires accepting the three axioms which are connected with Leibniz's principles (cf. Gerhard [11, p. 280], Janssen [17, p. 463]) and have the same scheme:

Leibniz's Axioms For $h = \iota, \mu, \delta$
If $e, e' \in Sen$ *and* $e' = e(p/p')$ *then*
if (*e is true in the model* $h(\boldsymbol{L})$ *iff* e' *is true in the model* $h(\boldsymbol{L})$), then $h(p) = h(p')$.

Leibniz's Axioms together state that: *If replacing in any sentence its constituent p by an expression p' never alters the truth value of the replaced sentence in the syntactic, in the intensional, in the extensional, respectively, model, then p and p' have the same categorial index, the same meaning, the same denotation, respectively.*

So, three theorems which follow from *Metatheorem 2* say that (cf. Hodges [14]: *Two expressions of the language L have the same categorial index (syntactic category, form), the same meaning (intension), the same denotation (extension), respectively, if and only if replacing one of them by another in any sentence never alters the truth value of the replaced sentence in the syntactic, intensional, extensional, respectively, model of L.*

4 The Role of the Formal Theory *TC* in Philosophy

Human thought, particularly 20$^{\text{th}}$-century thought, has given rise to a number of new scientific disciplines and theories that grew on the basis of certain old domains of knowledge. The specific language of the new disciplines is being enriched by new terms, while the meanings of many old ones have undergone appropriate transformation; their denotation and ontological reference have also changed. The functioning of old terms within the new developing branches of science obviously brings the risk of incoherence and inconsistence, hampering their development to a considerable degree. The development of new disciplines of knowledge and scientific theories is indeed tightly linked to the philosophical question of adequate linguistic representation, to shaping their languages in such a manner that they are able to perform the most significant function for science, i.e. the function of precise expression of knowledge adequate to the cognitive reality. The fulfilling of this function is conditioned by satisfying certain principles. What are these principles? What principles should be satisfied in particular so that the language of the old disciplines of knowledge could evolve as one capable of adequately expressing the knowledge which concerns the reality being learnt? Looking for an answer to the classical question of the attitude of the language towards cognition in the scientific sense, one can present the following hypothesis:

The principles of compositionality formulated in the present work are ones that every language of science should satisfy in the process of its formation.

The above hypothesis constitutes, at the same time, an answer to the second question posed above, that is the one concerning transformation of the language of the old fields of science into the language of the new science.

Similarly, with the formation of the language there is connected another problem that has been nurturing the philosophers of language: What is the mechanism that stabilizes the process of forming a language like? The theory of compositionality ***TC***, outlined in the present work, displays a triad-like mechanism of compositionality: language-cognition-reality. It is worth noticing that:

*A formal framing of **TC** and outlined of certain bases of its metatheory makes* *it possible to detect all inconsistencies that destabilize the process of language formation.*

The points of my results that show the benefits of the formal theoretical approach to the principles of compositionality, as proposed in the present work, can be expressed in the following form:

1. The formal approach to the principles of compositionality starts from the formation rules for well-formed language expressions whose meanings belong to the conceptual reality and denotations belong to the ontological reality. The formal framework allows us to discover on the level of metatheory of the language the unity of the formation of: (i) well-formed expressions, (ii) concepts and propositions corresponding to them in the conceptual reality and (iii) knowable beings corresponding to them in the ontological reality. It is possible because the principles of compositionality establish homomorphisms between structures of realities:
   - the language and the conceptual one,
   - the conceptual and the ontological one,
   - the language and the ontological one.
2. The unity of formation of entities in three related by compositionality realities is also the answer to the question as to what knowledge about the ontological reality can be expressed in the language.
3. The theory of compositionality also explains why the formation of concepts and propositions in the conceptual reality is not sufficient for expressing knowledge about ontological reality. The reason is the fact that the conceptual denotations of concepts and propositions could be incompatible with the denotations of their language counterparts. The compatibility of the denotations of language expressions and conceptual objects, which are the meanings of these expressions, is a criterion of the correctness of using the language knowing the ontological reality.
4. The metatheoretical results obtained on the basis of the formal theory ***TC*** are certain conditions of adequacy, that should be satisfied by objects of realities corresponding to one another: language, conceptual and ontological ones. As long as these conditions are met, the triad-like framing of relations between these realities unifies the aspects of research, to date, of two well-known trends in philosophy of language, called by Strawson [28] a communicative-intentional approach and a formal-semantic approach, respectively. The first trend concerns binary relations: language reality—conceptual reality (Grice, Austin) and conceptual reality—ontological reality (Husserl, Ajdukiewicz), whereas the other one deals with the relations of language reality—ontological reality. The bases of the metatheory of ***TC***, metatheory of syntax, meaning and denotation, formulated in the present paper, thus seem to soften the controversy surrounding the two trends in research related to an explanatory description of language.

Acknowledgements This paper was partially supported by a grant from The Netherlands Organisation for Scientific Research (NWO). I would like to thank NWO and all the people who helped me, in various ways, to conduct my scientific research, and who contributed to the preparation and improvement of this paper. I thank, in particular Professors Johan van Benthem, Jan van Eijck and Harrie de Swart.

I also appreciate the discussions I had with Dr. Edward Bryniarski, Dr. Filip Buekens, and Dr. Theo Janssen.

I want to thank Professor Wilfrid Hodges for his kind permission to use his work on the subject-matter and for the discussions.

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Chapter 8
On Meta-knowledge and Truth

Urszula Wybraniec-Skardowska

Abstract The paper deals with the problem of logical adequacy of language knowledge with cognition of reality. A logical explication of the concept of language knowledge conceived of as a kind of codified knowledge is taken into account in the paper. Formal considerations regarding the notions of meta-knowledge (logical knowledge about language knowledge) and truth are developed in the spirit of some ideas presented in the author's earlier papers (1991, 1998, 2001a,b, 2007a,b,c) treating about the notions of *meaning*, *denotation* and *truthfulness* of well-formed expressions (*wfes*) of any given categorial language. Three aspects connected with knowledge codified in language are considered, including: 1) syntax and two kinds of semantics: *intensional* and *extensional*, 2) three kinds of non-standard language models and 3) three notions of truthfulness of *wfes*. Adequacy of language knowledge to cognitive objects is understood as an agreement of truthfulness of sentences in these three models.

Key words: Meta-knowledge • Categorial syntax • Meaning • Denotation • Categorial semantics • Nonstandard models • Truthfulness • Language knowledge adequacy

Introduction

It is commonly realized that the term 'knowledge' is ambiguous. Speaking about knowledge, we disregard psychological knowledge offered through unit cognition, although it is from knowledge of that sort that verbal knowledge codified by means of language arose. Knowledge will be understood as an inter-subjective knowledge preserved in language, where it is formed and transferred to others in cognitive-

First published in: Makinson, D., Malinowski, J., Wanshing, H. (eds.) Towards Mathematical Philosophy. Trends in Logic, vol. 28, pp. 319—343. Springer, Berlin—Heidelberg (2009).

communicative acts. Representation of this knowledge is regarded as language knowledge.

For our purposes, in this paper we will consider three aspects of language knowledge: one syntactic and two semantic ones: *intensional* and *extensional*. The main aim of the paper is to answer the following well-known, classical philosophical problem:

`When is our language knowledge in agreement with our cognition of reality?`

In this paper, the problem is considered from a logical and mathematical perspective and is called: *the problem of logical adequacy of language knowledge*. We will consider it as:

1. an adequacy of syntax and two kinds of semantics,
2. concord between syntactic forms of language expressions and their two correlates: meanings and denotations, and
3. an agreement of three notions of truth: one syntactic and two semantic ones.

The main ideas of our approach to meta-knowledge (logical knowledge about language knowledge) and truthfulness of sentences in which knowledge is encoded will be outlined in Section 1. In Section 2 we will give the main assumptions of a formal-logical theory of syntax and semantics which are the basis for theoretical considerations, and in Section 3 we will define three notions of truthfulness of sentences. The paper ends with Section 4 containing a formulation of a general condition for adequacy of language knowledge with regard to these notions.

* * *

The paper is a result of many years of research conducted by the author and a summary of results obtained earlier [47–51, 59, 52–57]. The synthetic character of the article provides a strong motivation for the conceptual apparatus introduced further. The apparatus employs some formal-logical and mathematical tools. The synthesis being produced does not always allow detailed, verbal descriptions of particular formal fragments of the paper; nor can it allow for development of some formal parts. The author does, however, believe that the principal ideas and considerations in the paper will be clear to the reader.

1 Ideas

The notion of meta-knowledge is connected with the relationships defined by the triad: language-cognition-reality (see *Figure 1*).

Three different aspects, representing cognitively independent factors, are taken into account at constituting language L as a tool of communication in which knowledge is formed and transmitted. They are: syntactic, semantic and pragmatic factors.

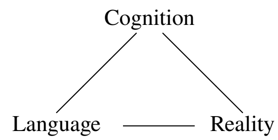

Fig. 1

Reliability of cognition of reality by means of language *L* and truthfulness of its sentences are given by an agreement of syntactic and two kinds of (*intensional* and *extensional*) semantic knowledge, which correspond to three levels of knowledge about the components of the triad (cf. Wybraniec-Skardowska 2007c).

According to *Figure 1*, following Frege [17], Husserl [25] and other modern followers of *gramatica speculativa*, the meta-knowledge is the *knowledge referring to three realities* (*spaces*):

1. *language reality* ***S*** (the set of all *well-formed expressions* of *L*), in which results of cognitive activities such as concepts and propositions are expressed,
2. *conceptual reality* ***C***, in which products of cognition of ordinary reality such as logical concepts and logical propositions (*meanings* of language expressions) are considered, and
3. *ontological reality* ***O*** which contains objects of cognition, among others, *denotations* of language expressions.

Applying the terms: 'lanuage reality', 'conceptual reality' and 'ontological reality' we aim at distinguishing some models of language *L* which are necessary to define three different notions of truthfulness of its sentences. Thus, we depart from the classical notion of 'Reality' as an object of cognitive research. In particular, speaking further about indexation reality ***I***, we mean certain metalinguistic space of objects (indices) serving the purpose of indication of categories of expressions of ***S***, categories of conceptual objects of ***C*** and ontological categories of objects of ***O***. The reality ***I*** forms categorial skeleton of language, conceptual and ontological realities.

Theoretical considerations are based on:

- ***syntax*** – describing ***language reality S*** related to *L*,
  and two kinds of semantics:
- ***intensional (conceptual) semantics*** – comprising the relationship between ***S*** and cognition – describing ***conceptual reality C***, and
- ***extensional (denotational) semantics*** – describing the relationships between *L* and ordinary reality – ***ontological reality O*** to which the language refers (see Wybraniec-Skardowska 1991, 1998, 2007a, b, c).

The theoretical consideration takes into account the **adequacy of the syntax and two kinds of semantics** of language *L*.

The ***language reality S*** is described by a theory of categorial syntax and the ***conceptual*** and ***ontological realities*** by its expansion to a theory of categorial semantics in which we can consider three kinds of ***models*** *of L*:

- one syntactic
  and
- two semantic (***intensional*** and ***extensional***).

For these models we can define three notions of ***truthfulness***:

- one syntactic
  and
- two semantic employing the notion of *meaning* (*intension*) and the notion of *denotation* (*extension*), respectively.

2 Main Assumptions of the Theory of Syntax and Semantics

2.1 Categorial Syntax and Categorial Semantics

Any syntactically characterized language L is fixed if the *set* $\boldsymbol{S}$ *of all well-formed expressions* (briefly *wfes*) is determined. L is given here on the *type*-level, where all *wfes* of $\boldsymbol{S}$ are treated as expression-*types*, i.e. some classes of concrete, material, physical, *identifiable* expression-*tokens* used in definite linguistic-situational contexts. Hence, *wefs* of $\boldsymbol{S}$ are here abstract ideal syntactic units of L.[1]

Language L can be exactly defined as a *categorial language*, i.e. language in which *wfes* are generated by a *categorial grammar* whose idea goes back to Ajdukiewicz (1935) and Polish tradition, and has a very long history.[2] Language L at the same time may be regarded as a linguistic scheme of ontological reality $\boldsymbol{O}$, keeping with Frege's ontological canons (1884), and of conceptual reality $\boldsymbol{C}$.

Considerations are formalized on the ground of author's general formal-logical theory of ***categorial syntax*** and ***categorial semantics*** (1985, 1991, 1998, 2001a,b, 2006).

Every compound expression of L has a ***functor-argument structure*** and both it and its constituents (the main part—the main ***functor*** and its complementary parts—***arguments of that functor***) have determined:

- the *syntactic*, the *conceptual* and the *ontological categories* defined by the *functions* ι_L, ι_C, ι_O *of the indications of categorial indices* assigned to them, respectively,

[1] Let us note that the differentiation *token-type* for linguistic objects originates from Charles Sanders Peirce (1931–1935). A formal theory of syntax based on this distinction is given in [49] and [51].

[2] The notion of categorial grammar originated from Ajdukiewicz (1935, 1960) was shaped by Bar-Hillel (1950, 1953, 1964). It was constructed under influence of Leśniewski's theory of semantic (syntactic) categories in his protothetics and ontology systems (1929, 1930), under Husserl's ideas of pure grammar (1900–1901), and under the influence of Russell's theory of logical types. The notion was considered by many authors: Lambek (1958, 1961), Montague (1970, 1974), Cresswell (1973, 1977), Buszkowski (1988, 1989), Marciszewski (1988), Simons (1989) and others. In this paper language L is generated by the so-called classical categorical grammar, the notion introduced and explicated by Buszkowski (1988, 1989) and the author (1985, 1989, 1991).

- *meanings* (*intensions*), defined by the *meaning operation* μ,
- *denotations* (*extensions*), defined by the *denotation operation* δ.

It should be underlined that since *wefs* of ***S*** are understood as some abstract syntactic units of *L*, meanings of *wfes* are not their mental signification and denotations of *wfes* are not the same as object references of their concrete, material expression-*tokens* (cf. [56]).

2.2 Three Referential Relationships of *Wfes*

We will concentrate on three referential relationships of *wfes of* ***S*** to three realities to which *wfes* refer:

- one syntactic: *metalinguistic* relationship connected with the above-mentioned ***indexation reality I***, and
- two semantic: ***conceptual*** (***intensional***) and ***denotational*** (***extensional***) relationships connected with realities ***C*** and ***O***, respectively. These relationships are illustrated in *Figure 2*.

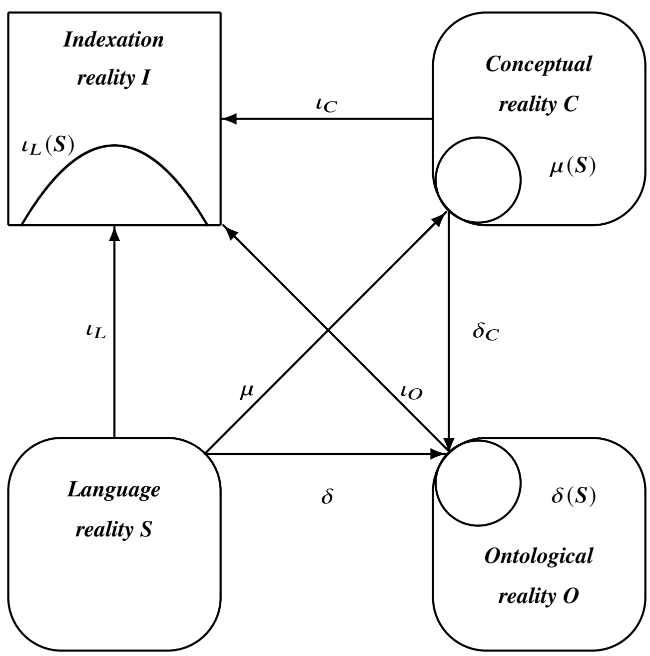

Fig. 2

2.3 Categorial Indices

The theory of categorial syntax is a theory formalising the basic principles of Leśniewski's theory of semantic (syntactic) categories (1929, 1930) improved by Ajdukiewicz (1935) by introducing *categorial indices* assigned to expressions of language L.

Categorial indices belong to the indexation reality $\boldsymbol{I}$ and are metalanguage expressions corresponding to expressions of language L. They serve to defining the set $\boldsymbol{S}$ of all *wfes* of L. The set $\boldsymbol{S}$ is defined according to the *principle* (SC) *of syntactic connection* referring to Ajdukiewicz's approach (1935).

(SC) is the rule establishing the correspondence between the index of any functor-argument expression of L and indices: the index of its main functor and indices of its successive arguments. It states that:

(SC) *The index of the main functor of a functor-argument expression is a complex (functoral) index formed of the index of that expression and the successive indices of the successive arguments of that functor.*

2.4 Syntactic Operations

In the theory the *functions:* ι_L*,* ι_C*,* ι_O *of the indications of categorial indices* are certain syntactic operations from reality $\boldsymbol{S}$ or fragments of realities $\boldsymbol{C}$ and $\boldsymbol{O}$ into reality $\boldsymbol{I}$, respectively, i.e.

- the syntactic operation $\iota_L : \boldsymbol{S} \to \boldsymbol{I}$,
- the ontological syntactic partial operation $\iota_O : \boldsymbol{O} \to \boldsymbol{I}$,
- the conceptual syntactic partial operation $\iota_C : \boldsymbol{C} \to \boldsymbol{I}$.

Categorial indices of $\boldsymbol{I}$ also serve to indicate *syntactic, conceptual* (*intensional*) and *ontological* (*denotational*) *categories*. These categories are included in realities $\boldsymbol{S}$, $\boldsymbol{C}$ and $\boldsymbol{O}$, respectively.

If $\xi \in \boldsymbol{I}$ then these categories are defined, respectively, as follows:

(1) $$Cat_{\xi} = \{e \in \boldsymbol{S} : \iota_L(e) = \xi\},$$

(2) $$Con_{\xi} = \{c \in \boldsymbol{C} : \iota_C(c) = \xi\},$$

(3) $$Ont_{\xi} = \{o \in \boldsymbol{O} : \iota_O(o) = \xi\}.$$

In order to define *semantic categories* indicated by categorial indices, and also by conceptual and ontological categories, we have to take into consideration two semantic relationships and use some semantic operations.

2.5 Semantic Operations

In the theory of categorial semantics such notions as ***meaning*** and ***denotation*** of a *wfe* of *L* are considered.

As it was illustrated in *Figure 2* we consider three semantic operations defining ***meanings*** and ***denotations*** *of wfes*:

- the *meaning operation* $\mu\colon \boldsymbol{S} \to \boldsymbol{C}$,
- the *denotation operation* $\delta\colon \boldsymbol{S} \to \boldsymbol{O}$,
- the *conceptual denotation operation* $\delta_C\colon \boldsymbol{C} \to \boldsymbol{O}$.

Let us note that the semantic functions: μ, δ and δ_C, are defined on abstract objects of $\boldsymbol{S}$ (on *wfes-types*) and of $\boldsymbol{C}$ (on meanings: logical concepts, logical propositions, operations on them, operations on these operations and so on), respectively.

The notion of ***meaning*** as a value of the *meaning operation* μ on any *wfe* of *L* is a semantic-pragmatic one and it is defined as a manner of using *wfes* of *L* by its users in connection to the concept of meaning deriving from L. Wittgenstein (1953) and, independently, from K. Ajdukiewicz (1931, 1934); see Wybraniec-Skardowska (2005, 2007 a,b). So, the notion of meaning of any *wfe* of *L* is an abstract entity.

We take the standpoint that any *wfe-type* of $\boldsymbol{S}$ has an established meaning which determines its denotation, even if such an expression is understood as an indexical one in natural language (e.g. 'he', 'this', 'today').[3] In this sense the approach presented here agrees with the classical Aristotelian position that the context has to be included somehow in the meaning; the manner of using *wfes* of *L* is in a way built into the meaning (cf. [56]).

The notion of ***meaning*** is differentiated from the notion of ***denotation*** in accordance with the distinction of G. Frege (1892) *Sinn* and *Bedeutung* and R. Carnap's distinction *intension-extension* (1947).

The *denotation operation* δ is defined as the composition of the operation μ and the *operation* δ_C *of conceptual denotation*, i.e.

$$(\delta_C) \qquad \delta(e) = \delta_C\big(\mu(e)\big) \quad \textit{for any } e \in \boldsymbol{S}.$$

So, we assume that denotation of the *wfe e* is determined by its meaning $\mu(e)$ and it is the value of the function δ_C of conceptual denotation for $\mu(e)$. Hence, we can state that:

If two wfes have the same meaning then they have the same denotation.

Formally:

[3] For example, let us note that the word-*type* 'today' understood as a class of all word-*tokens* identifiable with the word-*token*:

today

does not have a fixed meaning, but each of its sub-*types* consisting of *identifiable* tokens (utterances) of the word-*type* 'today' formulated on a given day is a meaningful *wfe-type* of English and determines by itself a denotation that is this day.

Fact 1 $\mu(e) = \mu(e') \;\Rightarrow\; \delta(e) = \delta(e') \quad$ *for any* $e, e' \in \boldsymbol{S}$.

It is well-known that the converse implication does not hold. So, the operation δ_C shows that something can differ ***meaning*** from ***denotation***.

2.6 Knowledge and Cognitive Objects

The image $\mu(\boldsymbol{S})$ of $\boldsymbol{S}$ determined by the meaning operation μ is a fragment of conceptual reality $\boldsymbol{C}$ and includes all meanings of *wfes* of language *L*, so all components of ***knowledge*** (logical notions, logical propositions, and operations between them, operations on the latter, and so on) and can be regarded as ***knowledge*** of relatively stable users of *L* about reality $\boldsymbol{O}$, codified by means of *wfes* of *L*.

The image $\delta(\boldsymbol{S})$ of $\boldsymbol{S}$ determined by the denotation operation δ is a fragment of ontological reality $\boldsymbol{O}$ and includes all denotations of *wfes* of language *L*, so all ***objects of cognition*** of $\boldsymbol{O}$ (things, states of things and operations between them) in cognitive-communicative process of cognition of reality $\boldsymbol{O}$ by relatively stable users of *L*.

We differentiate two kinds of semantic categories: intensional and extensional.

(4) $$Int_\xi = \{e \in \boldsymbol{S} \colon \mu(e) \in Con_\xi\}.$$

(5) $$Ext_\xi = \{e \in \boldsymbol{S} \colon \delta(e) \in Ont_\xi\}.$$

So, intensional categories consist of all meanings of *wfes* of the suitable conceptual categories, while extensional categories consist of all denotations of *wfes* belonging to suitable conceptual categories.

Adequacy of syntax and semantics required the syntactic and semantic agreement of *wfes* of *L*.

2.7 The Principles of Categorial Agreement

In accordance with Frege's-Husserl's-Leśniewski's and Suszko's understanding of the *adequacy of syntax and semantics of language L*, syntactic and semantic (*intensional and extensional*) categories with the same index should be the same (see Frege, 1879, 1892; Husserl, 1900–1901; Leśniewski, 1929, 1930; Suszko, 1958, 1960, 1964, 1968).

This correspondence of the ***categorial agreement*** (denoted by (*CA1*) and (*CA2*))—is here postulated by means of categorial indices that are the tool of coordination of language expressions and by two kinds of references that are assigned to them:

(*CA1*) $$Cat_{\xi} = Int_{\xi}.$$

(*CA2*) $$Cat_{\xi} = Ext_{\xi}.$$

From (1)–(5) and (*CA1*), (*CA2*) we get the following variants of the principles:

For any *wfe e*

(*C'A1*) $$e \in Cat_{\xi} \text{ iff } \mu(e) \in Con_{\xi}.$$

(*C'A2*) $$e \in Cat_{\xi} \text{ iff } \delta(e) \in Ont_{\xi}.$$

(*CA3*) $$\iota_L(e) = \iota_C\big(\mu(e)\big) = \iota_O\big(\delta(e)\big).$$

The condition (*C'A2*) is called the ***principle of categorial agreement*** and it is a formal notation the principle originated by Suszko (1958, 1960, 1964; cf. also Stanosz and Nowaczyk 1976).

So, according to innovative Frege's ideas, the problem of adequacy of syntax and semantics of L is solved if:

Well formed expressions of L belonging to the same syntactic category correspond with their denotations, and more generally—with their two kinds of references (meanings and denotations) that are assigned to them, which belong to the same ontological, and more generally—to the same conceptual and ontological category.

2.8 Algebraic Structures of Categorial Language and its Correlates

The essence of the approach proposed here is considering functors of language expressions of L as mathematical functions mapping some language expressions of $\boldsymbol{S}$ into language expressions of $\boldsymbol{S}$ and as functions which correspond to some set-theoretical functions on extralinguistic objects—indices, meanings and denotations of arguments of these functors.

All functors of L create the set $\boldsymbol{F}$ included in $\boldsymbol{S}$.

The systems:

$$\boldsymbol{L} = \langle \boldsymbol{S}, \boldsymbol{F} \rangle \quad \text{and} \quad \iota_L(\boldsymbol{L}) = \langle \iota_L(\boldsymbol{S}, \iota_L(\boldsymbol{f}) \rangle$$

are treated as some syntactic algebraic structures, while the systems:

$$\mu(\boldsymbol{L}) = \langle \mu(\boldsymbol{S}, \mu(\boldsymbol{F}) \rangle \quad \text{and} \quad \delta(\boldsymbol{L}) = \langle \delta(\boldsymbol{S}, \delta(\boldsymbol{F}) \rangle$$

can be treated as some semantic algebras.

All these algebras are partial algebras[4].

The functors of $\boldsymbol{F}$ differ from other, basic expressions of $\boldsymbol{S}$ in that they have indices formed from simpler ones.

If e is a complex functor-argument *wfe* with the index a and its main functor is $f \in \boldsymbol{F}$ and its successive arguments are $e1, e2, \ldots, en$ with indices $a_1, a_2, \ldots, a_n$, respectively, then the index b of f belonging to the set $\iota_L(\boldsymbol{F})$ is a *functoral* (complex) index formed from the index a and indices: $a_1, a_2, \ldots, a_n$ of its successive arguments.

The index b of the functor f can be noted as the quasi-fraction:

$$\iota_L(f) = b = a/a_1a_2 \ldots a_n = \iota_L(e)/\iota_L(e1)\iota_L(e2) \ldots \iota_L(en).$$

We will show that indices, meanings and denotations of functors of the set $\boldsymbol{F}$ are algebraic, partial functions defined on images $\iota_L(\boldsymbol{S})$, $\mu(\boldsymbol{S})$, $\delta(\boldsymbol{S})$ of the set $\boldsymbol{S}$, respectively.

First we will note that in accordance with the principle (*SC*) the main functor f of e can be treated as a set-theoretical function satisfying the following formula:

(*Catf*) $f \in Cat_{a/a_1a_2 \ldots a_n}$ iff

(f) $f\colon Cat_{a_1} \times Cat_{a_2} \times \ldots \times Cat_{a_n} \to Cat_a$ and $e = f(e1, e2, \ldots, en)$ and

(ι) $\iota_L(f)\colon \{(\iota_L(e1), \iota_L(e2), \ldots, \iota_L(en))\} \to \{\iota_L(e)\}$ and

(*PC1*) $\iota_L(e) = \iota_L(f(e1, e2, \ldots, en)) = \iota_L(f)(\iota_L(e1), \iota_L(e2), \ldots, \iota_L(en))$.

On the basis of the principles of categorical agreement we can state that semantic correlates of the functor f of the expression e are set-theoretical functions too, and they satisfy the following conditions:

(*Conf*) $\mu(f) \in Con_{a/a_1a_2 \ldots a_n}$ iff

(μ) $\mu(f)\colon Con_{a_1} \times Con_{a_2} \times \ldots \times Con_{a_n} \to Con_a$ and

(*PC2*) $\mu(e) = \mu(f(e1, e2, \ldots, en)) = \mu(f)(\mu(e1), \mu(e2), \ldots, \mu(en))$;

(*Ontf*) $\delta(f) \in Ont_{a/a_1a_2 \ldots a_n}$ iff

(δ) $\delta(f)\colon Ont_{a_1} \times Ont_{a_2} \times \ldots \times Ont_{a_n} \to Ont_a$ and

(*PC3*) $\delta(e) = \delta(f(e1, e2, \ldots, en)) = \delta(f)(\delta(e1), \delta(e2), \ldots, \delta(en))$.

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[4] Ideas about the algebraisation of language can already be found in Leibniz's papers. We can also find the algebraic approach to issues connected with syntax, semantics and compositionality in Montague's 'Universal Grammar' (1970) and in the papers of van Benthem (1980, 1981, 1984, 1986), Janssen (1996), Hendriks (2000). The difference between their approaches and the approach which we shall present here lies in the fact that carriers of the so-called *syntactic* and *semantic algebras* discussed in this paper include functors or, respectively, their suitable correlates, i.e. their ι_L- or some other semantic-function images. Simple functors and their suitable ι_L-, $\mu-$ or $\delta-$ images are partial operations of these algebras. They are set-theoretical functions determining these operations.

2.9 Compositionality

The conditions (*PC1*), (*PC2*) and (*PC3*) are called the principles of compositionality of syntactic forms, meaning and denotation, respectively (cf. Partee et al. 1990; Janssen 1996, 2001; Hodges 1996, 1998, 2001). They have the following scheme of *compositionality* (*Ch*) for the function h representing:

1) the function ι_L, 2) the operation μ and 3) the operation δ:

$$(Ch)\quad h(e) = h\big(f(e1, e2, \ldots, en)\big) = h(f)\big(h(e1), h(e2), \ldots, h(en)\big).$$

The scheme (*Ch*) says that: *1) the index, 2) the meaning and 3) the denotation of the main functor of the functor-argument expression e is a function defined on 1) indices, 2) meanings and 3) denotations of successive arguments of this functor.*

The suitable variants of compositionality are some requirement of homomorphisms between the mentioned partial algebras:

$$\boldsymbol{L} = \langle \boldsymbol{S}, \boldsymbol{F} \rangle \quad \xrightarrow[\iota_L]{hom} \quad \iota_L(\boldsymbol{L}) = \langle \iota_L(\boldsymbol{S}), \iota_L(\boldsymbol{F}) \rangle,$$

$$\boldsymbol{L} = \langle \boldsymbol{S}, \boldsymbol{F} \rangle \quad \xrightarrow[\mu]{hom} \quad \mu(\boldsymbol{L}) = \langle \mu(\boldsymbol{S}), \mu(\boldsymbol{F}) \rangle,$$

$$\boldsymbol{L} = \langle \boldsymbol{S}, \boldsymbol{F} \rangle \quad \xrightarrow[\delta]{hom} \quad \delta(\boldsymbol{L}) = \langle \delta(\boldsymbol{S}), \delta(\boldsymbol{F}) \rangle.$$

2.10 Concord Between Syntactic Forms and Their Correlates

On the level of metatheory, it is possible to show the agreement between syntactic structures of *wfes* of the language reality $\boldsymbol{S}$ and their correlates in the conceptual reality $\boldsymbol{C}$ and in the ontological reality $\boldsymbol{O}$.

As *wfes* have *function-argument form*: all the functors (all their correlates) precede their arguments (correlates of their arguments as appropriate). Then the algebraic approach to language expressions corresponds to the tree method.

Example 1 Let us consider two *wfes* of language of arithmetic:

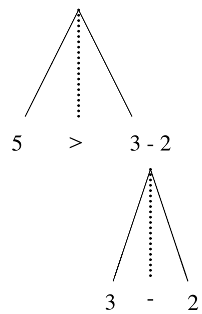
$$\text{a. } 5 > 3 - 2 \quad \text{and} \quad \text{b. } 3 - 2 > -1$$

First we present parenthetical recordings a'. and b'. for a. and b. and diagrams of trees meant to explicate them. Diagrams Ta and Tb show a natural, phrasal, natural *functorial analysis* of these expressions. The dotted lines show functors.

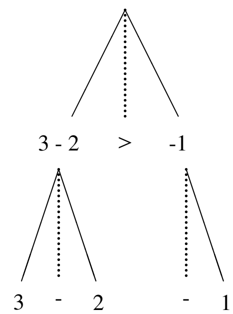
Appropriate function-argument recordings $\mathrm{a}_{f.}$ and $\mathrm{b}_{f.}$ and diagrams of trees: $T\mathrm{a}_{f.}$, $T\mathrm{b}_{f.}$ show "functional analysis" of expressions a., b. in Ajdukiewicz's prefix notation.

Let us note that the functorial analysis of a. and b. given here provides functional-argument expressions $\mathrm{a}_{f.}$ and $\mathrm{b}_{f.}$. It is unambiguously determined due to the semantic (denotational and intensional) functions of the signs '>' and '-': the first is a sign of

*T*a. 5 > 3 - 2



*T*b. 3 - 2 > -1

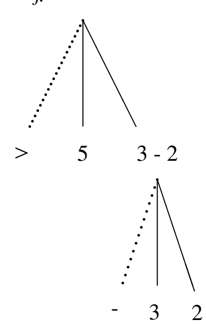


a’. (5) > ((3) - (2))

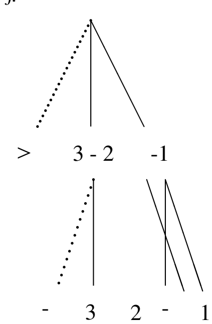
b’. ((3) - (2)) > (- (1))

two-argument operation on numbers, the second one in a. denotes a two-argument number operation, while in b. it also denotes a one-argument operation.[5] The mentioned signs, as functors, and thus as functions on signs of numbers, have as many arguments as their semantic correlates have.

*T*a$_{f.}$ 5 > 3 - 2



*T*b$_{f.}$ 3 - 2 > -1



a$_{f.}$ > (5, - (3, 2))

b$_{f.}$ > (- (3, 2), - (1))

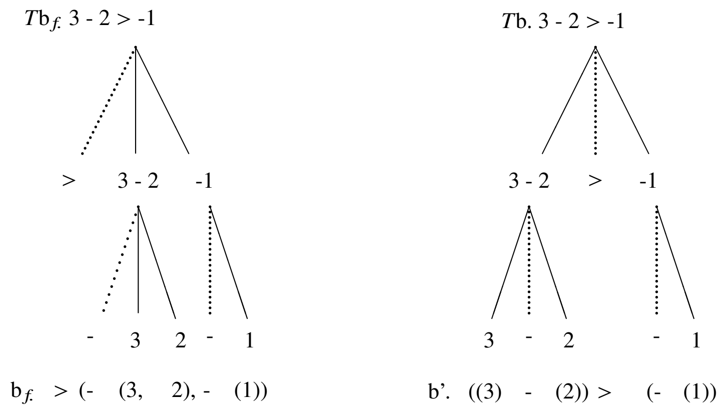
[5] Unambiguous “functorial analysis” is a feature of the languages of formal sciences. In relation to natural languages the analysis depends on linguistic intuition and often allows for a variety of possibilities (see e.g. Marciszewski 1981).

In this conception we do not state that “functorial analysis” of linguistic expressions must be determined unambiguously but we accept the statement that it is connected with expressions of a determined functor-argument structure.

Let us also note that traditional phrasal linguistic analysis, formalized by Chomsky (1957) in his grammars of phrasal structures, takes into consideration grammatical phrasal analysis and only two parts of functoral parsing of expressions.

Let us consider, for instance, the expression a. and its functorial analysis illustrated by a derivation tree in Chomsky’s sense.

Comparison of tree method and algebraic method based on compositionality shows one-to-one correspondence of constituents of any *wfe* of L with correlates in order to form and transmit our knowledge on reality $\boldsymbol{O}$ represented by L (see diagrams of trees Tb$_{f.}$ and Tb. of the expression b. and corresponding to them diagrams of trees of categorial indices $T_{\iota_L}(\mathrm{b}_f)$ and $T_{\iota_L}(\mathrm{b})$ of b.).



Let us note that from the principle (*PC1*) and in accordance with the principle (*SC*), for $e = f(e_1, e_2, \ldots, e_n) \in S$ and $\iota_L(e) = a$, $\iota_L(f) = b$, $\iota_L(e_i) = a_i$ ($i = 1, 2, \ldots, n$), we obtain, on the basis of our theory, the following reconstruction of the *rule of cancellation of indices* used by Ajdukiewicz (1935):

(*rc*) $\quad a/a_1a_2\ldots a_n(a_1, a_2, \ldots, a_n) = a$

The agreement between syntactic forms of *wfes* and their correlates is very important whenever we want to know whether our knowledge represented in language L is adequate to our cognition of reality.

Let e is any *wfe* of L and C_e is the *set of all constituents* of e. The concord between syntactic structure of e and its correlates is possible because the tree $\boldsymbol{T}(C_e)$ of constituents of e is isomorphic with trees:

$\boldsymbol{T}(\iota_L(C_e))$ of indices of all constituents of e,

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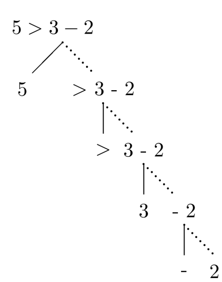

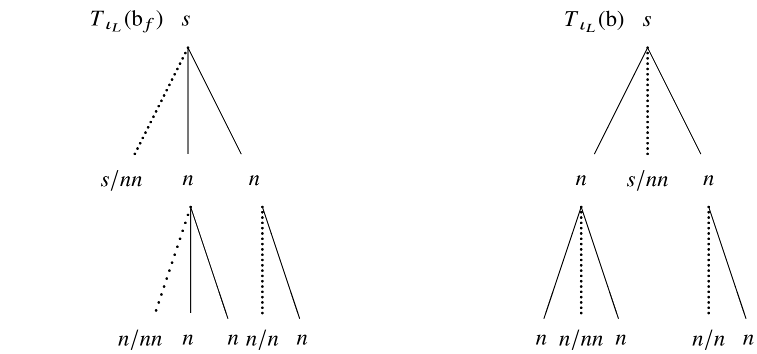

$\iota_L(\mathrm{b}_f)$. $s/nn(n/nn(n,n),(n/n,(n))$ $\iota_L(\mathrm{b'})$. $((n)n/nn(n))s/nn(n/n(n))$

$\boldsymbol{T}(\mu(C_e))$ of all meanings of all constituents of e and
$\boldsymbol{T}(\delta(C_e))$ of all denotations of those constituents.

These trees are formally defined as graphs by means of the set C_e and corresponding to it sets: $\iota_L(C_e)$, $\mu(C_e)$ and $\delta(C_e)$ of all constituents that are appropriate correlates of constituents of e. So,

$$\boldsymbol{T}(C_e) = \langle C_e, \approx> \rangle,$$
$$\boldsymbol{T}(h(C_e)) = \langle h(C_e), \approx>_h \rangle \ \text{ for } h = \iota_L, \mu, \delta,$$

where $\approx>$ is a linear ordering *relation of an earlier syntactic position in e* defined by means of the *relation* $\rightarrow$ *of syntactical subordination* (see Ajdukiewicz, 1960); $\approx>_h$ is h-image of the relation $\approx>$.

The mentioned isomorphisms of tree graphs are established by the functions h mapping every constituent of e in C_e that occupies in e a fixed syntactic position (place) onto its h-correlate that occupies in $h(e)$ the same position (place).

All notions introduced in this part can be defined formally.

Definition 1 (constituent of an expression *e*)

a. $t \in C_e^0 \iff e = t$.

A *constituent of order zero of a given wfe e* is equal to the expression.

b. $t \in C_e^1 \iff \exists_{n \geq 1} \exists_{f, t_0, t_1, \ldots, t_n \in S} \left(e = f(t_0, t_1, \ldots, t_n) \wedge \exists_{0 \leq j \leq n}(t = f \vee t = t_j)\right)$.

t is a constituent of the first order of a given expression e iff e is a functor-argument expression and t is equal to the main functor of the expression or to one of its arguments.

c. $k > 0 \Rightarrow \left(t \in C_e^{k+1} \Leftrightarrow \exists_{r \in C_e^k}\, t \in C_r^1\right)$.

A *constituent of* $k+1$*-th order* of e, where $k > 0$, is a constituent of the first order of a constituent of *k-th* order of e.

d. $t \in C_e \iff \exists_n\, t \in C_e^n$.

A *constituent of a given expression* is a constituent of a finite order of that expression.

Definition 2 (constituent of *e* with the fixed syntactic position)

a. $t \in C_e^{(j_1)} \;\Leftrightarrow\; e$ is a functor-argument expression $\wedge$ t is j_1-*th* constituent of C_e^1.

b. $k > 0 \;\Rightarrow\; \left(t \in C_e^{(j_1, j_2, \ldots, j_{k+1})} \Leftrightarrow t\right.$ is equal to the j_{k+1}-*th* constituent of a constituent of the set $\left.C_e^{(j_1, j_2, \ldots, j_k)}\right)$.

Definition 3 (relation of an earlier syntactic position in *e*)

a. $s \rightarrow s'$ iff $\exists_{k,j}\left(s \in C_e^k \wedge s' \in C_e^j \wedge k \leq j\right)$.

b. $s \approx> s'$ iff $s \rightarrow s' \vee$
$$\left(\exists_{j_1, j_2, \ldots, j_k, n, m}\left(s \in C_e^{(j_1, j_2, \ldots, j_k, n)} \wedge s' \in C_e^{(j_1, j_2, \ldots, j_k, m)} \wedge n < m\right)\right).$$

s has in e an earlier syntactic position than s' iff s, s' are constituents of e and either s has the order lesser than or equal to the order of s' or s and s' are simultaneously constituents of some part e' of e with the same order $k > 0$ but s has in e' the position n while s'—the position $m > n$.

On the basis of the principles of compositionality it is easy to prove

Theorem 1 *For $h = \iota_L, \mu, \delta$*

$$\boldsymbol{T}(C_e) = \langle C_e, \approx> \rangle \quad \xrightarrow[isom]{h} \quad \boldsymbol{T}(h(C_e)) = \langle h(C_e), \approx>_h \rangle.$$

Uniformity of algebraic approach and tree approach allows to compare knowledge reference to three kinds of realities and to take into account the problem of its adequacy. It is connected with the problem of truthfulness of sentences of L representing knowledge.

3 Three Notions of Truthfulness

3.1 Three Kinds of Models of Language and the Notion of Truth

We have treated the language reality $\boldsymbol{S}$ and corresponding to it ι_L-, $\mu-$ and $\delta-$images of $\boldsymbol{S}$, i.e. $\iota_L(\boldsymbol{S})$—a fragment of the indexation reality $\boldsymbol{I}$, $\mu(\boldsymbol{S})$—a fragment of the conceptual reality $\boldsymbol{C}$ and $\delta(\boldsymbol{S})$—a fragment of the ontological reality $\boldsymbol{O}$ as some algebraic structures, as some partial algebras.

Let us distinguish in $\boldsymbol{S}$ the set of all *sentences* of L. Models of L are non-standard ***models***. They are the three mentioned algebraic structures (partial algebras) given as homomorphic images of algebraic structure $\boldsymbol{L} = \langle \boldsymbol{S}, \boldsymbol{F} \rangle$ of language L:

$$\iota_L(\boldsymbol{L}) = \langle \iota_L(\boldsymbol{S}), \iota_L(\boldsymbol{F}) \rangle,$$
$$\mu(\boldsymbol{L}) = \langle \mu(\boldsymbol{S}), \mu(\boldsymbol{F}) \rangle,$$
$$\delta(\boldsymbol{L}) = \langle \delta(\boldsymbol{S}), \delta(\boldsymbol{F}) \rangle.$$

They are determined by the fragments $\iota_L(\boldsymbol{S})$, $\mu(\boldsymbol{S})$ and $\delta(\boldsymbol{S})$ of the realities $\boldsymbol{I}$, $\boldsymbol{C}$ and $\boldsymbol{O}$, respectively. The first of them $\iota_L(\boldsymbol{L})$ is *syntactic* one and the next two are *semantic*: $\mu(\boldsymbol{L})$—*intensional* and $\delta(\boldsymbol{L})$—*extensional*.

3.2 Three Notions of Truthfulness

For the three models $\iota_L(\boldsymbol{L})$, $\mu(\boldsymbol{L})$ and $\delta(\boldsymbol{L})$ of the language L we define three notions of truthfulness. For this purpose we distinguish three nonempty subsets $T\iota_L, T\mu, T\delta$ of realities $\boldsymbol{I}$, $\boldsymbol{C}$ and $\boldsymbol{O}$, respectively:

- $T\iota_L$ consists only of the index of any true sentences,
- $T\mu$ consists of all meanings of sentences of L that are true logical propositions and
- $T\delta$ consists of all denotations of sentences of L that are states of affairs that obtain.

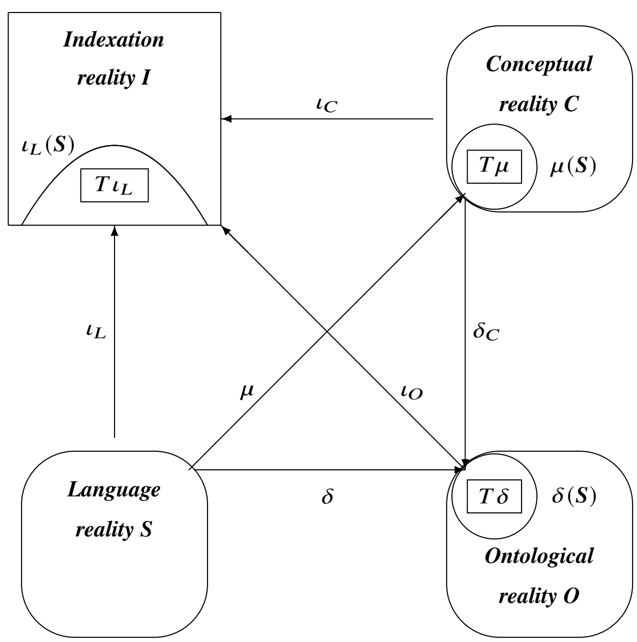

Fig. 3

All of the three definitions of a true sentence in one of the models $\iota_L(\boldsymbol{L})$, $\mu(\boldsymbol{L})$ and $\delta(\boldsymbol{L})$ of L are analogous and are substitutions of the following definition scheme:

Scheme of definitions (truthfulness): *For* $h = \iota, \mu$ and δ

The sentence e *is true in the model* $h(\boldsymbol{L})$ *iff* $h(e) \in Th$.

The definitions of a true sentence correspond to the ***truth value principle*** (cf. W. Hodges 1996). An expansion of the principle could be formulated as follows:

The correlate of a sentence (i.e. its index, meaning or denotation, respectively) *determines whether or not it is true in a suitable model.*

The three definitions of a true sentence can be given as follows:

- e *is syntactically true* iff $\iota_L(e) \in T\iota_L$,
- e *is intensionally true* iff $\mu(e) \in T\mu$,
- e *is extensionally true* iff $\delta(e) \in T\delta$.

From the above scheme of definitions of truthfulness of sentences we can easily get the following scheme of theorems:

Metatheorem 1 *For* $h = \iota_L, \mu, \delta$

If e, e' *are sentences and* $h(e) = h(e')$, *then* e *is true in* $h(\boldsymbol{L})$ *iff* e' *is true in* $h(\boldsymbol{L})$.

Metatheorem 1 is the scheme of the following three theorems our formal theory:

1. *If we have two sentences with the same index then they are syntactically true iff they have the same truth value in the syntactic model, i.e. their index is the index of all true sentences,*
2. *If two sentences have the same meanings then they are intensionally true iff they have the same truth value in the intensional model, i.e. their meanings are true logical propositions,*
3. *If two sentences have the same denotation then they have the same truth value in the extensional model, i.e. their denotations are the states of affairs that obtain.*

3.3 Reliability of Cognition of Reality

The main purpose of cognition is aiming at an agreement of truthfulness of sentences that are results of cognition in all three models: $\iota_L(\boldsymbol{L})$, $\mu(\boldsymbol{L})$ and $\delta(\boldsymbol{L})$ (cf. *Figure 3*).

Let us note that if a sentence is true in the extensional model $\delta(\boldsymbol{L})$ then it does not have to be true in the remaining models. So, in particular, a deductive knowledge that is included in the conceptual reality $\boldsymbol{C}$ cannot be in agreement with knowledge referring to the ontological reality $\boldsymbol{O}$. There can be true sentences in $\delta(\boldsymbol{L})$ that are not deduced from the knowledge accepted earlier and cannot be true in the intensional model $\mu(\boldsymbol{L})$.

Considerations outlined in this paper point to a new aspect of the importance of Gödel's Incompleteness Theorem (1931): it explains why language cognition of reality illustrated by *Figure 3* can be incomplete.

Justification of these statements requires introducing some new notions.

3.4 Operations of Replacement

The most important theorems which follow from the principles of compositionality (*PC1*), (*PC2*) and (*PC3*) use the syntactic notion of the three-argument *operation* π *of replacement of a constituent of a given wfe of L*. The operation π is defined by means of the *operation* π^n *of replacement of the constituents of n-th order*. The expressions $e' = \pi(p', p, e)$ and $e' = \pi^n(p', p, e)$ are read: *the expression* e' *is a result of replacement of the constituent p, respectively, the constituent p of n-th order, of e by the expression* p'. The definition of the operation π^n is inductive (see Wybraniec-Skardowska, 1991).

Definition 4 (operation of replacement) Let $e, e', p, p' \in \boldsymbol{S}$. Then

a. $e' = \pi^0(p', p, e)$ iff $p = e$ *and* $p' = e'$,

b. $e' = \pi^1(p', p, e')$ iff *e and* e' *are some functor-argument expressions of the set* $\boldsymbol{S}$ *with the same number of arguments of their main functors and differ from one another only by the same syntactic position when in e occurs the constituent p and in* e' *occurs the constituent* p',

c. $e' = \pi^{k+1}(p', p, e)$ iff $\exists_{q,q' \in S}\ \big(\pi^k(q', q, e)\ \ \&\ \ q' = \pi^1(p', p, q)\big)$,

d. $e' = \pi(p', p, e)$ iff $\exists_{n \geq 0}\ (e' = \pi^n(p', p, e))$.

We can define the operations of replacement $h(\pi)$ for the correlates *wfes* of $\boldsymbol{S}$ $(h = \iota_L, \mu, \delta)$ in an analogous manner.

3.5 The Most Important Theorems

In this part we will give some theorems of our deductive, formal-logical theory of syntax and semantics. They are logical consequences of the above-given definitions and principles of compositionality formulated earlier.

It is easy to justify three principles of compositionality with respect to the operation π. They are a substitution of the following metatheorem:

Metatheorem 2 (compositionality with respect to π) *For* $h = \iota_L, \mu, \delta$

$$(PC_\pi) \qquad h(\pi(p', p, e)) = h(\pi)(h(p'), h(p), h(e)).$$

We can also easily state that the theorems that we get from the next scheme are valid:

Metatheorem 3 (homomorphisms of replacement systems) *For* $h = \iota_L, \mu, \delta$

$$\langle \boldsymbol{S}, \pi, T\rangle \quad \xrightarrow[hom]{h} \quad \langle h(\boldsymbol{S}), h(\pi), h(T)\rangle,$$

where T is the set of all true sentences of L.

We can postulate that $T\iota_L = \iota_L(T)$, $T\mu = \mu(T)$ and $T\delta = \delta(T)$.
Now, we will present theorems called replacement theorems.

Fact 2 *For* $h = \iota_L, \mu, \delta$
If $e = f(e_1, e_2, \ldots, e_n)$, $e' = f'(e'_1, e'_2, \ldots, e'_n) \in \boldsymbol{S}$
then $h(e) = h(e')$ *iff* $h(f) = h(f')$ *and* $h(e_i) = h(e'_i)$ *for any* $i = 1, \ldots, n$.

By means of Fact 2 we can easily obtain the one *fundamental syntactic replacement theorem* and two *fundamental semantic replacement theorems* which are the suitable substitutions of the following metatheorem of our theory:

Metatheorem 4 (replacement principles) *For* $h = \iota_L, \mu, \delta$

$$\textit{If } e, e' \in \boldsymbol{S} \textit{ and } e' = \pi(p', p, e) \textit{ then } (h(p) = h(p') \textit{ iff } h(e) = h(e')).$$

So: *Two expressions have the same correlate* (*the same categorial index—the syntactic category, the same meanings, the same denotation, respectively*) *if and only if by the replacement of one of them by the other in any wfe of L we obtain a wfe of L which has the same correlate* (*the same categorial index—the same syntactic category, the same meaning, the same denotation, respectively*), *as the expression from which it was derived.*

Corollary 1 *If* $e, e' \in \boldsymbol{S}$ *and* $e' = \pi(p', p, e)$, *then*

$$\begin{aligned} \exists_\zeta(p, p' \in Cat_\zeta) \ \textit{ iff } \ \exists_\zeta(e, e' \in Cat_\zeta), \\ \exists_\zeta(p, p' \in Con_\zeta) \ \textit{ iff } \ \exists_\zeta(e, e' \in Con_\zeta), \\ \exists_\zeta(p, p' \in Ont_\zeta) \ \textit{ iff } \ \exists_\zeta(e, e' \in Ont_\zeta). \end{aligned}$$

The next theorems are connected with the true value principles.

Metatheorem 5 (referring to the truth value principles) *For* $h = \iota, \mu, \delta$

$$\textit{If } e, e' \textit{ are sentences of L and } e' = \pi(p', p', e) \textit{ and } h(p) = h(p'),$$
$$\textit{then } e \textit{ is true in } h(\boldsymbol{L}) \textit{ iff } e' \textit{ is true in } h(\boldsymbol{L}).$$

The three theorems that we get from the above metatheorem together state that:
Replacing in any sentence its constituent by an expression which has the same correlate (*the same index, the same meaning, the same denotation, respectively*), *never alters the truth value of the replaced sentence in the given syntactic, intensional, extensional, respectively, model.*

If we accept the following axiom:

Axiom *If e is a sentence and $\mu(e) \in T\mu$, then $\delta(e) \in T\delta$,*

then from the above metatheorem, for $h = \mu$, we get:

Fact 3 *If e, e' are sentences, $e' = \pi(p', p', e)$ and $\mu(p) = \mu(p')$, then if e is true in $\mu(\boldsymbol{L})$ then e' is true in $\delta(\boldsymbol{L})$.*

So: *Replacing in any true sentence in the intensional model its constituent by an expression that has the same meaning, we get a sentence which is true in the extensional model.*

Stronger Metatheorem (referring to truth value principles) *For $h = \iota, \mu, \delta$.*

If e, e' are sentences and $e' = \pi(p', p, e)$, then
$h(p) = h(p')$ iff (e is true in $h(\boldsymbol{L})$ iff e' is true in $h(\boldsymbol{L})$).

The recognition of the above metatheorem requires accepting the three axioms which are connected with Leibniz's principles (cf. Gerhard 1890, p. 280, Janssen 1996, p. 463) and have the same scheme:

Scheme of Leibniz's Axioms *For $h = \iota, \mu, \delta$.*

If e, e' are sentences and $e' = \pi(p', p, e)$, then
if (e is true in $h(\boldsymbol{L})$ iff e' is true in $h(\boldsymbol{L})$) then $h(p) = h(p')$.

Leibniz's Axioms together state that:

If replacing in any sentence its constituent p by an expression p' never alters the truth value of the replaced sentence in the syntactic, in the intensional, in the extensional, respectively, model, then p and p' have the same categorial index, the same meaning, the same denotation, respectively.

Three theorems which follow from *Stronger Metatheorem* (referring to truth value principles) together say that (cf. Hodges 1996):

Two expressions of the language L have the same correlates (*the same categorial index—syntactic category or form, the same meaning—intension, the same denotation—extension, respectively*), *if and only if replacing one of them by another in any sentence never alters the truth value of the replaced sentence in the syntactic, intensional, extensional, respectively, model of the language L.*

4 Final Remarks

- We have tried to give a description of meta-knowledge in connection with three references of knowledge to:
  - language,
  - conceptual reality and
  - ontological reality.

- Thanks to it we could define three kinds of models of language and three kinds of truthfulness in these models.
- These models are not standard models; in particular the notion of truth does not employ the notions of satisfaction and valuation of variables used for formalized languages.
- Adequacy of knowledge to cognitive objects of reality is understood as an agreement of truthfulness in these three models.
- It is possible to give a generalization of the notion of meta-knowledge in communication systems in order to apply it to knowledge in text systems but the solution of this problem requires more time and is solved by my co-worker Edward Bryniarski.

Acknowledgements I thank unknown referees and my colleagues Edward Bryniarski and Marek Magdziak for their stimulating comments, remarks and suggestions which allowed me to complete or improve some fragments of my paper.

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Chapter 9
On Language Adequacy

Urszula Wybraniec-Skardowska

Abstract The paper concentrates on the problem of adequate reflection of fragments of reality *via* expressions of language and inter-subjective knowledge about these fragments, called here, in brief, ***language adequacy***. This problem is formulated in several aspects, the most general one being: *the compatibility of the language syntax with its bi-level semantics*: *intensional and extensional*. In this paper, various aspects of language adequacy find their logical explication on the ground of the formal-logical theory of syntax ***T*** of any categorial language *L* generated by the so-called classical categorial grammar, and also on the ground of its extension to the bi-level, *intensional* and *extensional* semantic-pragmatic theory ***ST*** for *L*. In ***T***, according to the *token-type* distinction of Ch. S. Peirce, *L* is characterized first as a language of well-formed expression-*tokens* (*wfe-tokens*)—material, concrete objects—and then as a language of *wfe-types*—abstract objects, classes of *wfe-tokens*. In ***ST*** the semantic-pragmatic notions of meaning and interpretation for *wfe-types* of *L* of *intensional semantics* and the notion of denotation of *extensional semantics* for *wfe-types* and constituents of knowledge are formalized. These notions allow formulating a postulate (an axiom of categorial adequacy) from which follow all the most important conditions of the language adequacy, including the above, and a structural one connected with three principles of compositionality.

Key words: *Token-type* distinction • Categorial grammar • Intensional semantics • Meaning • Interpretation • Constituent of knowledge • Extensional semantics • Referring • Ontological object • Denotation • Categorization • Compatibility of syntax and semantics • Algebraic models • Truth • Compositionality • Communication

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First published in: Studies in Logic, Grammar and Rhetoric **40** (53), 257–292 (2015).

1 Introduction

In the process of cognizing reality, we acquire knowledge about it, gathering knowledge in a certain system and representing it in some sign system, usually a language-based one (see Diagram 1). In the language system of representation, this knowledge is processed, leading to a new knowledge about the reality of interest to us, thus to a better cognition of it.

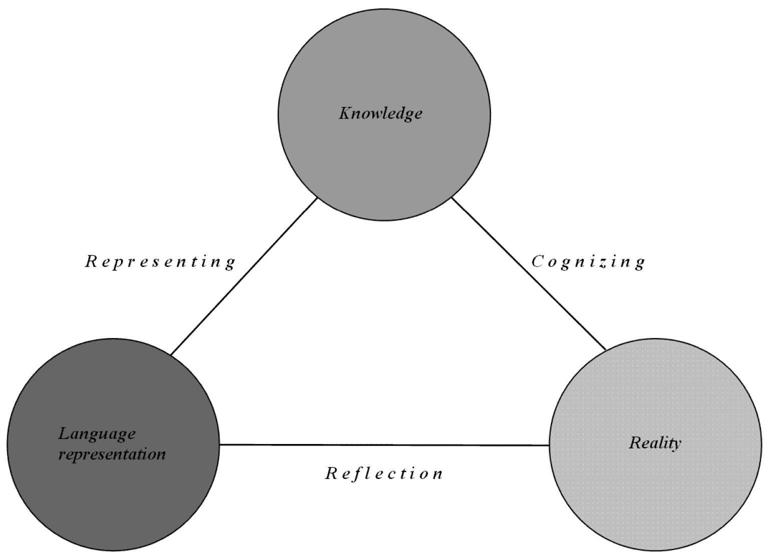

Diagram 1: Representation of knowledge

The effectiveness of cognition is dependent on mutual relations between the three elements of the triad:

Language – Knowledge – Reality.

This is obtained when the syntax of language reflects, in an adequate manner, its semantics, and thus the suitable fragment of the cognized reality, as well as the knowledge being the result of inter-subjective cognition.

2 The Problem Area of Language Adequacy

The problem of language adequacy in relation to cognition is, beside that of adequacy of cognition, one of the central, traditional philosophical problems. The question of adequate reflection of fragments of reality *via* expressions of language and inter-subjective knowledge about these fragments is called here, in brief, ***language adequacy***. This problem can be formulated in several aspects, the most general one being: *the compatibility of the language syntax with its bi-level semantics*:

intensional semantics,

in which to expressions of language correspond—as constituents of knowledge—their meanings (*intensions*),

and

extensional semantics,

in which to these expressions correspond—as ontological objects of reality—their object references (*references*) and denotations (*extensions*).

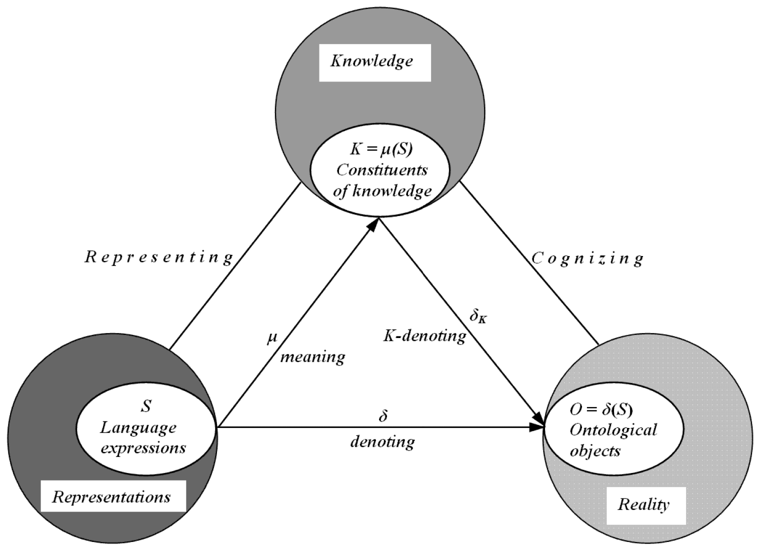

Diagram 2: Semantic adequacy

The problem area of *language adequacy* (discussed in Section 4 of this paper) will be considered formally on the ground of the logical theory of syntax ***T*** (outlined in Section 3.1 of this paper) and its extension to the semantic theory ***ST*** (characterized in Section 3.2 of this paper), describing the bi-level semantics of categorial language. The theories ***T*** and ***ST*** are presented in the author's papers ([65–68], [71–78]) and are built in the spirit of Leśniewski's [38, 39] and Ajdukiewicz's [3, 4] theories of syntactic (semantic) categories, with simultaneous retention of Frege's ontological canons [22].[1]

[1] Independently of Leśniewski, a theory of syntactic category was presented and developed for the needs of the so-called combinatory logics by Curry [19, 20]. A somewhat complementary theory to ***ST*** is the so-called *Transparent Intensional Logic* presented by Duži, Jaspersen and Materna [21].

In the theory of syntax ***T***, the notion of a *well-formed expression* (meaningful) and that of the *syntactic category* are defined. In the semantic theory ***ST***—with reference to Frege's [23] distinction: *Sinn-Bedeutung*, or Carnap's [16]: *intension–extension*—such notions as: *meaning* (*intension*) of a meaningful expression, its *interpretation*, its *object reference* (*reference*), as well as *denotation* (*extension*) are defined, and also two notions of *semantic category*: the notion of *intensional category* and that of *extensional category* are introduced.[2]

The *meanings* (*intensions*) of rational expressions are treated as certain constituents of inter-subjective knowledge: logical notions, logical judgments, operations on such judgments or on such notions, on the former and the latter, on other operations.

Object references (*references*) of language expressions, and also constituents of knowledge, are objects of the cognized reality: individuals, states of things, operations on the indicated objects, and the like. *Denotations* (*extensions*) of meaningful expressions of language and constituents of knowledge are sets of such objects. ***Semantic adequacy***—the agreement of these denotations—is illustrated in Diagram 2.

Semantic adequacy is one of the aspects of language adequacy, taking into account the bi-level semantic.

3 An Outline of the Theory of Categorial Language

In this paper, various aspects of language adequacy find their logical explication on the ground of the formal-logical theory ***T*** of any categorial language, describing its syntax, and also on the ground of its extension to the theory ***ST***, describing the bi-level semantics (*intensional* and *extensional*) for such a language. The theories ***S*** and ***ST*** are based on first order predicate logic and set theory.

Let *L* be any, yet—in our consideration—an established language characterized *categorially*. The language *L* is defined when the set ***S'*** of all its *well-formed expressions*, and its subset ***S*** of *meaningful expressions*, is determined, satisfying the requirements of *categorial syntax* and *categorial semantics*.

3.1 Categorial Syntax – Theory *T*

3.1.1 General Characteristics of the *Categorial Language*

The theory ***T*** of the syntax of the language *L* is built on the basis of Husserl's idea of pure grammar [32] and in accordance with the general assumptions of Leśniewski's [38, 39] and Ajdukiewicz's [3, 4] theories of syntactic (semantic) categories. The language *L*, syntactically characterized in it, can be precisely defined as a *categorial*

[2] Let us pay attention in this place to the fact that the notions of *intension* and *extension* introduced in the theory ***ST*** differ considerably from those introduced in Montague's pragmatics [**?**].

language; that is, as a language all of whose *well-formed expressions* of the set ***S'*** (briefly *wfes* of ***S'***) are generated by a *categorial grammar*, the idea of which originated from Ajdukiewicz [3, 4] and which has already had a long history (see Bar-Hillel [6–8]; Lambek [36, 37]; Hiż [26–28]; Montague [42, 43]; Geach [**?**]; Cresswell [17, 18]; Gamut [25]; Marciszewski [41]; Buszkowski [10–13]; van Benthem [62**?**]; Simons [50, 51]; Tałasiewicz [57]; Duži, Jespersen & Materna [21]; Wybraniec-Skardowska [65, 67, 73]; Wybraniec-Skardowska & Rogalski [79]). On the basis of the theory ***T***, it is possible to reconstruct the classical *categorial grammar*.

A characteristic feature of the categorial language *L*, generated by the classical categorial grammar, is that each *wfe* of the set ***S'*** has a *functor-argument* structure, that it is possible to distinguish in it the main part—the so-called *main functor*, and the other parts—called *arguments* of this functor, yet each constituent of a meaningful expression of ***S*** has a determined *syntactic category* and *semantic categories* (*extensional* and *intensional*), can have a *meaning* assigned to it, and thus also a *category of knowledge* (the *category of constituents of knowledge*), and also *denotation*, and thus—an *ontological category* (the *category of ontological objects*).

The syntactic categories of *wfes* of *L*, and also the indicated categories corresponding to them, are determined by attributing to them *categorial* indices (*types*) which were introduced by Ajdukiewicz [3] into logical semiotics with the aim of determining the syntactic role of expressions and of examining their *syntactic* connection, in compliance with the *principle of syntactic connection* (*Sc*), which will be discussed below.

The categorial indices are, however, useful not only while establishing and examining *syntactic connection* of *wfes* of *L*. They appear simultaneously in the role of a tool coordinating meaningful expressions and metalanguage objects (see Suszko [53–55]; Ajdukiewicz [4]; Stanosz & Nowaczyk [52]); they also serve to describe ***categorial adequacy***—a main aspect of language adequacy.

The principle of syntactic connection (*Sc*), which makes reference to the principle applied by Ajdukiewicz, can be formulated freely in the following way:

(*Sc*) *If e is a functor-argument expression of the language L, f is the main functor of the expression e, and $e_1, e_2, \ldots, e_n$ $(n \geq 1)$ are subsequent arguments of the functor f, then if a is a categorial index of the expression e, while $a_1, a_2, \ldots, a_n$ are categorial indices of subsequent arguments of the functor f, then the categorial index of the functor f is formed out of the index a of the expression e, which the functor forms, as well as out of the subsequent indexes $a_1, a_2, \ldots, a_n$ arguments of this functor.*

In the *quasi*-fractional notation applied by Ajdukiewicz, the index of the functor f is the following fraction:

$$a/a_1a_2\ldots a_n.$$

And thus, for example, the expression:

Warsaw is the capital of Poland,

in which 'is' is distinguished as its main functor, with the categorial index s assigned to sentences, satisfies the principle (*Sc*), since the functor 'is', with the subsequent arguments which are the names '*Warsaw*' and '*the capital of Poland*', and the categorial indices n and n, as a sentence-forming functor with arguments being names, has the categorial index s/nn, formed out of the index s and the indices of its subsequent arguments.

In the formal definition of a *wfe*, it is required that each complex functor-argument constituent of the given expression should satisfy the principle (*Sc*). As regards our instance of the sentence, this principle must also be satisfied by the expression '*the capital of Poland*'.

The set ***S'*** of all *wfes* of L is defined in the axiomatic theory $\boldsymbol{T}$ of categorial syntax, with the help of primitive notions of this theory.

3.1.2 Two Levels of Formalization of Categorial Syntax

Formalization of the theory $\boldsymbol{T}$ runs on two levels. In accordance with the distinction by Peirce [46]: *token-type* of signs,[3] the double ontological nature of signs of the language L is taken into account in it.

On the ground of the theory $\boldsymbol{T}$, the language L is syntactically characterized as:

- a language of expression-*tokens*—on the first level, the *level of tokens*

and

- a language of expression-*types*—on the other level, the *level of types*.[4]

Tokens of the signs of L are a starting point in formalization of the theory $\boldsymbol{T}$. They are intuitively understood as concrete, material, empirical, spanning over time and space, objects perceived through senses. Usually, though not necessarily, they are graphical signs. They can appear on paper, on a school blackboard, on computer screens. They can be illuminations of light on advertising billboards, smoke signals, arrangements of objects, e.g., configurations of stars, compositions of flowers, stones, and the like. The method of conceptualization, which leads to formalization of knowledge about language within an independently fixed temporal range of considerations and a freely-established area of language-based communication, allows isolating (extracting) a set-*universe* of sign-*tokens* which are used in this communication.

Types of the signs of the language L are its secondary objects. In the theory $\boldsymbol{T}$ they are defined by means of *tokens* of a determined *universe*. They are abstract objects, whose concrete realizations are *tokens*. The *types* are understood as set-theoretical sets, classes of *tokens* remaining in a broadly-understood *indentifiability relation* between one another (defined, obviously, on the given *universe*). The notion of *identifiability* is the result of the conceptualization process (*notioning*) of knowledge,

[3] In order to distinguish signs in such a way, Carnap [15] applies the terms "sign-event" and "sign-disign".

[4] The theory $\boldsymbol{T}$ can be, in an equivalent way, formalized—first—on the *level of types*, and then—on the *level of tokens* (see Wybraniec-Skardowska [66, 67], *Final Remarks*), representing the *Platonizing* approach to the description of language syntax.

with the same manner of use of sign-*tokens* (making use of these signs) in a selected fragment of the system of communication between human beings.

3.1.3 The Foundations of the Formal Theory *T* – the Level of Tokens

The theory ***T*** built on the level of tokens is an axiomatic theory, including the ***concretistic*** categorial characteristics of the language L. Its primitive notions on the *level of tokens* are:

- the universe U of all sign-*tokens* of L,
- the binary relation $\sim$ of *identifiability of tokens* of the set U,
- the ternary relation c of concatenation, defined on *tokens* of the set U,
- the initial vocabulary V_0^1 of L,
- the auxiliary initial vocabulary V_0^2 for L, containing a set of categorial indices,
- the binary relation i of indicating indices to word-*tokens* of L,
- the binary relation r_1 of forming functor-argument expression-*tokens* of L,
- the binary relation r_2 of forming indices of functor-*tokens* of L.

The system of axioms which characterize the primitive notions of the theory ***T*** are given in the author's works [65–67, 73]. It is postulated about the *universum* U of sign-*tokens* of L that it is a non-empty set, about the relation $\sim$ of *identifiability*—that it is an equivalence relation in the universe U. It is not assumed about the *concatenation* relation c that it is a function: a concatenation of two *tokens* is a complex *token*, formed out of two tokens *identifiable* with them, respectively, and also each token *identifiable* with it. For example, the *concatenation* of two word-*tokens*:

semiotics

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the right and the left ones, of different fonts, thickness and size of type, is both:
the complex word-*token*:

Logical Semiotics

and the word-*token*:

LOGICAL SEMIOTICS

and also each word-*token identifiable* with the two complex words.

As regards the initial vocabularies V_0^1 and V_0^2 of L, it is postulated that they are non-empty subsets of the *universe* U, out of which the set W^1 of all word-*tokens* of L and the set W^2 of all auxiliary word-*tokens* for I are formed, respectively. The initial vocabularies may contain structural symbols, e.g., brackets or punctuation marks.

Sets of word-tokens W^1 and W^2 are defined as set-theoretical intersections of all sets including, respectively, the vocabulary V_0^1 and the auxiliary vocabulary V_0^2, which are closed with respect to the concatenation relation c.

The relation i of indicating the indices of word-*tokens* of L (in short: the *indexation* or *typification* relation) is defined on the subset of the Cartesian product $W^1 \times W^2$:

$$i \subseteq W^1 \times W^2.$$

Its left domain is a set of word-*tokens* possessing categorial indices (*types*), the right one—the set I of indices of such words.[5] This relation is not a function—however, to a word-*token* there corresponds, with the accuracy to *identifiability*, one categorial index of the set I.

We read the expression $i(w, a)$: *a is a categorial index* (*type*) *of the word-token* w.

The proper vocabulary V^1 of L is defined as a set of word-*tokens* of the initial vocabulary V_0^1 possessing a categorial index (*type*), whereas the proper vocabulary V^2 auxiliary to L—as a set of auxiliary word-*tokens* of the vocabulary V_0^2, being indices of words of the vocabulary V^1.

The left domains of the relations r_1 and r_2 are, respectively, a set of finite tuples of word-*tokens* of the set W^1 possessing indices from the set I and a set of finite tuples of indices of such words. The relations r_1 and r_2 are not functions, but assign to any finite tuple of word-*tokens* possessing indexes, or, respectively, to any tuple of indices of word-*tokens*, with the accurate to *identifiability*, one complex word-*token* called *functor-argument expression-token*, or, respectively, one *index of the functor*.

We read the expression

$$(e) \qquad r_1(f, e_1, e_2, \ldots, e_n; e)$$

as follows: e is a functor-argument expression-*token* composed of the main functor f and its subsequent arguments $e_1, e_2, \ldots, e_n$.

The expression

$$(i) \qquad r_2(a, a_1, a_2, \ldots, a_n; af)$$

is read: a_f is an index of the functor f, formed out of the index a and subsequent indexes $a_1, a_2, \ldots, a_n$.

The expression e in (e) can be treated as a schema representing any expression-*tokens* of L, formed from the functor f and its subsequent arguments $e_1, e_2, \ldots, e_n$, irrespective of the concrete rules of the syntax of L, independent of the position which these constituents take in the expression e, and independent of the applied notation, type, etc.

Similarly, the expression a_f in (i) replaces any index of the functor formed from the index a and indices $a_1, a_2, \ldots, a_n$, irrespective of the applied notation of the functor indices, e.g., *quasi*-fractional, or with the use of brackets, or still any other, applied by researchers of categorial grammars.

The set E^1_{f-a} of all the functor-argument expression-*tokens* of the language L (complex expressions of L) is defined as the right domain of the relation r_1, and the set E^2_{f-a} of all the indices of functors (complex indices)—as the right domain of the relation r_2, contained in the set I of index-*tokens*.

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[5] In the literature dealing with categorial grammars, it is accepted to refer to the categorial indices introduced by Ajdukiewicz as *types*. The categorial indices should not, obviously, be mistaken for the indices introduced by Montague (1970b) and applied as the ordered tuples of agent's factors which constitute the context of usage of expressions.

The set E^1 of all the expression-*tokens* of L and the set E^2 of all their index-*tokens* are defined, for $k = 1, 2$ as the following sets:

$$E^k = V^k \cup E^k_{f-a}.$$

In the theory $\boldsymbol{T}$, the principle (*Sc*) of syntactic connection for the functor-argument expression e, satisfying the formula (e), is formalized by means of the formula:

$$(Sc_e) \qquad \forall_{1\leq j\leq n} \left(i(f, a_f) \ \wedge \ i(e_j, a_j) \ \wedge \ i(e, a)\right) \ \Rightarrow \ (i).$$

In accordance with axioms of the theory $\boldsymbol{T}$, for the expression e satisfying the formula (e) we obtain the following rule corresponding to that of cancelation of indices, applied by Ajdukiewicz [3] to examine the syntactic connection of expressions:

$$\forall_{1\leq j\leq n} \left((i) \ \wedge \ i(f, a_f) \ \wedge \ i(e_j, a_j)\right) \ \Rightarrow \ i(e, a).$$

In the notation applied by Ajdukiewicz to this formal rule there corresponds the following rule of cancelation indices (*types*):

$$a/a_1a_2 \ldots a_n(a_1, a_2, \ldots, a_n)) \to a.$$

In our given example of the expression:

Warsaw is the capital of Poland

and checking whether it is a sentence, the rule takes the form:

$$s/n \; n \; (n, n) \to s.$$

A reconstruction of the classical categorial grammar on the ground of the theory $\boldsymbol{T}$ is the system of notions:

$$\Gamma = \langle U, c, \sim, V^1, V^2, i, r_1, r_2, (Sc)\rangle,$$

generating the set *S*' of all *wfe-tokens* of *L*. The set *S*' is defined as follows:

Definition 1 (*the set of all well-formed expression-tokens*)

$$S\text{'} = \bigcap \left\{X \subseteq E^1 : V^1 \subseteq X \ \wedge \ \forall e \, \forall f, e_1, e_2, \ldots, e_n \in X(e) \ \wedge \ (Sc_e) \Rightarrow e \in X\right\}.$$

The set *S*' is, thus, the smallest set of expression-*tokens* containing the vocabulary V^1 of the language L and each of its functor-argument expression e such that, providing the structure (e) is preserved, satisfies the principle of syntactic connection (Sc_e).

Each *wfe-token* of *S*' possesses a categorial index which determines its *syntactic category*. On the *level of tokens*, the syntactic categories of *wfe-tokens* are determined by categorial indices of the set *I* and are defined as sets of *wfes* possessing, with the exactitude to *identifiability*, the same categorial index.

Definition 2 (*syntactic category with the index ξ*)

$$SC_{\xi} = \{e \in S\text{'}: i(e, a) \Rightarrow a \sim \xi\}.$$

It is assumed that the set S' is a sum of the set B of *basic expressions* of L (with simple indices (*types*) of the auxiliary vocabulary V^2) and the set of *functors* F (with complex indices of the set E^2_{f-a}).

The basic expressions of categorial languages are usually sentences and names. The category of sentences is typically indicated by means of the index s, and the category of names by means of the index n. Complex indices which are assigned to functors are formed from these indices. And so, for instance, the index s/nn is attributed to sentence-forming functors of two nominal arguments (thus, in particular, the functor '*is*' in the sentence: *Warsaw is the capital of Poland*); on the other hand, the index n/n—to name-forming functors of one nominal argument (thus, in particular, the functor '*the capital of*' in the name '*the capital of Poland*').

The semiotic-logical characteristics of L on the *level of tokens* is insufficient. *Tokens* of expressions indeed appear in the practice of human communication, in acts of language-based communication; nevertheless, in order to explain the very notion of language communication itself in logical pragmatics, it is necessary to have expression-*types*, and in logical semantics expression-*types* serve to define the notions of *meaning* and *denotation* of language expressions, in logical syntax—to describe grammatical rules.

3.1.4 Foundations of the Formal Theory *T* – the Level of Types

Each set of *tokens Set*, introduced into formalization of the theory ***T*** on the *level of tokens*, has—in the theory ***T*** on the *level of types*—its dual counterpart ***Set***, being a quotient family of equivalent classes of the $\sim$ *identifiability* relation, with representatives from the set *Set*. Thus:

$$\boldsymbol{Set} = Set/\!\sim\, = \{C\colon \exists e \in Set\ (C = [e]_{\sim})\}.$$

Each relation r, introduced into the theory ***T*** on the *level of tokens* and defined on the *tokens*, has—in the theory ***T*** on the *level of types*—its dual counterpart $\boldsymbol{r}$, determined on types and defined in the following way:

$$\boldsymbol{r}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n) \ \Leftrightarrow\ \exists e_1, e_2, \ldots, e_n$$
$$\big(\boldsymbol{e}_1 = [e_1]_{\sim}\ \wedge\ \boldsymbol{e}_2 = [e_2]_{\sim}\ \wedge \ldots \wedge\ \boldsymbol{e}_n = [e_n]_{\sim}\ \wedge\ r(e_1, e_2, \ldots, e_n)\big),\ n > 1.$$

We will give some characteristics of the theory ***T*** on the *level of types*. Let us note that on the *level of types*

- to the relation of *identifiability* $\sim$, determined on *tokens*, there corresponds the relation = of equality of *types* represented by these *tokens*,[6]
- to all the other relations of the *level of tokens*, on the *level of types*, there correspond relevant relations on *types*, being set-theoretical functions;
- all the dual counterparts of axioms, definitions and theorems of the theory $\boldsymbol{T}$, binding on the *level of tokens*, are theorems of the theory $\boldsymbol{T}$ on the *level of types*;
- the categorial language L on the *level of types* is characterized by categorial grammar

$$\boldsymbol{\Gamma} = \langle \boldsymbol{U}, \boldsymbol{c}, \boldsymbol{V}^1, \boldsymbol{V}^2, \boldsymbol{i}, \boldsymbol{r}_1, \boldsymbol{r}_2, (\boldsymbol{Sc})\rangle,$$

  the notions of which are sets of the *types* $\boldsymbol{U}$, $\boldsymbol{V}^1$, $\boldsymbol{V}^2$ and relation-functions $\boldsymbol{i}, \boldsymbol{r}_1, \boldsymbol{r}_2$ determined for *types*,
- the principle (***Sc***) of syntactic connection for functor-argument expression-*types* is defined in a way similar to that for principle (*Sc*) for expression-*tokens*;
- the set ***S***' of all *wfe-types* (the set of equivalence classes, of *identifiable wfe-tokens* of the set *S*') is generated by grammar Γ;
- the functor-argument expression-*type* $\boldsymbol{e}$ satisfying the formula:

$$(\boldsymbol{e}) \qquad \boldsymbol{r}_1(\boldsymbol{f}, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n; \boldsymbol{e}),$$

  and thus built from *types*: the main functor $\boldsymbol{f}$ and its arguments $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, can be written in the function-argument form:

$$(\boldsymbol{e}_{\boldsymbol{f}}) \qquad \boldsymbol{e} = \boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n),$$

  because each functor $\boldsymbol{f}$ can be treated as a set-theoretical function determined on finite tuples of word-*types* of the set $\boldsymbol{W}^1$, possessing categorial index-*types*, and taking values in this set (precisely in its subset $\boldsymbol{E}^1_{f-a}$);
- If the expression-*type* $\boldsymbol{e}$, having the form ($\boldsymbol{e}_{\boldsymbol{f}}$), is a *wfe-type* (belongs to the set ***S***'), then in compliance with the principle of syntactic connection ($\boldsymbol{Sc}_e$) the index of its main functor $\boldsymbol{f}$, formed out of the index $\boldsymbol{a}$ of the expression $\boldsymbol{e}$ and of the subsequent indices $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ of the subsequent arguments $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ of the functor $\boldsymbol{f}$, can be written in the quasi-fractional form:

$$(\boldsymbol{i}_{\boldsymbol{f}}) \qquad \boldsymbol{i}(\boldsymbol{f}) = \boldsymbol{i}(\boldsymbol{e})/\boldsymbol{i}(\boldsymbol{e}_1)\boldsymbol{i}(\boldsymbol{e}_2)\ldots\boldsymbol{i}(\boldsymbol{e}_n) = \boldsymbol{a}/\boldsymbol{a}_1\boldsymbol{a}_2\ldots\boldsymbol{a}_n.$$

- Syntactic categories of expression-*types* of the set ***S***' are determined by index-*types* and by the indexation function i restricted to the set ***S***'—the function $\boldsymbol{i}_{\boldsymbol{S}}$:

$$\boldsymbol{SC}_{\xi} = \left\{\boldsymbol{e} \in \boldsymbol{S}'\colon \boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}) = \xi\right\}.$$

The ***syntactic category with the index*** ξ is a set of all *wfe-types* which have the categorial index ξ.

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[6] Since, from the pragmatic point of view, equiform *tokens* may not be identifiable: equiform expression-*tokens* can have different functor-argument structures, then are treated as different language expressions of language. Thus, *types* of equiform expression-*tokens* do not have to be equal.

- If $\boldsymbol{e}$ is a complex *wfe-types* of the set $\boldsymbol{S}$', formed from the main functor $\boldsymbol{f}$ and its arguments $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, satisfying the formula ($\boldsymbol{i}_{\boldsymbol{f}}$), then the functor $\boldsymbol{f}$ and its index $\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{f})$ can be treated as set-theoretical functions which satisfy the equivalence:

(*R1*) $\boldsymbol{f} \in \boldsymbol{SCa}/\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ if and only if
($\boldsymbol{f}$) $\boldsymbol{f}: \boldsymbol{SCa}_1 \times \boldsymbol{SCa}_2 \times \ldots \times \boldsymbol{SCa}_n \rightarrow \boldsymbol{SCa} \wedge \boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n) = \boldsymbol{e} \wedge$
($\boldsymbol{i}$) $\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{f}): \{\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}_1)\} \times \{\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}_2)\} \times \ldots \times \{\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}_n)\} \rightarrow \{\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e})\} \wedge$
(*PCS*) $\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}) = \boldsymbol{i}_{\boldsymbol{S}}\big(\boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)\big) = \boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{f})\big(\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}_1), \boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}_2), \ldots, \boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{e}_n)\big)$.

We call the condition (*PCS*) the ***principle of syntactic compositionality***. Loosely speaking, this principle says that:

The syntactic category (*categorial index*) *of the well-formed functor-argument expression-types* $\boldsymbol{e}$ *of L is a function of syntactic categories* (*categorial indices*) *of arguments of its main functor* $\boldsymbol{f}$; *this function is* $\boldsymbol{i}_{\boldsymbol{S}}(\boldsymbol{f})$.

3.2 Categorial Semantics – the Theory *ST*

The theory ***ST*** is an axiomatic theory, built over the theory of syntax ***T***. It describes both the *intensional semantics* and the *extensional semantics* of the categorial language *L*.

3.2.1 Intensional Semantics

The basic notions of the *intensional categorial semantics* of *L* are the following:

- the notion of *meaning* (*intension*) of a *wfe-type* of *L*,
- the notion of a *category of knowledge* (constituents of knowledge), determined by means of the notion of *meaning*, and
- the notion of an *intensional semantic category*, defined by means of the previous notion.

In the semantic, formal characteristics of *L*, these notions are defined on the *level of types*. However, introducing into the formal theory ***ST*** the notion of *meaning* of a meaningful *wfe-type* of the set ***S***, and also that of *interpretation* of such an expression, as well as derivative notions, requires making references to some notions of the theory ***ST*** which are introduced on the *level of tokens*.

There exist various philosophical concepts concerning the nature of the *meaning* of a language expression, and also various theories of this notion. In the theory ***ST***, the formal concept of *meaning* is based on the general theory ***TM&I*** of meaning and interpretation, which were presented in the author's works [71, 72, 74, 75]. This concept is a logical pragmatic-semantic one and has certain connections with the understanding of *meaning* as a ***manner of using language expressions***. It takes into account the so-called ***functional approach to language analysis*** represented by Pelc [47, 48].

According to the approach proposed by Pelc, we can speak of a double ***manner of using language expressions***:

1. regarding the first of them, the ***manner of using*** (*use*) takes place only in given conditionings, in determined situational-language contexts and concerns solely expression-*tokens*,
2. regarding the other one, the ***manner of using*** (*usage*, *Use*) characterizes the *meaning* of an expression; this manner is built into the *meaning* of an expression, while the very expression itself can be treated as isolated, static, torn out of context, e.g., as a dictionary entry; then it is an expression-*type*, a class of its concrete occurrences, a class of expression-*tokens*, either applied to represent some object or used in acts of communication and in given situations, with reference to only one broadly-understood object, or with reference to more than one object, still one of the same kind.

The difference between these two *manners of using* of expressions manifests itself in that two persons can use—in the sense of *Use*—the same expression-*type* by means of its two different *tokens*, thus using its different *tokens* in the sense of *use*.

In the set-theoretical formalization of the theory ***ST*** it is accepted that *use* is a relation dealing with real or potential physical acts of object references of *wfe-tokens*, already performed, being performed, or ones that may be performed by users of *L* in a determined communication process by means of these expressions. The relation *use* is a primitive notion of the theory ***ST***, whereas the relation *Use*, concerning the usage of expression-*types* by users of *L*, is a secondary notion of this theory. It is defined by means of the relation *use* and appears useful in the proposed, formal concept of *meaning* and *interpretation*, which makes references to certain ideas of Wittgenstein [63] and Ajdukiewicz [1, 2]. This concept is connected with understanding the *meaning* of expression-*types* as the *Use* manner of using them.[7]

The primitive notions of the theory ***ST***, with which the theory of syntax ***T*** is enriched are the following:

- the set *User* of all users of *L*,
- the set *Ont* of all extra-language objects, described by *L*,
- the binary operation *use* of *wfe-tokens* of the set *S*'.

It is assumed only axiomatically about the sets *User* and *Ont* that they are non-empty. A user of *L*, belonging to the set *User*, can be not only a current, but also a past or future user of it. On the other hand, objects of the set *Ont* can be not only concrete, material objects, but also fictional or abstract creations described by *L*. We do not assume anything, either, about categorization of the set *Ont*. Ontological

[7] The convergence between Ajdukiewicz's ideas and those of Husserl regarding the question of meaning of expressions as a manner of their usage is drawn attention to by Olech [44]. The very concept of meaning, deriving from Ajdukiewicz, is discussed in the book by Wójcicki [64]. A review of different concepts of meaning and a discussion on Ajdukiewicz's concept can be found, among others, in Maciaszek's copious monograph [40].

categories can, but do not have to, be: a category of individuals, categories of sets of individuals, various categories of set-theoretical relations and functions, a category of situations (states of things), etc.

The relation *use* is understood in a very broad way, as well. It can be an operation of human production (not necessarily external) of expression-*tokens*, exposing them, or also interpreting with the aim to refer to determined objects of the set *Ont*. Such an operation conceived broadly—within a liberally fixed temporal space and any fixed area of language-based communication between people—is treated as all such physical activities of users of *L*, which are taking place currently, occurred in the past and may—potentially—happen in the future, and which are subject to referring concrete expression-*tokens* to determined objects of the set *Ont* in relevant situations. The operation *use* can be called a ***function of object reference of wfe-tokens of the language L by its users***.

We postulate that the operation *use* is a two-argument partial function, whose first domain is the set *User* of users, the second—some proper subset of the set *S*' of all *wfe-tokens* of *L*, while the counter-domain—the subset of objects of the set *Ont*, to which these expressions are referred. And thus:

Axiom 1 (sets: *User*, *Ont*)

$$User \neq \emptyset \text{ and } Ont \neq \emptyset.$$

Axiom 2 (*use*) *use is a partial function:*

$$User \times S' \rightarrow Ont,$$
$$D_1(use) = User \text{ and } D_2(use) \subset S'.$$

We read the expression: $use(u, e) = o$, where $u \in User$, $e \in S'$, $o \in Ont$ as follows:[8] *the user u uses*(*produces*,*exposes*) *the wfe-token e with reference to the object o*. The object o is called an ***object of reference*** or a ***referent*** or a ***correlate*** *of the expression e indicated by its user u*.

Thus, each user of *L* *uses* at least one *token* of an expression of this language with reference to some object, but not every language *token* must have some object reference (a referent, a correlate).

Let us note, formally, when an expression-*token* possesses an object reference:

Definition 3 (*possessing a referent*)

e **has an object reference** iff $e \in S' \wedge \exists u \in User\, \exists o \in Ont\, (use(u, e) = o)$.

Thus: *Object reference is possessed only by such a wfe-token that is used by some user of L with reference to an extra-language object*.

Definition 4 (*possessing the same manner of use of tokens*)

$e \approx e'$ iff $\exists o \in Ont\, \big(\exists u \in User\, \big(use(u, e) = o\big) \wedge \exists u \in User\, \big(use(u, e') = o\big)\big)$.

[8] The way in which the expression: $use(u, e) = o$ is read cannot be mistaken with the manner of interpreting this expression, in agreement with an intuitive, broad understanding of the operation *use*.

Thus: *Two wfe-tokens have the same manner of use* if and only if *they have the same object reference* (*they have the same referent*).

We introduce the ***relation of using expression-types*** in the sense *Use* in the following way:

Axiom 3 (*Use*)

$$\emptyset \neq Use \subseteq User \times \boldsymbol{S}\text{'},$$

Definition 5 (*Use*)

$$u\ Use\ \boldsymbol{e} \text{ iff } \exists e \in \boldsymbol{e}\ \exists o \in Ont\ \big(use(u, e) = o\big).$$

Therefore we postulate as follows: *There exists a user of L, who uses a wfe-type, and the user u uses the wfe-type* ***e*** *if and only if u uses a token of the expression* ***e*** *with reference to a referent.*

The notion of *meaning* of an expression-*type* is determined by means of the ***relation*** $\cong$ of ***possessing the same manner of Use of expression-types***. The notion of meaning is thus defined only for expressions which belong to $D_2(Use) = \boldsymbol{S} \subseteq \boldsymbol{S}$'. It is only to such expressions that meaning is assigned.[9] We will call the set $\boldsymbol{S}$ the *set of meaningful expressions of L.*

Definition 6 (*possessing the same manner of Use of types*)

$$\boldsymbol{e} \cong \boldsymbol{e}\text{' iff } \forall u \in User\big((u\ Use\ \boldsymbol{e}\ \Leftrightarrow u\ Use\ \boldsymbol{e}\text{'}) \wedge$$
$$\wedge\ \forall o \in Ont\ \big(\exists e \in \boldsymbol{e}\ (use(u, e) = o)\ \Leftrightarrow\ \exists e\text{'} \in \boldsymbol{e}\text{'}\ (use(u, e\text{'}) = o)\big)\big).$$

The above-given definition states that: *Two meaningful expression types* ***e*** *and* ***e***' *of* ***S*** *have the same manner of Use if and only if any user of L Uses one of them, when he/she Uses also the other of them and for each extra-language object it is a referent of some token of the wfe-type* ***e*** *if and only if this object is also a referent of some token of the other wfe-type* ***e***'.

The relation between the two different relations of possessing the same manner of using expressions of *L* is formulated by:

Theorem 1

$$\exists u \in User(u\ Use\ \boldsymbol{e})\ \wedge\ \boldsymbol{e} \cong \boldsymbol{e}\text{'}\ \Rightarrow\ \exists e \in \boldsymbol{e}\ \exists e\text{'} \in \boldsymbol{e}\text{'}\ (e \approx e\text{'}),$$

in compliance with which: *If the two used expression-types* ***e*** *and* ***e***' *of* ***S*** *have the same manner of using types* (in the other sense, the one of *Use*), *then there exist their relevant tokens e and e*', *which also have the same manner of using, but one that is proper to tokens* (the manner of using in the first sense, the one of *use*).

Let us note that in accordance with the introduced definition of the relation $\cong$ we can state that:

[9] In English, some not meaningful expressions are, for instance, sentences (well-formed expressions) like the following: *The computer gives the celling* or *The flowers are cooking dinner*.

Theorem 2 *Relation* $\cong$ *is an equivalence relation in the set* $\boldsymbol{S}$.

We define the basic notion of intensional semantics for *L*, i.e., the notion of ***meaning*** (*intension*) ***of any meaningful wfe-type e*** of *L* as the equivalence class of relation $\cong$, determined by this expression:

Definition 7 (*meaning of the expression-type e*)

$$\mu(\boldsymbol{e}) = [\boldsymbol{e}]_{\cong} \text{ for every } e \in \boldsymbol{S}.$$

The *meaning* $\mu(\boldsymbol{e})$ of *wfe-type* $\boldsymbol{e} \in \boldsymbol{S}$ may be intuitively understood as a common property of all these *wfe-types* which possess the same manner of using (*Use*) as that of $\boldsymbol{e}$. This common property can be called the ***manner of using*** *Use* ***of expression-type*** $\boldsymbol{e}$.

The *meaning of the wfe* $\boldsymbol{e} \in \boldsymbol{S}$ can be determined also as an equivalence class of all expression-*types* being ***synonyms*** of the expression $\boldsymbol{e}$, and thus having the same meaning as that of $\boldsymbol{e}$, the same manner of using (*Use*) as that of $\boldsymbol{e}$.

It follows from the definition of *meaning* of a meaningful expression-*type* that there is exactly one meaning—the *global meaning*—that corresponds to such an expression. It needs, however, to be observed that since a *wfe-type* is a class of all *identifiable* expression-*tokens* (the fixed universe *U*), used in any time interval considerations and any established area of language communication, its global meaning can consist of several meanings determined by its *subtypes*—its subsets of *identifiable tokens*. For example, in the English language, the global meanings of the individual word-*types*: "logic", "key", "profession", or "leak" treated as classes of equiform, identifiable tokens, consist of, at least, two meanings ascribed to certain of their subtypes. These words are ambiguous and as such do not have one fixed meaning.

The notion of ambiguity is introduced into the theory ***ST*** by means of that of *denotation*—a notion of *extensional semantics*. The notion of *not possessing an established meaning*, on the other hand, is determined by the definition:

Definition 8 (*not possessing an established meaning*)
e does not possess an established meaning iff

$$\neg\forall \boldsymbol{e}\text{'} \subseteq \boldsymbol{e} \ \big(\mu(\boldsymbol{e}\text{'}) = \mu(\boldsymbol{e})\big),$$

i.e.,

$$\exists \boldsymbol{e}\text{'} \subseteq \boldsymbol{e} \ \big(\boldsymbol{e}\text{'} \neq \boldsymbol{e} \ \wedge \ \mu(\boldsymbol{e}\text{'}) \neq \mu(\boldsymbol{e})\big).$$

There follows from the definition, in particular:

Theorem 3

a. *$\boldsymbol{e}$ does not have an established meaning iff* $\exists \boldsymbol{e}_1, \boldsymbol{e}_2 \left(\boldsymbol{e}_1 \subseteq \boldsymbol{e} \ \wedge \ \boldsymbol{e}_2 \subseteq \boldsymbol{e} \ \wedge \ \boldsymbol{e}_1 \neq \boldsymbol{e}_2 \wedge \mu(\boldsymbol{e}_1) \neq \mu(\boldsymbol{e}_2)\right)$*, i.e., there exist two subtypes of the wfe-type* $\boldsymbol{e}$ *with different meanings.*

b. *If* $\boldsymbol{e}$ *does not have an established meaning,*
then $\exists u \in User\, \exists e_1, e_2 \in \boldsymbol{e}\, \forall o \in Ont\, \neg\left(use(u, e_1) = o = use(u, e_2)\right)$*,*
i.e., there exists a user of L, who does not use at least two tokens of the expression $\boldsymbol{e}$ *with reference to the same extra-language object.*

c. *If* $\exists e_1, e_2 \in \boldsymbol{e} \left(\neg(e_1 \approx e_2)\right)$*, then* $\boldsymbol{e}$ *does not have an established meaning.*

In compliance with condition c. of Theorem 3: *Expression-type does not have an established meaning when some two of its tokens are not used in the same manner.*

The given definition of *meaning* of an expression-*type* determines at the same time the ***operation of meaning*** μ as the following mapping:

$$\mu\colon \boldsymbol{S} \to 2^{\boldsymbol{S}}$$

of the set $\boldsymbol{S}$ of all meaningful expression-*types* of the language L into a family of all of its subsets. We call the image of the set $\boldsymbol{S}$ under the operation μ the ***set of constituents of knowledge*** and denote it by $\boldsymbol{K}$. Thus:

$$\boldsymbol{K} = \mu(\boldsymbol{S}).$$

The operation of meaning μ corresponds to the ***operation of interpretation*** ι defined as mapping:

$$\iota\colon \boldsymbol{S}^* \to 2^{\boldsymbol{S}}$$

defined by the formula:

$$\iota(\boldsymbol{e}) = [e]_{\cong \mathbf{i}}, \text{ for any } \boldsymbol{e} \in \boldsymbol{S}^* \subseteq \boldsymbol{S},$$

where $\cong \mathbf{i}$ is a ***relation of possessing the same manner of interpreting meaningful expression-types*** and a sub-relation of the relation possessing the same manner of using (*Use*) such expressions.[10]

Interpretation of a meaningful expression-type can be intuitively understood as a common property of all the meaningful expression-*types* which possess the same manner of interpreting.

It is well-known that if an expression-*type* is intermediary in language communication, its *interpretation* can differ from its meaning. Let us note that formally we can merely state that for any meaningful expression-*type* $\boldsymbol{e}$

$$\iota(\boldsymbol{e}) \subseteq \mu(\boldsymbol{e}).$$

[10] Relation $\cong \mathbf{i}$ is defined by means of the binary relation *Int* of *interpreting* expression-*types* (corresponding to the relation *Use*) and the binary operation *int* of *interpreting wfe-tokens*, about which it is assumed axiomatically that it is a non-empty reduction of the operation *use* of using expression-*tokens* (see Wybraniec-Skardowska, 2007a, 2007b). The set $\boldsymbol{S}^* \subseteq \boldsymbol{S}$ is the set of all meaningful expression-*types* that can be *Interpreted* by *Users* of *L*.

We can divide the set of constituents of knowledge $\boldsymbol{K}$ into *categories of knowledge*, like we have divided the set of *wfe-types* of L into syntactic categories. In order to do so we make use of categorial indices of the set $\boldsymbol{I}$ and introduce the ***function of indexation*** $\boldsymbol{i_K}$ ***of components of knowledge***:

$$\boldsymbol{i_K} : \boldsymbol{K} \to \boldsymbol{I}.$$

We define the ***category of knowledge with the index*** ξ in the following way:

$$\boldsymbol{K}_\xi = \{\boldsymbol{k} \in \boldsymbol{K} \colon \boldsymbol{i_K}(\boldsymbol{k}) = \xi\}.$$

If, in L, we have sentences, names, and functors-functions defined on them, then their meanings—as constituents of knowledge—determine, respectively, the category of logical judgments, the category of logical notions, and categories of operations on logical judgments and/or logical notions.

In the semantic, *intensional* description of L, we count *wfe-types* of L to suitable *intensional semantic categories* determined by categorial indices. And so, we are introducing the following definition:

$$\boldsymbol{Int}_\xi = \left\{\boldsymbol{e} \in \boldsymbol{S} \colon \boldsymbol{i_K}\left(\mu(\boldsymbol{e})\right) = \xi\right\} = \left\{\boldsymbol{e} \in \boldsymbol{S} \colon \mu(\boldsymbol{e}) \in \boldsymbol{K}_\xi\right\},$$

i.e., the ***intensional semantic category with the index*** ξ is a set of all these meaningful expression-*types* of L, whose *meanings* belong to the category of knowledge with the index ξ.

One of the conditions of *language adequacy* is an agreement of syntactic categories with *semantic categories*, and this of both *intensional* and *extensional* ones. We will introduce the latter formally on the second level of language semantics of the language L.

3.2.2 Extensional semantics

In compliance with Frege's distinction [23]: *Sinn-Bedeutung* and Carnap's distinction [16]: *intension-extension*, we distinguish the *meaning* of expression-*type* of L from a *denotation* of such an expression. We introduce the notions of *denotation* (*extension*) *of an expression-type* and that of *denotation of a constituent of knowledge*, corresponding to this expression, formally on the basis of the theory $\boldsymbol{ST}$, by means of respective notions of *denoting* (*reference*). All these notions belong to the semantics of the second level—the *extensional semantics* of L.

Denoting (*reference*) Ref_1 is a binary relation that holds between expression-*types* and extra-language objects of the set *Ont*. The notion of *denoting* can, however, also be introduced as the relation Ref_2 holding between constituents of knowledge and extra-language objects of the set *Ont*. Therefore, formally:

$$Ref_1 \subseteq \boldsymbol{S} \times Ont \ \text{ and } \ Ref_2 \subseteq \boldsymbol{K} \times Ont,$$

and the definitions of these relations are as follows:

Definition 9 (*denoting*)

a. $\boldsymbol{e}\ Ref_1\ o$ iff $\exists u \in User\ \exists e \in \boldsymbol{e}\ \big(use(u, e) = o\big)$, where $\boldsymbol{e} \in \boldsymbol{S}$.
b. $\boldsymbol{k}\ Ref_2\ o$ iff $\exists \boldsymbol{e} \in \boldsymbol{S}\ \big(\boldsymbol{k} = \mu(\boldsymbol{e})\ \wedge\ \boldsymbol{e}\ Ref_1 o\big)$, where $\boldsymbol{k} \in \boldsymbol{K}$.

The ***expression-type e*** *denotes the object o* if and only if there exists a user of *L*, who *uses* any token of the expression $\boldsymbol{e}$ with reference to the object o, whereas the ***constituent of knowledge k denotes the object*** o, when there exists a meaningful expression of *L* determining $\boldsymbol{k}$ and denoting o. We will refer to the objects denoted by expression-*types* or constituents of knowledge as their ***denotates***.[11]

As an example, the *denotate* of the name "a computer" is each computer; any computer is also denoted by the notion 'a computer'; any computer is thus a *denotate* of this notion as well.

It is easy to notice that the *denotate* of an expression-*type* is, at the same time, an object reference of a *token*.

The set of all *denotates* of an expression-*type* or, respectively, a constituent of knowledge, is called its ***denotation*** or ***extension***. Thus:

Definition 10 (*denotation*)

a. $\delta(\boldsymbol{e}) = \{o \in Ont\colon \boldsymbol{e}\ Ref_1\ o\}$, where $\boldsymbol{e} \in \boldsymbol{S}$.
b. $\delta_{\boldsymbol{K}}(\boldsymbol{k}) = \{o \in Ont\colon \boldsymbol{k}\ Ref_2\ o\}$, where $\boldsymbol{k} \in \boldsymbol{K}$.

The *denotation* of a meaningful expression-*type* or a constituent of knowledge corresponding to it does not have to be a non-empty set. It is such a set when a user of *L uses* the same expression-*type*; that is, he/she *uses* any of its *tokens* with reference to an extra-language object. Hence, we have:

Theorem 4 (*the criterion of non-emptiness of denotation*)

$$\exists u \in User\ (u\ Use\ \boldsymbol{e}) \text{ iff } \delta(\boldsymbol{e}) \neq \emptyset \text{ iff } \delta_{\boldsymbol{K}}(\mu(\boldsymbol{e})) \neq \emptyset.$$

The definitions of *denotation* of a meaningful expression-*type* given below and the constituent of knowledge corresponding to it cover the so-called *global denotation*.

[11] Let us pay attention to the fact that—according to the assumptions of the theory ***ST***—the notion of a *denotate*, as an object of the set *Ont* denoted by an expression-*type*, is broader than the notion of a *designate of such an* expression, usually accepted in the logic of language. In particular, it is accepted in logical semiotics that *designates* of the so-called concrete names can be material objects only. Such objects can be *denotates* of such names then, but they do not have to be; they can also be intentional, fictional objects. This explains, in particular, certain misunderstandings connected with so-called empty names. Such names as for instance, "Zeus", "Sphinx" or "Smurf" are acknowledged—on the one hand—to be *empty names* (as ones which do not denote any *designate*), on the other one—as *non-empty names* (as ones denoting their *denotates*).

Inasmuch as an expression-*type* is ambiguous,[12] its *global denotation* is composed of *denotations* determined by its unambiguous *subtypes*.[13]

When the *denotation* of a meaningful expression-*type* or a constituent of knowledge corresponding to it is a one-element set (a singleton), we identify it sometimes, in practice, with its sole *denotate*. This is so, for instance, when we come to deal with proper names. Let us note that in situational semantics, *denotates* of logical sentences are conceived as situations and frequently identified with *denotations* of such sentences. Also, in Frege's traditional semantics [23], a *denotate* and—at the same time—*denotation* of a logical sentence is its logical value, i.e. truthfulness or falsity.

Let us note, too, that *denotations* of the so-called general names (predicative) and the logical notions corresponding to them, are called *scopes*, identifying the latter. For example, the scope (*denotation*, *extension*) of the name "a computer"—that is—the set of all computers, is identified with the scope (*denotation*) of the notion 'a computer'. This agreement of the denotations of names and notions corresponding to them is connected with *language adequacy*, and more precisely—with *semantic adequacy*, which is illustrated by Diagram 2.

In the theory ***ST***, there holds a theorem which frames this adequacy:

Theorem 5 (*semantic adequacy*)

$$\delta(\boldsymbol{e}) = \delta_{\boldsymbol{K}}(\mu(\boldsymbol{e})), \text{ for any } \boldsymbol{e} \in \boldsymbol{S}.$$

According to Theorem 5: *Denotations of any meaningful expression-type of L and the meaning* (*a constituent of knowledge*) *of this expression are in agreement.*

There follows immediately an important theorem from this theorem, pointing to the fact that the *meaning* of an *expression-type* determines its *denotation*:

Theorem 6 (*dependence between the meaning and denotation*)

$$\mu(\boldsymbol{e}) = \mu(\boldsymbol{e}') \;\Rightarrow\; \delta(\boldsymbol{e}) = \delta(\boldsymbol{e}'), \text{ for any } \boldsymbol{e}, \boldsymbol{e}' \in \boldsymbol{S}.$$

According to this theorem: *If two expression-types have the same meaning* (*intension*), *then they also have the same denotation* (*extension*). In other words: *If two expressions are synonymous, then they are extensionally equivalent.*

The reverse theorem does not hold: two expressions can have the same *denotation* (be extensionally equivalent), but may not have the same *meaning* (may not be synonymous). Instances of such expressions are: the "Morning Star" and the "Evening Star" (see Frege [23]).

The two conclusions below follow from the above theorem, in particular:

[12] The formal definition of an ambiguous expression-*type* was given in the author's earlier paper [74].

[13] The *global denotation* can also be seen as the upper approximation of denotation of a vague expression, yet in this paper the problem of vagueness of language expressions will not be dealt with.

Corollary 1

If $\exists o \in Ont\,(\boldsymbol{e}_1\ Ref\ o\ \wedge \neg\,\boldsymbol{e}_2\ Ref\ o\ \vee\ \boldsymbol{e}_2\ Ref\ o\ \wedge \neg\,\boldsymbol{e}_1\ Ref\ o)$*, then* $\mu(\boldsymbol{e}_1) \neq \mu(\boldsymbol{e}_2)$.

Corollary 2 *Any expression-type does not possess an established meaning, when there exist two such subtypes of it that an object is the denotate of only one of them.*

Following Corollary 1: *Two expression-types do not have the same meaning as long as an object is the denotate of only one of the expressions.*

In accordance with the other conclusion, for example, the ambiguous name "a key" does not possess an established meaning, since there exists a key which is the *denotate* of a certain subtype of this name, yet which is not the *denotate* of another subtype of this name.

The given definitions of the denotation of an expression-*type* and the denotation of a constituent of knowledge determine simultaneously two ***denotation operations***: δ and $\delta_{\boldsymbol{K}}$. They are the following mappings:

$$\delta\colon \boldsymbol{S} \to 2^{Ont} \ \text{ and } \ \delta_{\boldsymbol{K}}\colon \boldsymbol{K} \to 2^{Ont},$$

respectively: of the set $\boldsymbol{S}$ of all meaningful expression-*types* of L into the family of all subsets of the set of extra-language objects *Ont* and of the set $\boldsymbol{K}$ of all constituents of knowledge into this family.

Thus, it follows from Theorem 5 of semantic adequacy that the denotation operation δ is a composition of denotation operation $\delta_{\boldsymbol{K}}$ and the meaning operation μ that is (see Diagram 2): $\delta = \delta_{\boldsymbol{K}} \circ \mu$.

The image of the set $\boldsymbol{S}$ with respect to the operation δ and the set $\boldsymbol{K}$ with respect to the operation $\delta_{\boldsymbol{K}}$, are called *a set of ontological objects* and denoted by $\boldsymbol{O}$. Thus (see Diagram 2:

$$\boldsymbol{O} = \delta(\boldsymbol{S}) = \delta_{\boldsymbol{K}}(\boldsymbol{K}).$$

We can divide the set $\boldsymbol{O}$ of ontological objects into ontological categories, in a similar way as we divided the set $\boldsymbol{S}$ of meaningful expressions of L into syntactic categories, and the set of constituents of knowledge $\boldsymbol{K}$ into categories of knowledge. For this purpose we use the categorial indices of the set $\boldsymbol{I}$ and introduce the ***function of indexation*** $\boldsymbol{i_O}$ ***of ontological objects***:

$$\boldsymbol{i_O}\colon \boldsymbol{O} \to \boldsymbol{I}.$$

We define the ***ontological category with the index*** ξ in the following way:

$$\boldsymbol{O}_{\xi} = \{\boldsymbol{o} \in \boldsymbol{O}\colon \boldsymbol{i_O}(\boldsymbol{o}) = \xi\}.$$

If, in L, we have sentences, individual names, and functor-functions defined on them, then the ontological objects corresponding to them—as their denotations—determine, respectively, a category of states of things (in Frege's semantics—a category of logical values), a category of individuals, and a category of operations on states of things (resp., on logical values), on individuals, on the former and/or the latter, etc.

In the semantic, *extensional* description of L, the meaningful expression-*types* of this language count into respective *extensional semantic categories*, determined by categorial indices. And so:

$$\boldsymbol{Eks}_{\xi} = \{\boldsymbol{e} \in S\colon \boldsymbol{i_O}(\sigma(\boldsymbol{e})) = \xi\} = \{\boldsymbol{e} \in S\colon \sigma(\boldsymbol{e}) \in \boldsymbol{O}_{\xi}\},$$

i.e., the ***extensional semantic category with the index*** ξ is a set of all the expression-*types* of L, whose denotations (*extensions*) belong to the ontological category with the index ξ.

One of the conditions of *language adequacy* is an agreement of *syntactic categories* with applied *semantic categories*, and this both *intensional* as well as *extensional*. This agreement is not ensured by the agreement of both levels of the semantics of L: *intensional* and *extensional*.

In the next part of the paper, we will consider, with more precision, the problem area of *language adequacy*, discussing its various aspects.

4 Language Adequacy and its Aspects

In the *Introduction*, we defined the problem area of *language adequacy* in a most general manner, as a compatibility of language syntax and its bi-level semantics: *intensional semantics* and *extensional semantics*. Formal consideration of the problem of *language adequacy* can be conducted on the basis of the theory of syntax $\boldsymbol{T}$ and its extension to the semantic theory $\boldsymbol{ST}$ for the categorial language L. Taking into account the bi-level semantics of L, we have already established an important theorem which characterizes the ***semantic adequacy*** for this language and states that for any expression-*type* $\boldsymbol{e} \in \boldsymbol{S}$ of L:

$$\delta(\boldsymbol{e}) = \delta_{\boldsymbol{K}}(\mu(\boldsymbol{e})) \in \boldsymbol{O},$$

that is, the same object of the reality described by L corresponds to the denotation of any meaningful expression-*type* $\boldsymbol{e}$ of L and the denotation of its counterpart which is a constituent of knowledge (see Diagram 2). *Semantic language adequacy*, like certain *intensional* and *extensional* agreement with reality described by the language, is the starting point in the consideration of various aspects of *language adequacy*.

In compliance with the understanding of the adequacy of language syntax and semantics provided by Frege [22, 23], Husserl [32], Leśniewski [38, 39] and Suszko [53–56],[14] ***language adequacy*** assumes, primarily, that the categories of language expressions—syntactic and semantic (*extensional*), with the same indices—should be the same. Extending this agreement onto the identity of all distinguished kinds of categories of meaningful expression-*types* of L: syntactic, semantic extensional, as well as semantic intensional, with the same categorial indices, we will use the term ***categorial adequacy***. In order to determine it, we postulate the following:

[14] See also Stanosz and Nowaczyk [52].

Postulate (*categorial adequacy*)

$$\boldsymbol{SC}^{*}_{\xi} = \boldsymbol{Int}_{\xi} = \boldsymbol{Eks}_{\xi}, \text{ for any } \xi \in \boldsymbol{I},$$

where $\boldsymbol{SC}^{*} = \{\boldsymbol{e} \in \boldsymbol{S} : \boldsymbol{i_S}(\boldsymbol{e}) = \xi\}$.[15]

We can formulate the postulate of *categorial language adequacy* given above in two equivalent ways imposed by conditions a and b of the following theorem (see Diagram 3):

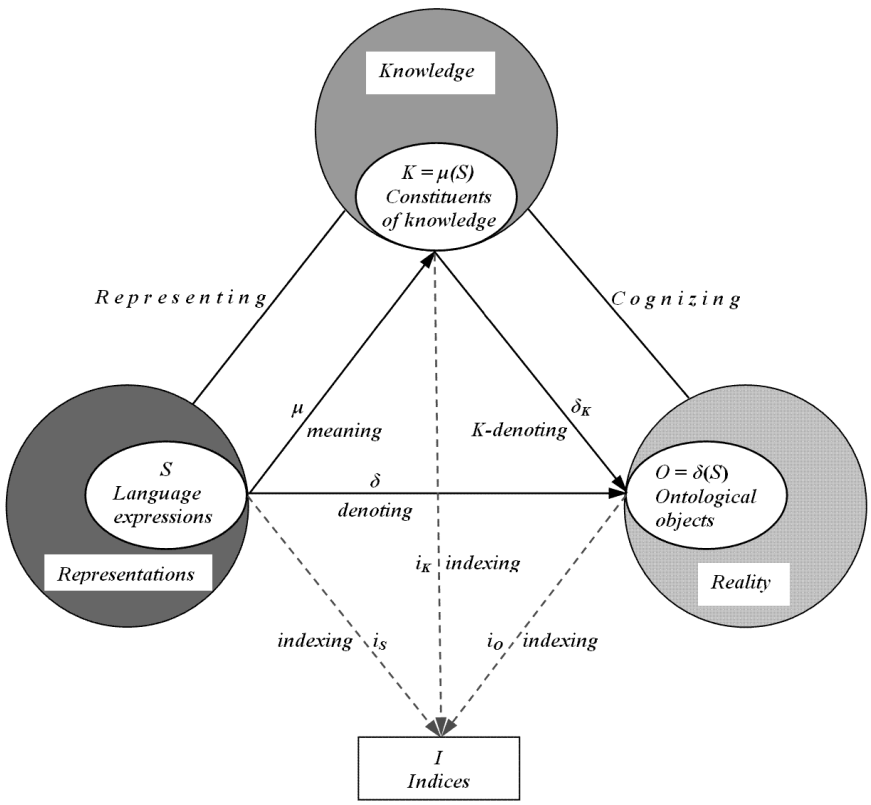

Diagram 3: Categorial adequacy

[15] Let us notice that a formalization of the notion of categorial adequacy does not require assuming that language expressions have to have a functor-argument structure. So if language is generated by another type of grammar than a categorial grammar, e.g. a phrase structure grammar or a dependency grammar (see Tesnière [58]), then the postulate could be adapted.

Theorem 7 (***categorial adequacy***)

a. $\boldsymbol{e} \in \boldsymbol{SC}^{*}_{\xi}$ *iff* $\mu(\boldsymbol{e}) \in \boldsymbol{K}_{\xi}$ *iff* $\sigma(\boldsymbol{e}) \in \boldsymbol{O}_{\xi}$, for any $\boldsymbol{e} \in \boldsymbol{S}$.
b. $\boldsymbol{i_S}(\boldsymbol{e}) = \boldsymbol{i_K}(\mu(\boldsymbol{e})) = \boldsymbol{i_O}(\sigma(\boldsymbol{e}))$, for any $\boldsymbol{e} \in \boldsymbol{S}$.

The categorial adequacy is therefore ensured by the identity of categorial indices: of any meaningful expression-*type* of *L*, its *meaning* and its *denotation*.

There follow from Theorem 7 of categorial adequacy theorem-equivalents of the theorem (*R1*), permitting one to state that it is not only a functor and its index, but also the semantic equivalents of the functor—its *meaning* and its *denotation*—that can be treated as set-theoretical functions (see Wybraniec-Skardowska [77]:

Theorem 8 (***meaning and denotation of functor***)

If $\boldsymbol{e} = \boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$ *is a meaningful expression of the set* $\boldsymbol{S}$*, satisfying the formula* ($\boldsymbol{i_f}$)*, then the following equivalences are satisfied:*

(*R2*) $\mu(\boldsymbol{f}) \in \boldsymbol{Ka}/\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ *iff*
($\mu(\boldsymbol{f})$) $\mu(\boldsymbol{f})\colon \boldsymbol{Ka}_1 \times \boldsymbol{Ka}_2 \times \ldots \times \boldsymbol{Ka}_n \rightarrow \boldsymbol{Ka} \wedge$
(*PCM*) $\mu(\boldsymbol{e}) = \mu\big(\boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)\big) = \mu(\boldsymbol{f})\big(\mu(\boldsymbol{e}_1), \mu(\boldsymbol{e}_2), \ldots, \mu(\boldsymbol{e}_n)\big)$

and

(*R3*) $\sigma(\boldsymbol{f}) \in \boldsymbol{Oa}/\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ *iff*
($\sigma(\boldsymbol{f})$) $\sigma(\boldsymbol{f})\colon \boldsymbol{Oa}_1 \times \boldsymbol{Oa}_2 \times \ldots \times \boldsymbol{Oa}_n \rightarrow \boldsymbol{Oa} \wedge$
(*PCD*) $\sigma(\boldsymbol{e}) = \sigma\big(\boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)\big) = \sigma(\boldsymbol{f})\big(\sigma(\boldsymbol{e}_1), \sigma(\boldsymbol{e}_2), \ldots, \sigma(\boldsymbol{e}_n)\big)$

We call the condition (*PCM*) the ***semantic principle of compositionality of meaning***, and the condition (*PCD*) the ***semantic principle of compositionality of denotation***. These principles were already known to Frege [23].[16]

Loosely speaking, these principles state, respectively, that:

The meaning (resp. *denotation*) *of a well-formed functor-argument expression of L is the value of the function of meaning* (resp. *denotation function*) *of its main functor, defined by meanings* (resp. *by denotations*) *of arguments of this functor*.

The categorial character of the language *L* under consideration allows speaking also about ***structural adequacy*** as an agreement of the structure of any expression composed of a functor and its arguments, with the structure of the constituent of knowledge that corresponds to it and with the structure of the object of the cognized reality that corresponds to it. ***Structural adequacy*** is obtained through holding three *principles of compositionality*:[17] one syntactic—the principle (*PCS*) *of compositionality of syntactic forms*—and two semantic principles: (*PCM*) and (*PCD*), *of compositionality of meaning and compositionality of denotation*.[18]

The three principles of compositionality mentioned above, for $h = \boldsymbol{i_S}, \mu, \sigma$ and any meaningful expression $\boldsymbol{e} = \boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$, have the following common schema:

[16] See also Gamut [25].

[17] See Wybraniec-Skardowska [70, 78].

[18] The problem of *semantic compositionality* is the subject of a heated discussion (see Montague [42]; Partee [45]; Janssen [33, 34]; Hodges [29–31]; Pelletier [49]; Kracht [35]).

$$h(\boldsymbol{e}) = h\big(\boldsymbol{f}(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)\big) = h(\boldsymbol{f})(h(\boldsymbol{e}_1), h(\boldsymbol{e}_1), \ldots, h(\boldsymbol{e}_1)),$$

which can be treated as a schema of three conditions of the homomorphism of partial algebra $\boldsymbol{L}$ of L in the algebra of its images $h(\boldsymbol{L})$), i.e.,

$$\boldsymbol{L} = \langle \boldsymbol{S}, \boldsymbol{F} \rangle \underset{h}{\rightarrow} h(\boldsymbol{L}) = \langle h(\boldsymbol{S}), h(\boldsymbol{F}) \rangle,$$

where $\boldsymbol{F}$ is a set of partial functor-functions defined by subsets of the set $\boldsymbol{S}$ and with values in the set $\boldsymbol{S}$, and $h(\boldsymbol{F})$, for $h = \boldsymbol{i_S}, \mu, \sigma$, is a set of operations corresponding to operations of the set $\boldsymbol{F}$.[19]

We call the algebra

$$\boldsymbol{i_S}(\boldsymbol{L}) = \langle \boldsymbol{i_S}(\boldsymbol{S}), \boldsymbol{i_S}(\boldsymbol{F}) \rangle$$

a ***syntactic model*** of L, and the algebras:

$$\mu(\boldsymbol{L}) = \langle \mu(\boldsymbol{S}), \mu(\boldsymbol{F}) \rangle = \langle \boldsymbol{K}, \mu(\boldsymbol{F}) \rangle \text{ and } \sigma(\boldsymbol{L}) = \langle \sigma(\boldsymbol{S}), \sigma(\boldsymbol{F}) \rangle = \langle \boldsymbol{O}, \sigma(\boldsymbol{F}) \rangle$$

the ***semantic models*** of this language; the first is called the ***intensional model***; the other—the ***extensional model***.

In the process of cognition of reality, *language adequacy* also consists in that sentences of language L should be true in its above mentioned models.

If for $h = \boldsymbol{i_S}, \mu, \sigma$ it is so that the sentence $\boldsymbol{e}$ of L is *true in the models* $h(\boldsymbol{L})$, then we can say that our *cognition is true* or that there occurs language ***cognitive adequacy***.

The notions of *truthfulness* in respective models are introduced in the theories $\boldsymbol{T}$ and $\boldsymbol{ST}$ by means of three primitive notions *Th*, satisfying at $h = \boldsymbol{i_S}, \mu, \sigma$ the axioms:

$$\emptyset \neq Th \subseteq h(\boldsymbol{S})$$

and understood intuitively, respectively, as: a singleton composed of the index of true sentences, a set of true judgments, a set of situations that take place (in Frege's semantics—a singleton composed of the value of truth).

Definition 11 (***truthfulness***) For $h = \boldsymbol{i_S}, \mu, \sigma$

The sentence $\boldsymbol{e}$ of L *is true in the model* $h(\boldsymbol{L})$ iff $h(\boldsymbol{e}) \in Th$.

Language-related knowledge is passed in the process of inter-human communication. The transmitting and proper reception of it are connected with the proper *interpretation* of language expressions and ***communication adequacy***, based on the agreement of *meaning* and *interpretation* of language expressions which mediate in the communication (see Diagram 4).

Thus, if the expression-*type* $\boldsymbol{e}$ mediates in the communication between its sender and its receiver, then ***communication adequacy*** is secured by the condition:

[19] Ideas connected with the algebraization of language can be found already in works by Leibniz. The algebraic approach to the syntax and the semantics of language can also be found in the works of Dutch logicians of language, especially in those by van Benthem [59–62]. However, the algebraic approach presented here differs significantly from that given by van Benthem.

$$\mu(e) = \iota(e).$$

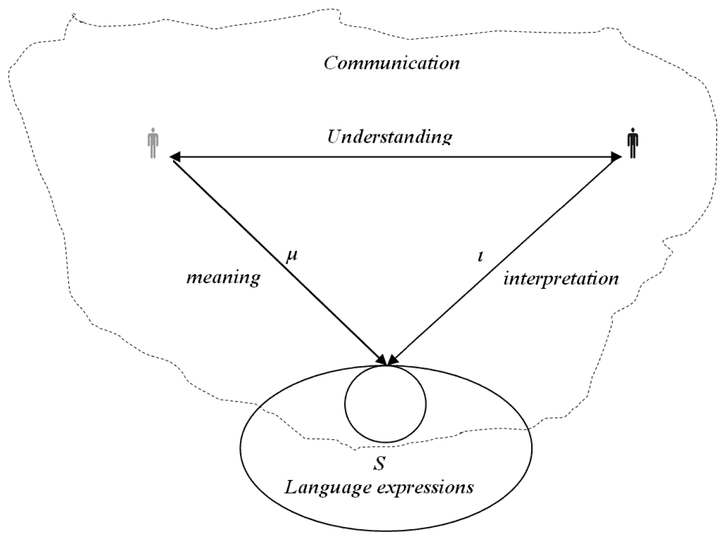

Diagram 4: Communication language adequacy

Let us pay attention to the fact that formal securing of the communication adequacy of *L* is based on such a formalization of it that uses the relations *Use* of using expression-*types* of this language, and therefore also the relation *use* of using its *tokens*, since in the presented theory ***ST*** the notions of *meaning* and *interpretation* of meaningful expression-*types* of *L* are defined by means of these relations. This fact implies the possibility of formalizing the notion of an inter-human *communication act* by means of *tokens* of language expressions and establishing formal conditions of its adequacy (see Wybraniec-Skardowska & Waldmajer [80]).

5 Summary

• This paper has offered a synthetic framework of the main ideas and theoretical considerations presented in earlier papers of the author, especially those dealing with the syntax and semantics of language characterized categorially:

- in the spirit of Husserl's idea of pure grammar [32],
- in compliance with the basic assumptions of the theory of syntactic categories of Leśniewski-Ajdukiewicz,
- according to Frege's [23] ontological canons, and also

- to Bocheński's [9] well-known statement: *syntax mirrors ontology*.

• The formal-logical framework of the theory of language syntax outlined in this work is based on Peirce's [46] distinction of sign-*tokens* from sign-*types*, on the assumption that expressions of language possess a double ontological nature: they can be physical concrete objects—*tokens*—or ideal abstract beings—*types*, classes of expression-*tokens*.
• Taking into account, in the theory of syntax, the double ontological character of language objects, as well as—following Pelc [47, 48]—the functional approach to the logical analysis of language, allows speaking about two manners of using language expressions. The first of them—applied in acts of inter-human language-based communication—concerns using expression-*tokens*; the other—using expression-*types* and determining on what, formally, correct *adequate* language communication depends. The other way allows also introducing, formally, basic semantic notions: the notion of *meaning* and that of *denotation* of expression-*type*, differentiating between them basically (similarly as was done by Frege [23] and Carnap [16]), and also using means of logical pragmatics.
• A formal characteristic of semantic notions takes place in the theory of semantics of language, built over this theory of syntax. The formal-logical theory of language which is presented in this paper, is a result of conceptualizing inter-subjective knowledge about language communication in a liberally established time range, as well as a liberally determined area of such communication. The conceptualization includes the bi-level semantics of language: *intensional* and *extensional*. On the first of them, the *intensional* level of theoretical considerations, the notions of *meaning* and *interpretation* are introduced, making reference to the use (*Use*) of expression-*types* (through the use of expression-*tokens* by users of language) and preserving certain intuitions of Wittgenstein [63] and Ajdukiewicz [1, 3], connected with the first of these notions. On the other, the *extensional* level of theoretical considerations, two notions of *denoting* and two notions of *denotation* are introduced. The notions of *denoting* and *denotation* (*extension*) of expression-*types* are differentiated from those of *denoting* and *denotation* of their *meanings* (*intensions*) treated as constituents of knowledge. All these notions are introduced as semantic-pragmatic ones, through referring to the two mentioned ways of using the expressions. This agreement of the two types of *denotation* is referred to as *semantic adequacy*, since it is connected with the bi-level semantics of language described by the theory under discussion in this paper.
• If—according to the ontological canons of Frege and Bocheński—language is to be a linguistic schema of ontological reality, and—at the same time—a tool of its cognition, its syntax should be in agreement with the bi-level semantics corresponding to it. This compliance has been called *language adequacy*, and its occurrence is guaranteed in the formal theory of language by accepting the respective postulate (axiom) of *categorial adequacy*. There follows from it an important condition of *structural* (*compositional*) *adequacy*, connected with the *principles of compositionality of meaning and denotation*, which were known already to Frege, and also with their syntactic counterpart introduced in papers by the author.

• In the outlined theory of language there are also formally considered other aspects of *language adequacy*. They are connected with the effectiveness of human cognition and inter-human communication by means of language expressions.

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Chapter 10
What Is the Sense in Logic and Philosophy of Language?

Urszula Wybraniec-Skardowska

Abstract In the paper, various notions of the logical semiotic sense of linguistic expressions—namely, syntactic and semantic, intensional and extensional—are considered and formalised on the basis of a formal-logical conception of any language *L* characterised categorially in the spirit of certain Husserl's ideas of pure grammar, Leśniewski-Ajdukiewicz's theory of syntactic/semantic categories and, in accordance with Frege's ontological canons, Bocheński's and some of Suszko's ideas of language adequacy of expressions of *L*. The adequacy ensures their unambiguous syntactic and semantic senses and mutual, syntactic and semantic correspondence guaranteed by the acceptance of a postulate of categorial compatibility of syntactic and semantic (extensional and intensional) categories of expressions of *L*. This postulate defines the unification of these three logical senses. There are three principles of compositionality which follow from this postulate: one syntactic and two semantic ones already known to Frege. They are treated as conditions of homomorphism of partial algebra of *L* into algebraic models of *L*: syntactic, intensional and extensional. In the paper, they are applied to some expressions with quantifiers. Language adequacy connected with the logical senses described in the logical conception of language *L* is, obviously, an idealisation. The syntactic and semantic unambiguity of its expressions is not, of course, a feature of natural languages, but every syntactically and semantically ambiguous expression of such languages may be treated as a schema representing all of its interpretations that are unambiguous expressions.

Key words: Logic and philosophy of language • Categorial language • Syntactic and semantic senses • Intensional semantics • Meaning • Extensional semantics • Denotation • Categorisation • Syntactic and semantic compatibility • Algebraic models • Truth • Structural compatibility • Compositionality • Language communication

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First published in: Bulletin of the Section of Logic **49** (2), 185–211 (2020).

1 Introduction

The word ‘sense’ has many meanings, and it appeals to us in many ways. On the basis of philosophy (and/or theology), it is for centuries that we have been trying to grasp and understand what the sense of our life is; likewise the sense of existence, the sense of our action and endeavour, and what the sense of the world is in general. From the point of view of philosophy, there are various visions and many theories regarding the sense of the world, the sense of life, our actions, etc. To discover their rational justifications, logical knowledge is needed, but, obviously, it is not enough. This philosophical meaning of the word ‘sense’ must clearly be distinguished from the logical, semiotic one. In the philosophical meaning, the word ‘sense’ is used as a certain property of extra-linguistic objects when it is said that something has or does not have sense, while referring to such objects. It derives from the basic, logical and semiotic meaning of this word, the meaning referring to linguistic objects, verbal signs. It should be noted, however, that it is not only the non-semiotic, but also the semiotic usage of the word ‘sense’ that is homogenous. Thus, one can speak of many notions of sense.

In this paper, we would like to characterise and formalise various notions of the *logical and philosophical sense* of linguistic expressions; from the viewpoint of logic, only these notions of sense can be of interest to us. The contemporary logic, logic of language (logical semiotics) can define the semiotic sense, *logical sense* strictly with regards to some general aspects of developing the cognition of the world and, at the same time, contributing to an explication of one of the most important traditional philosophical problems: *Language adequacy of our knowledge in relation to cognition of reality*, or, briefly: *language adequacy*. It is connected with the mutual relations between the three elements of the triad, reality–knowledge–language, and an adequate reflection of fragments of reality via expressions of language and inter-subjective knowledge of these fragments [68].

The above-mentioned adequacy requires, first of all, syntactic and semantic characterisation of language expressions as generalised by a grammar [57, 67]. Languages structured by grammar and logic are important tools of thinking, cognition of reality and knowledge acquisition, which stand for the foundations of our sense of existence [39]. In modern logic and philosophy of language, an approach based on Frege functions. It is implemented by the trend of formal and logical reflection on language and Fregean senses.

Logical sense, in its different variants, is considered and formalised on the basis of the conception of formalisation of language *L*, which is sketched below. The *syntactic sense* of these expressions is defined on the basis of language syntax and *semantic senses*—on the basis of bi-level language semantic: *intensional* and *extensional*.

From the logical point of view, the three notions of the sense of expressions of language *L* are understood as follows [62, 63, 69] (see Fig. 1):

- *syntactic sense* is found in expressions of *L* which are well-formed; it is their *essence*; it is defined in the syntax of *L*, and—in accordance with Carnap's distinction, *intension-extension* [18], or Frege's differentiation, *Sinn-Bedeutung* [19]—two kinds of *semantic sense*:

- *intensional sense* is proper to the expressions of L which have a meaning, *intension*; it is defined in intensional semantics of L,
- *extensional sense* is proper to the expressions of L which have a denotation, *extension*; it is defined in extensional semantics of L.

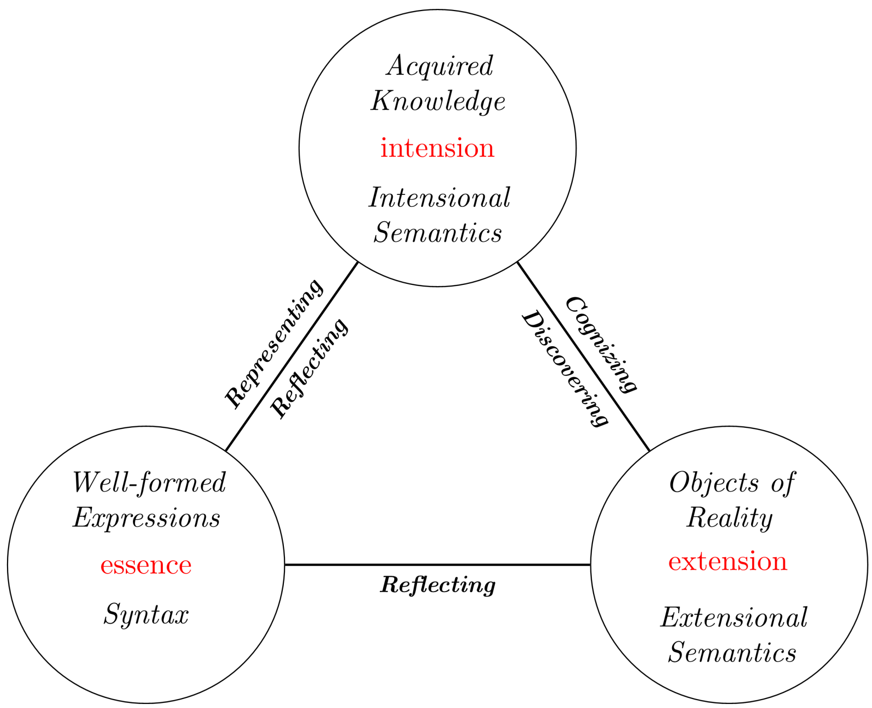

Figure 1: Three notions of linguistic sense: essence, intension, extension.

The syntactic and semantic notions of sense must be differentiated and explicated. This is possible through a conceptualisation of these notions that will lead to a formal-logical theory of syntax and semantics of language L, which specifies and describes these notions.

There are different points of view on the grammar of language, its syntax and semantics. In the paper, any language L, its syntax and bilevel semantic: intensional and extensional, is characterised and formalised *categorially* in the spirit of some ideas of Husserl (see [26]) and Leśniewski-Ajdukiewicz's theory of syntactic/semantic categories [3, 4, 32, 33], in accordance with Frege's ontological canons [18], Bocheński's motto, *syntax mirrors ontology* [53], and some ideas of Suszko:

language should be a linguistic scheme of ontological reality and simultaneously a tool of its cognition [46–49].

2 Main Ideas of the Formalisation of Categorial Language *L*

Categorial language *L* is defined if the set *S* of all *well-formed expressions* (*wfes*) of *L* is determined. These expressions must satisfy the requirements of categorial syntax and categorial semantics. The categorial syntax is connected with generating the set *S* by the classical categorial grammar, the idea of which originated from Ajdukiewicz [3, 4] under the influence of Leśniewski's theory of semantic (syntactic) categories in his systems of protothetics and ontology [32, 33], under Husserl's ideas of pure grammar (see [26]), and under the influence of Russel's theory of logical types. The notion of categorial grammar was shaped by Bar-Hillel (see [7–9]) and developed by Lambek, Montague, Cresswell, Buszkowski, Marciszewski, Simons, Tałasiewicz and others [10–15, 17, 18, 30, 31, 34–36, 41, 42, 48].The first formalisation of languages generated by the aforementioned classical categorial grammar, the notion introduced and explicated by Buszkowski was presented in the author's book in Polish [55] and its English translation, as well as some extension [56] (see also [64]).

In the categorial approach to language *L*, *wfes* of *S* should belong to appropriate syntactic categories. A characteristic feature of categorial syntax is that each composed *wfe* of the set *S* has a *functor-argument* structure, so that it is possible to distinguish in it the main part (the so-called *main functor*) and the other parts (called *arguments* of this functor), yet each constituent of the *wfe* has a determined syntactic category. Categorial intensional and extensional semantics is connected with *meaning* and *denotation* of *wfes* of *S* and with their membership in appropriate semantic categories: intensional and extensional, respectively (see [21, 61, 64–67]). Each constituent of the composed *wfe* has a determined semantic (intensional and extensional) category, can have a *meaning* (*intension*) assigned to it, and thus also a *category of knowledge* (the category of constituents of knowledge) and also *denotation* (*extension*), and thus—an *ontological category* (the *category of ontological objects*).

The meanings (intensions) of *wfes* of *L* are treated as certain constituents of inter-subjective knowledge: logical concepts, logical judgments, operations on such notions or judgments, on the former and the latter, on other operations.

Object references (*references*) of *wfes* of *L*, and also constituents of knowledge, are objects of the cognised reality: individuals (concretes or abstract), states of things, operation on the indicated objects, and the like. *Denotations* (*extensions*) of *wfes* of *L* and constituents of knowledge are sets of such objects. The compatibility of these denotations is called *semantic compatibility* of *L*.

3 General Assumption Concerning the Logical Sense of Expressions of Language L

In the logical conception of language *L* and the semiotic senses outlined in the paper, expressions of *L* have syntactic, intensional and extensional senses and satisfy some general conditions of the logical sense of these expressions. Baseline conditions apply to syntactic and semantic unambiguity expressions of language *L* and the subsequent—relate to categorial compatibility and structural compatibility.

3.1 Syntactic and Semantic Unambiguity

The starting point is the syntactic and semantic unambiguity of the language expressions of language *L*. They should be:

- syntactically coherent and *wfes* of the set *S* (its essences),
- structurally unambiguous: have one syntactic sense (essence), i.e. do not contain amphiboly and have the one *mentioned functor-argument structure*,
- semantically unambiguous: have one *intension* and one *extension*, thus, one meaning and one denotation.

Remark 1 Syntactic and semantic unambiguity is not, of course, a feature of natural languages and not often of languages of non-exact sciences, but every syntactically and semantically ambiguous expression of these languages may be treated as a schema representing all of its interpretations that are unambiguous expressions (with exactly one syntactic and/or semantic sense) and which serve for an adequate description of specified fragments of reality.
For example, the sentence:
Teachers are tired because they teach students in various schools and they have a lot of them.
is structurally ambiguous (contains amphiboly), but it can be treated as a schema of two unambiguous sentences:
Teachers are tired because they teach students in various schools and they have a lot of students.
and
Teachers are tired because they teach students in various schools and they have a lot of schools.
On the other hand, the structurally unambiguous sentences
I came back tomorrow on foot on the colourful black-and-white train of 25:66.
She laughed with sweet tears which fell weightlessly onto the ceiling.
have no meaning or intensional and extensional sense; they are semantic nonsense.

In the categorial approach to language *L*, generated by the classical categorial grammar, a categorial index (type) $i(e)$ of a certain set *T* of types is unambiguously assigned to every *wfe* e of the set S, and every composed *wfe* of S has the functor-argument structure. Categorial indices (types) were introduced into logical semiotics by Ajdukiewicz [3] with the goal of determining the syntactic role of expressions and to examine their *syntactic connection*, in compliance with the *principle of syntactic connection* (*Sc*) which, in a free formulation, says that:

(*Sc*) *The categorial type of the main functor of each functor-argument expression of language L is formed out of the categorial type of the expression which the functor forms together with its arguments, as well as out of the subsequent types' arguments of this functor.*

Every functor-argument expression e of L can be written in a functionalargument form as follows:

$$(e) \qquad e = f(e_1, e_2, \ldots, e_n),$$

where f is the main functor of e and $e_1, e_2, \ldots, e_n$ are its subsequent arguments. Then, assuming that t is the type of e and $t_1, t_2, \ldots, t_n$ are successive types of its arguments, the type of the functor f satisfying the principle (*Sc*) can be written in the following quasi-fractional form:

$$(i(f)) \qquad i(f) = i(e)/i(e_1)i(e_2)\ldots i(e_n) = t/t_1t_2\ldots t_n.)$$

Then, the set S of all *wfes* of L is defined as the smallest set including the vocabulary V of L and closed under the principle (*Sc*):

Definition 1

$$S = \bigcap \left\{X\colon V \subset X \ \wedge\ \forall e = f(e_1, e_2, \ldots, e_n)\ (Sc(e)) \rightarrow e \in X\right\} \textit{ where}$$
$$Sc(e) = \big(i(e) = t \ \wedge\ \forall j = 1, 2, \ldots, n\; i(e_j) = t_j\big) \rightarrow i(f) = t/t_1t_2\ldots t_n.$$

In the formal definition of set S, it is required that each functor-argument constituent of the given expression should satisfy the principle (*Sc*).

Every *wfe* e of S is a *meaningful expression* of L possessing one *intension*, i.e. one meaning $\mu(e)$, where μ is the *operation of indicating the meaning* defined on the set S:

$$\mu\colon S \rightarrow \mu(S) = K.$$

The ***meaning*** $\mu(e)$ of the *wfe* e of the set S may be intuitively understood, in accordance with the understanding of meaning of expressions by Ajdukiewicz [1, 2] and, independently, by Wittgenstein [55] as a common property of all the *wfes* of S which possess the same manner of using as does e by competent users of language L (cf. [38]). Formalisation of thus conceived notion of meaning (and related notions) is given by WybraniecSkardowska in [62, 63]. In [62], its different philosophical conceptions, in particular those originating from Richard Montague, Donald Davison or Michael Dummett, are sketched. In my approach to the meaning of an expression of L, it is treated as a constituent of knowledge $K = \mu(S)$.

Every *wfe* e of S is a *meaningful expression* of L possessing one denotation, *extension* $\delta(e)$, where δ is the operation of denoting defined on set S:

$$\delta\colon S \longrightarrow \delta(S) = O.$$

The notion of *denoting* can, however, be introduced also as the *operation of denoting* δ_K, defined on the set of constituents of knowledge K:

$$\delta_K\colon K \longrightarrow \delta_K(K) \subseteq O.$$

The denotation $\delta(e)$ of the meaningful expression e is defined as the set of all ontological objects (or the ontological object) of the set $O = \delta(S)$, whose occurrences the expression e refers to. The ***denotation*** $\delta_K(k)$ of the constituent k of knowledge K is defined as the set of all extra linguistic, ontological objects to which k refers. Semantic compatibility takes place iff $\delta(S) = \delta_K(K) = O$ (see Fig. 2).

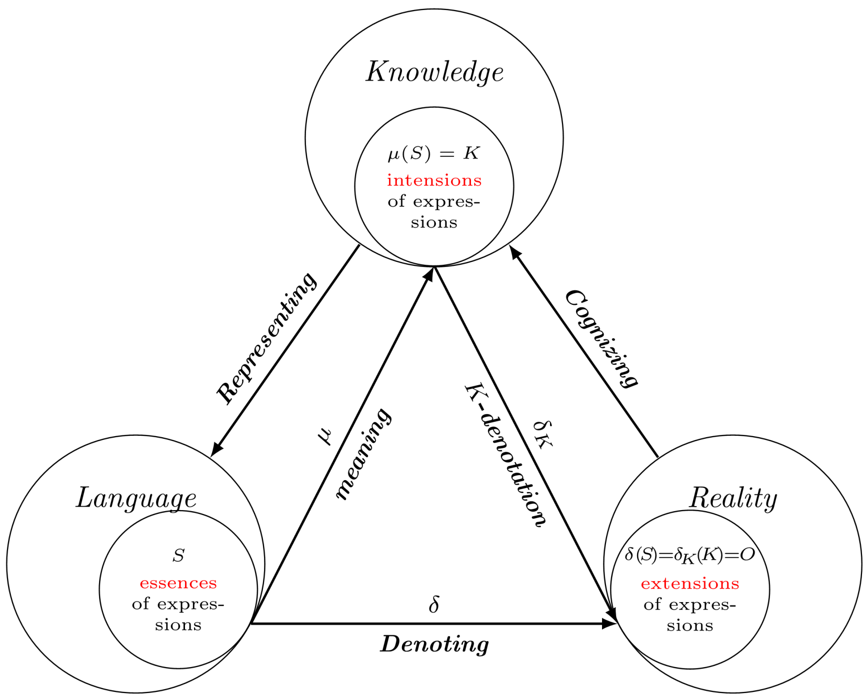

Figure 2: Semantic compatibility and language adequacy

3.2 Categorial Compatibility

In the logical conception of language L, the three distinguished kinds of logical sense of expressions of L must be compatible: any *wfe* of L having the syntactic sense, *essence* (belonging to a syntactic category of the defined kind), has a semantic, intensional sense (*intension*) and an extensional sense (*extension*) and is, simultaneously, a meaningful expression of L belonging to a defined intensional and, respectively, to a defined extensional semantic category. The logical sense of *wfes* of L is connected with the compatibility of their syntactic and semantic, intensional and extensional categories. In the categorial approach to language, the aforementioned categories of *wfes* of L are determined by attributing to them, as to their expressions, categorial indices (types) of the set T. Compatible categories have the same categorial type that unifies these three notions of sense (see Fig. 3).

Categorial types play here the role of a tool coordinating meaningful expressions and extralinguistic objects: *intensions* and *extensions* [4, 43, 46–48].

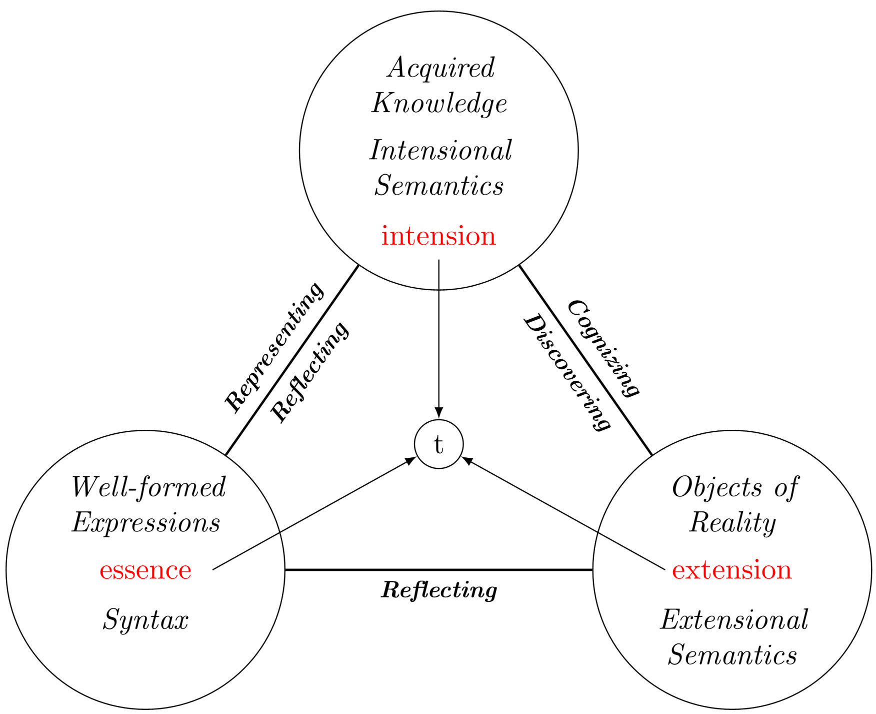

Figure 3: Type-unifying three notions of logical sense: essence, intension, extension.

3.2.1 Postulate of Categorial Compatibility

The postulate of categorial compatibility of syntactic and semantic categories is one of the most important conditions of the logical sense of *wfes* of language L. Here is a more formal description of this postulate. Let

1. S be the set of all *wfes* of L,
2. K—the set of all *intensions* of expressions of the set S; $K = \mu(S)$,
3. O—the set of all *extensions* of expressions of the set S; $O = \delta(S) = \delta_K(K)$.

The above-discussed syntactic and semantic categories of meaningful *wfes* of L are the following subsets of the set S:

Definition 2 $Syn_{\boldsymbol{t}} = \{\boldsymbol{e} \in S \colon \boldsymbol{i}(\boldsymbol{e}) = t\}$, where $\boldsymbol{i} \colon S \to T$,

Definition 3 $Int_{\boldsymbol{t}} = \{\boldsymbol{e} \in S \colon \boldsymbol{i_K}(\mu(\boldsymbol{e})) = t\}$, where $\boldsymbol{i_K} \colon K \to T$,

Definition 4 $Eks_{\boldsymbol{t}} = \{\boldsymbol{e} \in S \colon \boldsymbol{i_O}(\delta(\boldsymbol{e})) = t\}$, where $\boldsymbol{i_O} \colon O \to T$.

The syntactic (resp. **intensional**, resp. **extensional**) **category with the index** t is the set of all *wfes* of S that have the categorial index t (resp. *intensions* of which, resp. *extensions* of which have the index t).

The postulate of categorial compatibility defining an aspect of the logical sense of *wfes* of L has the following form [66–68]:

AXIOM(P) $\qquad Syn_{\boldsymbol{t}} = Int_{\boldsymbol{t}} = Eks_{\boldsymbol{t}}$ for any $t \in T$.

3.2.2 Type-unifying Logical Senses

The formal postulate (P) does not grasp the problem of the logical sense of language expressions of L adequately, because it does not show the relationships of the distinguished categories of *wfes* (*essences*) with the corresponding extra-linguistic categories of *intensions* and ontological categories of *extensions* in such a way that the mutual correspondence of elements of the triad: reality–knowledge–language, and the language adequacy of syntax with bi-level semantics, intensional and extensional, have been preserved (see Fig. 2).

As it was mentioned, unambiguous determined meanings (*intensions*) and denotations (*extensions*) should be assigned to *wfes* of L. They belong, respectively, to suitable extra linguistic categories of objects: *categories of meanings*, *intensions* (e.g. logical notions, logical judgments, operations on them) and *ontological categories of denotations, extensions* (e.g. individuals, set of individuals, states of affairs, or operations on them).

The categories of meanings, *intensions*, are subsets of the set K of constituents of knowledge, and ontological categories—subsets of the set O ontological objects. They are determined by categorial indices (types). And so, for any type $t \in T$:

Definition 5 $K_t = \{m \in K \colon i_K(m) = t\}$,

Definition 6 $O_t = \{o \in O : i_O(o) = t\}$.

Semantic categories (see Definitions 3 and 4) can by defined by formulas:

Corollary 1 $Int_t = \{e \in S : \mu(e) \in K_t\}$,

Corollary 2 $Ext_t = \{e \in S : \delta(e) \in O_t\}$,

stating that the semantic intensional (resp. extensional) category with the index t is the set of all *wfes* of L, the meanings, *intensions* (resp. denotations, *extensions*) of which belong to the category of constituents of knowledge (resp. to the ontological category) with the type t.

It is easy to prove that for any $e \in S$ and $t \in T$, by Corollaries 1 and 2 we can state that the Axiom (P) of categorial compatibility can be replaced by the following equivalent conditions:

Theorem 1 $e \in Syn_t$ *iff* $\mu(e) \in K_t$ *iff* $\delta(e) \in O_t$,

Theorem 2 $i(e) = i_K(\mu(e)) = i_O(\delta(e))$.

So, we see that categorial types serve also as a tool coordinating *wfes* of L and corresponding extra-linguistic objects, and that they unify the three notions of logical sense (see Fig. 4).

The idea of unification of the type of the logical term of the natural language, its intension and extension, is also one of the features of the type-theoretic object theory of E. Zalta [71, 72].

3.2.3 Semantic Compatibility

From Figures 2 and 4, we conclude that ontological objects of the set O are not only denotations of *wfes* of the set S (its *essences*), but also denotations of *intensions* of knowledge K corresponding to them.

Semantic compatibility for language L is defined by the following formula:

Definition 7 $\delta(e) = \delta_K(\mu(e)) \in O$, where δ_K is the operation on intensions, meanings of knowledge K.

From Definition 7, it immediately follows that if two expressions of L have the same meaning, then they have the same denotation:

Corollary 3 $\mu(e) = \mu(e') \rightarrow \delta(e) = \delta(e')$.

It is well known that the reverse implication does not hold. For example, the extensions of the terms ‘equilateral triangle’ and ‘equiangular triangle’ are the same, but their intensions are not.

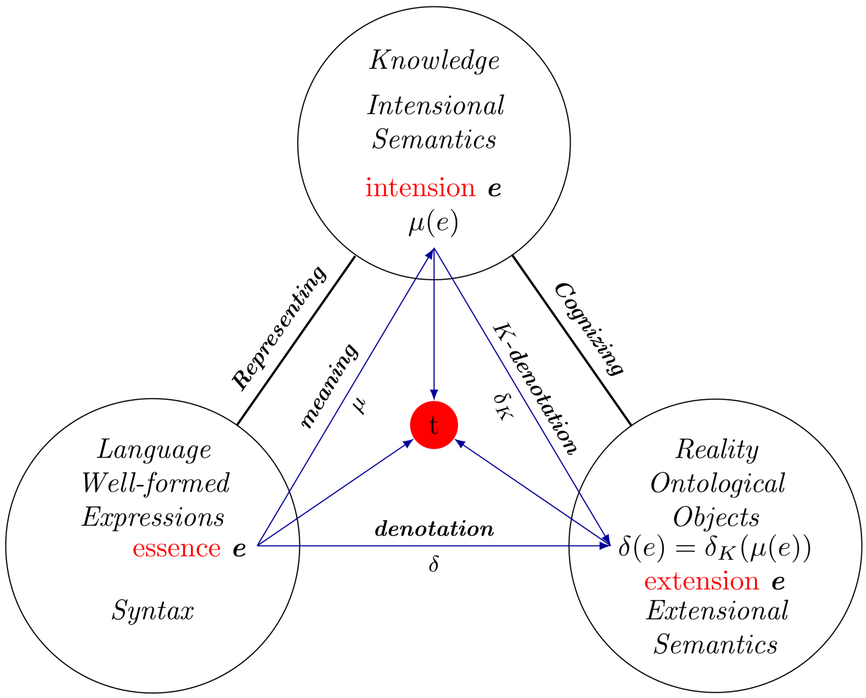

Figure 4: Semantic compatibility and type – unifying the three notions of sense.

3.3 Structural Compatibility

3.3.1 On the Structure of Expressions and Their Semantic Counterparts

The form of language expressions, their connectivity, well-formedness and logical sense, are connected with the structure of our knowledge and the structure of the cognising part of reality. Its language description is composed of parts that can be separated. Some of them are independent or relatively independent and are counted as basic language categories. In categorial languages, these are *names* and *sentences*. Others are auxiliary, dependent constituents of language expressions, which allow for the construction of more composed expressions from simpler ones. They are *functors*.

The categorial approach to L allows us to define the structural compatibility of its composed expressions and their corresponding meanings and denotations. Every *wfe* of L has one functor-argument structure. Functors of such expressions may be treated as partial functions defined on a proper subset of the set S and with the values in this set. Language L can then be characterised as the following partial algebra:

$$\boldsymbol{L} = \langle S, F\rangle, \text{ where } F \subset S, \text{ and } F \text{ is the set of all functors of } L.$$

As we mentioned in Sec. 3.1, (e), every composed expression e of the set S can be written in the functional-argument form:

$$e = f(e_1, e_2, \ldots, e_n),$$

where f is the main functor and $e_1, e_2, \ldots, e_n$ its subsequent arguments.

If the expression e is *wfe* of the set S, then—in accordance with the principle of syntactic connection (*Sc*)—the index of its main functor f, formed from the type t of e and successive types $t_1, t_2, \ldots, t_n$ of successive arguments $e_1, e_2, \ldots, e_n$ of the functor f, can be written in the following quasi-fractional form $(i(f))$:

$$(i(f)) \qquad i(f) = t/t_1 t_2 \ldots t_n.$$

The functor-function f corresponds to the function defined on meanings (*intensions*), respectively denotations (*extensions*) of arguments of this functor with subsequent types $t_1, t_2, \ldots, t_n$, the value of which is the intension, respectively the extension, of the expression e, which the functor f forms, with the type t.

If, in language L, we have two basic syntactic categories, names and sentences with respective types n and s, then meanings, *intensions*—logical notions with the type n—are assigned to names, and meanings, *intensions*—logical judgments with the type s—are assigned to sentences. Denotations of names are usually individuals or their sets, and denotations of sentences (in situational semantics) are states of affairs, situations. They also have, respectively, indices n and s.

Example 1 Let us consider the following sentence of a natural language:

(i) Robert practices football.

with the index s, the main functor of which is the word 'practices' of two name arguments, 'Robert' and 'football', with the index n. The expression (i) can be written in the following function-argument form:

(ii) practices(Robert, football).

The index of the functor 'practices' is s/nn. The meaning of the functor (with the same index) is the function which, being defined on the notions of 'Robert' and 'football' with the index n, as meanings (*intensions*) of these names in the sentence (ii), has, as the value, its meaning, i.e. the logical judgment with the index s stating that Robert practices football. Denotation of the functor is the mapping which, being defined on denotations (*denotates*) of names in (ii) with index n, so on person Robert and the sport discipline football, has, as its value, the state of affairs: the fact that Robert practices football, being the denotation of the sentence (ii); it has, like the sentence, the index s.

If somebody accepts, in accordance with Chomsky's phrase-structural grammar, that in (i) the main functor is 'practices football' (the predicate) of one argument 'Robert', then the function-argument form of (i) is as follows:

(iii) practices football (Robert) = (practices(football))(Robert)

The index of the composed functor ‘practices football’ is s/n, and the index of the functor ‘practices’ in it is $(s/n)/n$. Then the meaning and the denotation of the latter functor differ essentially from those used in (ii).

Remark 2 As we can see in a natural language, sentences may have a different functor-argument structure, thus different semantic senses: intensions and extensions. Therefore, they can be treated as skeletons, schemas which represent unambiguous expressions with one functor-argument structure, one meaning and one denotation.

3.3.2 Principles of Compositionality

From the axiom (P) of categorial compatibility, three principles of compositionality follow [64–67]: one syntactic (compositionality of essences, syntactic forms) and two semantic: compositionality of meaning (intension) and compositionality of denotation (extension). For every composed expression of L, the form $e = f(e_1, e_2, \ldots, e_n)$ and functions $h = i, \mu, \delta$, their common schema has the form:

$$(COMP_h) \qquad h(e) = h((f(e_1, e_2, \ldots, e_n)) = h(f)(h(e_1), h(e_2), \ldots, h(e_n)).$$

For $h = i$, we have the syntactic principle, for $h = \mu, \delta$ we obtain the semantic principles corresponding to the ones already known to Frege [19] (cf. also [21, 27, 28, 35–37, 39, 22–24, 29]).

Speaking freely, these principles state that:
The categorial type (*the syntactic form*), resp. *the meaning*, resp. *the denotation of a well-formed functor-argument expression of language L is the value of the function of the type,* resp. *the function of the meaning*, resp. *the function of the denotation, of its main functor defined on types,* resp. *on meanings,* resp. *on denotations subsequent arguments of this functor.*

3.3.3 Main Properties of Functions $h(f)$

The formulation of the principle ($COMP_h$) defines $h(f)$ as functions. Indeed, index $i(f)$ of functor f is the function:

$$i(f)\colon \{(i(e_1)\} \times \{i(e_2)\} \times \ldots \times \{i(e_n))\} \to \{i(e)\},$$

which, defined on n-tuple indices $(i(e_1), i(e_2), \ldots, i(e_n))$, has the value $i(e)$; hence there follows the syntactic principle of compositionality ($COMP_i$).

Similarly, the meaning and the denotation of the functor f, defined on meanings and, respectively, on denotations of its arguments, are functions whose values are, respectively, meanings and denotations of the expression e. However, let us remember that the same n-argument functor ($n \geq 1$), e.g. ‘practices’ in (ii) of Example 1, may have different arguments, though its meaning, respectively denotation, is uniquely determined.

Thus, for any *wfe* $e = f(e_1, e_2, \ldots, e_n)$ such that for types $i(e) = t$, $i(e_k) = t_k$, where $k = 1, \ldots, n$, $\mu(f)$, is the function:

$$\mu(f)\colon K_{t_1} \times K_{t_2} \times \ldots \times K_{t_n} \to K_t,$$

which for intensions of arguments of functor f has the value $\mu(e)$ compatible with the principle ($COMP_\mu$), and $\delta(f)$ is the function:

$$\delta(f)\colon O_{t_1} \times O_{t_2} \times \ldots \times O_{t_n} \to O_t,$$

which for denotations of arguments of functor f has the value $\delta(e)$ compatible with the principle ($COMP_\delta$).

Remark 3 Note that the logical sense of language expressions, including functors, assumes that they have both intensions and extensions. Thus, any functor f forming the complex expression e has the meaning $\mu(f)$ and at the same time denotation (reference) $\delta(f)$, and its meaning and denotation are functions that meet the conditions listed above in accordance with the semantic principles of compatibility.

In semiotic literature, however, we encounter some controversy regarding the sense of functors, which are predicates of name arguments in natural language sentences. Debate on Geach-Dummet controversy about the sense of a predicate is reconstructed by M. Tałasiewicz [49]. For Peter T. Geach sense of a predicate is its meaning and a function satisfying the principle of compositionality of meaning, while for Michael Dummet the sense is rather something that determines its denotation (reference). Tałasiewicz in [49] proposed a solution giving predicates both semantic senses as functions fulfilling the relevant conditions of semantic compositionality principles. This solution is interesting because it allows us to maintain semantic compatibility (see Def. 7).

3.3.4 Generalisation of Ajdukiewicz's Cancellation Principles

Just like the index of functor f in expression $e = f(e_1, e_2, \ldots, e_n)$ (see ($i(f)$) in Sec. 3.1, we write its meaning and denotation in a quasi-fractional form. The general quasi-fractional form of the functions $h(f)$, for $h = i, \mu, \delta$ is given as the schema:

$$(h(f)) \qquad h(f) = h(e)/h(e_1)h(e_2)\ldots h(e_n).$$

At the established quasi-fractional records ($\boldsymbol{i}(f)$): the type of the functor f,($\mu(f)$) of its meaning (intension) and ($\delta(f)$) of its denotation (extension), some counterparts of Ajdukiewicz's rules of cancellation of fractional indices (types) that serve to check the syntactic connection of complex expressions, correspond to the principles of compositionality ($COMP_h$). They follow from them. To justify these rules, it is sufficient to use the equality ($COMP_h$) from the left to the right and ($h(f)$). They allow us to calculate types, meanings (intensions) and denotations (extensions) of functor-argument expressions of L. Their schema, for $h(f)$, for $h = i, \mu, \delta$, can be written in the following way:

(CANCh) $$h(e)/h(e_1)h(e_2)\ldots h(e_n)(h(e_1), h(e_2), \ldots, h(e_n)) = h(e).$$

Example 2 For the functor 'practices' in the functor-argument sentence

(ii) practices(Robert, football)

the cancellation principles for $h = i, \mu, \delta$ have the forms:

$$s/nn(n,n) = s,$$
$$\mu((ii))/\mu(\text{Robert})\mu(\text{football})(\mu(\text{Robert}), \mu(\text{football})) = \mu((ii)),$$
$$\delta((ii))/\delta(\text{Robert})\delta(\text{football})(\delta(\text{Robert}), \delta(\text{football})) = \delta((ii)),$$

while for the functor 'practices football' in the sentence

(iii) practices football(Robert) = (practices(football))(Robert)

the cancellation principles for $h = i, \mu, \delta$ are the following:

$$((s/n)//n(n))(n) = s/n(n) = s,$$
$$\mu(\text{practices football(Robert)}) = ((\mu(\text{practices(football)})(\mu(\text{Robert})) =$$
$$= (\mu(\text{practices})(\mu(\text{football}))(\mu(\text{Robert})) =$$
$$= (((\mu(\text{iii})/\mu(\text{Robert}))//\mu(\text{football})(\mu(\text{football}))(\mu(\text{Robert})) =$$
$$= ((\mu(\text{iii})/\mu(\text{Robert}))(\mu(\text{Robert})) = \mu(\text{iii}).$$

Similarly, for $h = \delta$.

Let us observe that sentences (ii) and (iii) have the same categorial type s, and, according to Theorem 2, their intensions and extensions also have the type s. However, the appropriate constituents of these sentences and their intensions and extensions have different categorial types.

3.3.5 Models of L and the Notion of Truth

The principles of compositionality can be considered as some conditions of homomorphisms $h = i, \mu, \delta$ of the syntactic algebra of language $\boldsymbol{L}$ into algebras of its images $h(\boldsymbol{L})$, i.e.

$$\boldsymbol{L} = \langle S, F \rangle \quad \underset{h}{\rightarrow} \quad h(\boldsymbol{L})\langle h(S), h(F) \rangle,$$

where F is the set of all simple functor-partial functions mapping subsets of set S into set S, and $h(F)$, for $h = i, \mu, \delta$, is the set of functions corresponding to the functor-functions of set F.

Let us notice that the algebraisation of language can already be found in Leibnitz's papers. We can also find the algebraic approach to issues connected with syntax, semantics and compositionality in Montague's *Universal Grammar* [35] and in papers of Dutch logicians, especially in those by J. van Benthem [50–53] and T.M.V. Jansen [27, 28]. The difference between their approaches and the approach which is presented here lies in fact that carriers of the syntactic and semantic algebras include functors, or, respectively, their suitable correlates, i.e. their $i-$ or semantic-function $\mu-$ and $\delta-$ images; simple functors and their suitable $i-$, $\mu-$, $\delta-$ images are simultaneously partial operations of this algebras. They are set-theoretical functions, determining those operations.

The algebra $i(L) = \langle i(S), i(F)\rangle$ is called the *syntactic model of language L*, while the algebras

$$\mu(L) = \langle \mu(S), \mu(F)\rangle = \langle K, \mu(F)\rangle \text{ and } \delta(\boldsymbol{L}) = \langle \delta(S), \delta(F)\rangle = \langle O, \delta(F)\rangle$$

are the *semantic models* for L; the first is called the *intensional model* for *L*, the other one, the *extensional model* for *L*.

In the process of cognition of reality, we want the sentences of the language *L*, representing the knowledge acquired about it, to be the carriers of true information about cognised portion of reality; they should be true in the above-mentioned models of *L*. Language as a tool for describing reality must distinguish the category of *sentences* among its syntactic categories. True sentences have informative content and allow us to enrich our knowledge. If for $h = i, \mu, \delta$, it is the case that the sentence e of language L is true in models $h(\boldsymbol{L})$, we may say that our cognition by means of the sentence e is *true*.

The notions of truthfulness in appropriate models are introduced theoretically by means of three new primitive notions *Th*, satisfying for $h = i, \mu, \delta$ the schema of axioms:

AXIOM(*Th*) $\quad\quad \emptyset \neq Th \subseteq h(S)$

and are understood intuitively, respectively, as the singleton consisting of the index of true sentences, the set of all true logical judgments, the set composed of the states of affairs that take place (in situational semantics) or the singleton composed of the value of truth (in Frege's semantics).

For $h = i, \mu, \delta$, we assume that:

Definition 8 The sentence e of language L *is true in the model* $h(\boldsymbol{L})$ iff $h(e) \in Th$.

In particular, if $h = \delta$, then we may state that the sentence e of L *is true in the extensional model* iff its extension is the state of affairs that takes place (in situational semantics), or it is the value of truth (in Fregean semantics).

3.3.6 Some Remarks Concerning the Problem of Categories of First-Order Quantifiers

There is a well-known problem with determining syntactic and semantic categories, and therefore a problem with categorial types of quantifiers, and, in particular, of quantifiers of the first order language $L1$ and types of their intensions and extensions.

To solve this problem, we can apply the principles of compositionality and the cancellation rules. Some general findings relating to the solution to the problem of syntactic categories of quantifiers, their denotation or/and meaning are presented in the following papers: [58, 59, 70, 71]. In this work, I will limit myself to dealing with this problem for the quantifier in the simple formulas of $L1$.

Example 3 Let us consider the quantifier expressions:

$$(1)\ \forall x\ P(x) \qquad \text{and} \qquad (2)\ \exists x\ P(x),$$

in which P is an established one-argument predicate treated as a one-argument functor-function, and the quantifiers $\forall$ and $\exists$ are treated as two-argument functors-functions defined on a variable standing next to them and a sentential function with a free variable bound by the given quantifier. The categorial type for x is n_1, i.e. $i(x) = n_1$, the type for P is s_1/n_1, i.e. $i(P) = s_1/n_1$, because we assume that the type for the sentential function $P(x)$ is s_1, since $i(P(x)) = i(P)(i(x)) = s_1/n_1(n_1) = s_1$. The type of quantifiers $\forall$ and $\exists$ is then: s/n_1s_1, i.e. $i(\forall) = i(\exists) = s/n_1s_1$. Using the principles of compositionality and cancellation, we can 'compute' the type of the expression (1) in its functor-argument form:

$$\begin{aligned} i(\forall(x, P(x)) &= i(\forall)(i(x), i(P(x))) = i(\forall)(i(x), i(P)(i(x))) \\ &= s/n_1s_1(n_1, s_1/n_1(n_1)) = s/n_1s_1(n_1, s_1) = s. \end{aligned}$$

In a similar way, we 'calculate' the index of the expression $(2) = \exists(x, P(x))$. Thus, expressions (1) and (2) are sentences.

We will now define the denotation of the discussed quantifiers in Fregean semantics. We assume that $\delta(x) = U$, where U is the universe of individuals in an established model $\boldsymbol{M}_{L1}$; $\delta(P)\colon U \rightarrow \delta(P(x))$, where $\delta(P(x)) = \delta(P)(\delta(x)) = \{u \in U\colon \delta(P(x/u)) = 1\}$ and $P(x/u)$ is a sentence which we get for replacing in the sentential function $P(x)$ its free variable x by the name of the individual u, and 1 is the value truth. Then,

$$\delta(\forall x\ P(x)) = \delta(\forall)(\delta(x), \delta(P(x)) = \begin{cases} 1 & \text{if } \delta(x) = U = \delta(P(x)) \\ 0 & \text{if } \delta(x) = U \neq \delta(P(x)) \end{cases}$$

$$\delta(\exists x\ P(x)) = \delta(\exists)(\delta(x), \delta(P(x)) = \begin{cases} 1 & \text{if } \delta(x) \cap \delta(P(x)) \neq \emptyset \\ 0 & \text{if } \delta(x) \cap \delta(P(x)) = \emptyset. \end{cases}$$

So, *the denotation* $\delta(\forall)$ (resp. $\delta(\exists)$) *of the quantifier* $\forall$ (resp. $\exists$) *is the function which, for the universe U and the denotation of the scope of the quantifier, has the truth value iff the denotation of its scope is the universe* (resp. *the denotation of this scope has at least one individual of the universe*).

In a similar way, we define the meanings of the quantifiers $\forall$ and $\exists$ in (1) and (2).

Example 4 It is obvious that the quantifiers $\forall$ and $\exists$ are typically ambiguous in logic, depending on a type. In other contexts, e.g., in the expressions

$$(3)\ \forall x, y\ R(x, y) \quad \text{and} \quad (4)\ \exists x, y\ R(x, y) \quad \text{or}$$
$$(5)\ \forall x\ R(x, y) \quad \text{and} \quad (6)\ \exists y\ R(x, y),$$

they have other categorial types, intensions and extensions. Their categorial type in expressions (3) and (4) is $s/n_1 n_1 s_2$, where s_2 is the index of the sentential function of two individual variables, while in expressions (5) and (6) they have the type $s_1/n_1 s_2$. The predicate-functor's R categorial type is, of course, $s_2/n_1 n_1$.

It is easy to check and 'compute' that exemplary expressions are syntactically connective, therefore *wfes*. The first of them, (3) and (4), are sentences, because they have the index s, while the others, (5) and (6), are sentential functions with one free variable, because they have the index s_1.

4 Final Remarks

The logical sense of language expressions is, of course, a kind of idealisation. In the logical and categorial conception of language, the sense of its expressions, both syntactic and semantic, intensional and extensional, ensures their structural and semantic unambiguity and mutual syntactic and semantic compatibility. A natural language, and often also the scientific variation, is a living creature, still developing. The degree of syntactic and semantic senses of its expressions changes, it can be narrower or higher, depending on its skilful precision. However, structural or semantically ambiguous expressions can always be split into expressions having unambiguous syntactic and semantic senses and be categorially analysed. Also, expressions that are imprecise or vague can be replaced by sets of sentences with precise meanings and denotations. Moreover, they can be considered separately with respect to their categorial structure, because only expressions with a high degree of logical sense, syntactical and semantical (intensional and extensional), get closer to the sense and may, after a proper justification, become theorems of a given discipline of knowledge and be a base for satisfactory interpersonal communication about our world.

Acknowledgements The author wishes to thank the Reviewer of this paper for comments and suggestions. Several remarks that he made led to some additions or improvement in the text.

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Chapter 11
Categories of First-Order Quantifiers

Urszula Wybraniec-Skardowska

Abstract One well known problem regarding quantifiers, in particular the 1st-order quantifiers, is connected with their syntactic categories and denotations. The unsatisfactory efforts to establish the syntactic and ontological categories of quantifiers in formalized first-order languages can be solved by means of the so called *principle of categorial compatibility* formulated by Roman Suszko, referring to some innovative ideas of Gottlob Frege and visible in syntactic and semantic compatibility of language expressions. In the paper the principle is introduced for *categorial languages* generated by the Ajdukiewicz's classical categorial grammar. The 1st-order quantifiers are typically ambiguous. Every 1st-order quantifier of the type $k > 0$ is treated as a two-argument functor-function defined on the variable standing at this quantifier and its scope (the sentential function with exactly k free variables, including the variable bound by this quantifier); a binary function defined on denotations of its two arguments is its denotation. Denotations of sentential functions, and hence also quantifiers, are defined separately in Fregean and in situational semantics. They belong to the ontological categories that correspond to the syntactic categories of these sentential functions and the considered quantifiers. The main result of the paper is a solution of the problem of categories of the 1st-order quantifiers based on the principle of categorial compatibility.

Key words: 1st-order quantifiers • Categorial languages • Syntactic categories • Denotation • Ontological categories • Denotational Semantics • Compositionality • Categorial compatibility

First published in: Garrido, A., Wybraniec-Skardowska, U. (eds.) The Lvov-Warsaw School. Past and Present, pp. 763–777. Springer-Birkhäuser, Cham (2018).

1 Introduction

Around 1879 Frege and—independently—Charles Sanders Peirce developed a way to extend sentential logic by introducing symbols representing *determiners*, such as ‘all’, ‘some’, ‘no’, ‘every’, ‘any’, and so on.

Frege and Peirce used two symbols: the *universal quantifier* (which we will write $\forall$) corresponding roughly to the English ‘all’, ‘every’ and ‘each’ and the *existential quantifier* (which we will write $\exists$) corresponding to the English ‘some’, ‘a’, ‘an’.

In this paper we will consider only standard, Fregean quantifiers $\forall$ and $\exists$ of the 1st-order as individual variable-binding operators. They are used in formulas of predicate logic of the 1st-order and in formalized languages of elementary theories based on this logic. Their syntactic role and semantic references, i.e. denotation, extension, created some problems that have not been satisfactorily solved yet.

In the next part (Section 2) I shall partially explicate the problem of quantifiers. In Section 3, I’ll outline some intuitive foundations of my theory of categorial languages which gives the formal direction for justification of my solution of the problem of quantifiers. It corresponds to the principle (*CC*) of categorial compatibility based on some Frege’s ideas and was formulated by Roman Suszko [30]. The solution of the problem is presented in Section 4.

2 Problem of Quantifiers

The problem of quantifiers is connected with the difficulty pertaining to establishing their syntactic and semantic categories.

Leśniewski’s theory of semantic/syntactic categories [20, 21], which was improved by Ajdukiewicz [3] by introducing categorial indices, does not, obviously, solve this problem, which limits the universal character of the theory.

Leśniewski’s hierarchy of semantic/syntactic category does not include any variable-binding operators. Leśniewski, in his protothetics and ontology systems, allows only one operator—the universal quantifier, noting it as parentheses, Ajdukiewicz, on the other hand, indicates the difficulty of assigning to quantifiers the index s/s or s/ns.

Assigning to them the index s/s, i.e. the category of sentence-forming functors of one-sentence argument, would mean that the quantifiers belong to the same category as one-argument connectives, and assigning to them the index s/ns of sentence-forming functors of one-name and one-sentence arguments would mean that we include them into the same category as some expressions of indirect speech, e.g. ‘think that’, ‘know that’, etc.

It has been suggested that the categorial grammar, which Bar-Hillel derived from Ajdukiewicz’s version of the theory of semantic/syntactic categories, does not satisfactorily account for the role of bound variables and operators binding them.

Suszko [28, 30] assigns to them the index $s//s/n$, and thus the index of sentence-forming functor of the argument, which is a one-argument predicate. In this way, the index, for example in the sentence '$\forall x(x\, \textit{flows})$' pertains to the entire quantifier-variable pattern '$\forall x(x \ldots)$' (see Simons [20]) which corresponds to English word 'everything' (see also Cresswell [4], Simons [21]).

Suszko and many other researchers of language syntax treat quantifiers as expressions independent of the quantifier variable. Generally, researchers avoid bound variables in attempting to solve the problem, for example by means of combinators (Curry [5, 6], Curry and Feys [7], see e.g. Simons [20]).

But earlier, Suszko stated that mounting variable-binding operators into a syntactic scheme requires general principles other than the theory of syntactic/semantic categories.

The principle (*CC*) of categorial compatibility is one such principle. It allows us to assign to every expression of a formalized 1st-order language, which possesses an index symbolizing a syntactic category, a denotation whose ontological category (relative to the universe U of a given model of the language) is indicated by the same index.

Suszko assumes that

- the denotation of the entire expression $\forall x(e(x))$, where $e(x)$ is a sentential function with the free variable x, is ether the logical value 1 (of truth) or the logical value 0 (of falsity) which belong to the ontological category with the index s, and
- the denotation of the universal quantifier $\forall$ is the function of generalization which has the value 1 in only one case, if its argument is the universe U.

The function of generalization belongs to the ontological category with the index $s//s/n$ because its arguments are any sets belonging to the family $P(U)$ included into the ontological categories with the index *s/n*. In this way the principle (*CC*) holds although the principle of syntactic connection (*SC*) does not hold because no index is assigned to quantifier variable x, and the scope of the quantifier $\forall$ (here $e(x)$) is not one-argument predicate of the syntactic category with the index s/n.

In the next parts of this paper I explicate both the principle (*SC*) of syntactic connection and the principle (*CC*) of categorial compatibility on the basis of my theory of categorial languages [36, 31, 32, 37, 40] which allows us to give some solutions to the problem of quantifiers.

The essence of the approach proposed here is considering them to be typical syntactic notions: functors-functions mapping language expressions into language expressions that correspond to some functions on extralinguistic objects—on denotations of arguments of these functors.

Let us note that a standard background for research in the field of mentioned quantifiers assumes treating them as some functions or relations on extralinguistic objects, mostly functions with index $t//t/e$ (cf. Mostowski [18], Lindström [15], Montague [16, 17], Nowaczyk [19], van Benthem [27, 28], van Benthem and Westerståhl [29]).

3 Some Intuitive Foundations of the Theory of Categorial Languages

3.1 Main Ideas of Formalization of Categorial Language

In the paper, formal-logical considerations relate to syntax and extensional semantics of any language L characterized *categorially*:

- in the spirit of some ideas of Husserl [14] and Leśniewski–Ajdukiewicz's theory of syntactic/semantic category (see Leśniewski [20, 21], Ajdukiewicz [3, 2]),
- in accordance with Frege's ontological canons [13],
- in accordance with Bocheński's motto [8]: *syntax mirrors ontology*, and
- some ideas of Suszko [28–31]: *language should be a linguistic scheme of ontological reality and simultaneously a tool of its cognition*.

The paper includes developing and some explications of these authors' ideas. It also presents, in a synthetic form, some ideas presented in my papers published in [36, 31, 32, 37, 40].

Language L is there defined, if the set S of all *well-formed expressions* (briefly *wfes*) is determined. These expressions must satisfy requirements of categorial syntax and categorial semantics.

3.2 Categorial Syntax

The categorial syntax of L is connected with generating the set S by the classical categorial grammar and belonging *wfes* of S to appropriate syntactic/semantic categories.

A characteristic feature of categorial syntax is that each composed *wfe* of the set S has a *functor-argument* structure, in this sense that, in accordance with the principle originated by Frege [8], it is possible to distinguish in it its constituent called the *main functor*, and the other constituents — called *arguments* of that functor, yet each constituent of the *wfe* has a determined syntactic category.

If e is a functor-argument *wfe* of S, f is its main functor and $e_1, e_2, \ldots, e_n$ its subsequent arguments then e can be written in the functional-argument form:

$$(e) \qquad e = f(e_1, e_2, \ldots, e_n).$$

In categorial approach to the language L, syntactic categories of *wfes* of L are determined by attributing to them, like their expressions, categorial indices of a certain set I. To every *wfe* e of the set S is unambiguously assigned a categorial index (type) $i_S(e)$ of the set I; *wfes* belonging to the same syntactic category *CATa* have the same categorial index a.

Categorial indices were introduced by Ajdukiewicz [3] into logical semiotics with the aim to determine the syntactic role of expressions and to examine their syntactic connection, in compliance with the principle of syntactic connection (*SC*) discussed below.

The set S of all *wfes* of L is then intuitively defined as the smallest set including the vocabulary of L and closed with respect to the principle (*SC*), which in free formulation says that

(*SC*) *The categorial index of the main functor of each functor-argument expression of the language L is formed out of the categorial index of the expression which the functor forms together with its arguments, as well as out of the subsequent indices of arguments of this functor*.

In the formal definition of the set S it is required that each functor-argument constituent of the given expression should satisfy the principle (*SC*).

If the functor-argument expression $e = f(e_1, e_2, \ldots, e_n)$ is a *wfe* (it belongs to the set S), then in accordance to the principle of syntactic connection (*SC*) the index of its main functor f formed from the index a of e and successive indices $a_1, a_2, \ldots, a_n$ of successive arguments $e_1, e_2, \ldots, e_n$ of the functor f, can be written in the following quasi-fractional form:

$$(\boldsymbol{i_S}) \qquad i_S(f) = i_S(e)/i_S(e_1)i_S(e_2)\ldots i_S(e_n) = a/a_1a_2\ldots a_n.$$

3.2.1 An Algebraic Structure of Categorial Language

In categorial language L we can distinguish two sets: the set B of all basic *wfes* of S and the set F of all functors of S such that

$$S = B \cup F \text{ and } B \cap F = \emptyset,$$

where functors of the set F differ from basic expressions of B that they have indices formed from simpler ones. If the functor f has the functoral index of the form ($\boldsymbol{i_S}$), i.e. the index of the form $a/a_1a_2\ldots a_n$ then it belongs to the syntactic category $CATa/a_1a_2\ldots a_n$ and so to the category of functors forming expressions with the index a if their arguments are n expressions with successive indices $a_1, a_2, \ldots, a_n$. So the functor f can be treated as the following partial function defined on *wfes* of S:

$$f : CATa_1 \times CATa_2 \times \ldots \times CATa_n \rightarrow CATa$$

mapping of *wfes* from Cartesian product of syntactic categories $CATa_1$, $CATa_2, \ldots,$ $CATa_n$ into the category $CATa$. Then we have

$$(CAT_f) \qquad f \in CATa/a_1a_2\ldots a_n = CATa^{CATa_1 \times CATa_2 \times \ldots \times CATa_n}.$$

In this way we simultaneously can regard the categorial language L as an algebraic structure $\boldsymbol{L}$, partial algebra with the carrier S and the set $Fo \subseteq F$ of partial functions on S (simple functors of L):

$$\boldsymbol{L} = \langle S, Fo \rangle\,.$$

3.3 Categorial Semantics

Categorial extensional semantics is connected with *denotations* of *wfes* of S and with their belonging to an appropriate semantic extensional category. Each constituent of the composed *wfe* has determined a semantic extensional category and also a *denotation*, and thus—an *ontological category* (the *category of ontological objects*). *Denotations* (*extensions*) of *wfes* of L are sets of *object references* (*references*) of *wfes* of L, objects of the cognized reality, e.g.: individuals, sets of individuals, states of affairs, operation on the indicated objects, and the like.

We will concentrate only on referential relationships between expressions of L and reality to which they refer. We enrich the categorial grammar generating L by the *denotation operation* δ regarded as its semantic component. The denotation operation δ assigns to every *wfe* of the set S an object of ontological reality *ONT* describing by the language L — its denotation belonging to an ontological category. So

$$(\delta) \qquad \delta : S \to ONT,$$

where *ONT* is the sum of all ontological categories corresponding to *wfes* of S.

According to some innovative ideas of Frege [9, 13], Bocheński's (his famous motto: syntax mirrors ontology) and Suszko [28–30] who anticipated the research in categorial semantics and was the first to use categorial indices as a tool for coordination of expressions and their references, extralinguistic objects, the mutual dependence of syntactic and semantic formal description of L should be considered by keeping the *principle* (*CC*) *of categorial compatibility*, based on the compatibility of the syntactic category of each language expression of L with the ontological category assigned to its denotation. The principle (*CC*) of syntactic and semantic, i.e. also ontological categorial compatibility in Suszko's formulation can be given by keeping for any *wfe* e of categorial language L the relationship:

$$(CC) \qquad e \in CAT\iota \quad \text{iff} \quad \delta(e) \in ONT\iota,$$

where $CAT\iota$ and $ONT\iota$ are: the syntactic category and the ontological category, respectively, with the same categorial index ι, and δ is the operation of denotation.

From the principle (*CC*) it follows that for any $e = f(e_1, e_2, \ldots, e_n) \in S$ with the main functor-function $f \in CATa/a_1a_2 \ldots a_n$ satisfying the condition (CAT_f) the following conditions are satisfied:

$$(ONTf) \qquad \delta(f) \in ONTa/a_1a_2 \ldots a_n = ONTa^{ONTa_1 \times ONTa_2 \times \ldots \times ONTa_n}$$

and

$$(PCD) \qquad \delta(f(e_1, e_2, \ldots, e_n)) = \delta(f)(\delta(e_1), \delta(e_2), \ldots, \delta(e_n)).$$

The condition (*ONTf*) states that the denotation (object reference) of the main functor of the composed *wfe* e of the set S is the set-theoretical function mapping the Cartesian product of ontological categories $ONTa_1 \times ONTa_2 \times \ldots \times ONTa_n$ into the ontological category *ONTa* and it is defined by means of the condition (*PCD*) connected with some Frege's ideas and called the principle of compositionality of denotation.

3.3.1 An aAlgebraic Ontological Structure Corresponding to the Partial Algebra *L*

The operation δ assigns the following ontological structure $\boldsymbol{R_L}$ of a reality corresponding to language L to the algebraic structure $\boldsymbol{L}$:

$$\boldsymbol{R_L} = \langle ONT, ONT_{Fo} \rangle \,,$$

where ONT_{Fo} is the sum of all ontological categories corresponding to all functors of the set *Fo*. The structure $\boldsymbol{R_L}$ is a partial algebra similar to the algebra $\boldsymbol{L}$ and the principle (*PCD*) is simultaneously the condition of homomorphism of the algebra $\boldsymbol{L}$ into the algebra $\boldsymbol{R_L}$, i.e.

$$\delta : \langle S, Fo \rangle \xrightarrow[hom]{} \langle ONT, ONT_{Fo} \rangle \,.$$

A ***model of language*** L is the structure of homomorphic images of components of $\boldsymbol{L}$, i.e. the substructure $\boldsymbol{M_L} = \langle \delta(S), \delta(Fo) \rangle$ of the structure $\boldsymbol{R_L}$.

If we distinguish in the set B of basic *wfes* of S the category *CATs* of all sentences of language L, then the notion of *truthfulness* of any sentence $e \in CATs$ in the model $\boldsymbol{M_L}$ is defined as follows:

(T) $\quad e$ is a *true sentence* in the model $\boldsymbol{M_L}$ iff $\delta(e) \in T$,

where T is primitive notion of the considered theory intuitively understood either as the singleton with the true value (in Fregean semantics) or as the set of all states of affairs that take place (in situational semantics).

4 The Solution of the Problem of Quantifiers of 1st-Order

The unsatisfactory efforts to establish, in the sense of the principle (*CC*) of categorial compatibility, the category of quantifiers in formalized 1st-order languages can be solved by means of notions and statements of the above outlined theory of categorial languages.

Let L_1 be any 1st-order formalized language. Let us treat any standard quantifier of L_1 as a context-dependent functor of two arguments:

1. a quantifier variable (the variable accompanying this quantifier) and
2. its scope, i.e. a sentential function including as a free variable the same variable as the quantifier variable.

4.1 Different Types of the 1st-Order Quantifiers and Their Syntactic Categories

A standard, the 1st-order quantifier is a functor forming a new sentential function (in particular a sentence of L_1) in which there occur one free variable less than in the scope of this quantifier (the variable bound by the quantifier). As such a functor, a quantifier can be treated as a set-theoretical function relative to the number of free individual variables occurring in its scope. So, we should not speak of one existential $\exists$ or one universal quantifier $\forall$ but about different types of such quantifiers depending of the number of free variables in their scope. We will use numerical superscripts in order to point out these different types of quantifiers.

Let

- *Var* be the set of all individual variables for L_1, with categorial index n_1;
- $S = S_0$—the set of all its sentences, with the categorial index s;
- $S_k (k \geq 1)$—the set of all sentential functions in which exactly k free variables occur, with the index s_k.

For example, if $\alpha(x_1, x_2, x_3) \in S_3$, where $x_1, x_2, x_3 \in Var$, then the expressions:

$$\forall^3 x_2 \alpha(x_1, x_2, x_3) \in S_2,$$
$$\exists^2 x_3 \forall^3 x_2 \alpha(x_1, x_2, x_3) \in S_1,$$
$$\forall^1 x_1 \exists^2 x_3 \forall^3 x_2 \alpha(x_1, x_2, x_3) \in S_0$$

and quantifiers $\forall^3$, $\exists^2$, $\forall^1$ belong to different syntactic categories with indices $s_2/n_1 s_3$, $s_1/n_1 s_2$, $s/n_1 s_1$, respectively.

More generally, the quantifiers $\forall^k$ and $\exists^k$ ($k \geq 1$) are treated as the functors-functions:

$$\forall^k, \exists^k : Var \times S_k \to S_{k-1} \qquad (S_0 = S).$$

Thus, in accordance to $(CATf)$, for $k > 0$ we have

$$(CAT\forall^k, \exists^k) \qquad \forall^k, \exists^k \in CATs_{k-1}/n_1 s_k \qquad (s_0 = s),$$

and the principle of syntactic connection (*SC*) for them is satisfied.

Their denotations and ontological categories should be defined in such a way as to satisfied the principle (*CC*) of categorial compatibility (their denotations should belong to the ontological category $ONTs_{k-1}/n_1 s_k$) and the principle (*PCD*) of compositionality of denotation.

Let the denotation operation for the language L_1 be the function d in Fregean, standard semantics and the function $\underline{d}$ in the situational, non-standard semantics:

$$d, \underline{d} : S(L_1) \to ONT(L_1)$$

mapping the set $S(L_1)$ of all *wfes* of L_1 into the set $ONT(L_1)$ which is the sum of all ontological categories in the ontological structure $\boldsymbol{R}_{L_1}$.

We will give here two possible solutions of denotations of quantifiers of the 1st-order taking into account two different ways of understanding of the denotation of sentences and sentential functions presented below.

4.2 Denotations of 1st-Order Quantifiers and Their Ontological Categories

4.2.1 Fregean Semantics

We assume that if U is the universe of individuals in an established model $\boldsymbol{M}_{L_1}$ of L_1, 1 is the value of truth, 0—the value of falsity then

$$\begin{array}{ll} d(x) \in \{U\} = ONTn_1 & \text{for any } x \in CATn_1 = Var; \\ d(p) \in \{0,1\} = ONTs & \text{for any } p \in CATs = S; \\ d(sf) \in 2^{U^k} = ONTs_k & \text{for any } sf \in CATs_k = S_k (k \geq 1) \end{array}$$

and for any $x_1, x_2, \ldots, x_k \in Var$ and for any $sf = \alpha(x_1, x_2, \ldots, x_k) \in S_k$

$$\begin{aligned} & d(\alpha(x_1, x_2, \ldots, x_k)) = \\ & \quad \{(u_1, u_2, \ldots, u_k) \in U^k \mid d(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x_k/u_k)) = 1\}, \end{aligned}$$

where $\alpha^o(x_1/u_1, x_2/u_2, \ldots, x_k/u_k)$ is a sentence which we get from sentential function *sf* by replacement of its all free variables $x_1, x_2, \ldots, x_k$ of *Var* by suitable individual names of individuals $u_1, u_2, \ldots, u_k$ of the universe U, i.e. the denotation of *sf* is the set of all k-tuples from U^k which satisfy this sentential function.

Denotation for the quantifier $\forall^k$ of the type $k(k \geq 1)$ is defined by induction as follows:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$d(\forall^1 x\alpha(x)) = d(\forall^1)(d(x)), d(\alpha(x))) = \begin{cases} 1, & d(x) = U = d(\alpha(x)) \\ 0, & d(x) = U \neq d(\alpha(x)); \end{cases}.$$

According to a) the quantifier sentence obtained from any sentential function $\alpha(x)$ by preceding it with the universal quantifier $\forall^1$ is a true sentence in the established model $\boldsymbol{M}_{L_1}$ of L_1 with the universe of individuals U *iff* every object of the universe U satisfies the $\alpha(x)$ which is the scope of $\forall^1$.

b) for $k = j + 1 (j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned}&d(\forall^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) =\\&\quad = d(\forall^{j+1})(d(x), d(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) =\\&\quad = \{(u_1, u_2, \ldots, u_{j+1}) \in U^j \mid d(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_{j+1}/u_{j+1})) = 1\\&\qquad \text{for each } u \in U\}.\end{aligned}$$

According to b) the denotation of the sentential function $sf_{k-1} \in S_{k-1}$ obtained from the sentential function $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_k (k > 1)$ by binding the variable x by the universal quantifier $\forall^k (k = j + 1 > 1)$ is the set of all $j = (k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$ of individuals of U such that all sentences obtained by the substitution of all j free variables in sf_{k-1}, respectively, by names of individuals of these tuples and names of any individuals of U representing x are true; in other words the denotation of sf_{k-1} is the set of all such $(k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$ of individuals of U that for any individual u of U k-tuples $(u_1, u_2, \ldots, u, \ldots, u_{k-1})$ build from them satisfy the scope $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})$ of the quantifier $\forall^k$.

Thus for any $k \geq 1$

$$d(\forall^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

Similarly for $d(\exists^k)$:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$d(\exists^1 x\alpha(x)) = d(\exists^1)(d(x)), d(\alpha(x))) = \begin{cases}1, & d(x) \cap d(\alpha(x)) \neq \emptyset\\ 0, & d(x) \cap d(\alpha(x)) = \emptyset;\end{cases}.$$

b) for $k = j + 1 (j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned}&d(\exists^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) =\\&\quad = d(\exists^{j+1})(d(x), d(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) =\\&\quad = \{(u_1, u_2, \ldots, u_{j+1}) \in U^j \mid d(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_{j+1}/u_{j+1})) = 1\\&\qquad \text{for some } u \in U\}.\end{aligned}$$

According to a) the quantifier sentence obtained from any sentential function $\alpha(x)$ by preceding it with the existential quantifier $\exists^1$ is true sentence in the established model $\boldsymbol{M}_{L_1}$ of L_1 with the universe of individuals U *iff* at least one object of the universe U satisfies the $\alpha(x)$ which is the scope of $\exists^1$.

According to b) the denotation of the sentential function $sf_{k-1} \in S_{k-1}$ obtained from the sentential function $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_k (k > 1)$ by binding the variable x by the existential quantifier $\exists^k (k = j + 1 > 1)$ is the set of all $j = (k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$ of individuals of U such that all sentences obtained by the substitution of all j free variables in sf_{k-1}, respectively, by names of individuals of these tuples and the substitution some individual name of u for x are true; in other words the denotation of sf_{k-1} is the set of all such $(k-1)$-tuples $(u_1, u_2, \ldots, u_{k-1})$

of individuals of U that for some individual u of U k-tuples $(u_1, u_2, \ldots, u, \ldots, u_{k-1})$ build from them satisfy the scope $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})$ of the quantifier $\exists^k$.

Thus, for any $k \geq 1$

$$d(\exists^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

Moreover, the principle (CC) is also valid for $\forall^k$ and $\exists^k$ in situational semantics.

4.2.2 Situational Semantics

In situational semantic we assume that

$$\begin{array}{ll} \underline{d}(x) \in \{U\} = ONTn_1 & \text{for any } x \in CATn_1 = Var; \\ \underline{d}(p) \in \{St\} = ONTs & \text{for any } p \in CATs = S, \end{array}$$

where St is the set of all states of affairs, $St = T \cup F, T \cap F = \emptyset$ and T is the nonempty set of all states of affairs that take place and F—the nonempty set of remaining states of affairs. $St_k \subset St$ is the set of states of affairs with k individuals.

$$\underline{d}(sf) \in 2^{St_k} = ONTs_k \quad \text{for any } sf \in CATs_k = S_k$$

and for any $x_1, x_2, \ldots, x_k \in Var$ and for any $sf = \alpha(x_1, x_2, \ldots, x_k) \in S_k$

$$\begin{aligned} &\underline{d}(\alpha(x_1, x_2, \ldots, x_k)) = \\ &\{s \in St_k \mid s = \underline{d}(\alpha^0(x_1/u_1, x_2/u_2, \ldots, x_k/u_k)) \text{ for any } (u_1, u_2, \ldots, u_k) \in U^k\}. \end{aligned}$$

So, if the denotation operation is understood here as the operation $\underline{d}$ then the denotations of sentences are states of affairs and the denotation of any sentential function is the set of all states of affairs that are denotations all sentences represented by the sentential function.

Denotation for the quantifier $\forall^k$ is defined by induction as follows:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$\underline{d}(\forall^1 x\alpha(x)) = \underline{d}(\forall^1)(\underline{d}(x)), \underline{d}(\alpha(x))) \in T \text{ iff } \underline{d}(\alpha^o(x/u)) \in T \text{ for each } u \in U;$$

b) for $k = j + 1 (j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned} &\underline{d}(\forall^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) = \\ &\quad = \underline{d}(\forall^{j+1})(\underline{d}(x), \underline{d}(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) = \\ &\quad = \{s \in St \mid s = \underline{d}(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_k/u_k)) \\ &\qquad \text{for each } u \in U, \text{ any } (u_1, u_2, \ldots, u_{j+1}) \in U^j\}. \end{aligned}$$

According to a) the quantifier sentence obtained from any sentential function $\alpha(x)$ by preceding it with the universal quantifier $\forall^1$ is a true sentence in an established model $\boldsymbol{M}_{L_1}$ of the language L_1 with the universe of individuals U *iff* every sentence representing this sentential function is true (because their denotations are states of affairs that take place).

According to b) the denotation of sentential function $sf_{k-1} \in S_{k-1}$ obtained from the sentential function $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_k (k > 1)$ by binding the variable x by the universal quantifier $\forall^k$ is the set of all denotations of sentences (intuitively – the set of all states of affairs describing by these sentences) which can be obtained from sf_{k-1} by replacing all free variables in it with individual names of any individuals of U; in other words, it is the set of all denotations of sentences (all states of affairs) which can be obtained from $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})$ by replacement for the variable x binding by $\forall^k$ individual names of any individual of U (of the denotation of this variable) and for remaining variables in it also individual names of any individuals of U.

Thus, for any $k \geq 1$

$$\underline{d}(\forall^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

Similarly for $\underline{d}(\exists^k)$:

a) for $k = 1$ and any $\alpha(x) \in S_1$

$$\underline{d}(\exists^1 x\alpha(x)) = \underline{d}(\exists^1)(\underline{d}(x)), \underline{d}(\alpha(x))) \in T \quad \text{iff} \quad T \cap \underline{d}(\alpha(x)) \neq \emptyset$$

b) for $k = j + 1 (j > 0)$ and any $\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}) \in S_{j+1}$

$$\begin{aligned} &\underline{d}(\exists^{j+1} x\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1})) = \\ &\quad = \underline{d}(\exists^{j+1})(\underline{d}(x), \underline{d}(\alpha(x_1, x_2, \ldots, x, \ldots, x_{j+1}))) = \\ &= \quad \{s \in St \mid s = \underline{d}(\alpha^o(x_1/u_1, x_2/u_2, \ldots, x/u, \ldots, x_{j+1}/u_{j+1})) \\ &\qquad \text{for some } u \in U, \text{ any } (u_1, u_2, \ldots, u_{j+1}) \in U^j\}. \end{aligned}$$

Thus, for any $k \geq 1$

$$\underline{d}(\exists^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

4.3 The Syntactic and Semantic Compatibility of Quantifiers

In our categorial approach to syntax and semantics of the 1st-order formalized language L_1 its quantifiers have been treated as context-dependent two-argument functors-functions of different categorial types $k > 0$ (defined on the set *Var* of all its individual variables and the set of all its sentential functions S_k with exactly k free variables) and with values in the set of sentential functions S_{k-1} possessing one free variable less or, in particular, in the set of sentences S:

$$\forall^k, \exists^k : Var \times S_k \to S_{k-1} \qquad (S_0 = S).$$

Thus, according to the condition (*CATf*), quantifiers $\forall^k$, $\exists^k$ belong to syntactic categories:

$$(CAT\forall^k, \exists^k) \qquad \forall^k, \exists^k \in CATs_{k-1}/n_1 s_k = CATs_{k-1}^{CATn_1 \times CATs_k} \qquad (s_0 = s),$$

and it means that they satisfy the principle (*SC*) of syntactic connection.

It was also shown that for the denotation operations:

$$d, \underline{d} : S(L_1) \to ONT(L_1)$$

their denotations, according to the condition (*ONTf*), belong to ontological categories:

$$(ONT\forall^k, \exists^k) \quad d(\forall^k), \underline{d}(\forall^k), d(\exists^k), \underline{d}(\exists^k) \in ONTs_{k-1}/n_1 s_k = ONTs_{k-1}^{ONTn_1 \times ONTs_k}.$$

5 Conclusions

From the conditions $(CAT\forall^k, \exists^k)$ and $(ONT\forall^k, \exists^k)$ follow the following conclusions:

1. *the 1st-order quantifiers* $\forall^k, \exists^k (k > 0)$ *satisfy the principle of syntactic connection* (*SC*) *and the principle of categorial compatibility* (*CC*)
   and
2. *the problem of standard quantifiers is solved by employing the conceptual apparatus and statements of the outlined theory of categorial languages.*

It should also be noted that

3. *in languages with other operators biding variables the problem of their denotations can be solved in an analogous way,*
   but
4. *for branching quantifiers used in Independence-Friendly logic* (*see Hintikka* [11]) *the outlined here denotational (compositional) semantics does not work.*

However,

5. *according to Frege's ideas, the proposed categorial approach to language syntax and semantics can be developed in the same spirit for formalized languages of higher order than 1.*
6. *the proposed approach to semantics of the 1st-order formalized languages of differ from the standard in the Tarski's approach* [26] *and other improved versions; first of all it refers to the concept of denotation of any language expression instead to the concept of satisfaction—the crucial ancillary notion in the definition of truth; this notion may be omitted in the definition of the concept of a true sentence and probably replaced by the notion of denotation.*

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Chapter 12
Logic and the Ontology of Language

Urszula Wybraniec-Skardowska

Abstract The main goal of this paper is to outline a general formal-logical theory of language construed as a particular ontological being. The theory itself will be referred to as an ontology of language, because it is motivated by the fact that language plays a special role: it reflects ontology, and ontology reflects the world. Linguistic expressions will be regarded as having a dual ontological status: they are to be understood as either *concreta*—i.e. *tokens*, in the sense of material, physical objects—or *types*, in the sense of *classes of tokens*—i.e. abstract objects. Such a duality will then be taken into account in the logical theory of syntax, semantics and pragmatics presented here. We point to the possibility of constructing the latter on two different levels, one stemming from *concreta*, construed as linguistic *tokens* of expressions, the other from their classes—namely *types*, conceived as abstract, ideal beings. The aim of this work is not only to outline such a theory with respect to the dual ontological nature of the expressions of language in terms that take into account a functional approach to language itself, but also to show that the logic based on it is ontologically neutral in the sense that it is abstracted from the level at which certain existential assumptions relating to the ontological nature of these linguistic expressions and their extra-linguistic ontological counterparts (objects) would have to be embraced.

Key words: Formal logic • Ontology • Ontology of language • Syntax • Expression-*token* • Expression-*type* • Semantics • Meaning • Denotation • Ontic category

1 Introduction

This section has a preliminary character. It discusses the principal aspects and concepts pertaining to the descriptive, representational and referential functions of

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First published in: Skowron, B. (ed.) Contemporary Polish Ontology, pp. 109–132. De Gruyter, Berlin–Boston (2020).

language, and the dual ontological nature of its expressions, given certain assumptions and logical foundations. The theory of language thus construed is outlined in the main part of the paper (Section 2), and some summary results and conclusions are included in Section 3.

1.1 Knowledge-Language-Reality

This section introduces the issue of linguistic adequacy as this relates to the function of language as an ontological being used on the one hand to describe the world (which is what ontology deals with as the theory of being), and on the other to represent our knowledge of this world. I seek here to furnish a justification for thinking that our theory of language should be an ontological one. I also argue in favour of the logical conception of language.

For the most general definition of ontology, we shall refer to the definition proposed by Perzanowski:

> Ontology is the general theory of possibility, i.e. the realm of all possibilities—the ontological space. Metaphysics, on the other hand, is an ontology of the world, i.e. the reality of all existing items, called facts. [. . .] Real philosophy, however, is about being. [26, p. 45]

By *being*, we understand here everything that exists, that can exist, that is not contradictory in itself. The task of ontology—as we understand it here—is to describe the structure of being or reality. Language, while at the same time serving as a tool for constructing the theory of being itself, is put at the service of such a description.

Studies of language can certainly prove helpful when it comes to producing a descriptive account of this kind. For language to be able to perform this basic function of providing a faithful description of reality and its structure, a specific sort of compatibility obtaining between the elements of the following triad is called for,

Language – Knowledge – Reality,

which I shall refer to for short as *linguistic adequacy*, where this in turn is to be described within a theory of language.

Language serves to represent human knowledge acquired in the process of cognizing reality. It is simultaneously a means of describing that cognized reality. Operating with language in ways that involve logic and thinking enables us to transform and enrich our knowledge in order to better get to know and discover the world. It is thus also a tool for expanding our cognition of reality on the basis of the knowledge we already possess—which, by the way, need not be confined to the domain of ontological matters.

In order for language to fulfil its descriptive function, it should reflect the structure of being, reality, with its own structure. The structure of language is undoubtedly connected with that of the cognized world. It is conditioned by the formation of knowledge obtained in the process of cognizing the reality, framing the structure of this reality. This structure is described, as we know, by ontology. Knowledge of the structure of reality allows us to speak in a uniformly consistent way about the world, making inter-human communication more effective.

The relevant components of knowledge correspond to the elements that compose the reality. In language, we speak about both the former and the latter by means of its expressions. They have their counterparts in language: in its components, its expressions. The components of reality belong to appropriate *ontological categories*, and those of knowledge to appropriate *categories of components of knowledge*, whereas components of language belong to appropriate *syntactic categories* or *semantic categories*—i.e. to certain defined categories of linguistic expression. The language of such expressions serves to faithfully describe the world and the given domain of knowledge.

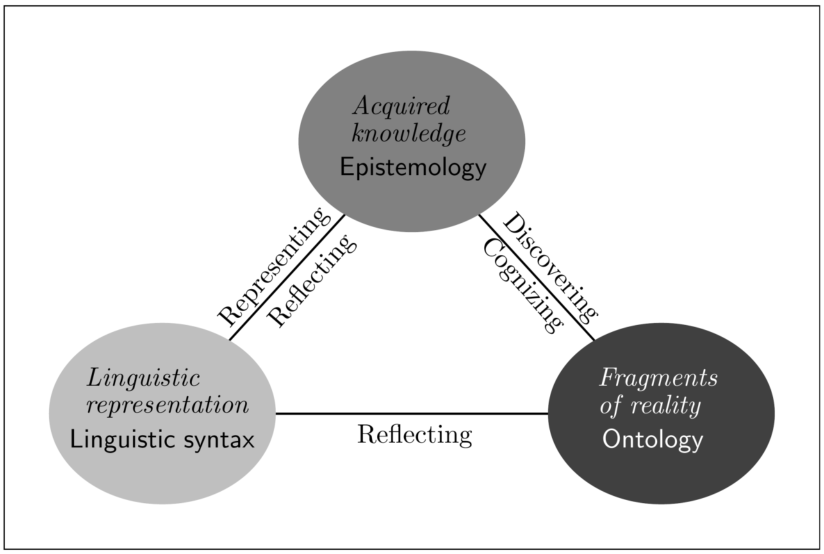

Diagram 1

Linguistic adequacy is achieved when the syntax of language faithfully reflects its bi-aspectual semantics: i.e. on the one hand the existing fragments of cognized or discovered reality (extensional semantics), and on the other the acquired knowledge resulting from the cognition or discovery of these (intensional semantics) (see Diagram 1). Language should thus reflect both some defined portion of reality and our knowledge of it—knowledge, that is, that has been acquired, but which is also in the process of being expanded.

As can be seen, language and its syntax are connected with both the ontology of the world—i.e. with everything that exists—and with epistemology, which deals with cognition of the world, whose result is the acquisition of knowledge.

Since language exerts such a considerable impact on ontology, it becomes vital to work out a general theory of language: language, that is, construed as a particular ontological being. This theory will be called the *ontology of language*. Using metalanguage, it will set out to describe the structure of language and its properties.

In the same way as there exist a great number of conceptions of being, so there exist plenty of conceptions of language, and many theories of language. Here, the conception of language in the framework proposed by Ajdukiewicz [6, p. 12, 13] will be of interest to us, and the theory to be constructed according to this conception will be the logical theory of language (logical semiotics) formalized on the basis of classical formal logic and set theory (see Diagram 2). Its assumptions are presented below.

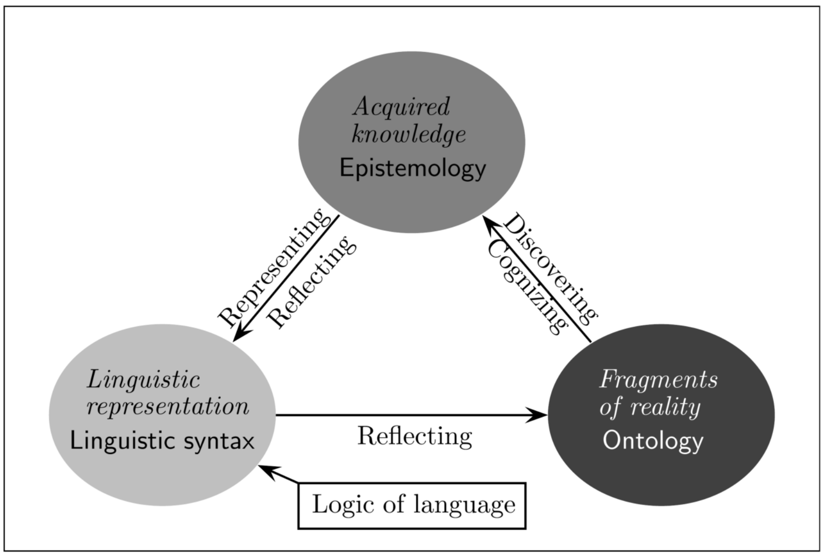

Diagram 2

1.2 The Logical Conception of Language

In the framework proposed by Ajdukiewicz [6, p. 13, note 6], the logical conception of language assumes that "in order to describe a language we have (i) to list its expressions, and (ii) *univocally* to assign specified meanings to these expressions." Ajdukiewicz [6, p. 13] also wrote that

> By drawing attention to the difference between the logical concept of language and those concepts of language which are being used by linguists we wish to emphasize that the logical concept of language is much simpler than the linguistic one, and that its analysis prepares that set of concepts which is indispensable to give clarity to the research done by linguists.

Ajdukiewicz's logical conception of language will figure in our own proposal for shedding light on how language should be understood on the basis of logic. In that context, language is to be conceived as a system of conventional signs. In compliance with such a conception, the following will be the basic elements which make up language (as a system of signs):

1. vocabulary,
2. rules of syntax:
   a. qualifying – establishing which objects qualify as simple expressions (words) of the vocabulary;
   b. constructive – determining how to form other signs from simpler signs – complex expressions of language;
3. semantic rules: settling down what the signs a) mean, b) designate and denote;
4. pragmatic rules: determining the relations between linguistic signs and their users when communicating and cognizing reality.

Language, on such a conception, is an ideal creation. All real languages known to us are "logically defective". By contrast, in this case it is an idealizing reconstruction of real languages. It is this that constitutes the subject of formal-logical description in the present work: a description that, as a matter of fact, does not apply to languages as they are normally used. The formal-logic theory of language that will be sketched in due course (in Section 2) sets out to frame problems pertaining to the foundations of the theory of language in the most general terms possible, narrowing the problem area while providing a simple set of concepts and solutions relevant to issues connected with the ontological nature of linguistic expressions, their meaning (*intension*), and their denotation (*extension*). In proceeding thus, the theory reflects various assumptions, including certain existential ones, which are not satisfied in full by authentic languages, since the actual conditions in which the latter function most often depend on extra-linguistic factors and are not entirely neutral.

Language characterized according to the logical conception will consist of verbal signs, these being expressions in compliance with the rules of syntax: i.e. so-called well-formed expressions, which have a single meaning and denotation assigned to them, and which at the same time perform the function of representing knowledge acquired about a cognized reality, all the while playing the role of an intermediary in the process of transferring and exchanging information. Because language, as an ontological being, consists of expressions, and their ontological nature in turn can be of two kinds, our theoretical considerations pertaining to language must include certain initial assumptions regarding the ontological status of the expressions of that language. This problem will be discussed in the following subsection of the present work.

1.3 The Dual Ontological Status of Linguistic Expressions

What is the ontological status of expressions—of the objects making up a language? The question of the ontological status of such objects comes down to the following two bipolar (yes-no) questions:

1) Are linguistic objects, including words or expressions, concrete, real objects of a defined shape, extended in time and space?
2) Are linguistic objects, including words or expressions, abstract, objects or ideal beings of some sort?

The ontological status of the linguistic objects figuring in the above-mentioned questions is different: they belong to two different ontological types. At the same time, in logico-semiotic practice they are treated as possessing equal status.

Most often, in compliance with the differentiation made by Peirce [23, Sec. 4.537], inscriptions, words or expressions are understood as either *concreta*, meaning *tokens* (events) that are material objects perceivable through the senses, or *types*, meaning *classes of* (uniform and, in a broad sense, identifiable) *tokens*, whose relevance is as abstract objects. Such a duality in respect of our understanding of linguistic inscriptions first showed up in the famous monograph by Tarski ([32, pp. 5, 6, note 5; p. 24, note 19; (1956) p. 156, note 1; p. 173, note 1], before gaining popularity particularly thanks to the work of Carnap in the 1940s (see Carnap [11, Sec. 3: Sign-Events and Sign-Designs]).

Expressions such as perform on the one hand the function of representing knowledge acquired about a cognized reality, and on the other hand the role of intermediaries in the process of transferring and exchanging information, are sign-*tokens*, meaning specimens of sign-*types*, which in turn are classes of sign-*tokens* that are in some respect *identifiable*—i.e. *equiform*.[1] Any *meaning* or *denotation* is, on the other hand, assigned only to expression-*types*, which in contrast to their tokens (these being their physical representations) are thus object-*concreta* (e.g. inscriptions or sounds), these being object-*abstracta*.[2] Here, it must be said immediately that although, in logical semantics, explanations of the notions of 'meaning' and 'denotation' require the use of expression-*types*, in the very defining of these notions themselves it is expression-*tokens* that are used.

We also encounter the dual ontological character of linguistic expressions when taking into account the so-called functional approach to language, as in the framework proposed by Jerzy Pelc [25]. This obliges us to take into consideration two manners of usage of expressions. As regards the first of these, the manner of usage (*use*) appears exclusively in defined conditions, in determinate language-situational contexts, and concerns expression-*tokens*, whereas in the case of the other the manner of usage (*use*) characterizes the meaning of an expression as an expression-*type* seen in

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[1] Equiformity is treated here as *cum grano salis* (Jadacki [17]). Carnap [10] refers to the relation as one of syntactical equality. (See the translation in 2001, p. 15).

[2] The differentiation between *sign-token* and *sign-type* was introduced into semiotics by Peirce ([22, pp. 506, 512]; [23, CP 4.537]; [24, pp. 125, 480, 488]).

isolation from any situational-context: e.g., as in an entry in a dictionary edited in traditional book form.

The logical theory of language should thus assume the existence of both expression-*tokens* (language-based *concreta*) and expression-*types* (language-based *abstracta*). The dual ontological character of *linguistic objects*, and their being employed in a dual manner, point to the necessity of giving a bi-aspectual characterization of language in the theoretical context of the logical conception of language: as the language of expression-*tokens* and of expression-*types*.

At the same time, no definite elaboration of a theory of language can remain uninfluenced by the two main currents of linguistic ontology that have emerged in the light of the fundamental opposition associated with the controversy over universals: i.e. nominalism and realism.

Taking the nominalistic and concretistic position, it will be assumed that the basic plane of language consists of expression-*tokens*, and thus *concreta*. Abstract expressions, that is types of expressions, are then constructs emerging from a secondary level of analysis. On the other hand, if we assume that the basis for linguistic studies consists of ideal objects, in the sense of *abstracta* understood as types of expression, with expression-*tokens* available thanks to cognition through the senses being seen as constructs emerging from a secondary level of analysis, then we are opting for a platonizing standpoint.

Nevertheless, when it comes to constructing a formal-logical theory of linguistic syntax (Section 2.1), we are obliged to determine whether the primary linguistic beings are sign-*concreta* and the secondary ones sign-*abstracta*, or the other way round. In this way, theoretical questions pertaining to the logic of language intertwine with problems of a philosophical nature, especially ontological ones. This concerns not only the logical syntax of language, but also its logical semantics and pragmatics, as well as the problem of the linguistic adequacy of knowledge itself as it relates to reality.

Any notions introduced and rendered precise within parts of a logical theory of language must have their existence secured by means of the relevant axioms and definitions. However, should logic settle anything as regards the existence of the extra-linguistic entities that linguistic expressions relate to? We shall attempt to answer this question in due course (Section 2.2), before discussing what the categories of linguistic expressions and of their extra-linguistic counterparts are.

1.4 Categories of Linguistic Expression and Ontic Categories of Object

The general idea of a category as a predicative subset of a given set, as articulated by Jadacki [18, pp. 109ff], permits one to speak of both linguistic categories (picked out from within the set of all expressions) and ontic ones (picked out from within the set of all objects).

Linguistic expressions performing a determinate syntactic function and constructed according to the rules of linguistic syntax (i.e. well-formed expressions) will occupy a place in the appropriate syntactic categories. Broadly speaking, expressions playing the same role in the construction of complex expressions will likewise belong to the same categories. At the same time, when we abandon the purely syntactic point of view on expressions and take into account their semantic counterparts (components of knowledge) or what count in turn as counterparts of the latter (i.e. beings which the expressions relate to or denote), then these expressions will also be included in the appropriate semantic categories (which will be intensional or extensional, respectively). The compatibility of the appropriate syntactic and semantic categories of linguistic expression is an indispensable condition of *linguistic adequacy of knowledge relative to cognized reality* (Wybraniec-Skardowska [40, Sec. 4]). This compatibility entails the compliance of linguistic categories with the appropriate ontological categories, where these latter include the extra-linguistic counterparts of linguistic expressions. This will, of course, be relative to whatever ontology is embraced. We thus embark on our investigation on the basis of a structure of prior ontological commitment, taking into account only the substantive counterparts of expressions of linguistic categories, meaning extra-linguistic objects—i.e. beings.[3] Such objects will be placed in appropriate ontic categories, the typology for the latter having been established by the ontology adopted. Depending on the ontological conception involved, one or several ontic categories will be distinguished. Hence, the following may serve as ontic categories: individuals, sets of individuals, properties, relations (in particular, single-argument or multi-argument operation-functions), periods, areas and states of affairs. Distinguishing these or other ontic categories is obviously connected with the issue of which beings we attribute existence to, as being real or not (e.g., as intentional or ideal).[4]

Beyond this, when it comes to presenting certain semantic foundations for the formal-logical theory of language (Section 2.2), we shall accept the postulate of the "democratic nature of beings": all beings are equally empowered, being treated in the same way when it comes to existence and deciding something about them.

2 Outline of a Formal-Logical Theory of Language

The formal-logical considerations we are seeking to address pertain to both syntax and a bi-aspectual, intensional and extensional semantics of language characterized categorially in the spirit of the theory of syntactic categories proposed by Leśniewski and Ajdukiewicz (see Leśniewski [20, 21] and Ajdukiewicz [3, 5]) while remaining in compliance with the ontological canons of Frege [13, pp. 31–36, 36–38; (1997)

[3] Following Jadacki [16], we can accept that everything that has at least one property vested in it counts as an object (an entity). (The relation of vesting here will be a binary relation, whose domain will be a set of properties and counter-domain just a set of objects; the relation of vesting is therefore probably a primitive notion of ontology.) Following Łukasiewicz, we can also assume that everything that can both have and not have a certain property, where this is non-contradictory, counts as an object.

[4] See Ajdukiewicz [4], Bocheński [9], Augustynek & Jadacki [7].

pp. 155–159, 159–171], the motto of Bocheński [8] to the effect that *syntax mirrors ontology*, and some ideas of Suszko [28, 29, 30, 31] asserting that language is supposed to function as a linguistic schema for ontological reality and, at the same time, as a tool for its cognition. The theory of language outlined here is intended to offer a framework for understanding the development (and some of the explication) of the ideas put forward in the works of the above-mentioned authors.[5] In Section 2.1 we outline our version of the logical theory of linguistic syntax, and in Section 2.2 its extension to the theory of linguistic semantics and pragmatics.

2.1 On the Logical Theory of Linguistic Syntax

Each and every language can be more or less adequately captured by a determinate grammar. In the Polish tradition it is categorial grammar that serves this purpose. The latter originated from the work of Kazimierz Ajdukiewicz [3, 5], and grew under the influence of Husserl's idea of pure grammar [14, 15], as well as Leśniewski's theory of semantic/syntactic categories [20, 21].

The logical syntax of any language L will be characterized formally on two dual levels, one of them concerning the language of expression-*tokens*, the other that of expression-*types*. If one were to accept the view that expression-*concreta*, i.e. expression-*tokens* (physical objects), make up the fundamental layer of language, whereas the secondary layer of L is made up of expression-*abstracta*, i.e. expression-*types* (ideal objects), then one would be adhering to a concretistic philosophical view about the nature of linguistic entities (as was endorsed by, among others, Leśniewski). On the other hand, in supporting the view that expression-*types* are the basic layer, while expression-*concreta* are secondary, we ourselves shall be adopting the opposite standpoint, which is a platonizing one.

2.1.1. In the first case, on the *level of tokens*, language L is generated in the most general way by grammar:

$$G = \langle U_L, \sim, c, V, \varrho, E; S\rangle, \text{ where}$$

U_L is a non-empty universum containing all sign-*tokens* of L,
$\sim$ is a two-argument relation of identifiability of signs of universum U_L,
V is a vocabulary of *word-tokens* of language L,
c is a three-argument relation of concatenation defined in U_L,
ϱ is an n-argument relation of forming complex expression-*tokens* ($n > 1$),
E is the smallest set of all expression-*tokens* containing V and closed under the relation ϱ,
S the set of all well-formed expression-*tokens* of language L.

[5] The theory will be constructed on the basis of classical logic and set theory. Its outline is based on ideas presented in my previous contributions [36–40]. *Tokens* of linguistic expressions will be represented by the variables $e, e', e_1, e_2, \ldots$, *types* of such expressions by the variables $t, t', t_1, t_2, \ldots$.

The notions $U_L, \sim, V, c, \varrho$, are primitive notions of the theory, characterized axiomatically. When G is a classical categorial grammar, each expression-*token* e of set S has a categorial index $i(e)$ of some non-empty set I assigned in an unambiguous way, and each complex expression of set S is constructed on a functor-argument basis, so that it is possible to distinguish within it a constituent, the so-called *functor*, which, together with the remaining constituents of that expression, called *arguments of the functor*, forms this expression. The notion of a constituent of a complex expression is defined inductively. Categorial indexes serve *inter alia* to establish the syntactical role of expressions and examine their *syntactic connectivity*. Set S is formally defined as the smallest set of expressions, containing vocabulary V and closed with respect to relations linked to Ajdukiewicz's *principle of syntactic connection*. All the sets and relations of system G are non-empty sets—hence the resulting primary existence of expression-*tokens* in particular.

On the second level, the *level of types* of expression, language L is characterized through a system of notions which is the dual of system G:

$$\underline{G} = \langle \underline{U}_L, =, \underline{c}, \underline{V}, \underline{\varrho}, \underline{E}; \underline{S} \rangle, \text{ where}$$

$\underline{U}_L$ is the set of all linguistic sign-*types* in language L,
$=$ is a relation of common identity of signs of universum $\underline{U}_L$,
$\underline{V}$ is a vocabulary of word-*types* of language L,
$\underline{c}$ is a relation of concatenation defined on types of sign of $\underline{U}_L$,
$\underline{\varrho}$ is a relation of forming complex expression-*types*,
$\underline{E}$ is the set of all expression-*types* of language L,
$\underline{S}$ is the set of all well-formed expression-*types* of language L.

All the notions of grammar $\underline{G}$ are derivative constructs, defined with reference to the dual notions of grammar G. Any set $\underline{Z}$ of types of system $\underline{G}$ is a quotient set of set Z of *tokens* of the first level, due to the relation of identifiability $\sim$, i.e.,

$$\underline{Z} = Z/\sim .$$

Thus, any set $\underline{Z}$ of *types* of expressions is composed of equivalence classes of *tokens* of set Z, i.e.,

$$t \in \underline{Z} \Leftrightarrow \exists e \in Z (t = [e]_\sim = \{e' \in Z \mid e' \sim e\}).$$

The relation of concatenation $\underline{c}$ on *types* of sign is defined by means of the relation of concatenation c on *tokens* of signs of language L:

$$\underline{c}(t_1, t_2; t) \Leftrightarrow \exists e_1, e_2, e \in U_J (t_1 = [e_1]_\sim, t_2 = [e_2]_\sim, t = [c(e_1, e_2; e)]_\sim).$$

The concatenation relation $\underline{c}$ is a two-argument function on types of sign in language L.

It is proved that each dual counterpart of the thesis of the theory of syntax initially constructed on the *level of concreta* is a thesis of this theory developed on the *level of types*—on the second level of formalization of the syntax of language L.

The concretistic approach to the formal-logical theory of the syntax of language L has been set out in previous works by the present author [36, 37].

2.1.2. The opposite standpoint—the *platonizing* one—is founded on the assumption that types of signs in language L are ideal signs, conceived as independent and objective beings, and are primary in relation to the linguistic *tokens* that are their representatives. The primitive notions of the syntactic theory are thus the following notions of system $\underline{G}$: $\underline{U}_L$, $\underline{c}$, $\underline{V}$. The other notions of this system are defined subsequently. Obviously, the axiom stating the existence of sign-*types*, assuming that any type t is a non-empty set, is then accepted.

On the other level of formalization, the *level of tokens*, we find *tokens* of signs of language L that are introduced through axioms and definitions:

(1) $e_1 \in t_1 \wedge e_1 \in t_2 \Rightarrow t_1 = t_2$,
(2) $e \in U_L \Leftrightarrow \exists t \in \underline{U}_L(e \in t)$.

The above definition (2) can be considered under the general schema of the definition of subsets Z of set U_L:

(DZ) $e \in Z \Leftrightarrow \exists t \in \underline{Z}(e \in t)$.

The relation of *identifiability* is defined as follows:

(D$_\sim$) $e \sim e' \Leftrightarrow \exists t \in \underline{U}_L(e, e' \in t)$.

The relation of concatenation on *tokens* of signs is determined by the following definition:

(Dc) $c(e_1, e_2; e) \Leftrightarrow \exists t_1, t_2, t \in \underline{U}_L(e_1 \in t_1, e_2 \in t_2, e \in t \wedge \underline{c}(t_1, t_2; t))$.

We determine relation ϱ in a similar fashion.

It is proved that each dual counterpart of the thesis of the syntactic theory initially constructed on the *level of types* is a thesis of this theory on the second level of its formalization—i.e. on the *level of tokens*.

2.1.3. The two dual approaches to the two-level syntactic theory of language given in Subsections 2.1.1 and 2.1.2 are logically equivalent (see [34]). Within the scope of the linguistic syntax, the two conceptions deriving from different existential assumptions are equivalent. This statement is of philosophical significance, as it proves that *in the context of theoretical syntactic considerations pertaining to language, the assumption of the existence of abstract linguistic beings can be passed over*.[6]

[6] The proof of this theorem (see Wybraniec-Skardowska [34, 35]) is, however, based on standard Platonic set theory. The applied formalism is not thus in fact ontologically neutral. This remark was formulated by Jerzy Perzanowski.

2.2 The Foundations of the Formal-Logical Theory of the Semantics and Pragmatics of Language

2.2.1. The logical theory of syntax allows us to determine sets S and $\underline{S}$ of all well-formed expressions of language L. Being in line with the logical conception, its characterization requires an unambiguous assignment of meanings to its expressions.

Only an efficient, precise and clear language can become a tool for describing the world, enabling us to properly transmit information and communicate about reality. The expressiveness of language consists specifically in the unambiguous character of its expressions, both as regards their structure and their meaning (*intension*) and denotation (*extension*). The absence of syntactic and semantic ambiguity where linguistic expressions are concerned is a condition of its logical meaningfulness. It entails the categorial compatibility of language, which is different from that mentioned earlier: i.e. the compatibility of syntactic categories of linguistic expression with semantic ones, be they semantic (*intensional*) or denotational (*extensional*). This compatibility, in turn, entails the syntactic and semantic structural compatibility of language, described in the form of three principles of compositionality for complex language expressions, mutually corresponding to one another. Of these, one is syntactic and two semantic, with the latter pair consisting of the compositionality of meaning and the compositionality of denotation.

However, since absence of ambiguity is such an important aspect of linguistic adequacy, we first need to establish what the *meaning* of the composed expressions of language L is, and what this unambiguous character of its expressions consists in.

There exist quite a number of conceptions relating to the nature of *meaning*, as well as different theories concerning this notion, in the literature dealing with the philosophy of language. So far, however, none of these has gained widespread acceptance. Moreover, none of them can be said to constitute a general theoretical conception. I myself have offered a sketch of such a conception in previous work [38] and will be making reference to that in this part of the present article.

2.2.2. Since the time of Frege, the notion of 'meaning' has been differentiated from that of 'denotation'. Frege [13, p. 31; (1997) p. 156] distinguished, respectively, *Sinn* (English: *intension*) and *Bedeutung* (English: *extension*), while we owe the *intension-extension* distinction to Carnap [12, Ch. I, Sec. 5, Sec. 6, pp. 26, 27; Sec. 9, pp. 40–41]. On the other hand, the literature devoted to linguistics and semiotics does not always differentiate between these two notions.

The notions of 'meaning' and 'denotation' are used with reference to expression-*types* of language L. They are "assignments" of meanings and denotations to these expressions, respectively. As such, they are operations (functions) on expressions of set $\underline{S}$, yet not on all expressions of the set—rather just their non-empty sub-types, meaning elements of the set. Thus:

$$\underline{S}^* = \{t' \subseteq t \mid t' \neq \emptyset \wedge t \in \underline{S}\},$$

with the "assignments" construed as functions on any non-empty sets of identifiable expression-*tokens* of set S.

We define these operations, making use of some ideas connected with the understanding of the notion of 'meaning' put forward by Ajdukiewicz [1, 2] and Wittgenstein [33, third edition (1967), paragraphs: 20, 349, 421-2, 508, p. 184, 190], as the *manner of usage* of an expression. In order to be able to determine what being the use of an expression consists in, we must invoke certain semantico-pragmatic notions.

2.2.3. We shall therefore enrich the theory of syntax of language L with some new primitive notions: the set *User* of all users of language L, the set *Ont* of all extra-linguistic objects which expressions of language L relate to, and the two-argument operation *use* of using expression-*tokens* of language L.

The sets *User* and *Ont* will be conceived in a very broad way. *User* may consist not only of current users of language L, but also past and future ones. Meanwhile, nothing will be assumed as regards the ontological nature or existence of objects of *Ont*: they may be *concreta*, *abstracta*, ideal, intentional (quasi-objects), fictional objects, etc. It will be merely axiomatically assumed about these objects that they are non-empty sets. Nothing will be assumed about the ontic categorization of *Ont*. The ontic categories can—but need not —consist of the following: the category of individuals satisfying certain properties, categories of various relations and functions, category of states of affairs, and the like.

The relation *use* of usage of expression-*tokens* will also be conceived in the broadest terms: e.g., as an operation of invoking, exposing and forming expression-*tokens* to indicate appropriate objects of the set *Ont*. The operation *use* will also be said to be a *function of the objective references of expression-tokens* made by users of language L. This function can also be conceived as the set of all physical activities of users of L that were, are or will continue to be activities used in determinate situations with the aim of referring concrete *tokens* of language L to objects of *Ont*. It will be axiomatically assumed that the function *use* is a set-theoretical function, partially mapping the Cartesian product $User \times S$ onto the set *Ont*, whose primary domain is the whole set *User*, while its secondary one will be a proper subset of the set of expression-*tokens* S.

We shall read the expression $use(u, e) = p$, where $u \in User$, $e \in S$ and $p \in Ont$, as follows: user u uses expression-*token* e with reference to object p. When $use(u, e) = p$ takes place, then object p is to be called the *object reference of token e* indicated by user u of language L. We say about expression e that it *has an object reference* when used by a user with reference to some object. Two expression-*tokens* have—at the same time—the same manner of usage *use*, when they have the same object reference.

The relation $\underline{use}$ of using expression-*types* is determined by means of the operation *use* of using *tokens of expressions*. It is axiomatically assumed about it that it is a non-empty relation defined in terms of the Cartesian product $User \times S^*$ and by the following formula:

D0. $u \, \underline{use} \, t \Leftrightarrow \exists e \in t \, \exists p \in Ont \, (use(u, e) = p)$.

It follows from the already accepted assumptions or definitions that each user of language L uses at least one expression-*token* with reference to any object, and hence $\underline{uses}$ at least one expression-*type* of set $\underline{S}^*$.

Defining the *meaning* of an expression-*type* as a common way of $\underline{using}$ types of expression requires that we introduce the notion of a *relation* $\approx$, *the same manner of usage of these expressions*. In the definition of this relation, however, it is necessary to employ the notion *use* of using expression-*tokens*.

2.2.4. The formal definition of relation $\approx$ is introduced in the following way:

D1. $t \approx t' \Leftrightarrow \forall u \in User[(u\, \underline{use}\, t \Leftrightarrow u\, \underline{use}\, t') \wedge \forall p \in Ont(\exists e \in t(use(u, e) = p) \Leftrightarrow \exists e' \in t'(use(u, e') = p))]$.

In accordance with definition D1, two expression-*types* will have the same manner of usage $\underline{use}$ if and only if each user of language L uses, in the sense of $\underline{use}$, one of them if and only if he or she uses the other of them, and uses, in the sense of *use*, a *token* of one of them with reference to any object if and only if he or she uses a *token* of the other of them with reference to the same object.

For instance, the word "rain" and the expression "an atmospheric fall in the form of drops of water falling down from a cloud" have the same manner of usage $\underline{use}$. Similarly, the expression "a public concert" and the expression "a public performance of pieces of music" have the same manner of usage $\underline{use}$.

It can easily be determined that if two expression-*types* have the same manner of usage $\approx$, then there exist *tokens* of one and of the other of them, respectively, which have the same manner of usage in the sense of *use*.

2.2.5. The relation $\approx$ of having the same manner of usage of *types* of expression is an equivalence relation in set $\underline{S}^*$ of expression-*types*. Operation *m of assigning a meaning* to these expressions can thus be defined as the function:

D2. $m : \underline{S}^* \to 2^{\underline{S}^*}$, where $m(t) = [t]_\approx$ for any $t \in \underline{S}^*$.

Thus, the *meaning* $m(t)$ of expression-*type* t is the equivalence class of relation $\approx$ of possessing the same manner of usage of *types* determined by *type* t. Intuitively, this may be conceived as the *common property of all expression-types having the same manner of usage as* t. It is this property which we shall call the *manner of usage of expression-type* t.

Meaning $m(t)$ *of expression-type* t *is thus an abstract being (a non-empty set), whose existence is guaranteed by set theory.*

2.2.6. In Ajdukiewicz's logical concept of language, each of its expressions is to have an unambiguously assigned meaning. Type t may, however, include subtypes, the meaning of which differs from the global meaning $m(t)$ as established by definition D2. For instance, the subtype "key^1" of the expression-*type* "key", composed only of the identifiable tokens of the expression-*type* "key" whose object references are musical clefs, has a meaning that differs from the global meaning of the word "key", which does not have an unambiguously assigned meaning.

D3. Expression-*type t has a meaning assigned unambiguously* $\Leftrightarrow$ no proper subtype of expression t has a meaning that differs from the meaning of expression t; i.e., symbolically:

$$\neg\exists t' \subseteq t(t' \neq t \wedge m(t') \neq m(t)),\ \text{i.e.}\ \forall t' \subseteq t(m(t') = m(t)).$$

2.2.7. An expression-*type* possessing an unambiguously assigned meaning in language L should be an *unambiguous expression* of this language. A formal definition of an unambiguous expression can be introduced by means of the notion of *denotation*, which makes reference to the idea of the designation of objects of set *Ont* by *types* of expression of language L.

D4. t *designates* $p \Leftrightarrow \exists u \in User\ \exists e \in t(use(u, e) = p)$, where $p \in Ont$.

Thus, expression-*type* $t \in \underline{S}^*$ designates an object p iff at least one user of language L uses some *token* of expression t with reference to the object p.

By way of example, the word "laptop" designates each and every laptop, and the expression "intention" each and every intention.

Objects designated by an expression-*type* are called *denotata* of this expression. When the denotata of some such expression are *object-concreta* (things, persons, etc.),[7] we shall call them *designata* of this expression.

We call *denotation* $d(t)$ of expression-*type* t the set of all its denotata. Formally, $d(t)$ is a value of *denotation* function d defined in the following way:

D5. $d\colon \underline{S}^* \to 2^{Ont}$ and $d(t) = \{p \in Ont \mid t \text{ designates } p\}$, for any $t \in \underline{S}^*$.

It follows from the already accepted assumptions or definitions that each expression-*type* which is used by someone in the sense <u>*use*</u> has (denotes) a non-empty denotation (a set of denotata), and therefore designates an object of set *Ont*. If, then, we speak about so-called *empty names* as having an empty denotation, we mean just that the set of designata (*concreta*) is an empty set. Such names are thus not used by users in the sense <u>*use*</u>, as their *tokens* do not make reference to any object (material, physical); the set *Ont* will consist for them exclusively of real *concreta*.

It should also be noted that not every well-formed expression-*type* has a non-empty denotation. For instance, the expression "the ceiling writes hot ice" is a syntactically correct one, but as a piece of semantic nonsense is not used and has no denotatum. Moreover, let us also take note of the fact that subtypes of a given expression-*type* can have a different denotation: one which is 'smaller' than the expression itself does.

Let us state here two theorems resulting from definitions D5 and D4, as well as from the theorems of algebra of sets:

T1. If t' is a subtype of expression-*type* t (i.e., $t' \subseteq t$), then $d(t') \subseteq d(t)$.
T2. If t_1, t_2 are subtypes of expression-*type* t and $t = t_1 \cup t_2$, then $d(t) = d(t_1) \cup d(t_2)$.[8]

[7] Here we should emphasize that we are thinking of actually existing *concreta*, in that one can also speak about non-existent *concreta* (e.g., *thought-based* ones; see Jadacki [16]).

[8] The proof of this theorem is given in the *Appendix* at the end of this article.

It can also be proved that the denotation of the sum of a finite number of subtypes forming a given type will be the sum of the denotations of these subtypes.

2.2.8. The relations between meaning, absence of ambiguity, and denotation are given in the theorems below.

The basic relation between meaning and denotation is described by the following theorem (cf. 38, pp. 127–128):

T3. $m(t) = m(t') \Rightarrow d(t) = d(t'),\quad$ for any $t, t' \in \underline{S}^*$.

According to T3, two expression-*types* have the same denotation when they have the same meaning; therefore, if the denotations of these expressions are different, their meanings will also be so.

The theorem that is the converse of T3 does not hold true as, e.g., the expressions "an equilateral triangle" and "an equiangular triangle" have the same denotation, yet different meanings.

The notion of an expression being unambiguous (i.e. having just one meaning) is introduced by means of the following definition:

D6a. t is *unambiguous* $\Leftrightarrow \neg\exists t' \subseteq t(d(t \setminus t') \neq \emptyset \wedge d(t') \cap d(t \setminus t') = \emptyset)$,
i.e. $\forall t' \subseteq t(d(t \setminus t') = \emptyset \vee d(t') \cap d(t \setminus t') \neq \emptyset))$.
D6b. t is *ambiguous* $\Leftrightarrow$ t is not unambiguous.

Thus, an expression-*type t* is *unambiguous* iff there does not exist any such subtype of t as would have some *denotatum* in common with a non-empty denotation of the difference between the expression t and this subtype;[9] when such a subtype does exist, expression t is *ambiguous*.

By way of example, the expression "a key" is ambiguous, as its subtype "a key^2", designating only keys to open doors, does not have *denotata* in common with the denotation of the expression "a key" \ "a key^2", designating all other keys (e.g., clefs in music, keys for decoding encrypted texts, or controls on mechanical devices).

We shall now state several theorems that serve to characterize unambiguous (and therefore also ambiguous) expressions with the help of the notions and theorems introduced earlier.

T4. t is *unambiguous* $\Leftrightarrow \neg\exists t' \subseteq t[(d(t \setminus t') \neq \emptyset \wedge d(t \setminus t') = d(t) \setminus d(t')]$.[10]

A direct conclusion following from Theorem T4 is

T5. $\forall t' \subseteq t(d(t') = d(t)) \Rightarrow t$ is *unambiguous*.

The implication that is the converse of T5 is not obviously true. For example, if t' is a *singleton* and has the following inscription as its only token,

laptop

[9] Let us remind ourselves that expression-*types* are sets of *tokens*; hence the difference between two expressions here will be that between two sets.

[10] The proof of this theorem is given in the *Appendix*.

whose object reference is my own laptop, while the expression t is a set of all *inscription-tokens* identifiable with this inscription (and acknowledged to be an unambiguous expression, according to D6a), then the denotation of subtype t' of expression t is not equal to the denotation of expression t.

T6. t has an unambiguously assigned meaning $\Rightarrow$ t is unambiguous.

Proof T6 follows directly from D3, T3 and T5.

Thus, the possession of unambiguously assigned meanings by expression-*types* of language L is a sufficient condition of their unambiguity.

Obviously, it also follows from T6 that ambiguous expressions do not have unambiguously assigned meanings, and that language as it figures in the logical conception should be free of ambiguous expressions.

The condition for the non-ambiguity of expression t is not, however, a sufficient one for t to have an unambiguously assigned meaning, as when, for example, t = "a book" is an unambiguous expression, in compliance with D6a, then there will exist some expression-*type* t' = "a book[1]" which is a set of tokens identifiable with words, whose object reference will be books by Jacek Jadacki, such that $t' \subset t$ and $d(t') \neq d(t)$ (because $d(t') \subset d(t)$), whence on the basis of theorem T3 we have $m(t') \neq m(t)$, and t does not have an unambiguously assigned meaning, as it does not satisfy definition D3.

3 On the Ontological Neutrality of Logic

In this part I offer a summary, recapitulating the basic assumptions and results presented in Section 2 with the aim of showing the extent to which the principal objectives pursued have been realized.

3.1. Subsections 2.2.1 and 2.2.2 of this work have sketched and discussed formal-logical theories of language, constructed in accordance with the logical conception of language. These theories are based on classical logic, together with set theory. The theoretical considerations addressed have been rather general, but also quite far-reaching. They do not depend on any particular symbolism or notations of expressions, or concrete grammatical rules, of the language being described.

3.2. In discussing in Subsection 2.2.1 the theory of linguistic syntax, we pointed to the possibility of building it on two different levels, one of which stems from *concreta*, i.e. linguistic *tokens* of signs, the other from their classes, i.e. *types* of linguistic sign, conceived as abstract beings.

- The outcome of our theoretical considerations has been a statement of complete analogousness obtaining between the syntactic notions of the two levels.
- Thus, logic does not settle here which view pertaining to the nature of linguistic objects—the concretistic one or the idealistic, platonizing one—is correct.

- Since, however, the two dual-aspected theoretical approaches to linguistic syntax are equivalent, in formalizing language initially on the level of concreta we are not impoverishing the resources offered in the form of theorems for the linguistic syntax being described, and we do without postulating the existence of ideal beings of the sort that types of language expression are.
- Hence, a philosophical thesis is entailed, concerning the possibility of eliminating assumptions regarding the existence of ideal beings in the context of considerations pertaining to syntax, as long as these beings are treated as classes of identifiable sign-*tokens* (linguistic *concreta*).[11]

3.3. By sketching, in Subsection 2.2.2, the semantic-pragmatic theory of language, we showed that:

- a meaning can be assigned to its well-formed expression-*types* (through function m),
- these expressions have a meaning (D2),
- a meaning (D3) can be unambiguously assigned to them,
- while being used, they designate some objects (D4),
- they denote (have a denotation), since
- a denotation can be assigned to them (through function d),
- designated objects belong to the set *Ont*.

As regards the set *Ont* of *extra-linguistic objects* (beings) designated by expression-*types*, we have simply assumed that it is a non-empty set, inseparably bound up with the structure of its beings and their ontological categorization. A full characterization of language, considered from an ontological point of view and in terms that comply with Ajdukiewicz's logical conception, can be furnished using this formal-logical theory.

For the purposes of our description we have not made use of any other existential assumptions, apart from those imposed by set algebra: neither when it came to the existence of linguistic expressions, nor for their extra-linguistic counterparts, was this the case. In this regard we may assert that the logic applied here (using set theory) has been ontologically neutral.

Appendix

We give proofs here of theorems T2 and T4, using the (assumptive) method of natural deduction put forward in the work of J. Słupecki and L. Borkowski [27].

T2. $t = t_1 \cup t_2 \wedge t_1 \subseteq t \wedge t_2 \subseteq t \Rightarrow d(t) = d(t_1) \cup d(t_2)$.

Proof

[11] The formalism leading to this statement, however, is based on Platonist set theory, and so is not really ontologically neutral (see note 7).

1. $t = t_1 \cup t_2$	{assum.}
2. $t_1 \subseteq t \wedge t_2 \subseteq t$	{assum.}
3. $d(t_1) \subseteq d(t) \wedge d(t_2) \subseteq d(t)$	{2, T1}
4. $d(t_1) \cup d(t_2) \subseteq d(t)$	{3}
1.1. $p \in Ont \wedge p \in d(t)$	(additional assumption)
1.2. $p \in d(t_1 \cup t_2)$	{1, 1.1}
1.3. $\exists u \in User\, \exists e \in t_1 \cup t_2 (use(u, e) = p)$	{1.2, D5, D4}
1.4. $u_1 \in User \wedge (e_1 \in t_1 \vee e_1 \in t_2) \wedge use(u1, e1) = p$	{1.3}
1.5. $e_1 \in t_1 \Rightarrow \exists e \in t_1\, \exists u \in User(use(u, e) = p) \Rightarrow$ $p \in d(t_1) \Rightarrow p \in d(t_1) \cup d(t_2)$	{1.4, D5, D4}
1.6 $e_1 \in t_2 \Rightarrow \exists e \in t_2\, \exists u \in User(use(u, e) = p) \Rightarrow$ $p \in d(t_2) \Rightarrow p \in d(t_1) \cup d(t_2)$	{1.4, D5, D4}
1.7. $e_1 \in t_1 \vee e_1 \in t_2 \Rightarrow p \in d(t_1) \cup d(t_2)$	{1.5, 1.6}
1.8. $p \in d(t_1) \cup d(t_2)$	{1.4, 1.7}
5. $p \in Ont \wedge p \in d(t) \Rightarrow p \in d(t_1) \cup d(t_2)$	{1.1 → 1.8}
6. $\forall p \in Ont(p \in d(t) \Rightarrow p \in d(t_1) \cup d(t_2))$	{5}
7. $d(t) \subseteq d(t_1) \cup d(t_2)$	{6}
$d(t) = d(t_1) \cup d(t_2)$	{4, 7}

□

T4. t is *unambiguous* $\Leftrightarrow \neg\exists t' \subseteq t[(d(t \setminus t') \neq \emptyset \wedge d(t \setminus t') = d(t) \setminus d(t')]$.

Proof Proof by contradiction (⇒).

1. t is *unambiguous*	{assum.}
2. $t_1 \subseteq t \wedge d(t \setminus t_1) \neq \emptyset \wedge d(t \setminus t_1) = d(t) \setminus d(t_1)$	{indirect assump.}
3. $d(t_1) \cap (d(t) \setminus d(t_1)) = \emptyset$	{set algebra}
4. $t_1 \subseteq t \wedge d(t \setminus t_1) \neq \emptyset \wedge d(t_1) \cap d(t \setminus t_1) = \emptyset$	{2, 3}
5. $\exists t' \subseteq t[d(t \setminus t') \neq \emptyset \wedge d(t') \cap d(t \setminus t') = \emptyset]$	{4}
6. t is *not unambiguous*	{D6, 5}
contradiction	{1, 6}

Proof by contradiction (⇐).

In the proof, we use the following theorem of set algebra:

T(*). If $A = A' \cup B \wedge A' \cap B = \emptyset \wedge A = A' \cup C \wedge A' \cap C = \emptyset$, then $B = C$.

1. $\neg\exists t' \subseteq t[(d(t \setminus t') \neq \emptyset \wedge d(t \setminus t') = d(t) \setminus d(t')]$	{assum.}
2. t is *not unambiguous*	{indirect assump.}
3. $t_1 \subseteq t \wedge d(t \setminus t_1) \neq \emptyset \wedge d(t_1) \cap d(t \setminus t_1) = \emptyset$	{D6a, 2}
4. $d(t_1) \subseteq d(t) \wedge t = t_1 \cup (t \setminus t_1)$	{3, T1}
5. $d(t) = d(t_1) \cup d(t \setminus t_1) \wedge d(t_1) \cap d(t \setminus t_1) = \emptyset$	{4, T2, 3}
6. $d(t) = d(t_1) \cup (d(t) \setminus d(t_1)) \wedge d(t_1) \cap (d(t) \setminus d(t_1)) = \emptyset$	{4}
7. $d(t \setminus t_1) = d(t) \setminus d(t_1)$	{5, 6, T(*)}
8. $\exists t' \subseteq t[(d(t \setminus t') \neq \emptyset \wedge d(t \setminus t') = d(t) \setminus d(t')]$	{3, 7}
contradiction	{1, 8}

□

Acknowledgements The author expresses her sincere gratitude to her colleagues Gabriela Besler, Alex Citkin and Zbigniew Bonikowski for an access to source publications and data needed to successfully complete this work.

The author would also like to thank Bartłomiej Skowron—the editor of this volume—and the Referees who offered a number of suggestions which greatly enhanced this paper.

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Chapter 13
A Logical Conceptualization of Knowledge on the Notion of Language Communication

Urszula Wybraniec-Skardowska

Abstract The main objective of the paper is to provide a conceptual apparatus of a general logical theory of language communication. The aim of the paper is to outline a formal-logical theory of language in which the concepts of the *phenomenon of language communication* and *language communication in general* are defined and some conditions for their adequacy are formulated. The theory explicates the key notions of contemporary syntax, semantics, and pragmatics. The theory is formalized on two levels: *token*-level and *type*-level. As such, it takes into account the dual—*token* and *type*—ontological character of linguistic entities. The basic notions of the theory: *language communication*, *meaning* and *interpretation* are introduced on the second, *type*-level of formalization, and their required prior formalization of some of the notions introduced on the first, *token*-level; among others, the notion of an *act of communication*. Owing to the theory, it is possible to address the problems of adequacy of both empirical acts of communication and of language communication in general. All the conditions of adequacy of communication discussed in the presented paper, are valid for one-way communication (sender-recipient); nevertheless, they can also apply to the reverse direction of language communication (recipient-sender). Therefore, they concern the problem of two-way understanding in language communication.

Key words: Act communication • Language communication in general • Token-type distinction • Meaning • Interpretation • Problem of adequacy of communication • Formal-logical theory of language communication

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First published in: Studies in Logic, Grammar and Rhetoric **52** (65), 247–269 (2017). See also [28–30]

1 Introduction

The key issue of modern pragmatics as a part of semiotics is communication, whose main task is the transmission, processing, and transformation of information. It does not mean, however, that we fully understand what communication is and what the conditions of its proper flow are. The problem of communication is as old as mankind and has been present in many different fields ever since, for example: in cultural systems, sign systems (including language systems), but also in market systems, bank systems, and recently emerged computer networks.

The discovery and cognition of reality is best realized through the processes of cognition, whose result is knowledge of a conceptual space. It is expressed and represented in language and transferred to others in *acts of communication* by means of concrete, material language expressions—*token-expressions* (see Diagram 1).

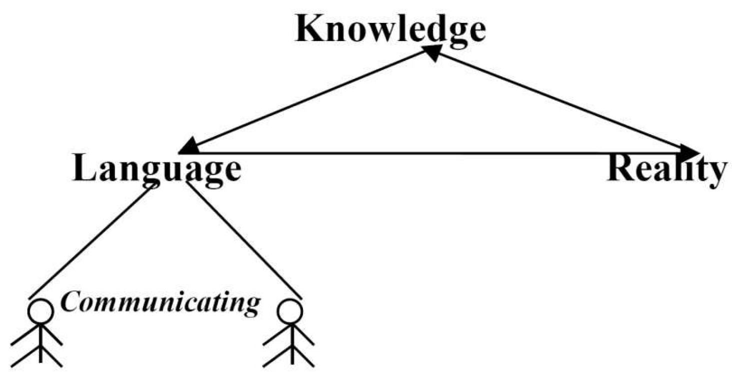


Diagram 1

Acts and processes of communication take place not only among people, but also among any communication channels, organization units, which are the subjects of this communication, for example: groups of people, firms, political parties, governments and so on. In communication acts, a very important role is played by the knowledge of objects represented by means of words and other signs. It can also be influenced by cultural, psychological, sociological, political, and technical factors. In this paper I concentrate on the representation of knowledge that takes place in language systems of communication.

The aim of the paper is to outline a logical theory of language in which the *phenomenon of language communication* and *language communication in general* are defined and some conditions for their adequacy are formulated.

Assimilation and transfer of knowledge about objects to other people is possible owing to the cognitive-communicative function of language. The transfer of verbal knowledge takes place in *acts of communication* by means of concrete, material language expressions (*token*-expressions). In formal considerations, first, we want to provide definitions of an *act of language communication* and the related notions such as: *using linguistic tokens* and *interpreting linguistic tokens* in order to formulate some general conditions for the correct course of the act of communication, i.e. to consider its *adequacy* and indicate some general causes of *verbal miscommunication*.

The notion of an *act of communication* has to be differentiated from the notion of *language communication* which is a basic concept of logical pragmatics and of the logic of language in general.

Answering the following questions:

what is language communication as such?

and

what are the conditions for correct communication?

i.e. considering the problem of its adequacy is a primary task for a general theory of language communication.

A logical conceptualization of the knowledge on the notion of *language communication* and such related notions as *meaning* and *interpretation* of language expressions involved in communication cannot be performed unless certain philosophical assumptions concerning the nature of these notions and of the expressions themselves are adopted, and unless some prior assumptions are made on the selection of primitive notions and the method of defining.

In the paper, an axiomatic theory of language communication *TLC*, as a semantic-pragmatic theory, independent of extra-logical factors, is outlined. First, in Section 2, some aspects that we take into account in formalization of the theory *TLC* will be discussed. The theory has to be based on a theory of syntax T. Some foundations of the syntax theory T will be presented in Section 3. According to the *token-type* distinction of language objects originated from Ch. S. Peirce [19] it is formalized on two levels: *token* and *type*. The proposed theory *TLC* will be developed in Section 4 as an expansion of the syntax theory T to the semanticpragmatic theory in which—on the *token*-level—the concept of *act of language communication* by means of *token*-expressions (understood as physical, material, empirical, enduring through time-and-space objects) will be defined, and the problem of adequacy of communicating by means of such expressions will be considered, while—on the *type*-level—the notion of *language communication* by means of *type*-expressions (understood as abstract, ideal objects, classes of *token*-expressions) and such related notions as *meaning* and *interpretation* will be defined and some conditions of adequacy for such communication will be formulated. The paper ends with Section 5, in which we differentiate the earlier given conditions for adequacy and general, logical factors for verbal miscommunication and misunderstanding from the extra-logical (e.g., psychological, sociological, political) ones.

2 Three Aspects in Formalization of the *TLC* Theory

The presentations of an axiomatic formal-logical theory *TLC* as a semantic-pragmatic theory, independent of any extra-logical factors, psychological or sociological or communication channels, which—on the one hand—can enhance understanding, but—on the other—can interfere with it, will be however based on some assumptions.

Although *TLC* will concern communication by means of expressions of any language, it will take into consideration, to a certain degree, the following three aspects:

1. the cognitive-communicative function of natural language, according to its genesis,
2. the so-called functional approach to logical analysis of this language, and the one connected with it:
3. two understandings of a manner of use and a manner of interpreting language expressions in communication.

Let us expand on these aspects.

2.1 The Cognitive-communicative Function of Natural Language According to its Genesis

Given the genesis of natural language, one can easily observe that it was formed in the process of cognition and communication between people who made use of material, concrete signs. Accordingly, we make the assumption that the primitive linguistic entities applied in communication acts between their senders and recipients are material creations, e.g. given sounds, written signs, physical objects somehow placed in time and space, concrete objects which have some referents attributed to them, and which are called *tokens*. According to the well-known token-*type* distinction made by CH. S. Peirce [19], we differentiate *token*-signs (signs-examples) from *type*-signs, which are abstract, ideal linguistic objects, and whose physical representations are just *tokens*.

In acts of communication (see Diagram 2) the sender *s* calls, uses a token *e* of a sign with reference to a broadly conceived object *o*, while the recipient *r* interprets it in compliance or in discordance with the sender's intention, as object *o* or another object *o*'. Compliance produces understanding, while discordance produces misunderstanding.

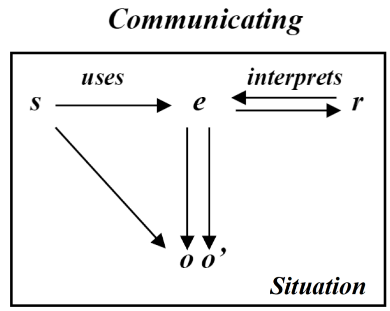


Diagram 2

2.2 The Functional Approach to Natural Language Analysis

As we have already seen, in order to explain the notion of communicating we had to introduce the terms *using* and *interpreting*, which entailed the use of concrete entities, i.e. *tokens*, and the inclusion of situational contexts accompanying them. This shows how *tokens* function in communication acts. Even though we are not going to refer to situational contexts in our theoretical considerations, the context is always present in such acts.

In the proposed theory *TLC*, the basic semantic-pragmatic notions, including the notion of *language communication* and the related concepts—*meaning* and *interpretation*—are defined by means of *expression*-types, and yet their definitions involve such primitive notions of the theory as *using* and *interpreting* expression-*tokens*.

The formal conception of language communication has some connections with the understanding of meaning as a manner of use of expressions and interpretation as a manner of interpreting expressions.

Speaking about the functional approach to natural language analysis, we have to take into consideration the *manner of use* and the *manner of interpreting* language expressions. The latter will be regarded as a special case of the former.

2.3 Two Understandings of *manner of use* and *manner of interpreting* Language Expressions in Communication

The functional approach to natural language analysis involves speaking about two meanings of the terms: '*a manner of use*' and '*a manner of interpreting*'.

After the approach of J. Pelc [20, 21], we distinguish two understandings of these terms:

- in the first of them, the *manner of using* (*use*) and the *manner of interpreting* (*int*) occur only in given circumstances, in specific languagesituational-contexts, and concern expression-*tokens* only,

- in the second—the *manner of Use* (*usage*) and the *manner of interpreting* (*Int*) characterize the meaning of the expression and the interpretation of the expression, respectively; these manners are somehow built into this meaning and this interpretation, respectively. In this case an expression can be treated as isolated, static, out of context, e.g. as an entry in a dictionary. It is then an expression-*type*, a class of its concrete occurrences, a distributive set of expression-*tokens* used either to represent a given object, or in concrete acts of communication in specific linguistic-situational contexts, with reference to only one broadly conceived object or to a set of objects of the same kind.

For example, two single tokens of the expression-*type* ‘scientist’, having an established meaning (the *manner of Use*) or a specific interpretation (the *manner of Int*) in English, can be used in a similar linguistic-situational context either with reference to a given scientist, e.g. the one which I am pointing to, or with reference to two different scientists, e.g. in a situation of teaching a student the meaning of the word ‘scientist’ through a definition and pointing to two different scientists.

The relation *use* and its sub-relation *int*, concerning all the relations of physical object-based reference of expression-*tokens* made by users of language, will be primitive notions of the theory *TLC* proposed here. The relation *Use* (resp. the relation *Int*) is, on the other hand, a relation defined by means of the relation of *use* (resp. the relation *int*) and applied by users of language for expression-*types*. The difference between these relations is explained by the fact that two persons can *Use* the same expression-*type* by means of its two different *tokens*.

The notion of an expression is a syntactic one and must be defined on the basis of a theory of syntax.

3 Language Syntax; Theory T

3.1 Two Levels of Formalization of Syntax of Language

The theory of syntax T is formalized on two levels: *token*-level and *type*-level. According to the *token-type* distinction by Peirce [19], any language L is characterized as a construct of a double ontological nature: both as

- a language of expression-*tokens* (at the *token*-level)

and as

- a language of expression-*types* (at the *type*-level).

The theory T is first formalized on the *token*-level as the theory of *token*-syntax describing L as a language of expression-*tokens*, and then, on the *type*-level, as the theory of *type*-syntax describing L as a language of expression-*types*. The theory of *type*-syntax is an extension of the theory of *token*-syntax.

Tokens are primitive objects of the theory T. They are intuitively understood as concrete, material, empirical objects, enduring through time and space and perceived by sight. They are usually inscriptions, but do not have to be inscriptions. They can be on paper, a notice board, a blackboard, a computer screen, a stone, etc.; they may be configurations of such things as jigsaw-puzzle pieces, leaves, stones, stars, or smoke signals, or illuminated advertisements, and so on.

Types are derived objects of the theory T defined by means of *tokens*. They are understood as sets (classes) of *tokens* bearing an *identifiability* relation to each other, i.e. *types* are ideal, abstract entities.

3.2 Identifiability of linguistic *tokens*

The *relation of identifiability* $\sim$ *of tokens* (a primitive notion of the theory T) is determined by pragmatic factors and not by physical similarity, and it is understood very broadly. For instance, inscriptions printed in different types but consisting successively of the same letters of the alphabet may be *identifiable*, e.g. the word-*tokens*:

DUBROVNIK **Dubrovnik**

Dubrovnik

written in capital letters, in bold with bigger typeface or in italics, respectively, can be regarded as *identifiable* words.

We will assume that the relation of the *identifiability* $\sim$ of tokens is an equivalence relation.

The *expressions* of language L are defined separately on the *token*-level and on the *type*-level. They are suitable *concatenations* of *tokens* or *types*. The *relation of concatenation of tokens* is another primitive notion of the theory T.

3.2.1 Concatenations

Concatenations of *tokens* are complex words of language L obtained from two words of the *vocabulary* of language L—the next primitive notion of theory T. Concatenations on the *token*-level may be, but do not have to be, sequences of two *tokens*. Intuitively, a concatenation of two written *tokens* a and b, for example in a European language, is a written *token* c that is made up by adding the written *token* b^*, identifiable with b, to the *token* a^*, identifiable with a, on the right.

For example, the concatenations of the following word-*tokens*:

C
o
n FORMAL METHODS
f
e
r
e n c

e

the second and the first, is the name-token:

Formal Methods Conference

and any name-*token* identifiable with it, in particular the token aligned vertically:

FORMAL METHODS
CONFERENCE

or any token written on each poster on the conference.

So, the relation of *concatenation* defined by *tokens* is not a set-theoretical function and the relation of *identifiability* is not a relation of physical similarity. These two relations and the *vocabulary* of *tokens* are primitive notions of the theory of words which is included in the theory of syntax T. They are formalized on the *token*-level. All of them satisfy some specific axioms of the theory.

3.2.2 Well-formed Expressions

The most important notion of the theory of syntax T is the notion of a *well-formed expression* of language L (for short: *wfe*). The theory T can be built as a theory of language syntax in which (see WybraniecSkardowska [23]) all *wfes* are generated by a categorial grammar (see K. Ajdukiewicz [3]; Y. Bar-Hillel [4–6]; J. Lambek [12, 13]; R. Montague [15–18]; M. J. Cresswell [10, 11]; W. Marciszewski [14]; W. Buszkowski [7, 8] and others). On the basis of the theory T we can reconstruct such a grammar. The notion of a *wfe* is defined firstly on the *token*-level and then on the *type*-level. Then the set S *of all wfe-tokens* is formally defined as the smallest set including the vocabulary of tokens and closed with respect to syntactic connection rules.

The set $\underline{S}$ of all well-formed expression-*types* (for short: *wfe-types*) is defined as the quotient family of the set S of all *wfe-tokens* determined by the relation $\sim$ of *identifiability*:

$$\underline{S} = S/\!\sim .$$

Hence, we get that:

$$\underline{p} \in \underline{S} \text{ iff } \exists p \in S \left(\underline{p} = [p]_{\sim} = \{q \in S\colon q \sim p\}\right).$$

So, any well-formed expression-type $\underline{p}$ is an equivalence set of all *wfe-tokens* identifiable with a *wfe*-token p.

In the following sections, we will use *wfe-types* not only as elements of the set $\underline{S}$ but also all non-empty subtypes of *wfe-types* of this set. By *wfe-types* of L we will mean all elements of the set $\underline{S}^*$:

$$\underline{S}^* = \{\underline{e} \subseteq \underline{p}\colon \underline{e} \neq \emptyset \ \wedge \ \underline{p} \in \underline{S}\}$$

i.e. all non-empty sets of identifiable *wfe-tokens*.

4 A Theory of Language Communication – Theory *TLC*

4.1 *Token*-level

Because the formal theory *TLC* should define the notion of *language communication*, its conceptual apparatus has to refer to the notions of *meaning* and *interpretation* of language expressions and to empirical *acts of communication* among people. So, on the *token*-level its conceptual apparatus has to include the notions of *using* and *interpreting token*-expressions by users of language L. Thus, we accept the postulate that in communication acts the sender, in order to send the message, applies the function *use* connected with the object reference of a *wfe-token*, whereas the recipient, in order to receive the message, applies another function—the function *int* of interpreting tokens.

4.1.1 Primitive notions of *TLC*

Primitive notions of the theory *TLC* are:

- the *set* User *of all users of a given language* L,
- the *set* Ont *of all extra-linguistic objects described by* L,
- the two-place *operation* use *of using the wfe-tokens of* L.
- the two-place *operation* int *of interpreting the wfe-tokens of* L.

The first two primitive notions are understood very broadly. The set User of users of language L can be composed of current as well as former or future users of this language. We do not make any assumptions, either, about the ontological nature of objects of the set Ont. They can be not only material objects, but also, for instance, fictional or abstract creations described by language L.

Of course, the sets User and Ont are non-empty sets:

Axiom (*sets*: User, Ont) *$\mathsf{User} \neq \emptyset$ and $\mathsf{Ont} \neq \emptyset$.*

We understand the operation (relation) use as an operation producing, calling, using, exposing or interpreting *wfe-tokens* in order to refer them to corresponding objects of the set Ont. We can also call this operation a *function of object reference of wfe-tokens* by users of language L.

The operation int occurs when we speak about communication by means of expression-tokens. This operation will be a restriction of the former one.

The operations use and int satisfy the following axioms:

Axioms (*using*) *use is a partial function of*

$$\mathsf{User} \times S \to \mathsf{Ont},$$
$$Dom_1(\mathsf{use}) = \mathsf{User} \text{ and } Dom_2(\mathsf{use}) \subset S.$$

Axiom (*interpreting*) *int is a partial function of the function use, i.e.*

$$\emptyset \neq \mathsf{int} \subseteq \mathsf{use} \text{ and } Dom_2(\mathsf{int}) \subseteq Dom_2(\mathsf{use}) \subset S.$$

The expression: $\mathsf{use}(u, e) = o$, where $u \in \mathsf{User}$, $e \in S$, $o \in \mathsf{Ont}$ is read: *the user u uses* (*makes or exposes*) *the wfe-token e to refer to the object o*. This object o is called the *referent* of the *wfe-token e* assigned by its user u. Similarly, the expression $\mathsf{int}(u, e) = o$ is read: *the user u interprets* (*understands*) *the wfe-token e as a sign-token of the object o*. The object o is called the *interpretandum* of the *wfe-token e*.

It follows from the second axiom that every user of L uses at least one *wfe-token* of L to refer to an object. Not every *wfe-token* must have a referent. From the third axiom it follows that the operation *int* of *interpreting tokens* is narrower than the operation *use* of *using tokens*. This is because the pair ⟨a user, a token⟩, which has a referent, may have no corresponding interpretandum when, for instance, this *token* cannot be received or was used with the intention of being interpreted by a recipient, but he/she cannot interpret it. The fact is, however, that each pair that has an interpretandum also has the same referent.

The notion *int* of interpreting tokens emerges when we speak about communication by means of expression-*tokens*. From the axioms, we immediately get:

Corollary 1

a. $\forall u \in Dom_1(\mathsf{int})\, \forall e \in Dom_2(\mathsf{int})\, (\mathsf{int}(u, e) = \mathsf{use}(u, e))$,
b. $\exists u \in \mathsf{User}\, \exists e \in S\, \exists o \in \mathsf{Ont}\, (\mathsf{use}(u, e) = o = \mathsf{int}(u, e))$,
c. $Dom_1(\mathsf{int}) \subseteq Dom_1(\mathsf{use}) \subseteq \mathsf{User}$.

Thus (see part *c*.), *interpreting tokens* is a particular case of *using tokens*.

On the basis of part *a*. we can state that if we limited both domains of the operation *use* using *wfe-tokens* to the domain of operation *int* interpreting *wfe-tokens* of L, then these two operations would not be discernible; then every user using any expression-*token* to refer to an object is a person who also interprets this expression as this object. Such a situation is not specific of communicating by means of *tokens*, but it follows from part *b*. that there exists at least one user of L who uses and interprets a *token* in a given act of communication by means of this *token* as the same object.

4.1.2 Act of Communication

The notion of *communication act* is new in *TLC*. An *act of communication* is defined as a triple satisfying of some conditions:

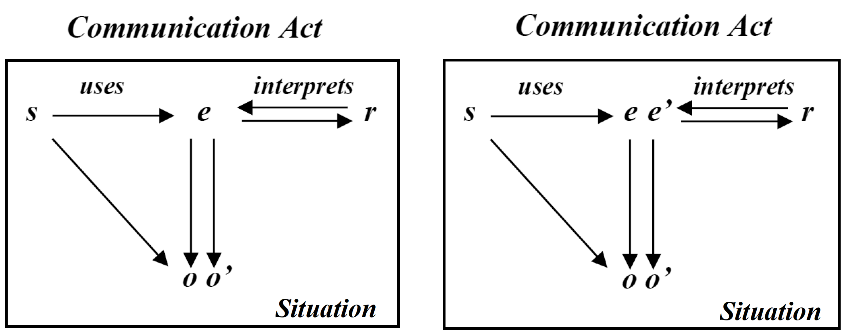
Definition 1a (*act of communication*)

$\langle s, e, r\rangle \in acom$ iff

$$s, r \in \mathsf{User} \ \wedge\ e \in S \ \wedge\ \exists o, o' \in \mathsf{Ont}\ (\mathsf{use}(s, e) = o \ \wedge\ \mathsf{int}(r, e) = o').$$

Its first element s (the sender) and the third of its elements r (the recipient) are users of language L, the second element e is a *wfe-token* of L and there exist objects $o, o' \in \mathit{Ont}$ such that the sender s of the expression e uses the expression e to refer to the object o (the referent) and the recipient r of the expression e interprets this expression as a sign-token of the object o' (the interpretandum) (see Diagram 3a).

Communication acts can be carried out by means of two different expression-tokens of the same *wfe-type* (see Diagram 3b), if the sender *uses* a token and the recipient interprets another *token* the same expression-*type*; this is so in e-mail, microphone or telephone communication.



(a) (b)

Diagram 3

So, the more general definition of an *act of communication* is in accordance with:

Definition 1b (*act of communication*)

$$\langle s, e, r\rangle \in \mathit{ACom} \text{ iff } s, r \in \mathit{User} \land$$
$$\land \exists \underline{e} \in \underline{S}^{*} \ (e \in \underline{e} \ \land \ \exists e' \in \underline{e} \ \exists o, o' \in \mathit{Ont} \ (\mathit{use}(s, e) = o \ \land \ \mathit{int}(r, e') = o')).$$

It is easy to see that any act of communication by means of one *token* is also an act of communication by means of two expression-*tokens*. So, we have

Corollary 2 $\langle s, e, r\rangle \in \mathit{acom} \Rightarrow \langle s, e, r\rangle \in \mathit{ACom}$.

Examples of communication acts include: making an announcement, this present paper, a specific question, e.g. in a discussion, etc.

4.1.3 Adequacy of Communication Acts

The problem of adequacy of an act of communication by means of a *wfe-token* consists in its effectiveness. A communication act is effective if using the *token* by its sender and interpreting the *token* or a *token* identifiable with that *token* by its recipient are in agreement, i.e. the referent to which the sender uses the *token* and the interpretandum as an object of interpreting the *token* or a *token* identifiable with that token by its recipient, are the same. In other words, a communication act is effective when an *understanding* takes place between its sender and its recipient.

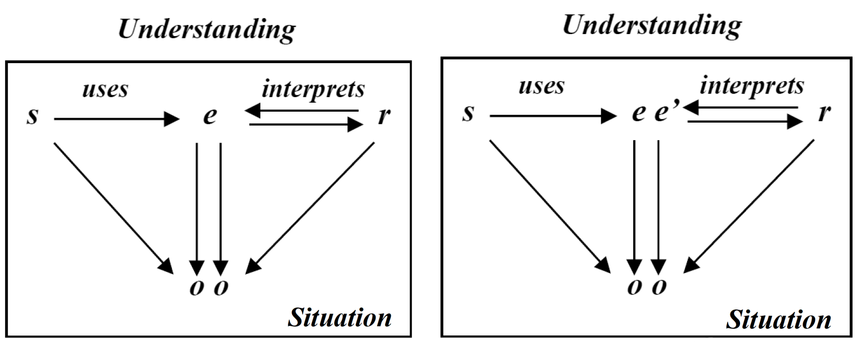
Two definitions of an act of communication by means of *wfe-tokens* will bring us to two definitions of the notion of *understanding* (see Diagrams 4a and 4b).

Definition 2a (*understanding*)

$und_e(s, r)$ iff $s, r \in \mathit{User} \ \wedge \ e \in S \ \wedge \ \exists o \in \mathit{Ont}(\mathit{use}(s, e) = o = \mathit{int}(r, e))$.

Definition 2b (*understanding*)

$Und_e(s, r)$ iff
$s, r \in \mathit{User} \ \wedge \ \exists \underline{e} \in \underline{S}^* \ (e \in \underline{e} \ \wedge \ \exists e' \in \underline{e} \ \exists o \in \mathit{Ont} \ (\mathit{use}(s, e) = o = \mathit{int}(r, e')))$.



(a) (b)

Diagram 4

Abbreviations '$und_e(s, r)$' and '$Und_e(s, r)$' are used here for the expressions: 'Between s and r in an act of communication by means of the *wfe-token* e or the *tokens*: e and some identifiable *token* e', respectively, there exists understanding'. The object which is both the referent and the interpretandum in the act of communication determined by $Und_e(s, r)$, is called the *object of understanding*.

It is quite obvious that if there exists understanding in the first sense, then there exists understanding in the second sense, and the following conclusions are valid:

Corollary 3

a) $und_e(s, r) \Rightarrow Und_e(s, r)$,
b) $und_e(s, r) \Rightarrow \langle s, e, r\rangle \in acom$,
c) $Und_e(s, r) \Rightarrow \langle s, e, r\rangle \in ACom$,
d) $\exists u \in \mathit{User} \, \exists e \in S(\langle u, e, u\rangle \in acom \ \wedge \ und_e(u, u))$,
e) $acom \neq \emptyset \ \wedge \ ACom \neq \emptyset$.

Point *d*) of the above corollary states that there exists at least one user of the language L who takes part in an act of communication by means of an expression-token simultaneously as the sender and the recipient, and understanding takes place in the act. So, we have *e*): the sets of all communication acts, in both senses, are nonempty.

4.1.4 Miscommunication: Misunderstanding

If, in an act of communication by means of a *wfe-token*, understanding does not take place between its sender and its recipient, then the act of communication is *not adequate* and we may speak about *miscommunication*. It occurs if *misunderstanding* takes place in this act or if an attempted act of communication fails because of *non-understanding* between the sender and the recipient.

From two definitions of a communication act, we will obtain two definitions of *misunderstanding* and two definitions of *non-understanding*.

Definition 3a (*misunderstanding*)

$misund_e(s,r)$ iff

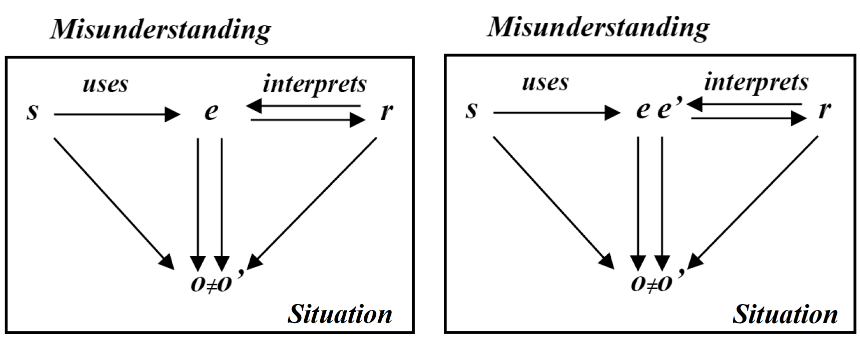
$$s, r \in \mathsf{User} \ \wedge \ e \in S \ \wedge \ \exists o, o' \in \mathsf{Ont}\ (\mathsf{use}(s,e) = o \neq o' = \mathsf{int}(r,e)).$$

Definition 3b (*misunderstanding*)

$Misund_e(s,r)$ iff $s, r \in \mathsf{User}\ \wedge$

$$\wedge\ \exists \underline{e} \in \underline{S}^*\ (e \in \underline{e}\ \wedge\ \exists e' \in \underline{e}\ \exists o, o' \in \mathsf{Ont}\ (\mathsf{use}(s,e) = o \neq o' = \mathsf{int}(r,e'))).$$

If the sender of the expression-*token e* uses this expression to refer to an object and the recipient interprets this or another expression-*token e*' of the same type as another object, then there exists a *misunderstanding* between the sender and the recipient in the act of communication by means of the expression *e* (see Diagrams 5a and 5b).



(a) (b)

Diagram 5

4.1.5 Miscommunication: Non-understanding

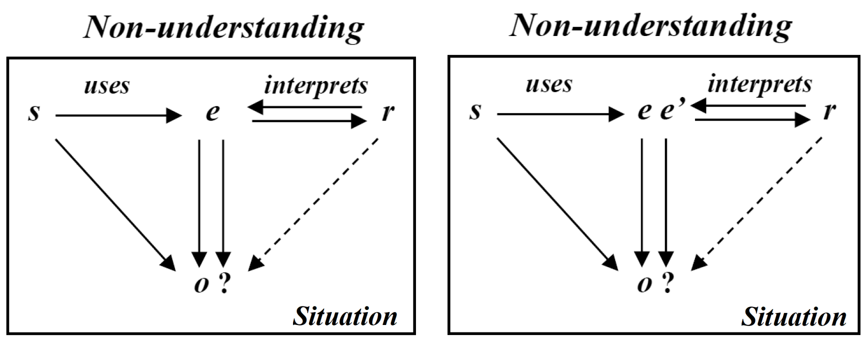
If the sender of the expression e uses e to refer to a referent but the recipient is unable to interpret the expression e or an expression identifiable with that expression, then there follows a *non-understanding* (see Diagrams 6a and 6b). Thus, symbolically:

Definition 4a (*non-understanding*)

$$\begin{aligned} &\textit{non-und}_e(s,r) \text{ iff } s,r \in \mathit{User} \ \wedge\ e \in S \ \wedge \\ &\qquad \wedge\ \exists o \in \mathit{Ont}\ \big(\mathit{use}(s,e) = o\ \wedge\ \forall o' \in \mathit{Ont}\ (\neg \mathit{int}(r,e) = o')\big). \end{aligned}$$

Definition 4b (*non-understanding*)

$$\begin{aligned} &\textit{Non-und}_e(s,r) \text{ iff } s,r \in \mathit{User} \ \wedge\ \exists \underline{e} \in \underline{S}\ (e \in \underline{e} \ \wedge \\ &\qquad \wedge\ \exists e' \in \underline{e}\ \exists o \in \mathit{Ont}\ \big(\mathit{use}(s,e) = o\ \wedge\ \forall o' \in \mathit{Ont}\ (\neg \mathit{int}(r,e') = o')\big). \end{aligned}$$



(a) (b)

Diagram 6

4.2 *Type*-level

4.2.1 Communication by Means of Expression-*types*

Empirical communication by means of expression-*tokens* has to be distinguished in a given community of *Users* from communication by means of *wfe-types*. On the *type*-level we expand the conceptual apparatus of the *TLC* with new notions. The most important one is the notion of *communication by means of types*. It is determined as a value of an *operation communication C* defined on expression-*types*.

The *operation communication C* is a function defined as follows:

Definition 5 (*operation communication*)

$$C : \underline{S}^{*} \rightarrow 2^{\mathit{User} \times S \times \mathit{User}}$$

$C(\underline{e}) = \{\langle s, e, r\rangle\colon\ s, r \in \mathit{User}\ \wedge\ e \in \underline{e}\ \wedge\ \langle s, e, r\rangle \in \mathit{ACom}\}$
for every *wfe-type e* of language L.

The value $C(\underline{e})$ of the function C for the expression-*type* $\underline{e}$ is called communication by means of the expression-type $\underline{e}$. *Communication* $C(\underline{e})$ by *means of the expression-type* $\underline{e}$ is the relation $\mathit{User} \times S \times \mathit{User}$ consisting of all ordered triples, such that the first element (the sender) uses a *wfe-token* of $\underline{e}$ and the third component (the recipient) interprets a *token* of $\underline{e}$ in an act of communication. So, communication $C(\underline{e})$ by means of the expression-*type* $\underline{e}$ is the set of all communication acts by means of expression-*tokens* of the type $\underline{e}$.

It includes the set of all communication acts by means of only one *token* of the *type*. Moreover, it follows from earlier corollaries that there exists a *wfe-type* $\underline{e}$ such that communication $C(\underline{e})$ by means of type $\underline{e}$ is a nonempty set. Thus we arrive at:

Corollary 4

a) $\{\langle s, e, r\rangle\colon\ s, r \in \mathit{User}\ \wedge\ e \in \underline{e}\ \wedge\ \langle s, e, r\rangle \in \mathit{acom}\} \subseteq C(\underline{e})$,
b) $\exists \underline{e} \in \underline{S}^* \ (C(\underline{e}) \neq \emptyset)$.

4.2.2 Using *types* and Interpreting *types*

Users that participate in acts of communication belonging to language communication by means of an expression-*type* $\underline{e}$ are also *Using* the expression-*type* $\underline{e}$: senders *Use* this *type* while recipients *Interpret* it. The relation *Use* of *Using* expression-*types* and its sub-relation *Int* of *Interpreting* expression-*types* are new notions of *TLC*. They are binary relations satisfying some axioms and defined by means of relations *use* and *int* for tokens, respectively:

Axiom (*Use*) $\mathit{Use} \subseteq \mathit{User} \times \underline{S}^*$,

Axiom (*domain of Int*) $\mathit{Dom}_1(\mathit{Int}) \subseteq \mathit{Dom}_1(\mathit{int}) \subseteq \mathit{User} = \mathit{Dom}_1(\mathit{use})$.

The relation *Use* is defined as follows:

Definition 6 (*Using types*)

$$u\ \mathit{Use}\ \underline{e} \text{ iff } \exists e \in \underline{e}\ \exists o \in \mathit{Ont}(\mathit{use}(u, e) = o).$$

According to this definition, the user u *Uses* the *wfe-type* $\underline{e}$ iff the user u *uses* a *wfe-token* of the type $\underline{e}$ to refer to some referent.

The definition of relation *Int* is dual to the definition of the relation *Use*,

Definition 6^i (*Interpreting types*)

$$u\ \mathit{Int}\ \underline{e} \text{ iff } \exists e \in \underline{e}\ \exists o \in \mathit{Ont}(\mathit{int}(u, e) = o).$$

and it says that the user u *Interprets* the *wfe-type* $\underline{e}$ iff the user u interprets a *wfe-token* of the type $\underline{e}$ as some interpretandum.

Because $\mathit{int} \subseteq \mathit{use}$, i.e. the relation *int* of interpreting tokens is included in the relation *use* of using tokens, the relation *Int* of interpreting *types* is included in the relation *Use* of using types (see Corollary 5a); however, from the Axiom given above for the relation *Int* for types, it follows that the user who *Uses* a *type* does not need to be the one who *Interprets* it.

Because communication $C(\underline{e})$ by means of the type $\underline{e}$ is a nonempty set, the above definitions lead to Corollary 5b) and the comment found at the top of this subsection is justified:

Corollary 5

a) $\mathit{Int} \subseteq \mathit{Use}$,
b) $\mathit{int} = \mathit{use} \Rightarrow \mathit{Int} = \mathit{Use}$,
c) $\langle s, e, r\rangle \in C(\underline{e}) \Rightarrow s\ \mathit{Use}\ \underline{e}\ \wedge\ r\ \mathit{Int}\ \underline{e}$
d) $\mathit{Use} \neq \emptyset\ \wedge\ \mathit{Int} \neq \emptyset$.

Point *d*) of the above corollary immediately follows from point *c*).

4.2.3 Problem of Adequacy of Language Communication

Adequate, effective, successful *communication* in a community of *Users* by means of the expression-type e is based on the agreed *meaning* $\mu(\underline{e})$ of the expression-type $\underline{e}$ used by users who are senders of *tokens* of $\underline{e}$ in acts of communication, and based on the correlation $\mu(\underline{e})$ with the *interpretation* $\iota(\underline{e})$ of the expression-type $\mu(\underline{e})$ interpreted by users who are recipients of these tokens in the acts (cf. Wybraniec-Skardowska [26]). Compatibility of the meaning and the interpretation of the expression-type $\mu(\underline{e})$ leads to *understanding* between senders and recipients (see Diagram 7).

A disagreement between the meaning and the interpretation of the expression-*type* leads to *misunderstanding*, while ignorance of the interpretation of the expression-type leads to *non-understanding*.

4.2.4 Notions Relating to Language Communication

It is obvious that the conceptual apparatus of the theory *TLC* has to be enriched by notions concerning meaning and interpretation of language expression-*types*.

As we said before, these notions will be characterized in relation to the understanding of *meaning* as a *manner of Using* (*usage*) expression-*types* and *interpretation* as a *manner of Interpreting* (*Int*) these expressions; these manners are in a way built into this *meaning* and this *interpretation*, respectively.

Interpretation indicates the *meaning* or *meanings* of a given expression-*type* and cannot be identified with its *meaning*. Let us also note that the notion of *interpretation*

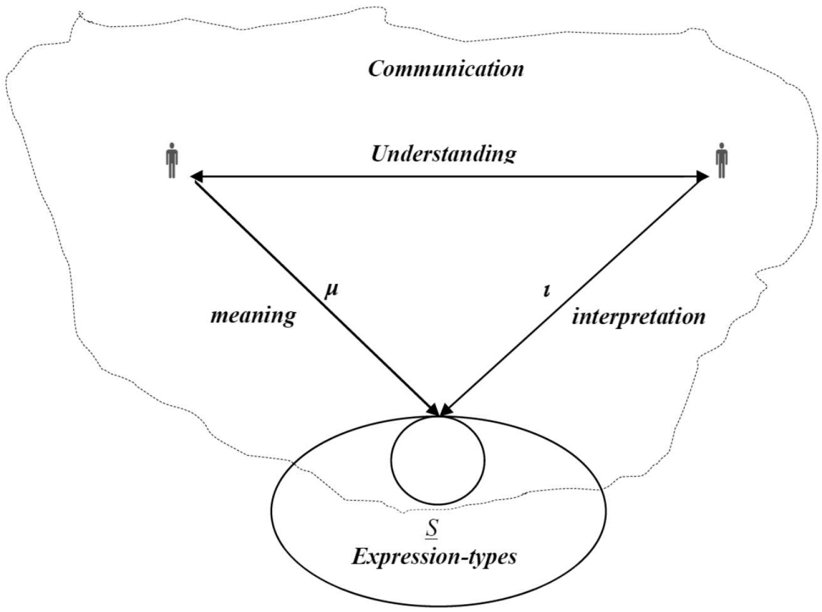


Diagram 7

does not need to be connected with sign-based systems of communication only; in semantics, it plays a special, central role.

The notion of *meaning* is defined by means of the *relation* $\cong$ *of having the same manner of Using wfe-types* and the notion of *interpretation*—by means of the *relation* $\cong_i$ *of having the same manner of Interpreting* (*understanding*) *wfe-types* (see Wybraniec-Skardowska [24–27]). The definitions of these relations are as follows:

Definition 7 (*having the same manner of Using types*)

$$\underline{e} \cong \underline{e}\text{'} \text{ iff } \forall u \in \mathsf{User}\ \big((u\ \mathsf{Use}\ \underline{e} \ \Leftrightarrow\ u\ \mathsf{Use}\ \underline{e}\text{'}) \wedge \\ \wedge \forall o \in \mathsf{Ont}\ (\exists e \in \underline{e}\ (\mathsf{use}(u, e) = o) \ \Leftrightarrow\ \exists e\text{'} \in \underline{e}\text{'}\ (\mathsf{use}(u, e\text{'}) = o))\big).$$

Definition 7^i (*having the same manner of Interpreting types*)

$$\underline{e} \cong_i \underline{e}\text{'} \text{ iff } \forall u \in \mathsf{User}\ \big((u\ \mathsf{Int}\ \underline{e} \ \Leftrightarrow\ u\ \mathsf{Int}\ \underline{e}\text{'}) \wedge \\ \wedge \forall o \in \mathsf{Ont}\ (\exists e \in \underline{e}\ (\mathsf{int}(u, e) = o) \ \Leftrightarrow\ \exists e\text{'} \in \underline{e}\text{'}\ (\mathsf{int}(u, e\text{'}) = o))\big).$$

Two wfe-types $\underline{e}$ and $\underline{e}$' *have the same manner of Using* (resp. of *Interpreting*) *wfe-types* if and only if every user of language *L Uses* (resp. *Interprets*) the other one every time he/she *Uses* (resp. *Interprets*) either of them, and every object is a referent (resp. an interpretant) of some *token* of the *type* $\underline{e}$ (*used*/*interpreted* by the user) iff it is a referent (resp. an interpretant) of some *token* of the other *type* $\underline{e}$' (*used*/*interpreted* by the user).

The *relation* $\cong_i$ *having the same manner of Interpreting types* is given if its arguments belong to $Dom_2(\mathsf{Int})$. So, we adopt the following axiom:

Axiom (*domain of* $\cong_i$) $\cong_i \subseteq \big(Dom_2(\mathit{Int}) \times Dom_2(\mathit{Int})\big) \cap \cong$.

And, the relation $\cong_i$ is a sub-relation of the relation $\cong$, and it can easily be proved that it is a nonempty relation (from Corollary 5c: $\mathit{Int} \neq \emptyset$, and because it is a reflexive relation).

Theorem 1 *The relations $\cong$ and $\cong_i$ are equivalence relations in the set $\underline{S}^*$.*

Definitions of meaning and interpretation of the *wfe-type* $\underline{e}$ are the following:

Definitions 8 (*meaning* and *interpretation*)

$$\text{a) } \mu(\underline{e}) = [\underline{e}]_{\cong} \ \text{ and } \ \text{b) } \iota(\underline{e}) = [\underline{e}]_{\cong_i}.$$

The definition of interpretation $\iota(\underline{e})$ of the *wfe-type* $\underline{e}$ is dual to the definition of meaning $\mu(\underline{e})$ of the expression. According to these definitions: Meaning $\mu(\underline{e})$ and interpretation $\iota(\underline{e})$ of the *wfe-type* $\underline{e}$ is the equivalence class of all expressions possessing the same manner of *Using* or, respectively, *Interpreting* (*understanding*), as the expression $\underline{e}$, and can be intuitively understood as a common property of all *wfe-types* having the same manner of *Using* or, respectively, *Interpreting* as the expression-type $\underline{e}$. The property can be called the *manner of using* or, respectively, the *manner of interpreting* of the expression-*type* $\underline{e}$. In this way, we are referring here to ideas originating from Ludwig Wittgenstein [22] and Kazimierz Ajdukiewicz [1, 2], that is to understanding of the meaning as a manner of its *Use*/*Interpreting*.

It is easy to see that we have:

Theorem 2

a) $\iota(\underline{e}) \subseteq \mu(\underline{e})$,
b) $\mathit{int} = \mathit{use} \Rightarrow \iota(\underline{e}) = \mu(\underline{e})$,

So, the notion of meaning is stronger than the notion of interpretation.

4.2.5 Dual Conceptual Counterparts

It should be observed that the notions of the system:

$$(*) \qquad \mathit{use}, \mathit{Use}, \cong, \mu,$$

have, within *TLC*, dual counterparts in the system:

$$(**) \qquad \mathit{int}, \mathit{Int}, \cong_i, \iota.$$

All the notions of the system (marked with two asterisks) have dual definitions towards the corresponding definitions of the theory *TLC* concerning the notions of the first system (∗).

So, all theorems of the theory *TLC* formulated for the notions of (∗) remain valid if we replace the notions of this system (∗) with their dual counterparts of (∗∗). The close relationships between the semantic-pragmatic notions of the systems ($*$) and ($**$) cause these notions to be often regarded as identical. However, each relation or function of the system ($**$) is only a sub-relation of its counterpart in the system ($*$) and not of all theorems of *TLC* concerning the notions of this system have their dual counterparts.

The meaning $\mu(\underline{e})$ of a wfe-type $\underline{e}$ and the interpretation $\iota(\underline{e})$ of the type $\underline{e}$ may differ. If that is the case, the communication $C(\underline{e})$ by means of the wfe-type $\underline{e}$ does not have to be *adequate*.

Using the notions of *meaning* and *interpretation* we can define the notion of *adequacy of language communication*.

4.2.6 Adequacy of Language Communication

As it has already been mentioned, in language communication, interpretation indicates the meaning or meanings of the expression-*type* which intermediates in this communication. An expression-*type* may have more than one meaning. If it has more meanings, they are determined by subtypes of the expression, as for example, for the terms: ‘key’ or ‘bank’.

We will adopt the following definition of *adequacy of communication*:

Definition 9 (*adequacy of language communication*)
If $\underline{e}$ has n ($n \geq 1$) meanings determined by its subtypes $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$, then
$C(\underline{e})$ is an *adequate communication* iff

$$\forall k = 1, \ldots, n \ (\underline{e}_k \text{ has determined interpretation and } \iota(\underline{e}_k) = \mu(\underline{e}_k)).$$

From the definition of adequacy of communication by means of wfe-*type* we obtain some conditions of adequacy of language communication:

Corollary 6

a) *If $\underline{e}$ has n ($n \geq 1$) meanings determined by its subtypes $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$, then* $C(\underline{e})$ is not an adequate communication *iff* $\exists k = 1, \ldots, n$ ($\underline{e}_k$*does not have a determined interpretation or* $\iota(\underline{e}_k) \neq \mu(\underline{e}_k)$).

b) *If $\underline{e}$ has an established meaning and $\underline{e}$ has a determined interpretation then* $C(\underline{e})$ is an adequate language communication *iff* $\iota(\underline{e}) = \mu(\underline{e})$.

c) *If $\underline{e}$ has an established meaning and $\underline{e}$ does not have a determined interpretation, then* $C(\underline{e})$ is not an adequate language communication.

d) *If $\underline{e}$ has an established meaning, $\underline{e}$ has a determined interpretation and* $\iota(\underline{e}) \neq \mu(\underline{e})$, *then* $C(\underline{e})$ is not an adequate language communication.

We see that the accord of meaning and interpretation is a necessary condition of adequate language communication by means of expression-*type* of L.

The next theorem provides us with some sufficient condition for adequacy of communication by means of types.

Theorem 3 *If* $\mathsf{int} = \mathsf{use}$ *and $\underline{e}$ has an established meaning and a determined interpretation then* $C(\underline{e})$ is an adequate language communication.

The above theorem follows from Theorem 2b and Corollary 6b.

5 Summary

The main objective of the work presented was to provide a conceptual apparatus of a general logical theory of language communication. The outlined axiomatic theory explicates the key notions of contemporary syntax, semantics and pragmatics.

The theory is formalized on two levels: *token*-level and *type*-level. As such, it takes into account the dual—*token* and *type*—ontological character of linguistic entities.

The basic notions of the theory: language communication, meaning and interpretation are introduced on the second, *type*-level of formalization, and they require prior formalization of some of the notions introduced on the first, *token*-level; among others, the notion of an act of communication.

Owing to the theory, it is possible to address the problems of adequacy of both empirical acts of communication and of language communication in general.

However, so far it has not been possible to theoretically capture the intuitive relationships between the adequacy of language communication and the correctness of its communication acts.

The paper is only an attempt at providing a conceptual apparatus for the theory. One cannot expect it to offer strong theorems as yet, although it seems that the theorems concerning the relationships between adequacy of language communication and adequacy of its communication acts should function well enough.

All the general conditions of adequacy of language communication discussed in the presented paper were shown as if they were valid for one-way communication (sender–recipient); nevertheless, they can also apply to the reverse direction of language communication (recipient–sender). Therefore, they concern the problem of two-way understanding in language communication.

Finally, it can be noted that the conceptual apparatus of the theory can be enriched through the introduction of notions concerning some specific forms of communication, such as discourse and dialog.

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