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On Certain Axiomatizations of Arithmetic of Natural and Integer Numbers

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Abstract: The systems of arithmetic discussed in this work are non-elementary theories. In this paper, natural numbers are characterized axiomatically in two different ways. We begin by recalling the classical set P of axioms of Peano's arithmetic of natural numbers proposed in 1889 (including such primitive notions as: set of natural numbers, zero, successor of natural number) and compare it with the set W of axioms of this arithmetic (including the primitive notions like: set of natural numbers and relation of inequality) proposed by Witold Wilkosz, a Polish logician, philosopher and mathematician, in 1932. The axioms W are those of ordered sets without largest element, in which every non-empty set has a least element, and every set bounded from above has a greatest element. We show that P and W are equivalent and also that the systems of arithmetic based on W or on P, are categorical and consistent. There follows a set of intuitive axioms PI of integers arithmetic, modelled on P and proposed by B. Iwanuś, as well as a set of axioms WI of this arithmetic, modelled on the W axioms, PI and WI being also equivalent, categorical and consistent. We also discuss the problem of independence of sets of axioms, which were dealt with earlier.

Keywords: axiomatizations of arithmetic of natural and integers numbers; second-order theories; Peano's axioms; Wilkosz's axioms; axioms of integer arithmetic modeled on Peano and Wilkosz axioms; equivalent axiomatizations; metalogic; categoricity; independence; consistency

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1. Introduction

The notion of natural numbers counts amongst the oldest, being one of the most universal abstract notions. Natural numbers belong to the fundamental subject of study of theoretical arithmetic concerned with defining all kinds of numbers, as well as studying their properties and relations between numbers of the same or different kinds. Theoretical arithmetic deals with examination of different types of numbers and their axiomatization, including that of natural numbers and integers. While defining its notions, we base ourselves on second-order logic and set theory.

In the paper, we will discuss some different axiomatizations of arithmetic of natural numbers **NA** and arithmetic of integer numbers **IA** (the presentation is based on results originally published in Polish by various authors, and which, as a consequence of their being available only in Polish, are not known among the vast majority of mathematical logicians). Presented theories will be non-elementary second-order theories and alphabet of languages which will assume two sorts of variables: individual variables and variables ranging over sets of individuals, i.e., natural numbers or integers, respectively.

We will start with the original axiomatization of **NA** proposed by Giuseppe Peano [1] by the set $\boldsymbol{P}$ of axioms on which is based the deductive system $\boldsymbol{PA}$ (the axiomatic non-elementary deductive theory; for short: the system $\boldsymbol{PA}$) and will compare it with the little known axiomatization of the arithmetic **NA** by the set $\boldsymbol{W}$ of axioms, which was provided by Witold Wilkosz [2], a Polish logician, mathematician and philosopher of Kraków. The deductive system based on Wilkosz's set $\boldsymbol{W}$ of axioms will be denoted by $\boldsymbol{WA}$.

Then, we will expand both sets ***P*** and ***W*** of axioms to the sets of axioms of arithmetic of integer numbers **IA**, which are modeled on them: the set of axioms ***PI*** by Iwanuś [3] and mine ***WI*** [4,5], which will be compared with each other and also with the set ***SI*** of axioms given by Sierpiński [6].

We will also give several metalogical theorems of the systems of arithmetic, which are presented.

2. Two Simple Axiomatizations of NA

2.1. *Peano's Axioms for **PA***

Historically, the first axiomatic system of arithmetic of natural numbers, which is characterized by unique simplicity, was that presented by Italian mathematician Giuseppe Peano in 1889, in his book in Latin [1]. The essential ideas of axiomatization of NA were first published by Dedekind [7] in 1888. Peano's axioms specify the ideas, but there can be no doubt concerning the originality of Peano's work (see [8], p. 101). Peano's original formulation of the axioms assumes the following three primitive notions: *number* (*positive integer*) N, *unity* 1 and the *successor of a number*; the modern set memberships relation ∈ comes from Peano's relation ε (*is*) that he used in [1] (see [8], chapter VII). The most modern formulations of Peano's axioms use 0 as the "first" natural number instead of 1 and the set of all natural numbers as N. In this work the successor of a number is a unary function defined on natural numbers and denoted by the symbol *. In modern presentations, Peano's axioms are written using the symbolism of mathematical logic and set theory. They are axioms of a non-elementary theory of natural numbers, including set theory. Recollecting them, we use the convention that the individual variables m, n, k, l, ... ranging over the set N, while those of X, Y, Z ... over subsets of the set N. Peano's axioms of the system ***PA*** are the following:

P1. $0 \in N$,
P2. $n^* \in N$,
P3. $n^* \neq 0$,
P4. $m^* = n^* \Rightarrow m = n$,
P5. $0 \in X \wedge \forall k \in X\ (k^* \in X) \Rightarrow N \subseteq X$ (the induction principle).

Axioms P1–P4 are elementary ones, whereas axiom P5, called *induction axiom*, is an axiom of the 2nd order, a non-elementary one. In the first-order Peano arithmetic (elementary arithmetic) which is weaker than ***PA*** (see, e.g., [9], chapter II, section 7, chapter III, section 5 and [10], chapter 5), it is reformulated by the induction axiom schema. The induction axiom is applied in inductive proofs of theorems of the form T(n), where n denotes a natural number.

If T(n) with the free variable n is an expression of arithmetic **NA** and T(0) is its true expression and from the assumption that T(k) is true for $k \geq 0$ it follows that T(k*)—its truthfulness for number k*—then T(n) is true for any natural number of set N. Such proofs are based on the following schema of the rule of inductive proof of theorem T(n) for all n:

$$\frac{\begin{array}{l} T(0) \\ T(k) \Rightarrow T(k^*) \text{ for any } k \geq 0 \end{array}}{T(n) \text{ for any } n \in N}$$

In proofs of theorems based on the set of axioms ***P***, the following generalized theorems on induction are also made use of:

T1. $m \in X \wedge \forall k \in X\ (m \leq k \Rightarrow k^* \in X) \Rightarrow \forall n \in N\ (m \leq n \Rightarrow n \in X)$,
T2. $\forall m\ (\forall k < m\ (k \in X) \Rightarrow m \in X)) \Rightarrow \forall n \in N\ (n \in X)$.

In compliance with T1, if any set of natural numbers to which m belongs satisfies the condition that for each number k of the set, which is not smaller than m, its successor k* also belongs to this set, then to this set belong all the natural numbers not smaller than m. With m = 0, T1 = P5. With theorem T1 corresponds the rule of inductive proof based on the following schema:

T(m) for m ∈N
T(k) ⇒T(k*) for k ≥ m

T(n) for all n ≥ m, n ∈ N

In compliance with T2, if each natural number m satisfies the condition: if any number smaller than m belongs to a given set of natural numbers, then m also belongs to this set, then each natural number does. With theorem T2 corresponds the rule of inductive proof based on the following schema:

for each m ∈N
T(k) ⇒ T(m) for any k < m

T(n) for all n ∈ N

The proof of theorem T1 is based on the induction axiom P5 and the elementary theorems of system ***PA***, whereas in the proof of theorem T2 the minimum principle and the elementary theorems of system ***PA*** are made use of. The former follows from the induction principle P5 and requires introducing additional definitions into system ***PA***, including the definition of relation of less than, <, and that of non-greater than, ≤.

It can be proved that the induction principle P5, the maximum principle Pmax and the minimum principle Pmin are equivalent to one another on the basis of the elementary theorems of system ***PA***, since the following relations of implication hold:

$$\text{P5} \rightarrow \text{Pmax} \rightarrow \text{Pmin} \rightarrow \text{T2} \rightarrow \text{P5}$$

where

Pmax. In any non-empty set of natural numbers for which there is an upper bound element, there is the greatest number. Symbolically:

$$\exists k\ (k \in X) \wedge \exists n \forall m{\in}X\ (m \leq n) \Rightarrow \exists n{\in}X \forall m{\in}X\ (m \leq n).$$

Pmin. In any non-empty set of natural numbers, there is the least number. Symbolically:

$$\exists k\ (k \in X) \Rightarrow \exists n{\in}X \forall m{\in}X\ (n \leq m).$$

Thus, we obtain the first metalogical theorem:

MT1. The principles P5, Pmax and Pmin are mutually equivalent on the basis of the elementary theorems of system ***PA***.

Remark. *Each of these non-elementary expressions could then be the only non-elementary axiom of arithmetic of natural numbers* ***NA*** *if—from it and suitably selected elementary axioms—each elementary theorem follows (cf. Słupecki et al. [11] and Sierpiński [6]).*

The principles Pmax and Pmin are noted in system ***PA*** by means of relations of less than, <, or non-greater than, ≤, but the former is defined by means of the operation of addition +.

The definitions of the relation < and that of ≤ in system ***PA*** are the following:

D3. $m < n \Leftrightarrow \exists k \in N \setminus \{0\}\ (m + k = n)$,
D4. $m \leq n \Leftrightarrow m = n \vee m < n$.

The definitions of the operations of addition + and multiplication · are recursive in ***PA***:

D1a. $m + 0 = 0$,
b. $m + n^* = (m + n)^*$.

D2a. $m \cdot 0 = 0$,
b. $m \cdot n^* = m \cdot n + m$.

These operations satisfy the well-known properties of a commutative semi-ring with unity ($1 = 0^*$), and it can be proved that the relation $<$ (relation $\leq$) in ***PA*** well-orders set N (we differentiate two well-known notions of a relation ordering a set: strict ordering ($<$) a set and weak ordering ($\leq$) a set).

The structure $< N, +, \cdot, 0, 1, \leq >$ is an ordered commutative semi-ring.

2.2. *Wilkosz's Axioms for System **WA***

The primitive notions in Wilkosz's axiomatic system ***WA*** are: the set of natural numbers N and the relation of less than, $<$. We write Wilkosz's axioms, accepting that the variables m, n, l, k, . . . run over set N, X is a subset of N.

W1. $\exists n\ (n \in N)$—there is a natural number,
W2. $m \neq n \Rightarrow m < n \vee n < m$—trichotomy,
W3. $(m < n \Rightarrow \sim (n < m))$—anti-symmetry of relation $<$,
W4. $m < n \wedge n < k \Rightarrow m < k$—transitivity of relation $<$,
W5. $m < n \Rightarrow m, n \in N$—the field of relation $<$ is set N,
W6. $\exists k\ (k \in X) \Rightarrow \exists n{\in}X \forall m{\in}X\ (n \leq m)$—the minimum principle,
W7. $\exists k\ (k \in X) \wedge \exists n \forall m{\in}X\ (m \leq n) \Rightarrow \exists n{\in}X \forall m{\in}X\ (m \leq n)$—the maximum principle,
W8. $\sim\exists m \forall n\ (m \neq n \Rightarrow n < m)$—there is not the greatest number in set N.
It is easy to see that in system ***WA***, the relation $<$ well-orders set N (in the sense of strict order).
Relation $<$—a primitive notion in Wilkosz's system ***WA***—is a notion defined in Peano's system ***PA*** (see D3), and the primitive notions of system ***PA***, which are not primitive ones in Wilkosz's system ***WA***, are defined in it in the following way:

(1) $k = 0 \Leftrightarrow \forall n\ (k \leq n)$—0 is the least natural number,
(2) $k = n^* \Leftrightarrow k \in \{m \in N \mid n < m\} \wedge \forall i \in \{m \in N \mid n < m\}(k \leq i)$—$n^*$ is the least natural number among numbers greater than n.

Relation $\leq$ less than or equal (not greater) has the following definition:
$m \leq n \Leftrightarrow m = n \vee m < n$.

It can be proved that the definitions (1) and (2) are correct: there is precisely one natural number k satisfying the definiens of definition (1) and there is precisely one number k (the successor of number n), which satisfies the definiens of definition (2). In the proofs the axioms W2–W4 are used.

Relying on, in Wilkosz's system ***WA***, the definitions of zero and the successor function of a natural number, we can define the operations of addition + and multiplication, in the same way as in system ***PA*** (by means of definitions D1a,b and D2a,b).

2.3. *Equivalence of the Deductive Systems **PA** and **WA***

It needs reminding that, in accordance with Tarski's inferential definition of two equivalent sets of sentences of a deductive system (see [12]), two sets of sentences are *equivalent* if sets of all their consequences (deduced from them sentences) are equal. Thus, most often, for the equivalence of axiomatic deductive system the following definition (cf. [13,14]) is used:

(*) Two axiomatic deductive systems are *equivalent* if the set of axioms and definitions of one of them is *equivalent* to the set of axioms and definitions of the other system, i.e., if each axiom and definition of one of them is a theorem or definition of the other system and the other way round—each axiom and definition of the other system is a theorem or definition of the first system.

Let us note that

(i1) Axioms of system ***WA*** are theorems of system ***PA*** since W1 follows directly from P1; W8 follows from the fact that in system ***PA*** there holds the theorem that $n < n^*$ for any $n \in N$, and $n^* \in N$ (P2); W3–W5 in ***PA*** follow from the theorem that N is a set ordered by the relation <; W6 and W7 (principles minimum and maximum, respectively) follow from the induction axiom P5 (see MT1).

(i2) Definitions (1) and (2) of zero and the successor of a natural number in ***WA*** are theorems in system ***PA***.

(i3) Definitions of the operations addition and multiplication in system ***WA*** are the same as in system ***PA***.

(i4) Each axiom and definition of system ***WA*** is a theorem or definition in system ***PA***.

On the other hand

(j1) Axioms of system ***PA*** are theorems of system ***WA***, since from the correctness of definitions (1) and (2) in ***WA*,** in particular axioms P1 and P2 follow; also P3 is a theorem in ***WA***, because if it would be possible that $n^* = 0$, then it would follow from (1) that $n^* \leq n$ and from (2) that $n < n^*$, and hence that $0 \leq n$ and $n < 0$, that is on the basis of W3: $\sim 0 < n$ and $n = 0$, that is $0 < 0$, which leads to contradiction according to W3; next, P4 follows from (2) and from the property of relation <, as one ordering set N. Axiom P5—the induction principle follows from those of maximum and minimum (W7 and W6; see MT1).

(j2) Definition D3 of relation < in ***PA*** can be derived from axioms and definitions of system ***WA***.

(j3) Each axiom and definition of system ***PA*** is a theorem of system ***WA***.

From (i4) and (j3), in compliance with (*), we obtain the following metatheorem:

MT2. The systems of ***PA*** and ***WA*** are equivalent.

This equivalence was sketched in the booklet by Wilkosz [2] under the title *Arytmetyka liczb całkowitych* (The Arithmetic of Integers). Equivalence of Wilkosz's and Peano's systems was the subject of my MA thesis.

*2.4. Independence of Axioms in Systems **PA** and **WA***

As is well known from Gödel's first incompleteness theorem given in 1931 [15], no finite set of axioms of natural numbers is complete, or even each infinite, countable set of axioms of arithmetic is incomplete. There arises the problem, however, whether it is possible to reduce the number of axioms of ***PA*** and ***WA*** without depleting the set of theorems which can be proved about natural numbers.

It can be shown that

MT3a. The set of axioms of ***PA*** arithmetic system is independent (Sierpiński [6]).

b. The set of axioms of ***WA*** system is dependent and can be reduced to the set:

{W1, W3, W5, W6, W7, W8}.

Axiom W2 follows from axiom W6, while axiom W4 follows from axioms W3, W5 and W6.

Hence, it follows that the axioms of the theory of well-ordered sets in regard to relation < can be based on axioms W3, W5 and W6, while Wilkosz's system *WA* can be based on the axioms:

W1′. $\exists n\ (n \in N)$—there is a natural number,

W2.′ $\forall m \exists n\ (m \leq n)$—in set N there is not the greatest number,

W3.′ $\forall m \forall n\ (m < n \Leftrightarrow \sim (n < m))$—asymmetry of relation < in N,

W4.′ $\exists k\ (k \in X) \Rightarrow \exists n{\in}X \forall m{\in}X\ (n \leq m)$—the minimum principle,

W5.′ $\exists k\ (k \in X) \wedge \exists n \forall m{\in}X\ (m \leq n) \Rightarrow \exists n{\in}X \forall m{\in}X\ (m \leq n)$—the maximum principle.

To prove independence of the axioms one can, as it is known, use the method of interpretation, which consists in finding such an interpretation of primitive terms of the given system that makes all the axioms, apart from one, e.g., Ai, true at the interpretation. If we find it, then the given axiom Ai is independent from the others.

2.5. *Categoricity of Arithmetic Systems* ***PA*** *and* ***WA***

Let us recall the definition of the notion of categoricity (see, e.g., [16–18]):

(**) A deductive system is *categorical* if and only if all its models are isomorphic.

As we mentioned, ***PA*** and ***WA*** as second-order, non-elementary systems, as well as elementary Peanos arithmetic, are not complete, yet we can show that they are categorical (cf. [7,10,18,19], chapter 8).

A ***model*** of Peano's arithmetic system ***PA*** is each triple $<N, 0, S>$ assigned to the triple $<\mathrm{N}, 0, *>$ of primitive terms of system ***PA***, where N is an infinite set, $0 \in N$, and $S: N \to N$, which satisfies Peano's axioms P1–P5.

A ***model*** of Wilkosz's arithmetic system ***WA*** is each tuple $<N, \lessdot>$ assigned to the tuple $<\mathrm{N}, <>$ of primitive terms of ***WA*** system, where N is an infinite, countable set and $\lessdot$ a binary relation with the field N, which satisfies Wilkosz's axioms W1–W8 (W1′–W5′).

(m1) Two models of ***PA***: $P_1 = <N_1, 0_1, S_1>$ and $P_2 = <N_2, 0_2, S_2>$ *are isomorphic* if and only if there is bijection f: $N_1 \to N_2$ such that f is homomorphism from P_1 to P_2, that is $f(0_1) = 0_2$ and $f(S_1(m)) = S_2(f(m))$ for any $m \in N_1$.

(m2) Two models of ***WA***: $W_1 = <N_1, \lessdot_1>$ and $W_2 = <N_2, \lessdot_2>$ *are isomorphic* if and only if there is bijection f: $N_1 \to N_2$ being homomorphism from W_1 to W_2, that is $m \lessdot_1 n \Rightarrow f(m) \lessdot_2 f(n)$ for any $m, n \in N_1$. Dedekind already in [7] proved that

(m3) Each two models of arithmetic system ***PA*** are isomorphic. In the book by Słupecki et al. [11], there is a proof that

(m4) Each two models of arithmetic system ***WA*** are isomorphic.

Hence, we have the metalogic corollary:

MT4. The deductive systems ***PA*** and ***WA*** of natural numbers arithmetic **NA** are categorical; they are in power $\aleph_0$, so they are aleph-null categorical systems.

Thus, Peano's and Wilkosz's second-order systems have only one model, up to isomorphism.

This is not so when we consider the systems of arithmetic of natural numbers as systems (elementary theories) of the first-order. According to the upward Löwenheim–Skolem's theorem, there are non-standard models of Peano's elementary arithmetic system of all infinite cardinality (see e.g., [9], chapter III, section 5, [20], chapterVI).

2.6. *Set-Theoretical Models for* ***PA*** *and* ***WA***

Peano's arithmetic possesses a "natural" set-theoretical model deriving from Frege.

Let $\mathbb{N}$ be an infinite set of all cardinal numbers of finite subsets of any (infinite) set U, i.e.,

$$\mathbb{N} = \{\mathrm{card}(X) \mid X \in Fin(U)\},$$

where $Fin(U)$ is the smallest family of sets to which the empty set $\emptyset$ belongs and which is closed under the relation S:

$$X\, S\, Y \Leftrightarrow \exists x \in U \setminus X\, (Y = X \cup \{x\}) \text{ for any } X, Y \in Fin\,(U).$$

The formal definition of the set $Fin\,(U)$ is the following:

$$\mathrm{Fin}\,(U) = \cap\{A \subseteq \mathrm{P}(U) \mid \varnothing \in A \wedge \forall X \in A\, \exists Y \in A\, (X\, S\, Y \Rightarrow Y \in A)\}.$$

(mP) The set-theoretical model for ***PA*** is the triple $< \mathbb{N}, \mathrm{card}\,(\emptyset), S^* >$,

where for $m = \mathrm{card}(X)$ and $X\, S\, Y$, $S^*(m) = m + 1 = \mathrm{card}(Y)$, for $X, Y \in Fin\,(U)$.

(mW) The set-theoretical model for ***WA*** is the triple $< \mathbb{N}, < >$,

where $<$ is the relation of less than for the cardinal numbers of set $\mathbb{N}$:

$$m < n \Leftrightarrow m \leq n \wedge n \neq m,$$

$$m \le n \Leftrightarrow \exists X,Y\ (\mathrm{card}(X) = m \wedge \mathrm{card}(Y) = n \wedge \exists Z \subseteq Y(\mathrm{card}\ (Z) = m)).$$

From (mP) and (mW) we get two next metalogic corollaries:

MT5. ***PA*** and ***WA*** systems are consistent.

(since it follows from the theorem of categoricity (MT4) that all theorems of these systems are true, because they are true in each model of these systems).

MT6. Systems ***PA*** and ***WA*** are (treated as) fragments of set theory.

As we know, the theorem MT6 is of great importance to studies on the foundations of mathematics.

3. Simple Axiomatizations of Arithmetic of Integers, Based on Systems *PA* and *WA*

Axiomatic systems for integer arithmetic **IA** are most often based on notions of operations of addition and multiplication defined on the set I of integers. In this part of the work, we will give an axiomatization of integer arithmetic **IA** modelled on the systems ***PA*** and ***WA*** respectively for the arithmetic of natural numbers **NA**, extending these systems accordingly and comparing them with Sierpiński's system ***SIA*** [6], including addition and multiplication as its primitive notions.

3.1. Iwanuś's Axioms for IA, Modelled on the Axioms of System PA

We will give here two systems of axioms proposed by Bolesław Iwanuś [3] for **IA** system. They are interesting due to their intuitive character. The first system based on them will be denoted as $\boldsymbol{P^1IA}$, and the other one—$\boldsymbol{P^2IA}$. The primitive notions of $\boldsymbol{P^1IA}$ are: set N* of all non-negative integers, set *N of all non-positive numbers, integer 0 and two unary operations in N*∪*N of successor and predecessor of an integer. The successor and the predecessor of integer i will be denoted as i* and *i, respectively. In the intuitive meaning, i* = i + 1 and *i **=** i − 1.

We assume that i, j, k, l, . . . are variables ranging over the set N*∪*N, while variables A, B, C . . . range over the subsets of this set.

3.1.1. Axioms of System $\boldsymbol{P^1IA}$ Are the Symmetric Axioms for Numbers of the Sets N* and *N:

A*1. 0 ∈ N*,	*A1. 0 ∈*N,
A*2. i ∈ N* ⇒ i* ∈ N*,	*A2. i ∈ *N ⇒ *i ∈ *N,
A*3. i ∈ N* ⇒ i* ≠ 0,	*A3. i ∈ *N ⇒ *i ≠ 0,
A*4. 0 ∈ A ∧ ∀i ∈ A (i* ∈ A) ⇒ N* ⊆ A,	*A4. 0 ∈ A ∧ ∀i ∈ A (*i ∈ A) ⇒ *N ⊆ A.

A5. i **∈** N*∪*N ⇒ *(i*) = i = (*i)*.

Axioms A*1–A*3 and *A1–*A3 are modelled on those of Peano (P1–P3). Axioms A*4 and *A4 correspond to that of induction P5. Axiom A5 establishes relations between the successor and the predecessor operation and does not allow identification of these notions with each other, nor identification of sets N* and *N. The set I of all integers is defined as follows:

D0. I = N* ∪ *N.

The counterparts of Peano's axiom P4

i, j ∈ N* ∧ i* = j* ⇒ i = j and i, j ∈ *N ∧ *i = *j ⇒ i = j

are direct consequences of A5.

It is easy to notice that with the assumption that the set I of integers is a primitive notion of the system of arithmetic **IA**, the symmetrical axioms of $\boldsymbol{P^1IA}$ can be replaced by weaker ones, deriving from Słupecki in [11]:

A1. 0 ∈ I,

A2. i ∈ I ⇒ i*,*i ∈ I,

A3. i ∈ I ⇒ i* ≠ i,

A4, A ⊆ I ∧ 0 ∈ A ∧∀i ∈ A (i*, *i ∈ A) ⇒ I =A,

A5. i ∈ I ⇒ (*i)* = *(i*) = i,

In this system, there are theorems that, to a certain extent, are similar to Peano's axiom A3, which are in force:

$(i \in I \wedge i \neq {}^*0) \Rightarrow i^* \neq 0,$
$(i \in I \wedge i \neq 0^*) \Rightarrow {}^*i \neq 0.$

In Iwanuś's system $\boldsymbol{P^1IA}$, there are the following definitions of the operations of: addition +, subtraction – and multiplication:

D^I1 a. $i + 0 = i,$	D^I2 a. $i - 0 = i,$	D^I3a. $i \cdot 0 = i,$
b. $i + j^* = (i + j)^*,$	b. $i - j^* = {}^*(i - j),$	b. $i \cdot j^* = i \cdot j + i,$
c. $i + {}^*j = {}^*(i + j),$	c. $i - {}^*j = (i - j)^*,$	c. $i \cdot {}^*j = i \cdot j - i.$

It is assumed that $1 = 0^*$ and it is proved that

I1. ${}^*N = (I - N^*) \cup \{0\},$
I2. $k = j - i \Leftrightarrow i + k = j,$
I3. $i^* = i + 1,$
I4. ${}^*i = i - 1.$

In proofs of the theorems of system $\boldsymbol{P^1IA}$ the following meta-theorem is made use of:

MT7. If α is an expression of system $\boldsymbol{P^1IA}$, in which—beside primitive notions—there are exclusively the defined terms + and $\cdot$, then α is a theorem of this system if expression α^d, dual with respect to α, is a thesis of this system; expression α^d is *dual* to α, when the terms:

$$N^*, {}^*N, (\ )^*, {}^*(\ ), +,$$

which occur in it, are substituted in each place of their appearance with the following ones, respectively:

$$^*N, N^*, {}^*(\ ), (\ )^*, +, .$$

In proofs of theorems on the basis of axioms A*4 and *A4, the following rules of mathematical induction for integers based on the given below schemata are applied:

$$\frac{T(0) \qquad T(k) \Rightarrow T(k^*) \text{ for any } k \geq 0}{T(i) \text{ for any } i \in N^*}$$

$$\frac{T(0) \qquad T(k) \Rightarrow T(k^*) \wedge T({}^*k) \text{ for any } k}{T(i) \text{ for any } i \in I}$$

Remark 1. *On the basis of system* $\boldsymbol{P^1IA}$ *one can prove all the axioms of the commutative ring.*

The inequality relation less-than, <, in I is determined by the following definition added to $\boldsymbol{P^1IA}$:

$D^I4.\ i < j \Leftrightarrow \exists k \in N^* \setminus \{0\}\ (i + k = j).$

Remark 2. *In system* $\boldsymbol{P^1IA}$*, one can prove all the theorems of arithmetic of integers* ***IA*** *relating to relation <.*

3.1.2. The Other System of Arithmetic of Integers Built by Iwanuś [3] and Modelled on *System* ***PA***

The system is denoted by $\boldsymbol{P^2IA}$ and based only on the following three primitive notions:
set I of all integers, the function of successor * and number 0.
The following formulas are the axioms of system $\boldsymbol{P^2IA}$:

(I1) $i \in I \Rightarrow \exists j \in I\ (i = j^*),$
(I2) $i, j \in I \wedge i^* = j^* \Rightarrow i = j,$
(I3) $\exists A \subseteq I\ (0 \in A \wedge \forall i \in A\ (i^* \in A \wedge i^* \neq 0),$
(I4) $A \subseteq I \wedge 0 \in A \wedge \forall i \in A\ (i^* \in A \wedge \exists j \in A\ (i = j^*)) \Rightarrow I \subseteq A.$

Axiom I3 assumes the existence of a certain subset of set I, about which—on the base of the above accepted set of axioms—it can be proved that it is isomorphic due to function * to the set of all natural numbers. Axiom I4 is a postulate of induction in the set of integers.

If we introduce into system $\boldsymbol{P^2IA}$ still one more primitive term N (as a name of a subset of set I which is isomorphic to the set of natural numbers), then axiom I3 can be substituted with the following set of axioms:

I3a. $N \subseteq I$,
b. $0 \in N$,
c. $i \in N \Rightarrow i^* \in N$,
d. $i \in N \Rightarrow i^* \neq 0$.

Axiom I3 is weaker than axioms I3a–d, because I3 follows from these axioms, although not all of I3a–d follow from I3.

In $\boldsymbol{P^2IA}$ system the primitive notions of $\boldsymbol{P^1IA}$ system are defined in the following way:

D^I1'. $i, j \in I \Rightarrow ({}^*i = j \Leftrightarrow i = j^*)$,
D^I2'. $i \in N^* \Leftrightarrow \forall A \subseteq I\ (0 \in A \wedge \forall j \in A\ (j^* \in A)) \Rightarrow i \in A$,
D^I3'. $i \in {}^*N \Leftrightarrow \forall A \subseteq I\ (0 \in A \wedge \forall j \in A\ ({}^*j \in A)) \Rightarrow i \in A$.

All the remaining definitions of system $\boldsymbol{P^1IA}$ are the same in system $\boldsymbol{P^2IA}$.

Iwanuś proves that

MT8. Systems $\boldsymbol{P^1IA}$ and $\boldsymbol{P^2IA}$ are equivalent.

B. Iwanuś also proves in [3] that these systems are equivalent to Sierpiński's system of arithmetic of integers SIA [6], based on primitive notions: the set I, operations of addition + and multiplication ·, zero 0, one 1 and the set N*, satisfying the axioms of the ring without zero divisors:

R1. $i, j \in I \Rightarrow i + j \in I \wedge i \cdot j \in I$,
R2. $i, j, k \in I \Rightarrow i + j = j + i \wedge i \cdot j = j \cdot i \wedge (i + j) + k = i + (j + k) \wedge (i \cdot j)\cdot k = i\cdot(j \cdot k) \wedge i\cdot(j + k) = i \cdot j + i \cdot k$,
R3. $\forall i, j \in I\ \exists k \in I(i + k = j)$,
R4. $\forall i \in I\ (i + 0 = i) \wedge \forall i \in I\ (i\cdot 1 = i) \wedge 1 \in I$,
R5. $i, j \in I \wedge i \cdot j = 0 \Rightarrow i = 0 \vee j = 0$,
R6. $N^* \subset I$, R7. $0 \in N^*$, R8. $i \in N^* \Rightarrow i + 1 \in N^*$,
R9. $0 \in A \wedge \forall i \in A\ (i^* \in A) \Rightarrow N^* \subseteq A$,
R10. $\forall i \in A \backslash N^*\ \exists j \in N^*\ (i + j = 0)$.

Definitions of primitive terms of system $\boldsymbol{P^1IA}$ are introduced into system ***SIA*** as follows:

D^S1. ${}^*N = (I \backslash N^*) \cup \{0\}$,
D^S2. $k = j - i \Leftrightarrow i + k = j$,
D^S3. $i^* = i + 1$,
D^S4. ${}^*i = i - 1$.

Definition D^I4 of relation < of system $\boldsymbol{P^1IA}$ is the same as in system ***SIA***.

MT9. System $\boldsymbol{P^1IA}$ ($\boldsymbol{P^2IA}$ system), modelled on Peano's system of natural numbers arithmetic ***PA***, and system ***SIA*** are equivalent.

*3.2. Axioms of the System of Integer Arithmetic **WIA** Modelled on Wilkosz's System **WA***

The primitive notions of the system of integer arithmetic ***WIA*,** modelled on Wilkosz's system ***WA***, are the following: set I of all integers, integer zero 0 and less-than relation < in set I. The relation of weak inequality ≤ is determined by the definition (i, j, k, . . . run over I):

D0. $i \leq j \Leftrightarrow i < j \vee i = j$.

The axioms of system ***WIA*** which are presented by Wybraniec-Skardowska [4,5] are the following expressions:

W'1. $0 \in I$,
W'2. $i, j \in I \Rightarrow (i < j \vee i = j \vee j < i)$,

W'3. $i, j \in I \Rightarrow (i < j \Rightarrow \sim (j < i))$,
W'4. $i, j, k \in I \wedge (i < j \wedge j < k) \Rightarrow i < k$,
W'5. $\forall i \in I\, \exists j \in I\, (i < j)$ – in I there is not the greatest number,
W'6. $\forall i \in I\, \exists j \in I\, (j < i)$ – in I there is not the smallest number,
W'7. $A \subseteq I \wedge \exists i \in A\, \exists i \in I\, \forall j \in A\, (i < j) \Rightarrow \exists i \in A \forall j \in A\, (i \leq j)$,
W'8. $A \subseteq I \wedge \exists i \in A\, \exists i \in I\, \forall j \in A\, (j < i) \Rightarrow \exists i \in A \forall j \in A\, (j \leq i)$.

According to W'7, in each non-empty set of integers, which has a lower bound, there is the smallest number, while, according to W'8, in each non-empty set of integers, which has an upper bound, there exists the greatest number.

The content of axioms W'7 and W'8 is close to the principles of minimum and maximum of arithmetic ***WA***. The axioms of system ***WIA*** state that relation < orders set I, yet do not state that it well-orders the set.

In system ***WIA*** one can define the notion of successor and that of predecessor of an integer as well as the notions of sets N* and *N, which are primitive notions in Iwanuś's system $\boldsymbol{P^1IA}$. Let us note first that in system ***WIA*** it is possible to prove the theorem:

$$A \subseteq I \wedge \exists i \in A \wedge \exists i \in I \forall j \in A\, (i < j) \Rightarrow \exists_1 k \in A \forall j \in A\, (k \leq j). \tag{1}$$

Condition (1) allows introducing correctly the definition of minimum in set A:
D^W1. $A \subseteq I \wedge \exists i \in A \wedge \exists i \in I \forall j \in A\, (i \leq j) \Rightarrow (k = \min(A) \Leftrightarrow k \in A \wedge \forall j \in A\, (k \leq j))$.
It follows from Condition (1) and D^W1 that there is a unique minimum, min (A), when $A \subseteq I$, $A \neq \emptyset$ and set A has a lower bound.
Let

$$G(i) = \{j \in I \mid i < j\}.$$

Hence $G(i) \neq \emptyset$ (see W'5) and $\exists_1 k \in G(i) \forall j \in G(i)\, (k \leq j)$ (see Condition (1)); then on the basis of D^W1

$$\exists_1 k \in I\, (k = \min(G(i)). \tag{2}$$

The successor of an integer i is introduced by means of the definition:
D^W2. $i^* = \min(G(i))$ – i^* is the smallest integer which is greater than i.
D^W3. $N^* = \{i \in I \mid 0 \leq i\}$.
The following corollary which is dual to Condition (1):

$$A \subseteq I \wedge \exists i \in A \wedge \exists i \in I \forall j \in A\, (j < i) \Rightarrow \exists_1 k \in A \forall j \in A\, (j \leq k). \tag{3}$$

permits introducing the definition of maximum of a certain set of integers:
D^W4. $A \subseteq I \wedge \exists i \in A \wedge \exists i \in I \forall j \in A\, (j \leq i) \Rightarrow (k = \max(A) \Leftrightarrow k \in A \wedge \forall j \in A\, (j \leq k))$.
Let $L(i) = \{j \in I \mid j < i\}$.
Hence $L(i) \neq \emptyset$ (see W'6) and $\exists_1 k \in L(i) \forall j \in L(i)\, (j \leq k)$ (see Condition (3)) then on the basis of D^W4 the predecessor of integer i is defined as the greatest integer less than i, that is
D^W5. $^*i = \max(L(i))$,
and set *N is defined as follows
D^W6. $^*N = \{i \in I \mid i \leq 0\}$.

3.3. Equivalence of Systems $\boldsymbol{P^1IA}$ *and* ***WIA***

Remark 3. *With the definitions of the primitive notions of system* $\boldsymbol{P^1IA}$*, given in system* ***WIA****, all the axioms and definitions* $\boldsymbol{P^1IA}$ *become theorems of definitions in system* ***WIA****.*

The definitions of addition and multiplication, which are accepted in ***P*^1^*IA*** are the same in system ***WIA***, while definition D^I4 of relation < accepted in system ***P*^1^*IA*** is a theorem in system ***WIA***.

Thus, it follows from Remarks 3 and 2 and **MT8** that

MT10. System ***WIA*** is equivalent to those of Iwanuś $\boldsymbol{P^1IA}$ and $\boldsymbol{P^2IA}$.

It follows from the above and **MT9** that

MT11. All the systems of integer arithmetic: ***WIA***, $\boldsymbol{P^1IA}$, $\boldsymbol{P^2IA}$ and ***SIA*** are mutually equivalent.

In particular,

MT12. System $\boldsymbol{P^1IA}$ modelled on Peano's system ***PA*** and system ***WIA*** modelled on Wilkosz's system ***WA*** are equivalent.

3.4. Independence of the Axioms in $\boldsymbol{P^1IA}$ *and* ***WIA***

The axioms of the integer arithmetic system $\boldsymbol{P^1IA}$ can, as Iwanuś proved, be reduced by one axiom A*3 or *A3. If we found an axiom system of $\boldsymbol{P^1IA}$ on those of A*1–A*4 and *A1, *A2 and *A4, then axiom *A3 can be proved. It follows from A*1, A*2, and A5 as well as from theorems of this system:

0 ∉ N and i ∈N* ⇒ (i ∉ *N ∨ i = 0).

MT13. The set of axioms of system $\boldsymbol{P^1IA}$ can be based on an independent set of axioms A*1–A*4 and *A1, *A2, *A4, and A5.

The independence of these axioms was proved by interpretation in integer arithmetic **IA**. The primitive terms of the tuple <N*, *N, i*, *i, 0> correspond to the elements of a tuple in the form <*A*, *B*, f(i), G(i), a^0>, respectively, which does not satisfy only one axiom of $\boldsymbol{P^1IA}$. When we apply the denotation:

"ℕ ⁺ " denotes a set of non-negative integers,
"ℕ ⁻ " denotes a set of non-positive integers,
"E^+ " denotes a set of even non-negative integers;
"E^- " denotes a set of even non-positive integers, then the tuple:
<$\mathbb{N}^+ \setminus \{0\}$, $\mathbb{N}^-$, i + 1, i – 1, 0> does not satisfy A*1,
<$\mathbb{N}^+$, $\mathbb{N}^- \setminus \{0\}$, i + 1, i – 1, 0> does not satisfy *A1,
<$\mathbb{N}^+ \setminus \{1\}$, $\mathbb{N}^-$, i + 1, i - 1, 0> does not satisfy A*2,
<$\mathbb{N}^+$, $\mathbb{N}^- \setminus \{-1\}$, i +1, i - 1, 0> does not satisfy *A2,
<{0, 1}, {0, 1}, $f_1(i)$, $g_1(i)$, 0>, where $f_1(i) = g_1(i) = \begin{cases} 1 & for\ i=0 \\ 0 & for\ i\neq 0 \end{cases}$ does not satisfy A*3,
<$\mathbb{N}^+$, E^-, i + 2, i – 2, 0> does not satisfy A*4,
<E^+, $\mathbb{N}^-$, i + 2, i – 2, 0> does not satisfy *A4,
<{0, 1}, $\mathbb{N}^-$, $f_2(i)$, i – 1, 0>, where $f_2(i) = \begin{cases} i+1 & for\ i\leq 0 \\ 1 & for\ i>0 \end{cases}$ does not satisfy A5a,
<$\mathbb{N}^+$, {0, 1}, i + 1, $g_2(i)$, 0>, where $g_2(i) = \begin{cases} i-1 & for\ 0\leq i \\ -1 & for\ i<0 \end{cases}$ does not satisfy A5b,

when A5 is substituted by two axioms:

A5a. i ∈ N* ∪ *N ⇒ *(i*) = I; A5b. i ∈ N* ∪ *N ⇒ (*i)* = i.

It is also possible to reduce the system of the primitive notions of $\boldsymbol{P^1IA}$ system by one primitive notion—zero 0—since the following expression:

i = 0 ⇔ i ∈ N* ∧ i ∈*N

is a theorem of $\boldsymbol{P^1IA}$.

On the other hand,

MT14. The set of axioms I1—I4 of Iwanuś's $\boldsymbol{P^2IA}$ system is an independent set.

It is so, since applying the following interpretation:

1. I → $\mathbb{N}^+$, i* → |i | +1, 0 → 0, – I1 is not satisfied,
2. I → E^+ ∪ {1}, i*→ i + 1, if i ≠ 0, and i* → 1, for i = 1, 0 → 0 – I2 is not satisfied,
3. I → {0,1}, i* → 1 - |i |, 0→ 0, – I3 is not satisfied,

4. I $\rightarrow$ set of integers $\mathbb{Z}$, i* $\rightarrow$ i + 2, 0 $\rightarrow$ 0, – I4 is not satisfied.

It can also be justified that

MT15. The set of axioms of ***WIA*** system is an independent set.

3.5. Categoricity of the Axiomatic Systems of Integers Arithmetic IA

The classical model of $\boldsymbol{P^2IA}$ system is the triple $<\mathbb{Z}, *, 0>$, where $\mathbb{Z}$ is the set of all integers. The classical model of ***WIA*** system is the triple $<\mathbb{Z}, 0, <\,>$.

It can be proved (cf. [11]) that

MT16. Every two models of ***WIA*** system are isomorphic, therefore ***WIA*** system is categorical in power $\aleph_0$.

A model of ***WIA*** system is every triple $<\vartheta, 0, \prec\,>$ corresponding to that of $<\mathrm{I}, 0, <\,>$ of the primitive terms of ***WIA***, in which ϑ is an infinite set of cardinality $\aleph_0$, $0 \in \vartheta$, and $\prec$ is a binary relation satisfying axioms A′1–A′8 of ***WIA*** system.

The following theorem is true:

If an axiomatic system has the property that all its models are isomorphic, then each equivalent system has the same property.

Thus, from meta-theorems MT11 and MT16 follows the conclusion:

MT17. All the systems of integer arithmetic, which are presented in this work, are categorical in power of $\aleph_0$.

Thus, it is not only system ***WIA*** modelled on Wilkosz's system ***WA*** which is categorical, but also Iwanuś's system $\boldsymbol{P^1IA}$ ($\boldsymbol{P^2IA}$) modelled on Peano's system ***PA*** is categorical.

A separate proof that system ***SIA*** is also categorical is given in the book by Sierpiński [6].

All deductive systems of integer arithmetic presented in this paper have a standard model and all their models are isomorphic (MT17), so all the theorems of these systems are true. Hence, it follows that

MT18. The systems $\boldsymbol{P^1IA}$, $\boldsymbol{P^2IA}$, ***SLA*** and ***WIA*** of integer arithmetic are consistent.

All these systems are mutually equivalent.

4. Final Comments

➢ Theorems of categoricity of the systems of natural numbers and the integer systems answer—in a sense—the following question: To what extent do our axioms characterize natural numbers (respectively, integers)? It follows from them that each set which has properties expressed in our axioms is the same as the set of natural numbers (resp., integers), that is it is isomorphic.

The axioms given for systems ***PA*** and ***WA*** as well as, respectively, $\boldsymbol{P^1IA}$ ($\boldsymbol{P^2IA}$) and ***WIA***, characterize very strongly natural numbers (respectively, integers).

➢ It follows from the given considerations that from the point of view of set theory, the set of axioms of integer arithmetic systems $\boldsymbol{P^1IA}$ and $\boldsymbol{P^2IA}$, modelled on Peano's axioms of system ***PA*** of natural numbers arithmetic, and the set of axioms of system ***WIA*** modeled on Wilkosz's axioms of system ***WA*** of natural numbers arithmetic, have equal rights, similarly as the set of axioms of systems ***PA*** and ***WA***. The subject of the discussion can be—as it may seem—solely one problem: Which of the set of axioms is more intuitive or more useful in the didactic process?

Wilkosz's system ***WA*** and system ***WIA*** of the similar axiomatic character seem to be of certain greater value. The former (***WA***) can be acknowledged to arise as a result of studies of the natural model—one that forms the primary study of teaching in the early years of elementary school. As it appears, though, the problem of which of the systems discussed here can play its role in a better way as the curriculum of early education may be settled exclusively through psycho-sociological research in schools.

- ➢ Let us also note that the built systems of integer arithmetic, modelled on the systems of arithmetic of natural numbers of Peano and Wilkosz, were treated as respective extensions of the latter, since the set of natural numbers N in Peano's system ***PA***, with the function of successor * and zero 0 ∈ N, is isomorphic with a proper subset of set I of integers in the system of integer arithmetic ***P^{1}IA***, that is to set N*⊂ I, function * in N* and zero 0 ∈ N*, whereas the set of natural numbers N in Wilkosz's system of natural numbers arithmetic ***WA***, with zero 0 ∈ N and relation less-than < in N, is isomorphic with the proper subset N* ⊂ I in the system of integers arithmetic ***WIA***, with zero 0 ∈ N* and relation < in N*.
- ➢ It follows from the remark above that arithmetic of integers can be defined not only through giving a set of axioms, but as an extension of arithmetic of natural numbers by the well-known method of construction, as well.
- ➢ So, it follows from MT6 that both integer systems ***P^{1}IA*** (***P^{2}IA***) and ***WIA*** can be treated as fragments of set theory.

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