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**Logica Universalis**



# Operator Counterparts of Types of Reasoning

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**Abstract.** Logical and philosophical literature provides different classifications of reasoning. In the Polish literature on the subject, for instance, there are three popular ones accepted by representatives of the Lvov-Warsaw School: Jan Łukasiewicz, Tadeusz Czeżowski and Kazimierz Ajdukiewicz (Ajdukiewicz in Logika pragmatyczna [Pragmatic Logic]. PWN, Warsaw (1965, 2nd ed. 1974). Translated as: Pragmatic Logic. Reidel & PWN, Dordrecht, 1975). The author of this paper, having modified those classifications, distinguished the following types of reasoning: (1) deductive and (2) non-deductive, and additionally two types of them in each of the two, depending on the manner of combining their premises with the conclusion through the relation of classical logical entailment. Consequently, the four types of reasoning:

1.1. unilateral deductive (incl. its sub-types: deductive inference and proof),
1.2. bilateral deductive (incl. complete induction), and
2.1. reductive (incl. the sub-types: explanation and verification),
2.2. logically nonvaluable (incl. inference by analogy, statistic inference),

correspond to four operators of derivability. They are defined formally on the ground of Tarski's axiomatic theory of deductive systems, by means of the consequence operation $Cn$ (Tarski in Monatshefte Math Phys 37:361–404, 1930a, C R Soc Sci Lett Vars 23:22–29, 1930b). Also, certain metalogical properties of these operators are given, as well as their relations with Tarski's consequence operations $Cn^{+}$ ($Cn^{+} = Cn$) and dual consequences $Cn^{-1}$ (Słupecki in Zeszyty Naukowe Uniwersytetu Wrocławskiego Seria B Nr 3:33–40, 1959, Słupecki et al. in Stud Log 29:76–123, 1971, Wybraniec-Skardowska, in: Wybraniec-Skardowska, Bryll (eds) Z badań nad teorią zdań odrzuconych [Studies in the Theory of Rejected Propositions], Series B, Studia i Monografie, Zeszyty Naukowe Wyższej Szkoły Pedagogicznej w Opolu, Opole, 1969), and $Cn^{-}$ (Wójcicki in Bull Sect Log 2(2):54–57, 1973)).

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## 1. Introduction

Let us begin with giving the definition of reasoning which is of interest to us. *Reasoning* (i.e. inference in a broad sense) is a thought process in which—on the basis of sentences that have already been acknowledged or assumed, called *premises* of reasoning—one arrives at acknowledgment or justification of another sentence called a *conclusion* of reasoning.

In logic we do not study reasoning as a process of thinking but only its forms, schemata of its premises and conclusion.

Western philosophical literature provides a variety of types of reasoning. Among them three major kinds of inference are distinguished, namely: deduction, abduction and induction (see entry "Abduction" by Douven [3] in the *Stanford Encyclopedia of Philosophy*). The problem area of classification of reasoning has been dealt with chiefly in texts by Polish researchers. Accordingly, three classifications accepted by representatives of the Lvov-Warsaw School: Jan Łukasiewicz [4,5], Tadeusz Czeżowski [2] and Kazimierz Ajdukiewicz [1] are the most popular. All of them make use of the so called "classical" notion of logical entailment to some extent. This notion is defined here only for classical logic *CL* (classical propositional logic *CL** and classical predicate logic). Its definition is as follows:

The sentences $s_1, s_2, \ldots, s_n$ *entail logically* by virtue of *CL*, the sentence $s$ (in other words: from the sentences $s_1, s_2, \ldots, s_n$ there follows logically by virtue of *CL*, the sentence $s$) if and only if the implication in the form:
$s_1 \to (\ldots \to (s_{n-1} \to (s_n \to s)\ldots))$ or $s_1 \wedge (s_2 \wedge \ldots \wedge (s_{n-1} \wedge (s_n \to s))\ldots)$ is the substitution of a logical law of *CL*.

The sentences $s_1, s_2, \ldots, s_n$ from which there follows the sentence $s$ ($n \geq 1$), taken jointly (or their conjunction), are called *the reason*; the sentence $s$ is then called *the consequence*. Transition from the reason to the consequence determines the direction of the relation of logical entailment. We denote this relation by $\models_{CL}$.

The classification of reasoning proposed in this work (Sect. 2) is a modification of the above-mentioned classifications of the Polish authors. It also makes use of the classical concept of logical entailment, yet in a slightly different way. It serves to assign appropriate operators to the distinguished types of reasoning. The definitions of these operators will be given in Sect. 3. Their certain formal properties are formulated in Sect. 4, while some relations of these operators with the operations of Tarski's consequence and dual consequences are described in Sect. 5.

## 2. Proposed Classification of Reasoning

The set of all reasonings is dichotomously divided into:

1. set of deductive reasonings, and
2. set of non-deductive reasonings.

*Deductive reasoning* is one whose conclusion is logically entailed by its premises (i.e. its conclusion *logically follows* by virtue of *CL* from its premises). *Non-deductive reasoning* is one in which its conclusion does not logically follow from its premises.

Premises of deductive reasoning are *a reason*, whereas the conclusion is *a consequence*. The direction of the deductive reasoning is in agreement with that of the relation of logical entailment.

The schema of deductive reasoning can be presented in the form of a quasi-fraction: $\frac{R}{C}$, where above the bar, as its premises, the symbol of the reason, $R$, is placed, while under the bar, as its conclusion—the symbol of consequence, $C$.

In classical approach to reasoning deductive reasoning is reliable, i.e. it always leads from true premises to true conclusions. Non-deductive reasoning is unreliable: it can lead from true premises to a false conclusion.

Depending on the manner of connecting premises with the conclusion due to the relation of logical entailment within both types of reasoning, we can distinguish their two sub-types in each category: i.e., in the type of deductive reasoning:

1.1. unilateral deductive (incl. deductive inference and proof), and
1.2. bilateral deductive (incl. complete induction, proof of equivalence);

and in the type of non-deductive reasoning:

2.1. reductive (incl. verification, explanation), and
2.2. logically nonvaluable (incl. inference by analogy, statistic inference).

*Unilateral deductive reasoning* (1.1) is one in which from premises there logically follows the conclusion, but not conversely—from the conclusion there do not logically follow premises.

*Bilateral deductive reasoning* (1.2) is reasoning in which not only from premises there logically follows the conclusion, but also from the conclusion there logically follow premises.

*Reductive reasoning* (2.1) is reasoning in which from premises the conclusion does not follow logically, but from the conclusion (and enthymeme (i.e. an argument in which one premise is not explicitly stated)) there logically follow premises. The direction of the reductive reasoning is not in agreement with that of the relation of logical entailment.

*Logically nonvaluable reasoning* (2.2) is one in which neither from the premises there logically follows the conclusion nor from the conclusion there logically follow premises.

The distinguished four classes of reasoning, pairwise disjoint (see Fig. 1) will correspond to four operators of derivability, respectively.

However, before we define them (Sect. 3), let us draw attention to the best-known sub-types of the four distinguished types of reasoning.

The proposed fuller classification of reasoning can be illustrated by means of Fig. 2.

The type of deductive reasoning (unilaterally deductive 1.1) with the schema: $\frac{R}{C}$ is usually set against the type of reductive reasoning 2.1 with the schema: $\frac{C}{R}$.

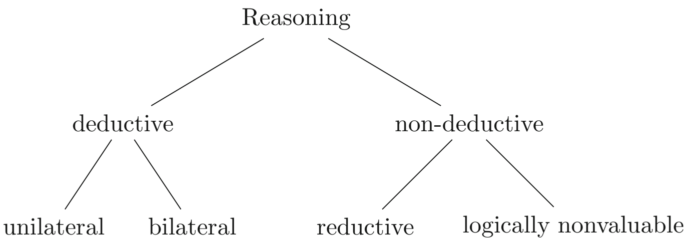


FIGURE 1. The main four types of reasoning

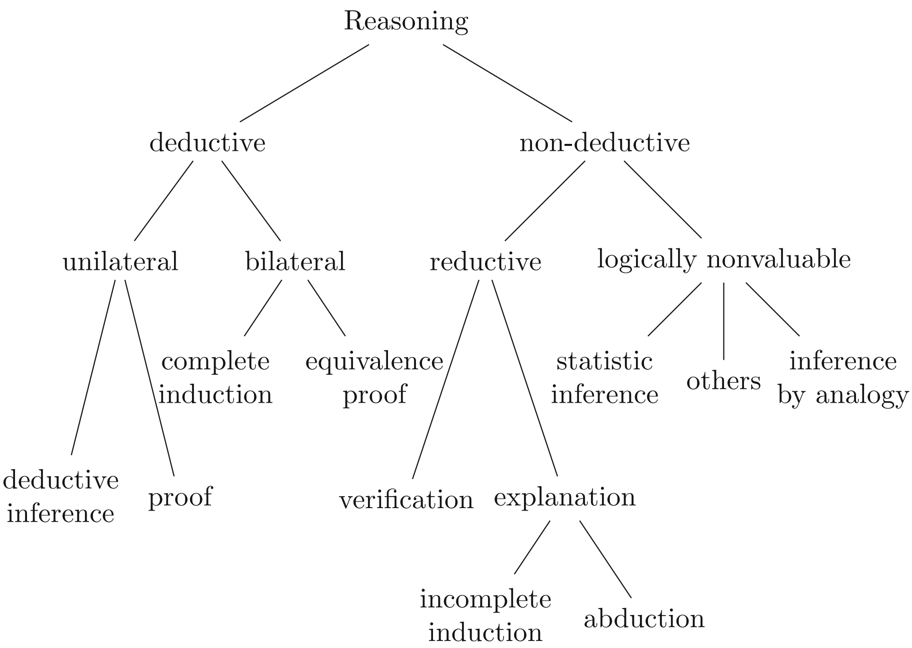


FIGURE 2. The best known sub-types of the main types of reasoning

(Premises in the last ones are Consequence $C$, while the conclusion—Reason $R$).

Due to the manner of selecting (searching) for the reason of a consequence or for the consequence of a reason in both types of reasoning mentioned above, we distinguish two sub-types in each category. Thus, the type of deductive reasoning is divided into sub-types: *deductive inference* and *proof*, whereas the type of reductive reasoning into sub-types: *verification* and *explanation*. The distinguished types of reasoning can be presented in the form of schemata, in which the arrows ↓, ↑ denote the direction of fitting the consequence to the reason, or conversely—the reason to the consequence:

| ded. inference | proof | verification | explanation |
|---|---|---|---|
| $\downarrow \frac{R}{C}$ | $\uparrow \frac{R}{C}$ | $\uparrow \frac{C}{R}$ | $\downarrow \frac{C}{R}$ |

Thus, *deductive inference* is a deductive reasoning in which we fit a consequence not known to be true to a reason acknowledged to be true, while *proof* is the deductive reasoning in which we search a reason acknowledged to be true to a consequence that is not known to be true. On the other hand, *verification* stands for the reductive reasoning in which we fit consequences known to be true to a reason not known to be so (often a hypothesis), whereas *explanation* is the reductive reasoning in which we fit reasons not acknowledged to be true to the consequence known to be true. The explanation includes, in particular, *incomplete induction* and *abduction* (i.e. explanatory reasoning in *generating* hypotheses or explanatory reasoning in *justifying* hypotheses).

## 3. Definitions of Operators Corresponding to Main Types of Reasoning

Reasoning can be defined as an operation on sentences (propositions): a proposition is justified or acknowledged through making reference to other propositions. Therefore, certain operators (operations) defined on sets of propositions with values in sets of propositions will correspond to types of reasoning.

### 3.1. Entailment Relations and Tarski's Consequence Operation

The considerations in this section intend to take into account a classical approach to reasoning. On the ground of classical logic $CL$ the relation of logical entailment $\models_{CL}$ is equivalent to the classical consequence relation $\vdash_{CL}$ (inferential entailment, deducibility), i.e. a relation defined as follows:

The sentence $s$ is a consequence of (is deducible from) the sentences $s_1, s_2, \ldots, s_n$ (symbolically: $\{s_1, s_2, \ldots, s_n\} \vdash_{CL} s$) iff there exists a proof of the sentence $s$ on the basis of the set $\{s_1, s_2, ...s_n\}$ and logic $CL$.

The relation $\vdash_{CL}$ is equivalent to the Tarski's operation of the classical consequence $Cn = Cn^+$, i.e. for any set of propositions $X$ and any proposition $x$ we have:

$$X \vdash_{CL} x \text{ iff } x \in Cn^+X.$$

Thus, it is evident that

$$X \models_{CL} x \text{ iff } x \in Cn^+X.$$

Therefore, certain operators (operations) that can be defined by means of the consequence operator $Cn$ will correspond to the types of reasoning described in Sect. 2. The properties of these operators will be thus established on the basis of the properties entitled to the classical consequence operation $Cn$ ($Cn = Cn^+$). The last properties, on the other hand, are established on the ground of Tarski's axiomatic theory of deductive systems based on the classical propositional logic $CL^*$ (see Tarski [14–16], Wybraniec-Skardowska [19]). Let us recall these properties now.

### 3.2. Basic Properties of the Consequence Operation *Cn*

Let $S$ be the set of all propositions of an arbitrary, but fixed language $L$ and the consequence operation $Cn$ be a function:

$$Cn\colon P(S) \to P(S),$$

which to any set of propositions $X$ of the family $P(S)$ of all subsets of the set $S$ assigns the set $CnX \subseteq S$ of all propositions deducible from the propositions of the set $X$ (i.e. the consequences of the propositions of the set $X$).

We assume that the variables $X, Y, Z, \ldots$ run over the elements of the family $P(S)$ and the variables $x, y, z, \ldots$ run over the propositions of the set $S$.

The consequence operation $Cn$ satisfies the following axioms of Tarski's general theory of deductive systems (cf. Tarski [14]):

A1. $\mathrm{card}(S) \leq \aleph_0$ — denumerability of the set $S$,
A2. $X \subseteq CnX$ — the consequence $Cn$ is *reflexive*,
A3. $CnCnX = CnX$ — the consequence $Cn$ is *idempontent*,
A4. $X \subseteq Y \Rightarrow CnX \subseteq CnY$ — the consequence $Cn$ is *monotonic*,
A5. $CnX \subseteq \bigcup\{CnY : Y \in Fin(X)\}$ — the consequence $Cn$ is *finitistic*,

where $Fin(X)$ denotes the family of all finite subsets of the set $X$.

### 3.3. Basic Properties of the Classical Consequence $Cn^+$

If the classical consequence operation $Cn$ ($Cn = Cn^+$) is based on the classical propositional logic $CL^*$ with primitive notions corresponding to the symbols:

$$\to, \neg, \wedge, \vee$$

representing the propositional connectives, respectively: implication, negation, conjunction, disjunction, then their metalogical counterparts for the language $L$ are, respectively:

$$c, n, k, a$$

and the classical consequence $Cn$ ($Cn = Cn^+$) satisfies additionally the following specific axioms of the so-called reached theory of deductive systems based on the logic $Cn^*$ (cf. Tarski [15], Pogorzelski and Słupecki [8]):

A6$^+$. $\mathrm{c}xy, \mathrm{n}x, \mathrm{k}xy, \mathrm{a}xy \in S$,
A7$^+$. $\mathrm{c}xy \in Cn^+X \Leftrightarrow y \in Cn^+(X \cup \{x\})$,
A8$^+$. $Cn^+\{x, \mathrm{n}x\} = S$,
A9$^+$. $Cn^+\{x\} \cap Cn^+\{\mathrm{n}x\} = Cn^+\emptyset$,
A10$^+$. $Cn^+\{\mathrm{k}xy\} = Cn^+\{x, y\}$,
A11$^+$. $Cn^+(X \cup \{\mathrm{a}xy\}) = Cn^+(X \cup \{x\}) \cap Cn^+(X \cup \{y\})$.

On the basis of the condition A10$^+$ we can easily state that deducibility from a finite set of propositions of the set $S$ is the same as deducibility from the conjunction of these propositions.

Conjunction of the finite number of propositions $x_1, x_2, \ldots, x_m$ ($m \geq 1$) can be defined inductively as follows:

Dk. 1. $\mathrm{k}(x_1) = x_1$,
2. $\mathrm{k}(x_1, x_2, \ldots, x_n, x_{n+1}) = \mathrm{k}(\mathrm{k}(x_1, x_2, \ldots, x_n), x_{n+1})$, for any $n \geq 1$.

A generalization of A10$^+$ is the valid formula

Tk. $Cn^+\{\mathrm{k}(x_1, x_2, \ldots, x_n)\} = Cn^+\{x_1, x_2, \ldots, x_n\}$, for any $n \geq 1$.

Proof of Tk is inductive. Tk follows from Dk, A10$^+$ and A7$^+$ $\leftarrow$, A7$^+$ $\rightarrow$.

Further in this work, we will also use the notion of set K$X$ *of all conjunctions* built from different sentences of the set $X$. It is defined by the following formula:

$$\text{DK. } x \in \mathrm{K}X \Leftrightarrow \exists x_1, x_2, \ldots, x_n \in X(x = \mathrm{k}(x_1, x_2, \ldots, x_n)).$$

From DK it directly follows that

K1. $\mathrm{K}\{x\} = \{x\}$.
K2. $\mathrm{k}(x_1, x_2, \ldots, x_n) \in \mathrm{K}\{x_1, x_2, \ldots, x_n\}$,
K3. $X \subseteq \mathrm{K}X$,
K4. $X \subseteq Y \Rightarrow \mathrm{K}X \subseteq \mathrm{K}Y$.

Let us note that most often a deduction from a finite set of propositions can be reduced to a deduction from a single proposition—a conjunction of the propositions of the finite set (see Tk).

Using the consequence $Cn^+$, we can define an operation $Cn^{+1}$:

$$\mathrm{D}Cn^{+1}.\ Cn^{+1}X = \{y : \exists x \in X(y \in Cn^+\{x\})\},$$

which is also a consequence operation (see Wybraniec-Skardowska [18]) called the *unit consequence operation induced by the consequence operation* $Cn^+$, and on the basis of A5, Tk, DK, A4 we can indeed easily state (cf. [11,18]) that the following theorem holds:

$$\text{T1. } X \neq \emptyset \Rightarrow Cn^+X = Cn^{+1}\mathrm{K}X \text{ and } Cn^+\emptyset = Cn^{+1}\{\mathrm{c}xx\}.$$

We will also make use the following generalized theorem Tk:

Tka. If $v_1, v_2, \ldots, v_m \in \mathrm{K}X$ then $Cn^+\{v_1, v_2, \ldots, v_m\} = Cn^+\{x_1, x_2, \ldots, x_n\}$ $(n \geq m > 0)$, where $\{x_1, x_2, \ldots, x_n\}$ is the set of all propositions of $X$ which are elements of the conjunctions $v_1, v_2, \ldots, v_m$.

The proof of Tka is inductive and based on DK, Tk, A7$^+$, given below MT1(a) and the fact that the $S$-substitution of the following law of $CL^*$:

$$p_1 \rightarrow (p_2 \rightarrow (\ldots \rightarrow (p_{n-1} \rightarrow (p_n \rightarrow p))\ldots) \leftrightarrow p_1 \wedge (p_2 \wedge \ldots \wedge (p_{n-1} \wedge (p_n \rightarrow p))\ldots)$$

belongs to $Cn^+\emptyset$.

$S$-substitution of law $\alpha$ of language $CL^*$ is the "translation" of $\alpha$ in the formula of language $L$, obtained by replacing in $\alpha$ all symbols of language $CL^*$ with the corresponding symbols of language $L$.

Let us remind about of the important metatheorem on adequacy of axioms A6$^+$–A11$^+$ for the classical consequence $Cn^+$ with respect to the classical propositional logic $CL^*$:

MT1. Let $\boldsymbol{L_{Cl}}$ be the set of all $S$-substitutions of the laws of $CL^*$.Then

(a) The expression '$\mathbf{L_{Cl}} \subseteq Cn^+\emptyset$' follows from the axioms A1–A5, A6$^+$ –A11$^+$,

(b) If the expression '$\alpha \in Cn^+\emptyset$' follows from the axioms A1–A5, A6$^+$ –A11$^+$, then $\alpha \in \boldsymbol{L_{Cl}}$.

### 3.4. Operators Corresponding to Main Types of Reasoning

As we have already mentioned, certain operators (operations) defined on sets of propositions (premises) and with the values in sets of propositions (conclusions) correspond to the four main types of reasoning discussed in Sect. 2. Thus, they are the operators $O^{\mathrm{ab}}$ (a, b $\in \{+, -\}$):

$$O^{\mathrm{ab}}\colon P(S) \to P(S)$$

defined, respectively, by means of the consequence operation $Cn^{+}$.

Similarly, as it was in the case of deducibility, we assume that derivability from a finite set of propositions (premises) of the set $S$ is the same as derivability from the conjunction of these propositions (premises).

The counterpart of unilateral deductive reasoning is the unilateral deductive operator $O^{+-}$ defined in the following way:

$$\mathrm{D}^{+-}.\ y \in O^{+-}(X) \Leftrightarrow \exists x \in \mathrm{K}X(y \in Cn^{+}\{x\} \wedge x \notin Cn^{+}\{y\}).$$

*The proposition y is a conclusion unilaterally deductive derivable from the set of propositions (premises) X* iff *y is a consequence of some conjunction of propositions (premises) of the set X but this conjunction is not a consequence of the conclusion y.*

The counterpart of bilateral deductive reasoning is the bilateral deductive operator $O^{++}$ defined as follows:

$$\mathrm{D}^{++}.\ y \in O^{++}(X) \Leftrightarrow \exists x \in \mathrm{K}X(y \in Cn^{+}\{x\} \wedge x \in Cn^{+}\{y\}).$$

*The proposition y is a conclusion bilaterally deductive derivable from the set of propositions (premises) X* iff *y is a consequence of some conjunction of propositions (premises) of the set X and this conjunction is a consequence of the conclusion y.*

The counterpart of reductive reasoning is the reductive operator $O^{-+}$ defined as follows:

$$\mathrm{D}^{-+}.\ y \in O^{-+}(X) \Leftrightarrow \exists x \in \mathrm{K}X(y \notin Cn^{+}\{x\} \wedge x \in Cn^{+}\{y\}).$$

*The proposition y is a conclusion reductive derivable from the set of propositions (premises) X* iff *y is not a consequence of any conjunction of propositions (premises) of the set X but this conjunction is a consequence of the conclusion y.*

The counterpart of logically nonvaluable reasoning is the operator $O^{--}$ defined in the following way:

$$\mathrm{D}^{--}.\ y \in O^{--}(X) \Leftrightarrow \exists x \in \mathrm{K}X(y \notin Cn^{+}\{x\} \wedge x \notin Cn^{+}\{y\}).$$

*The proposition y is a conclusion logically nonvaluable derivable from the set of propositions (premises) X* iff *neither y is a consequence of a conjunction of propositions (premises) of the set X nor this conjunction is a consequence of the conclusion y.*

## 4. Properties of the Derivability Operators Corresponding to Types of Reasoning

In this section, theorems and metatheorems establishing the properties of the operators defined in Sect. 3.4 are formulated. More difficult proofs of some of these theorems will be given in the Appendix.

From the definitions of operators $O^{\mathrm{ab}}$ (a, b $\in \{+,-\}$) we easily get the following corollaries:

$$\text{O1. } O^{\mathrm{ab}}(\emptyset) = \emptyset, \text{for a, b } \in \{+,-\}.$$

*We cannot derive any conclusion from the empty set.*

$$\text{O2. } y \in O^{\mathrm{ab}}(X) \Leftrightarrow \exists x \in \mathrm{K}X(y \in O^{\mathrm{ab}}(\{x\})), \text{for a, b } \in \{+,-\}.$$

*A proposition is a conclusion of any derivability operator* $O^{\mathrm{ab}}$ (a, b $\in \{+,-\}$) iff *it is the conclusion of only one proposition which is a conjunction of some premises of that operator.*

$$\text{O3. } O^{\mathrm{ab}}(\{\mathrm{k}(x_1, x_2, \ldots, x_n)\}) \subset O^{\mathrm{ab}}(\{x_1, x_2, \ldots, x_n\}),$$
$$\text{for } n > 1 \text{ and a, b } \in \{+,-\}.$$

*Derivability from conjunction of propositions is derivability from the set of all conjuncts of the conjunction.*

$$\text{O4. } O^{\mathrm{ab}}(X) \subset \bigcup\{O^{\mathrm{ab}}(Y) \colon Y \in Fin(X)\}, \text{for a, b } \in \{+,-\}$$
$$-\text{derivability operators are finitistic.}$$

Corollary O4 follows from the corollaries O2 and O3.

From the definitions of operators $O^{\mathrm{ab}}$ (a, b $\in \{+,-\}$) and the fact K3 we immediately state that

$$\text{O5. } X \subseteq Y \Rightarrow O^{\mathrm{ab}}(X) \subseteq O^{\mathrm{ab}}(Y), \text{for a, b } \in \{+,-\}$$
$$-\text{derivability operators are monotonic.}$$

Let us establish now for what derivability operators properties reflexivity and idempotency hold. Proofs of the properties below are given in the Appendix.

$$\mathrm{O}^{+-}1. \ \mathrm{Card}(X) \neq 1 \Rightarrow X \subseteq O^{+-}(X).$$

So, the unilateral deductive operator is reflexive for every set of premises which is not a singleton.

$$\mathrm{O}^{++}1. \ X \subseteq O^{++}(X) \ - \text{the bilateral deductive operator is reflexive.}$$

The reductive and logically nonvaluable operators are neither reflexive nor irreflexive. In particular, for the reductive operator $O^{-+}$ we have only the following corollary:

$\mathrm{O}^{-+}1.$ a. $X = \{x\} \Rightarrow \neg(X \subseteq O^{-+}(X))$,

b. If $y \in X$ and $Z = \{z : z = \mathrm{k}(x_1, x_2, \ldots, x_n) \in \mathrm{K}X \wedge \exists i = 1, 2, \ldots, n(x_i = y)\}$, then $\neg(Z \subseteq O^{-+}(Z))$ and $\neg(X \subseteq O^{-+}(Z))$.

The similar corollary, corollary $\mathrm{O}^{--}1$, is valid for the logically nonvaluable operator $O^{--}$.

$\mathrm{O}^{+-}2.$ $O^{+-}(O^{+-}(X)) \subseteq O^{+-}(X)$
−the unilateral deductive operator is idempotent.

$\mathrm{O}^{++}2.$ $O^{++}(O^{++}(X)) \subseteq O^{++}(X)$
−the bilateral deductive operator is idempotent.

$\mathrm{O}^{-+}2.$ $O^{-+}(O^{-+}(X)) \subseteq O^{-+}(X)$
−the reductive operator is idempotent.

On the basis of the corollaries given in this section we can formulate the following metatheorems:

MT2. If the operation $O^{+-}$ of unilateral deductive derivability is defined on any set of propositions different from a singleton, then it is a finitistic consequence operator in Tarski's sense.

MT3. The operator $O^{++}$ of bilateral deductive derivability is a finitistic consequence operator in Tarski's sense.

MT4. The operators $O^{+-}$ of reductive derivability and $O^{--}$ of logically nonvaluable derivability are not consequences operators in Tarski's sense.

MT2 follows from corollaries $\mathrm{O}^{+-}1$ (A2), $\mathrm{O}^{+-}2$ (A3), O5 (A4) and O4 (A5).

MT3 follows from corollaries $\mathrm{O}^{++}1$ (A2), $\mathrm{O}^{++}2$ (A3), O5 (A4) and O4 (A5).

MT4 follows from the fact that operators $O^{+-}$ and $O^{--}$ are not reflexive (see corollaries $\mathrm{O}^{+-}1$ and $\mathrm{O}^{--}1$).

## 5. Relationships Between Derivability Operations and Classical and Dual Consequences

In Sect. 1 we defined the unit consequence $Cn^{+1}$ by means of Tarski's consequence $Cn$ ($Cn = Cn^{+}$):

$$\mathrm{D}Cn^{+1}.\ Cn^{+1}X = \{y : \exists x \in X(y \in Cn^{+}\{x\})\},$$

and we formulated the theorem

$$\mathrm{T1}'.\ X \neq \emptyset \Rightarrow Cn^{+1}\mathrm{K}X = Cn^{+}X.$$

Thus, the deduction from a finite set of propositions is the deduction from only one proposition which is a conjunction of propositions of the finite set.

The consequence $Cn^{+1}$ is normal ($Cn^{+1}\emptyset = \emptyset$) and unit because it satisfies the condition:

$$\mathrm{C1}.\ y \in Cn^{+1}X \Leftrightarrow \exists x \in X(y \in Cn^{+1}\{x\}).$$

Using the consequence $Cn$, Słupecki [9] defined the operation $Cn^{-1}$ and proved that it is also a consequence operation—the so-called *rejection consequence* (see [12,18,20,21]). The definition of the operation $Cn^{-1}$ is the following:

$$\mathrm{D}Cn^{-1}.\ Cn^{-1}X = \{y : \exists x \in X(x \in Cn\{y\})\}.$$

The consequence $Cn^{-1}$ is normal ($Cn^{-1}\emptyset = \emptyset$) and unit because it satisfies the condition:

$$\text{C1}'.\ y \in Cn^{-1}X \Leftrightarrow \exists x \in X(y \in Cn^{-1}\{x\})\}.$$

The unit rejection consequence $Cn^{-1}$ is dual to the unit consequence $Cn^{+1}$ (see [13,22]).

In a similar way as by means of the unit consequence $Cn^{+1}$ we can define the finitistic consequence $Cn^{+}$ (see T1'), with the help of the unit rejection consequence $Cn^{-1}$ we can define a dual, with respect to $Cn$, finitistic consequence $Cn^{-}$. The theorem analogous to T1' is the theorem:

$$\text{T2.}\ X \neq \emptyset \Rightarrow Cn^{-1}\text{A}X = Cn^{-}X,$$

where A$X$ is the set of all disjunctions formed of different sentences of the set $X$, defined by DA in a analogous way as the set K$X$ (see DK, Dk and K1-K4) and $Cn^{-}$ is the operation defined by Wójcicki [17] in the following way:

$$\text{D}Cn^{-}.\ Cn^{-}X = \{y : \exists Y \in Fin(X)\{\bigcap\{Cn^{+}\{x\} : x \in Y\} \subseteq Cn^{+}\{y\}\}.$$

*Proof.* In the proof T2, we use the following lemma Ta for finite disjunction (similar to the lemma Tk for finite conjunction) following from Da, the axiom A$^{+}$11 (for $X = \emptyset$) and the axiom A7$^{+}$ $\leftarrow$, A7$^{+}$ $\rightarrow$:

$$\text{Ta.}\ Cn^{+}\{\text{a}(x_1, x_2, \ldots, x_n)\} = Cn^{+}\{x_1\} \cap Cn^{+}\{x_2\} \cap \ldots \cap Cn^{+}\{x_n\}, \text{for } n \geq 1.$$

Let $X \neq \emptyset$. Then

$$\begin{aligned} y \in Cn^{-1}\text{A}X &\Leftrightarrow \exists x \in \text{A}X(x \in Cn^{+}\{y\}) \\ &\Leftrightarrow \exists x_1, x_2, \ldots, x_n \in X(\text{a}(x_1, x_2, \ldots, x_n) \in Cn^{+}\{y\}) \\ &\Leftrightarrow \exists x_1, x_2, \ldots, x_n \in X(Cn^{+}\{\text{a}(x_1, x_2, \ldots, x_n)\} \subseteq Cn^{+}\{y\}) \\ &\Leftrightarrow \exists Y \in Fin(X)\{\bigcap\{Cn\{x\} : x \in Y\} \subseteq Cn^{+}\{y\}\} \Leftrightarrow y \in Cn^{-}X. \end{aligned}$$

□

Wójcicki's operation $Cn^{-}$ is a finitistic consequence in the usual sense (it satisfies Tarski's axioms A1-A5).

In accordance with T2, we can state that the rejection of a proposition on the basis of a finite set of propositions is rejection of this proposition on the basis of only one proposition which is the disjunction of propositions of the finite set.

Since the unit consequences are normal, T1', T2 hold and we have: $X \subseteq \text{K}X$, $X \subseteq \text{A}X$, it is easy to notice that the above-given unit consequences are weaker than the finitistic consequences which are defined by them. Thus,

$$\text{M5.}\ Cn^{+1} \leq Cn^{+} \text{ and } Cn^{-1} \leq Cn^{-}.$$

By means of the definitions of the operators given in Sect. 2 it is also easy to justify that for the operators corresponding to reasonings we have:

M6$a$. $O^{+-} \leq Cn^{+}$ – the (almost) consequence, the unilateral deductive operator is weaker than the consequence $Cn^{+}$.

Thus, if a proposition is unilaterally deductive derivable from a set of propositions (premises), then it is deducible from this set (is a consequence of this set) of propositions (premises).

M6b. $O^{++} \leq Cn^{+}$ and $O^{++} \leq Cn^{-}$ – the consequence, the bilateral deductive operator is weaker than the consequence $Cn^{+}$ and it is weaker than the dual consequence $Cn^{-}$.

Thus, if a proposition is bilaterally deductive derivable from a set of propositions (premises), then it is deducible from this set (it is a consequence of this set) of propositions and, simultaneously, it is a dual consequence of that set of propositions (premises).

M6c. $O^{-+} \leq Cn^{-}$ – the reductive operator is weaker than the dual consequence $Cn^{-}$.

Thus, if a proposition is reductive derivable from a set of propositions (premises), then it is a dual consequence of the set of propositions (premises).

Proofs of these metatheorems are given in the Appendix, in which we also give proofs of two further theorems that show the intuitive meaning of the consequences $Cn^{+}$, $Cn^{-1}$, $Cn^{-}$ and operators weaker than them.

The names *rejection consequence* given to $Cn^{-1}$ and *dual consequence* given to $Cn^{-}$ are related to the following two theorems provable by means of Tarski's axioms A1-A5, definitions D$Cn^{-1}$, D$Cn^{-}$ and theorem T2:

T3. $\forall X(X \subseteq Y \Rightarrow Cn^{+}X \subseteq Y) \Rightarrow \forall X(X \subseteq S \setminus Y \Rightarrow Cn^{-1}X \subseteq S \setminus Y)$.
T4. $\forall X(X \subseteq Y \Rightarrow Cn^{+}X \subseteq Y) \Rightarrow \forall X \neq \emptyset(X \subseteq S \setminus Y \Rightarrow Cn^{-}X \subseteq S \setminus Y)$.

If $Cn^{+}$ $(Cn = Cn^{+})$ is the usual consequence, a *reliable consequence* which yields true (or accepted as true) conclusions for true (or accepted as true, respectively) premises, then taking $Y$ to be the set of true (or accepted as true) propositions, $S \setminus Y$ is the set of false (or nonaccepted as true) propositions, and by T3 and T4, the expressions rejected on the basis of false propositions (or not accepted as true) are also false (or not accepted as true, respectively). Thus, if we denote a *reliable consequence*, or in other terms—a consequence with respect to acceptance by $Cn^{+}$ and assume that $Cn = Cn^{+}$, then this consequence and the corresponding unit consequence $Cn^{+1}$ yield true expressions (or accepted as true) when applied to true ones (or accepted as true), whereas the rejection consequence $Cn^{-1}$ and the dual consequence $Cn^{-}$ with respect to $Cn^{+}$ always yields false (or nonaccepted as true) expressions when applied to false (or nonaccepted as true) premises.

Since the operator $Cn^{+}$ of classical consequence is reliable, its weaker operators of unilateral and bilateral deductive derivability $O^{+-}$ and $O^{++}$ (see M6a,b) are also reliable (they lead from true premises to true conclusions). From T4, M6b and M6c it also follows that the bilateral deductive operator $O^{++}$ and the reductive operator $O^{-+}$, as weaker than the dual consequence operator $Cn^{-}$, have the property that from false (or nonaccepted as true) premises lead to false (or not accepted as true) conclusions.

## Final Remarks

- Derivability operators offer a handy tool serving to establish some general properties of types of reasoning corresponding to them.
- While settling these properties, we accepted the assumption that the operator consequence $Cn$ ($Cn = Cn^+$) satisfies Tarski's axioms A1-A5 and axioms A6$^+$, A7$^+$, A10$^+$, A11$^+$ characterizing only the functors c, k and a, corresponding to connectives of the classical propositional logic $CL$*: implication, conjunction and disjunctions, respectively, (we did not make use of axioms for the functor of negation n). Thus, these properties do not include some specific properties that are available to certain known sub-types of reasoning, e.g. indirect proof.
- All the above-mentioned axioms are satisfied by the consequence operators based on some nonclassical logics assuming classical propositional calculus $CL$* and on intuitionistic or minimal propositional calculi (for example, see [6,7,19]).
- Hence all the properties and relations established for the operator $Cn^+$ in Sects. 4 and 5 are valid for mentioned nonclassical operators of consequences characterized only by axioms for functors c, k and a.
- The properties are transferable thus not only onto the reasonings corresponding to them, which are run on the basis of the classical logic $CL$, but also onto reasonings based on the mentioned nonclassical logics.


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## Appendix

We give proofs here of corollaries and theorems formulated in Sects. 4 and 5 using the (assumptive) method of natural deduction put forward in the book of Słupecki and Borkowski [10]. When referring to the axioms A3 and A4 we use the abbreviation AT.

$\mathrm{O}^{+-}1$. $\mathrm{Card}(X) \neq 1 \Rightarrow X \subseteq O^{+-}(X)$.

*Proof.*

| | | |
|---|---|---|
| 1. | $\mathrm{Card}(X) \neq 1$ | {assum.} |
| 2. | $X = \emptyset \vee \mathrm{Card}(X) > 1$ | {1} |
| 3. | $X = \emptyset \Rightarrow X \subseteq O^{+-}(X)$ | {O1} |
| 1.1. | $\mathrm{Card}(X) > 1 \wedge y \in X$ | {additional assumption} |
| 1.2. | $x_1 \in X \wedge x_1 \neq y$ | {1.1} |
| 1.3. | $\{x_1, y\} \subseteq X \wedge \mathrm{k}(x_1, y) \in \mathrm{K}X$ | {DK} |
| 1.4. | $y \in Cn\{x_1, y\} = Cn\{\mathrm{k}(x_1, y)\}$ $\wedge \mathrm{k}(x_1, y) \notin Cn\{y\}$ | {A2,A10$^+$, A7$^+$, 1.2, MT1(b)} |
| 1.5. | $\exists x \in \mathrm{K}X(y \in Cn^+\{x\} \wedge x \notin Cn\{y\})$ | {1.3, 1.4} |
| 1.6. | $y \in O^{+-}(X)$ | {D$^{+-}$, 1.5} |
| 4. | $\mathrm{Card}(X) > 1 \Rightarrow X \subseteq O^{+-}(X)$ | {1.1 → 1.6} |
| | $X \subseteq O^{+-}(X)$ | {2, 3, 4} |

□

$\mathrm{O}^{++}1$. $X \subseteq O^{++}(X)$

*Proof.*

| | | |
|---|---|---|
| 1.1. | $y \in X$ | {additional assum.} |
| 1.2. | $y = \mathrm{k}(y) \wedge \mathrm{K}\{y\} \subseteq X$ | {Dk1,DK, 1.1} |
| 1.3. | $\mathrm{k}(y) \in \mathrm{K}X \wedge y \in Cn\{\mathrm{k}(y)\} \wedge \mathrm{k}(y) \in Cn\{y\}$ | {1.2, K3,A1} |
| 1.4. | $\exists x \in \mathrm{K}X(y \in Cn^+\{x\} \wedge x \in Cn^+\{y\})$ | {1.3} |
| 1.5. | $y \in O^{++}(X)$ | {D$^{++}$, 1.4} |
| 1. | $\forall y(y \in X \Rightarrow y \in O^{++}(X))$ | {1.1 → 1.5} |
| | $X \subseteq O^{++}(X)$ | {1} |

□

$\mathrm{O}^{-+}1$.a. $X = \{x\} \Rightarrow \neg(X \subseteq O^{-+}(X))$,

b. If $y \in X$ and $Z = \{z : z = \mathrm{k}(x_1, x_2, \ldots, x_n) \in \mathrm{K}X \wedge \exists i = 1, 2, \ldots, n(x_i = y)\}$, then $\neg(Z \subseteq O^{-+}(Z))$ and $\neg(X \subseteq O^{-+}(Z))$.

*Proof.*

a.

| | | |
|---|---|---|
| 1. | $X = \{x\}$ | {assum.} |
| 2. | $\mathrm{K}\{x\} = X \wedge \neg\exists x \in \mathrm{K}X(x \notin Cn\{x\} \wedge x \in Cn\{x\})$ | {1, K1} |
| 3. | $\exists y \in X(y \notin O^{-+}(X))$ | {1, 2, D$^{-+}$} |
| | $\neg(X \subseteq O^{-+}(X))$ | {3} |

b.

| | | |
|---|---|---|
| 1. | $y \in X$ | {assum.} |
| 2. | $Z = \{z\colon z = \mathrm{k}(x_1, x_2, ..., x_n) \in \mathrm{K}X \wedge \exists i = 1, 2, ..., n(x_i = y)\}$ | {assum.} |
| 3. | $y \in Z$ | $\{2, 1, \mathrm{Dk}, \mathrm{K2}\}$ |
| 4. | $\neg\exists x \in \mathrm{K}Z(y \notin Cn\{x\} \wedge x \in Cn\{y\})$ | $\{2, \mathrm{DK}, \mathrm{Tk}, \mathrm{Tka}\}$ |
| 5. | $\exists y \in Z(y \notin O^{-+}(Z)) \wedge \exists y \in X(y \notin O^{-+}(Z))$ | $\{3, 4, 1, \mathrm{D}^{-+}\}$ |
| | $\neg(Z \subseteq O^{-+}(Z))$ and $\neg(X \subseteq O^{-+}(Z))$ | $\{5\}$ |

□

$\mathrm{O}^{+-}2$. $O^{+-}(O^{+-}(X)) \subseteq O^{+-}(X)$

*Proof.*

| | | |
|---|---|---|
| 1. | $y \in O^{+-}(O^{+-}(X))$ | {assum.} |
| 2. | $x_1 \in \mathrm{K}O^{+-}(X) \wedge y \in Cn^+\{x_1\} \wedge x_1 \notin Cn^+\{y\}$ | $\{1, \mathrm{D}^{+-}\}$ |
| 3. | $x_1 = \mathrm{k}(z_1, z_2, \ldots, z_n) \wedge z_i \in O^{+-}(X)$ for all $i = 1, 2, \ldots, n$ | $\{2, \mathrm{DK}\}$ |
| 4. | $v_i \in \mathrm{K}X \wedge z_i \in Cn^+\{v_i\}$ for all$i = 1, 2, \ldots, n$ | $\{3, \mathrm{D}^{+-}\}$ |
| 5. | $\{z_1, z_2, ..., z_n\} \subseteq Cn^+\{v_1, v_2, ..., v_n\} = Cn^+\{x'_1, x'_2, ..., x'_m\}$, where every $x'_j \in X$ for$j = 1, 2, \ldots, m$ and $m \geq n$ | $\{4, \mathrm{AT}, \mathrm{Tka}\}$ |
| 6. | $Cn^+\{\mathrm{k}(z_1, z_2, \ldots, z_n)\} \subseteq Cn^+\{\mathrm{k}(x'_1, x'_2, \ldots, x'_m)\}$ | $\{5, \mathrm{AT}, \mathrm{Tk}\}$ |
| 7. | $Cn^+\{x_1\} \subseteq Cn^+\{v'\} \wedge v' = \mathrm{k}(x'_1, x'_2, \ldots, x'_m) \in \mathrm{K}X$ | $\{3, \mathrm{DK}, 5\}$ |
| 1.1. | $v' \in Cn^+\{y\}$ | {add.assum.} |
| 1.2. | $Cn^+\{v'\} \subseteq Cn^+\{y\}$ | $\{1.1, \mathrm{AT}\}$ |
| 1.3. | $z_i \in Cn^+\{v'\}$ for all $i = 1, 2, \ldots, n$ | $\{5, \mathrm{Tk}, 7\}$ |
| 1.4. | $z_i \in Cn^+\{y\}$ for all $i = 1, 2, \ldots, n$ | $\{1.3, 1.2\}$ |
| 1.5. | $Cn^+\{z_1, z_2, \ldots, z_n\} \subseteq Cn^+\{y\}$ | $\{1.4, \mathrm{AT}\}$ |
| 1.6. | $\mathrm{k}(z_1, z_2, \ldots, z_n) \in Cn^+\{y\}$ | $\{1.5, \mathrm{Tk}, \mathrm{A2}\}$ |
| 1.7. | $x_1 \in Cn^+\{y\}$ | $\{1.6, 3\}$ |
| 1.8. | contradiction | $\{1.7, 2\}$ |
| 8. | $v' \notin Cn^+\{y\}$ | $\{1.1 \rightarrow 1.8\}$ |
| 9. | $y \in Cn^+\{v'\} \wedge v' \in \mathrm{K}X$ | $\{2, 7\}$ |
| 10. | $y \in O^{+-}(X)$ | $\{\mathrm{D}^{+-}, 9, 8\}$ |
| | $O^{+-}(O^{+-}(X)) \subseteq O^{+-}(X)$ | $\{1 \rightarrow 10\}$ |

□

$\mathrm{O}^{++}2$. $O^{++}(O^{++}(X)) \subseteq O^{++}(X)$

*Proof.*

| | | |
|---|---|---|
| 1. | $y \in O^{++}(O^{++}(X))$ | {assum.} |
| 2. | $x_1 \in \mathrm{K}O^{++}(X) \wedge y \in Cn^+\{x_1\} \wedge x_1 \in Cn^+\{y\}$ | $\{1, \mathrm{D}O^{++}\}$ |
| 3. | $x_1 = \mathrm{k}(z_1, z_2, \ldots, z_n) \wedge z_i \in O^{++}(X)$ for all $i = 1, 2, \ldots, n$ | $\{2, \mathrm{DK}\}$ |
| 4. | $v_i \in \mathrm{K}X \wedge z_i \in Cn^+\{v_i\} \wedge v_i \in Cn^+\{z_i\}$ for all $i = 1, 2, \ldots, n$ | $\{3, \mathrm{D}^{+-}\}$ |
| 5. | $\{z_1, z_2, ..., z_n\} \subseteq Cn^+\{v_1, v_2, ..., v_n\} = Cn^+\{x'_1, x'_2, ..., x'_m\}$, where every $x'_j \in X$ for $j = 1, 2, \ldots, m$ and $m \geq n$ | $\{4, \mathrm{AT}, \mathrm{Tka}\}$ |

| | | |
|---|---|---|
| 6. | $Cn^+\{\mathrm{k}(z_1, z_2, \ldots, z_n)\} \subseteq Cn^+\{\mathrm{k}(x'_1, x'_2, \ldots, x'_m)\}$ | $\{5, \mathrm{AT}, \mathrm{Tk}\}$ |
| 7. | $Cn^+\{x_1\} \subseteq Cn^+\{v'\} \wedge v' = \mathrm{k}(x'_1, x'_2, \ldots, x'_m) \in \mathrm{K}X$ | $\{6, 3, \mathrm{DK}, 5\}$ |
| 8. | $y \in Cn^+\{v'\}$ | $\{2, 7\}$ |
| 9. | $\{v_1, v_2, \ldots, v_n\} \subseteq Cn^+\{z_1, z_2, \ldots, z_n\}$ | $\{4, \mathrm{AT}\}$ |
| 10. | $Cn^+\{v_1, v_2, \ldots, v_n\} \subseteq Cn^+\{\mathrm{k}(z_1, z_2, \ldots, z_n)\}$ | $\{9, \mathrm{AT}, \mathrm{Tk}\}$ |
| 11. | $Cn^+\{x'_1, x'_2, \ldots, x'_m\} \subseteq Cn^+\{x_1\}, x'_j \in X$ for all $j = 1, 2, \ldots, m$ and $m \geq n$ | $\{10, 5, 3\}$ |
| 12. | $Cn^+\{v'\} \subseteq Cn^+\{x_1\}$ | $\{11, \mathrm{Tk}, 7\}$ |
| 13. | $v' \in Cn^+\{y\}$ | $\{12, \mathrm{A2}, 2, \mathrm{AT}\}$ |
| 14. | $y \in O^{++}(X)$ | $\{\mathrm{D}^{++}, 7, 8, 13\}$ |
| | $O^{++}(O^{++}(X)) \subseteq O^{++}(X)$ | $\{1 \rightarrow 14\}$ |

□

$\mathrm{O}^{-+}2$. $O^{-+}(O^{-+}(X)) \subseteq O^{-+}(X)$

*Proof.*

| | | |
|---|---|---|
| 1. | $y \in O^{-+}(O^{-+}(X))$ | {assum.} |
| 2. | $x_1 \in \mathrm{K}O^{-+}(X) \wedge y \notin Cn^+\{x_1\} \wedge x_1 \in Cn^+\{y\}$ | $\{1, \mathrm{D}O^{-+}\}$ |
| 3. | $x_1 = \mathrm{k}(z_1, z_2, \ldots, z_n) \wedge z_i \in O^{-+}(X)$ for all $i = 1, 2, \ldots, n$ | $\{2, \mathrm{DK}\}$ |
| 4. | $v_i \in \mathrm{K}X \wedge z_i \notin Cn^+\{v_i\} \wedge v_i \in Cn^+\{z_i\}$ for all $i = 1, 2, \ldots, n$ | $\{3, \mathrm{D}^{-+}\}$ |
| 1.1. | $x_1 \in Cn^+\{v_j\} \wedge j \in \{1, 2, \ldots, n\}$ | {add.assum.} |
| 1.2. | $\mathrm{k}(z_1, z_2, \ldots, z_n) \in Cn^+\{v_j\}$ | $\{1.1, 3\}$ |
| 1.3. | $Cn^+\{z_1, z_2, \ldots, z_n\} \subseteq Cn^+\{v_j\}$ | $\{1.2, \mathrm{AT}, \mathrm{Tk}\}$ |
| 1.4. | $z_j \in Cn^+\{v_j\} \wedge j \in \{1, 2, \ldots, n\}$ | $\{1.3, \mathrm{A2}, 1.1\}$ |
| 1.5. | contradiction | $\{1.4, 4\}$ |
| 5. | $x_1 \notin Cn^+\{v_j\} \wedge j \in \{1, 2, \ldots, n\} \wedge v_j \in \mathrm{K}X$ | $\{1.1 \rightarrow 1.5, 4\}$ |
| 6. | $v_j \in Cn^+\{z_1, z_2, \ldots, z_n\}$ | $\{4, \mathrm{AT}, 5\}$ |
| 7. | $v_j \in Cn^+\{x_1\}$ | $\{6, \mathrm{Tk}, 3\}$ |
| 8. | $y \in Cn^+\{v_j\} \Rightarrow Cn^+\{y\} \subseteq Cn^+\{v_j\} \Rightarrow x_1 \in Cn^+\{v_j\}$ | $\{\mathrm{AT}, 2\}$ |
| 9. | $y \notin Cn^+\{v_j\} \wedge v_j \in Cn^+\{y\} \wedge v_j \in \mathrm{K}X$ | $\{8, 5, 7, 2, \mathrm{AT}\}$ |
| 10. | $y \in O^{-+}(X)$ | $\{9, \mathrm{D}^{-+}\}$ |
| | $O^{-+}(O^{-+}(X)) \subseteq O^{-+}(X)$ | $\{1 \rightarrow 10\}$ |

□

T3. $\forall X(X \subseteq Y \Rightarrow Cn^+X \subseteq Y) \Rightarrow \forall X(X \subseteq S \setminus Y \Rightarrow Cn^{-1}X \subseteq S \setminus Y)$

*Proof.*

| | | |
|---|---|---|
| 1. | $\forall X(X \subseteq Y \Rightarrow Cn^+X \subseteq Y)$ | {assum.} |
| 2. | $X_1 \subseteq S \setminus Y \wedge y_1 \in Cn^{-1}X_1 \wedge y_1 \in Y$ | {indirect assumption} |
| 3. | $x_1 \in X_1 \wedge x_1 \in Cn^+\{y_1\}$ | $\{2, \mathrm{D}Cn^{-1}\}$ |
| 4. | $x_1 \notin Y \wedge \{y_1\} \subseteq Y$ | $\{3, 2\}$ |
| 5. | $Cn^+\{y_1\} \subseteq Y$ | $\{4, 1\}$ |
| 6. | $x_1 \in Y$ | $\{3, 5\}$ |
| | *Contradiction* | $\{6, 4\}$ |

□

**Lemma**. $\forall X(X \subseteq Y \Rightarrow Cn^{+}X \subseteq Y) \Rightarrow \forall X(X \subseteq S \setminus Y \Rightarrow Cn^{-1}\mathrm{A}X \subseteq S \setminus Y)$

*Proof.*

| | | |
|---|---|---|
| 1. | $\forall X(X \subseteq Y \Rightarrow Cn^{+}X \subseteq Y)$ | {assum.} |
| 1.1. | $X \subseteq S \setminus Y$ | {additional assumption} |
| 1.2. | $\mathrm{A}X \subseteq S \setminus Y$ | {1.1, DA (compare DK)} |
| 1.3. | $\mathrm{A}X \subseteq S \setminus Y \Rightarrow Cn^{-1}\mathrm{A}X \subseteq S \setminus Y$ | {T3, 1} |
| 1.4. | $Cn^{-1}\mathrm{A}X \subseteq S \setminus Y$ | {1.3, 1.2} |
| 2. | $X \subseteq S \setminus Y \Rightarrow Cn^{-1}\mathrm{A}X \subseteq S \setminus Y$ | {1.1 → 1.4} |
| | $\forall X(X \subseteq S \setminus Y \Rightarrow Cn^{-1}\mathrm{A}X \subseteq S \setminus Y)$ | {2} |

□

T4. $\forall X(X \subseteq Y \Rightarrow Cn^{+}X \subseteq Y) \Rightarrow \forall X \neq \emptyset(X \subseteq S \setminus Y \Rightarrow Cn^{-}X \subseteq S \setminus Y)$

*Proof.* T4 follows from the above given Lemma and T2. □

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