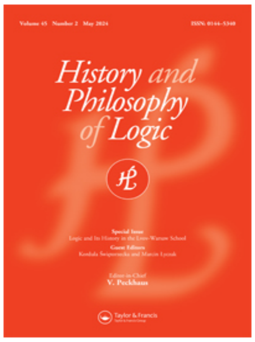

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The Pioneering Proving Methods as Applied in the Warsaw School of Logic – Their Historical and Contemporary Significance

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ABSTRACT

Justification of theorems plays a vital role in any rational human activity. It is indispensable in science. The deductive method of justifying theorems is used in all sciences and it is the only method of justifying theorems in deductive disciplines. It is based on the notion of *proof*, thus it is a method of proving theorems. In the Warsaw School of Logic (WSL) – the famous branch of the Lvov-Warsaw School (LWS) – two types of the method: *axiomatic deduction method* and *natural deduction method* were developed and practiced. In this paper, both of these methods are briefly discussed with an emphasis on their historical, groundbreaking significance for logic. The axiomatic method by means of rejection (proposed by Jan Łukasiewicz – a co-creator of the WSL), which is the method of the so-called *rejection proof* (*rejection/refutation method*) in logical systems and the proving method of *generalized natural deduction*, which is a hybrid deduction–refutation method of proving theorems, are also outlined in the paper. The author discusses their historical significance. This paper also contains a brief mention of the most significant results which the application of the discussed methods introduced into contemporary scientific research, not only logical one.

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1. Introduction

The whole Lvov-Warsaw School (LWS) was governed by the postulate of clarity, accuracy and precision of thinking as well as expressing thoughts in a language. It was connected with another postulate accepted by the LWS – that of convincing and appropriate justification of propositions and theorems.

Justification of theorems plays a vital role in any rational human activity. It is indispensable in science. As far as natural sciences and humanities are concerned, the starting point is *direct justification* of propositions, based on external observations or introspection. Here, observable propositions are the foundation on which other theorems of these sciences rely. Consequently, the latter are *indirectly justified* by means of *reasoning* (inference in the broad sense) and this both *substantiating inference* (in which the truthfulness of its premises does not guarantee that of the conclusion, but only a degree of its truthfulness) and *deductive*

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inference (in which the conclusion *follows logically from premises*, and the truthfulness of premises secures that of the conclusion).

Deductive inference finds its application in all sciences and in *deductive disciplines* – which include mathematical sciences and systems of formal logic – it is used as the sole method of justifying theorems. The notion of logical entailment, forms and schemata of deductive inference are established in logic. Deductive sciences (theories, systems) are thus disciplines in which the deductive method of reasoning, i.e. *deduction method*, is used exclusively.

Generally speaking, deductive science (theory, system) is a set of propositional expressions which are derivable, deducible from certain data, asserted or assumed expressions of this set by means of rules of deduction, rules of deductive inference. Expressions deducible from given defined expressions of this set are thus *theorems* of science; they have a *proof*. At the same time, the deduction method of justifying theorems of such a science (theory, system) is therefore a proving method of its theorems, a *method by means of proof*, in short – a *proof method*. It is the basic method that is discussed and applied in building deductive systems.

Applying the deduction method obviously requires specialist logical competence and knowledge of two basic types of the proof method: *axiomatic deduction method* and *natural deduction method* – both of which were developed and practiced in the Warsaw School of Logic (WSL) – the famous branch of the LWS – by the co-creator of the School – Jan Łukasiewicz and its important representative – Stanisław Jaśkowski, respectively. The methods will be discussed briefly in Section 2 and their breakthrough role in the history of logic will be indicated as well.

Section 3 will outline the axiomatic method by means of rejection or refutation proposed by Łukasiewicz, which is the method of the so-called *rejection proof* (*rejection/refutation method*) in logical systems and also the method of generalized natural deduction that is a hybrid deduction–refutation method in the formalization theory. We will discuss their historical significance there, too.

In Section 4, we will briefly mention the most significant results which the application of these methods brought in contemporary scientific research, not only logical one.

2. Two Methods of Deduction

The two deduction methods applied in building a deductive science are simultaneously the fundamental methods of forming deduction systems and theories. We will recall first the axiomatic method and next the – discovered much later – natural deduction method.

2.1. The Axiomatic Method

2.1.1. Characteristics of the Axiomatic Method and Formalization of Science

The axiomatic method is the most often applied one in constructing logical systems and mathematical sciences as deductive disciplines. It leads to building them as *axiomatic deduction systems*. Axiomatization of such a system consists – as it is well-known – in that a set of its expressions, called axioms, is chosen. They are acknowledged as the primitive theorems which within a given axiomatic system are assumed not to be proved on the basis of other theorems. With the help of these axioms, as well as the initial *primitive inference* *rules* accepted in the system, the remaining (derivative) theorems are proved. Axioms of the system should be selected, at the same time, in such a way that all the other theorems of it can be deduced, proved, by their means.

Contemporary understanding of science as a theory of high degree of exactness and precision requires treating it as a deduction system. Deductive disciplines have not always been built as axiomatic systems, though. Depending on the degree of methodological precision, the following three stages of constructing a deductive science as a system are distinguished: pre-axiomatic, non-formalized axiomatic, and formalized axiomatic (see *Ajdukiewicz 1966*, *1978*). They differ as far as the stages of deduction are concerned. The first stage of deduction is characterized by the fact that all general evidentiary theses (i.e. so-called certainties) are admitted as premises (assumptions) of the system and every scientific statement of the pre-axiomatic system is either evident or is related by evident entailment relations with evident statements. The second level of deduction takes place in an axiomatic system in which the underlying premises (axioms) of the system are explicitly listed, although the rules of inference are usually applied intuitively. The third stage is reached by the deductive system at formalization. Clearly formulated axioms are not only the main premises, but also axiomatic definitions constituting the meaning of the specific terms of the formalized system, and the rules of inference are also clearly formulated.

At the *formalized axiomatic* stage of a deductive science, being the most exact stage of building a deductive discipline as a system, the following are clearly made precise:

- the notion of a well-formed sentential expression, (in short: *wfe*) of the system,
- set of its axioms,
- primitive inference rules,
- rules of introducing definitions, if the system possesses such definitions at all.

The notion of *proof* is precisely defined on the ground of a given formalized axiomatic system. A *proof* of a given sentential expression of such a system is a finite sequence of *wfes*, whose last expression is this expression and in which the first element (elements) of the sequence is (are) its axioms or definitions, while the successive elements of the sequence are obtained from the preceding expressions of the sequence by means of some primitive inference rule (or meta-system definition that is a rule of replacing some expressions of the system by others and which contains new symbols included in the language of the system); let us add that among the elements of the sequence there are always some axioms.

A sentential expression of the given formalized axiomatic system is a *theorem* (*thesis*) of this system when there exists a proof on the ground of the system.

2.1.2. Łukasiewicz's Discovery

Formalized axiomatic systems are rooted in the tradition derived from Frege (*1891*, *1893*, *1903*). However, the axiomatic method had been well recognized before Frege and attributed to Euclid and his geometry. It had been practiced for a long time in a semi-axiomatic form, because Euclid distinguished two kinds of its theorems, namely, theorems accepted without proof (certainties, postulates) and theorems proved on their basis. The postulates (axioms) adopted by him, however, were not sufficient to prove all theorems of this geometry. The complete axiomatics was given only by D. Hilbert in 1899. Let us also note that the axiomatics of arithmetic of natural numbers, one of the oldest deductive sciences, was given by G. Peano in 1889.

However, one should bear in mind that it was Jan Łukasiewicz who proved that Aristotle's syllogistic was the first axiomatic (non-formalized) system in the history of European thought. He demonstrated this in his monograph on Aristotle's Syllogistic (*1951*) (and earlier, in his monograph written in Polish before the outbreak of the War (see *Łukasiewicz 1939*) and which got burnt during the Warsaw Uprising). The historic discovery by Łukasiewicz was revolutionary and contributed to formalization of different systems of syllogistic cultivated through centuries as the main tool of logical deduction.

Łukasiewicz construed the first formalized system of Aristotle's syllogistic (*1939*, *1951*), therefore a logical system satisfying contemporary requirements. The axioms are two true syllogistic modes: *Barbara* and *Datisi* and two expressions: 'Every S is S' and 'Some S are S'. They are recorded with the symbols of the system (*variable* symbols are capital letters S, P, M representing nonempty terms). Inference rules are the following: detachment rule (*modus ponens*) R1, substitution rule R2 and the definitional replacement rule R3 in the forms, respectively:

R1. If α then β, and α, thus β;
R2. α, thus Sub(α);
R3. If $\alpha =_{df} \beta$ is a definition of syllogistic, then β can be replaced by α and reversely, and every substitution of β can be replaced by analogical substitution of α and reversely;

α, β are sentential expressions of syllogistic, while expression Sub(α) is a substitution of expression α made by replacing any of its term variables with the letters used as term variables.

Łukasiewicz initiated also construction of other axiomatic systems of syllogistic, which satisfy contemporary requirements. Today various systems are known. Most of them are formalized term calculi built over classical propositional calculus which is the basic logical system.

2.1.3. The Axiomatic Proving Method as Applied in the WSL

In 1924, Łukasiewicz provided also a simple axiom system of classical two-valued propositional calculus. It was formalized as an implicative-negative axiomatic system. The primitive inference rules of this system are the detachment rule R1 and the substitution rule R2. The system was included in the authorized academic book elaborated by M. Presburger, brought out under the title *Elementy logiki matematycznej* (Elements of mathematical logic; see *Łukasiewicz 1929*; the first English edition came out in 1948). In the formalization of this, and later also of other propositional calculi, Łukasiewicz applied the world-famous now parenthesis-free symbolism – very economical, allowing omission of such punctuation means as brackets or dots. The manner was later called 'Polish notation'.

Łukasiewicz's disciples: B. Sobociński, J. Słupecki and M. Wajsberg made a considerable contribution later to the development of standard axiomatization of non-classical propositional calculi. As it is well known, Łukasiewicz was also one of the two – beside Emil Post – creators of non-classical logics – many-valued logics: first – three-valued propositional calculus (presented for the first time in Łukasiewicz's lecture delivered in 1920), and then – its generalization covering any finite- and infinite-valued propositional calculi (similarly presented for the first time in his lecture in 1922). The calculi were constructed by Łukasiewicz with the use of the semantic method: *matrix*, table, non-axiomatic. Wajsberg (*1935*), Sobociński (*1936*) and Słupecki (*1939*) worked on solving the axiomatizability problem of these logics. We owe the axiomatization of Łukasiewicz's three-valued logic to Wajsberg and Słupecki. The latter gave axiomatization for any finite n-valued propositional calculus of distinguished $k \geq 1$ values.

The discovery of many-valued logics and their axiomatization belong to the most significant logical developments in the twentieth century.

Historically, among the most meaningful studies in the field of standard formalization of logical systems, which were conducted by representatives of the WSL – apart from Łukasiewicz and his disciples – were those by Stanisław Leśniewski.

Leśniewski created three axiomatic logical systems: *protothetic* (*1927–31*, *1929*), *ontology* (*1930*) and *mereology* built over the latter and outlined in (*1916*) (see also Leśniewski *1927-31*; *Leśniewski 1938*); the original reprints of the works are found in *Pisma zebrane* (Collected Works) edited by J.J. Jadacki (see *Leśniewski 2015*), whereas their English translation in *Leśniewski 1992*; see also *Leśniewski 1988*. *Protothetic* is a generalization of the classical propositional calculus, Leśniewski's *ontology* is – on the one hand – generalization of the classical predicate calculus, and on the other one – the broadest-framed contemporarily – term calculus (calculus of names), while *mereology* is a system built over ontology, in which the axiomatic properties of being part of, relation part-whole are established as well as the concept of a set in the collective sense is defined.

Leśniewski – the creator himself – and his co-workers were taking part in perfecting and detailing the formalization as well as simplifying the axioms of his systems: A. Tarski, M. Wajsberg, B. Sobociński, J. Słupecki and Cz. Lejewski. Sobociński (*1934*) simplified, in particular, Leśniewski's axiomatics of ontology (in 1945 he also gave the shortest axiom of protothetic; see *Sobociński 1960*).

At many points of the formalization of the above-mentioned systems, Leśniewski's works were seminal regarding the times in which they appeared. They were times when efforts were made to build grand systems covering the whole deductive knowledge and it was attempted to reduce mathematics to logic. Leśniewski's logical systems were, by his intention, some realization of that trend. Those systems exerted a great impact on not only logical-mathematic research carried out by representatives of the WSL but also on studies and views expressed by logicians–philosophers of the LWS, especially those of Tadeusz Kotarbiński, who – according to Leśniewski's nominalistic preferences – adapted some of the assumptions and concepts in his philosophical works and views.

The methodological studies into formalization of the very deductive systems themselves that were conducted in the Interwar period in the twentieth century are also of historical importance. They were taken up by Alfred Tarski – probably the best-known representative of the WSL in the world and this is not just because of his most famous work on the classical notion of truth (*1933*). In 1930, Tarski published two works presenting – for the first time ever – axiomatization of the general theory of deductive systems (*1930a*) and the theory of deductive systems based on classical logic (*1930b*). These theories are called consequence theories and their axioms are used in methodological studies.

The formalized mathematical axiomatic systems can be founded on logical systems built by means of not only the axiomatic method but also with the use of the natural deduction method.

2.2. The Natural Deduction Method

The natural deduction method (ND method) was determined for logical systems called systems of natural deduction (ND systems, suppositional systems). It was officially presented for the first time by the outstanding representative of the WSL, Stanisław Jaśkowski, at the First Mathematical Convention in Lvov in 1927 (see *Jaśkowski 1929*); prior to that, it was announced in Łukasiewicz's seminar in 1926. The English version of the method, published by Jaśkowski (*1934*), coincided with the publication by German logician Gerhard Gentzen (*1934*). Therefore, Jaśkowski and Gentzen are regarded as two independent creators of ND systems. The systems are built as ones in which the proving method is based solely on inference rules, deductive inference rules, without assuming axioms (the set of axioms is empty). It is thus a method of deduction from assumptions only; the assumptions here are forms of premises of the accepted inference rules.

The ND method is close to the natural practice of running deduction or proof in the natural language, in a common discourse and science, and – as a formal method of proving – it is close to the natural proof method applied by mathematicians in research practice. The ND method is a formal reconstruction of the traditional manner of reasoning.

Systems built with the use of the ND method (ND systems) assume certain opening primitive inference rules at first. They are intuitive rules considered as deductive ones and can be verified easily.

The ND systems allow the following (see *Indrzejczak 2010*, *2018*):

- introduction of additional assumptions and creating sub-proofs and also their elimination,
- introduction of the inference rule exclusively instead of axioms,
- admittance of various forms of proof: direct, indirect, ramified, etc.

The ND systems include, apart from the inference rules, special rules of construing proofs thanks to which the construction of a proof is intuitive and much easier than construing proofs with the use of axioms.

The method of semantic tables, the tableaux method, belongs to the ND systems, as well, since it corresponds to the suppositional indirect proof.

The works by Jaśkowski and Gentzen were a breakthrough in the history of logic. Admittedly, as Łukasiewicz revealed it, it is already in antiquity that Stoics, while justifying theorems, accepted five opening 'non-provable' rules of argumentation, today known as the so-called *modus* rules of deduction. Still, it was not until Jaśkowski (and independently Gentzen) built the first ND systems that new ideas and methods were introduced into science and could be applied in building different deductive systems, mainly logical ones.

3. The Rejection Methods and Their Historical Significance

The rejection methods (rejection/refutation methods) complement the deductive ones discussed earlier, i.e. the axiomatic and ND methods. They were initiated by the works of Łukasiewicz and Jaśkowski. There exist two different rejection methods: axiomatic and generalized natural deduction; a detailed discussion of these methods can be found in the book by Bryll (*1996*). They will be discussed in Sections 3.1 and 3.2 of this paper.

3.1. The New Axiomatic Method by Means of Rejection Discovered by Łukasiewicz

The idea of rejection originated from Aristotle. The notion of rejection was introduced into formal logic by Łukasiewicz (*1921*), who applied it to complete syntactic characterization of deductive systems using the axiomatic method of rejection of propositions. Łukasiewicz formulated it first for Aristotle's syllogistic (*1939*, *1951*), and then applied it to some propositional calculi. Intuitively, it differs from the axiomatic deduction method in that while the common method applied to prove acceptable expressions, true for the given system, the rejection method is applied to refute or reject non-acceptable expressions, false ones for the system.

3.1.1. Łukasiewicz's Bi-Level Formalization of Syllogistic Logic

Still before World War 2 (see *Łukasiewicz 1939*), Łukasiewicz, upon constructing the axiomatic system of Aristotle's syllogistic, where, among others, all true syllogistic forms are proved, explored the problem of systematic refutation of false syllogistic forms.

As Łukasiewicz pointed out in his article (*1939*) and the monograph prepared before the outbreak of the War (which got burnt during the ravages of war and which he reproduced after the War and had it published in English): 'Aristotle, in his systematic investigations of syllogistic forms, not only proves the true ones but also shows that all the others are false, and must be rejected' (*Łukasiewicz 1951*, 67).

Further on (*Łukasiewicz 1951*, 67), Łukasiewicz observed:

> Aristotle rejects invalid forms by exemplification through concrete terms. This procedure is logically correct, but it introduces into the systems terms and propositions not germane to it. There are, however, cases where he applies a more logical procedure, reducing one invalid form to another already rejected. On the basis of this remark, a rule of rejection could be stated corresponding to the rule of detachment by assertion; this can be regarded as the commencement of a new field of logical inquiries and of new problems that have to be solved.

In that situation, Łukasiewicz formulated the axiomatic method of rejecting false syllogistic forms. He axiomatically refuted two syllogistic forms and attached to these rejection axioms the following rejection rules:

- $R1^{-1}$. *The rule of rejection by substitution*: any expression α can be rejected if its substitution $Sub(\alpha)$ is rejected;
- $R2^{-1}$. *The rule of rejection by detachment*: If the implication: 'If α then β', is asserted and its consequent β is rejected, then its antecedent α can be rejected.

The rejection rules $R1^{-1}$ and $R2^{-1}$, in contrast to the assertive rules of detachment R1 and substitution R2 (leading from true expressions to true expressions), run from false expressions to false expressions. Thus, by means of the axiomatic rejection method it is proved that all false forms of Aristotle's syllogistic belong to rejected expressions. Łukasiewicz accepted, at the same time, that each expression of his syllogistic system possesses a *proof by means of rejection* (*rejection proof*). The *rejection proof* of the given expression α is a sequence of expressions, whose last element is the expression α, and each element of which is either a rejection axiom or is obtained from preceding elements of this sequence or thesis of the system by means of any of the rejection rules $R1^{-1}$, $R2^{-1}$.

As we can see, Łukasiewicz, by applying the new rejection axiomatic method, beside the common axiomatic method of proving theorems, formalized Aristotle's syllogistic logic simultaneously on two levels: on the first one – as the *assertion system* and on the other one – as the *rejection/refutation system*.

Schematically, if a set of axioms of Aristotelian syllogistic logic is denoted by A^{+}, the set of its inference rules by R^{+}, while the set of its rejection axioms by A^{-}, and the set of its rejection rules accepted by Łukasiewicz by $R^{-Ł}$, then the *assertion syllogistic system* is formally characterized as the pair (A^{+}, R^{+}), and its *rejection/refutation system* as the pair $(A^{-}, R^{-Ł})$.

There arose a problem, though. Since the set of sentential expressions of Aristotle's syllogistic system is infinite, the question is as follows:

Is each sentential expression of this system either its theorem (thesis) or a rejected expression?

Therefore, more formally: If we denote by *S* the set of all sentential expressions of syllogistic and the set of all theorems (theses) of the assertion system (A^{+}, R^{+}) by T^{+}, while the set of all rejected expressions of the rejection/refutation system $(A^{-}, R^{-Ł})$ by $T^{-Ł}$, then the above-asked question can be formulated as follows:

Are the following conditions:

(1) sets T^{+} and $T^{-Ł}$ are disjoint.
(2) the union of the sets T^{+} and $T^{-Ł}$ is equal to the set *S*

satisfied?

It turned out that there exist sentential expressions of this system (of the set *S*), which neither are its theorems, theses (do not belong to the set T^{+}), nor belong to rejected expressions (do not belong to the set $T^{-Ł}$). Thus in his seminar conducted in 1938 Łukasiewicz brought up the following problem:

How to expand on Aristotle's logic in such a way that each of its expressions was its theorem (thesis) or a rejected expression?

A solution to this problem was soon found by Jerzy Słupecki.

3.1.2. Słupecki's Discovery

Solving the problem put forward by Łukasiewicz, Słupecki – apart from the rejection rules $R1^{-1}$, $R2^{-1}$ accepted by the former – introduced into the syllogistic system a completely new rule specific of this system; the rule – due to its historic and not only, significance – is called *Słupecki* rule. Słupecki rule $R^{-1Sł}$ states that:

If the implications:

'if α then γ', and: 'if β then γ', are rejected, then the implication:
'if α and β, then γ' is rejected,

if some conditions imposed on the form of the expressions α, β and γ are satisfied.

In compliance with Słupecki rule $R^{-1Sł}$, if the expression γ, which is included in the rule, does not follow from both negative sentences α and β, then it does not follow from their conjunction, either.

After attaching the rule $R^{-1Sł}$ to Łukasiewicz's rejection rules of the set $R^{-Ł}$, Słupecki proved that it is enough to accept only one of Łukasiewicz's assertive axioms and each sentential expression of syllogistic is either its theorem or a rejected expression. Thus, more formally, Słupecki proved that the following two conditions are satisfied:

(1) The sets T^{+} and $T^{-Sł}$ are disjoint,
(2) The union of the sets T^{+} and $T^{-Sł}$ is equal to the set *S*;

(the set T^{+} is here a set of all theorems (theses) of the assertion system (A^{+}, R^{+}), whereas the set $T^{-Sł}$ – a set of all rejected expressions of the rejection system (A^{-}, R^{-}), where R^{-} is the set composed of Łukasiewicz's rejection rules $R1^{-1}$, $R2^{-1}$ as well as Słupecki rule $R^{-1Sł}$).

Łukasiewicz acknowledged the result obtained by Słupecki as revelatory (see *Łukasiewicz 1939*) and – as he himself wrote in his article (*1939*) and then in the monograph (*1951*):

> The author regards this as the most significant discovery made in the field of syllogistic since Aristotle.

Słupecki failed to publish his results before the war. After the war, though, he managed to publish them in his article (*Słupecki 1946*) and, later, in a monograph *Słupecki 1948* (see the English version *Słupecki 1951*). In the latter, he used the following definition of the *rejection proof* equivalent to the definition accepted by Łukasiewicz: an expression is rejected (possesses a *rejection proof*), when, from this expression and the theses of the system T^{+} one can derive some axiomatically rejected expression of the set A^{-} by means of proving rules of the set R^{+}.

This definition and also its extension were applied by Słupecki to bi-level formalization of other logical systems (*Wybraniec-Skardowska 2005*, *2018*). He called the satisfying of Conditions (1) and (2) for any logical system Ł-*decidability* of the system (see *Słupecki, Bryll and Wybraniec-Skardowska 1971a*; *Słupecki 1972*); comparatively, Łukasiewicz used the term saturated system with reference to such a system.

The bi-level formalization of Aristotle's syllogistic introduced by Łukasiewicz and developed by Słupecki, the pioneers of this approach, gave rise to the studies, which were commonly conducted later, into application of the two axiomatic methods of proving to the bi-aspectual formalization of logical systems.

The bi-level formalization concerned systems built with the use of the ND method, as well.

3.2. The Generalized Method of Natural Deduction

The ND systems deriving from Gentzen and Jaśkowski differ in a significant way. They are, however, similar to each other in that – as we know well – they are non-axiomatic systems based solely on inference rules and proof construing rules.

There have been conducted many studies on them and the natural method of proving theorems applied in them was generalized and called the *generalized method of natural deduction* (GND method). It differs from the ordinary one since in formalization of the system it assumes not only ordinary assertive proving rules but also rejection/refutation ones, and the only rule of construing a proof is the rule of forming an indirect proof.

The method is similar to the *tableaux* method found in the literature on the subject or the *trees* method. Here, proofs have the shape of trees, whose branches are formed by means of both accepted types of rules (being rules of decomposition of complex expressions of the system at the same time) in compliance with the principle of serial or parallel connection; the tops of the trees are the simplest expressions of the system. When there appear two contradictory expressions (assertive and rejected), this expression is a theorem of the system.

The GND method, like the ordinary ND method, has a great number of applications.

4. The Impact and Significance of the Presented Methods on Subsequent and Contemporary Studies

We have already mentioned the historical significance of the axiomatic and ND methods initiated by Łukasiewicz and Jaśkowski. They exerted an enormous influence on studies and results of other outstanding representatives of the WSL, their disciples, disciples of the latter and many others.

These methods are used not only in logic but also in different scientific disciplines, such as mathematics, formal philosophy, physics, chemistry, biology, sociology, philosophical and psychological sciences, information sciences, discursive sciences, computer science and some technical sciences. They have contributed to their practical implementation in these domains, therefore not only to constructing logical systems (see, e.g. *Woodger 1937*; *Citkin and Wybraniec-Skardowska 2020*).

We will now discuss theoretical and practical aspects of these methods, necessarily giving their brief outline only.

4.1. The Influence and Applications of the Axiomatic Method

The axiomatic method, leading to presentation or characterization of logical and mathematical theories as deductive systems made the most commonly applied method in the WSL.

Here, different fully formalized logical systems, classical and non-classical, were created or developed. At the same time, it was endeavored at the WSL – successfully obtaining valuable results – to achieve possibly perfect axiomatization of the given calculus through formulating possibly the smallest number of axioms, with the use of possibly the smallest number of primitive terms. As regards the WSL, the significance of such a precise axiomatic framing of a logical system was firmly stressed, which was not always considered of importance by logicians from outside the School.

Apart from the above, a practical or intuitive notation was applied. Łukasiewicz elaborated on his parenthesis-free notation known today as Polish notation. It is useful in particular in automatic proving of theorems. Leśniewski, while building his three logical systems, applied another original intuitive notation. All of the three Leśniewski's systems were axiomatically formalized and this with great pedantry from the point of view of requirements of correct and precise formalization. Although they did not meet their creator's expectations, today they make the object of explorations by many logicians and philosophers throughout the world.

The methodological approach to the bi-aspectual axiomatic method of characterization of deductive systems as acceptance (asserted) systems and rejection (refutation) systems, introduced by Łukasiewicz and developed by his disciple Słupecki – the pioneers of the method – has proved relevant in modern approaches to logic. Bi-level formalizations of propositional deductive systems and systems of names evoked a broad response in the literature after the Second World War. A detailed discussion of the results relating to this formalization and Ł-decidability (cf. Conditions 1 and 2) of some important logical systems is presented in *Wybraniec-Skardowska 2005*, *2018*. Below, the most important of them are presented.

- Łukasiewicz, while staying abroad, in Dublin, mentioned in *Łukasiewicz 1953* that the classical propositional calculus with his inference rules: R1 and R2 and rejection rules: $R1^{-1}$, R^{-1} is Ł-decidable. A complete refutation system for it determines one rejected axiom, namely the sentential variable *p* and his rejection rules.
- Łukasiewicz used the bi-level formalization in his research on intuitionistic logic (*1952*), but he did not prove Ł-decidability of this logic. On the other hand, he proved Ł-decidability of a four-valued modal system of propositional calculus (*1953*), built by himself.

At the same time, in Poland, further studies inspired by Słupecki on Ł-decidability of deductive systems and the very notion of rejected proposition itself were carried out. Słupecki's research on Aristotle's syllogistic was continued mainly by Bogusław Iwanuś. Using the Słupecki–Łukasiewicz's methods,

- Iwanuś (*1973*) gave a proof of Ł-decidability of the whole traditional calculus of names, i.e. the system of Aristotle's syllogistic enriched by nominal negation.

Much later,

- Iwanuś (*1992*) gave a proof of Ł-decidability of the system of Aristotle's syllogistic built by Słupecki (*1946*); in this system, unlike in the original Aristotle's system, it is permissible for variables to represent empty names.
- Iwanuś also, in *Iwanuś 1972*, gave a proof of Ł-decidability of a certain version of elementary ontology, distinguished from the system of Leśniewski's ontology (both terms were invented by Słupecki (*1946*)). In elementary ontology, it is possible to interpret both the assertion system of Aristotle's syllogistic and other, richer asserted syllogistic systems, with nominal negation.

In studies on Ł-decidability of propositional logic, beside the results obtained by Łukasiewicz, it is possible to list the following achievements:

- Skura (*1993*) formulated rejected axioms and rejected rules for the classical first-order calculus
- Scott (*1957*) and, independently, Dutkiewicz (*1989*) and Skura (*1989*; *1999*), using different methods, gave the proof of Ł-decidability for the intuitionistic propositional logic
- Słupecki initiated research on Ł-decidability of Lewis modal system S5; the proof is given in the paper by Słupecki & Bryll (*1973*)
- Skura (*1992*, *1993*, *1995*) gave a simpler proof of Ł-decidability of Lewis system S5
- Skura (*1992*, *1996*, *1999*) using the algebraic method proved the Ł-decidability for modal logic S4 and for its extension (see also *Skura 2013*).

A little earlier

- Goranko (*1991*, *1994*) formulated a complete refutation system for some normal modal propositional logics (including S4)
- Bryll and Maduch (*1968*) formulated a uniform method of rejection of formulas in an $n+1$-valued implicational, implicative-negative and definitionally complete Łukasiewicz's calculus
- Skura (*1993*) built a complete refutation system for $\aleph_0$-valued Łukasiewicz's calculus
- Sochacki (*2007*, *2010*) gave complete refutation systems for all invariant Łukasiewicz's many-valued logics (in which Łukasiewicz's rule of rejection by substitution was eliminated)
- Goranko, Pulcini and Skura (*2020*) presented origins and new research on refutation methods including the method of axiomatic rejection.

To obtain the above-listed results, specific rules typical of the given system were often added to Łukasiewicz's rejection rules. They are also applied in newer studies concerning Ł-decidability of deductive systems. It is worth noting that some studies on the notion of rejection/refutation and application of it were presented at the special workshop 'Refutation', which was held at the VI World Congress of Universal Logic in Crete in April 2022.

The axiomatic method and its two variants: the ordinary method and the rejection method, found their application in methodological studies relating to the theory of deductive systems.

The axiomatic theories built by Tarski: the general theory of deductive systems (*1930a*) (called the theory of consequence operation) and the theory of deductive systems based on classical logic (*1930b*) (called the theory of classical consequence) built over it, gave not only an impulse to provide an axiomatic system for consequence theories founded on non-classical logics (theories of non-classical consequences), which upon Słupecki's inspiration were presented in *Pogorzelski and Słupecki* (*1960a*, *1960b*) (see also *Wybraniec-Skardowska 2004*), but also contributed to expanding both Tarski's theories that characterize rejection consequences.

Słupecki (*1959*) generalized the notion of rejection and, on the basis of Tarski's generalized consequence theory, defined the concept of *rejection consequence*, as well as gave its general properties. Out of the inspiration and under the supervision of Słupecki, the axiomatic theory of rejected prepositions was formed, built over Tarski's theory of classical consequence (see *Wybraniec-Skardowska 1969*; *Bryll 1969a*, *1969b* and *Słupecki, Bryll and Wybraniec-Skardowska 1971a*, *1971b*).

Some applications of this theory in the methodology of empirical sciences can be found in *Słupecki, Bryll and Wybraniec-Skardowska 1971b*. They are connected with the method of rejecting/refuting hypotheses that is commonly used in these disciplines. As it is well known, they are also commonly applied in judicature and investigation practice.

Słupecki's rejection consequence is the so-called unit consequence and can be generalized to a finitistic rejection consequence, referred to as the *dual consequence* with respect to Tarski's ordinary consequence (see *Wójcicki 1973*). Intuitively, both of these rejection consequences – in contrast to the ordinary consequence operation (leading from true sentences of a system to true sentences of this system) – lead from false or unacceptable expressions of a system to false or unacceptable expressions of this system.

Rejection consequences were applied to construct the so-called dual calculi. Studies on these calculi were conducted mainly by Ryszard Wójcicki's disciples.

It is worth noting, too, that with reference to Popper's conception of comparing scientific theories, Jan Woleński (*1992*) found an interesting application of the consequences to the examination of true (asserted) content and false (rejected) content of the given theory.

At the end, we would like to observe that Tarski (*1933*), in his famous work on the notion of truth, made use of the axiomatic method, characterizing the language of metascience, in which he defined this important notion. Giving axioms for metascience, he proceeded from the notion of concatenation. His axioms for concatenation are binding today in all formal languages and his theory is known in the world as the strings theory or the theory of concatenations. It also had an influence on constructing the theory of categorial languages presented in the books by Wybraniec-Skardowska (*1991*, *2022*).

4.2. The Impact and Application of the ND Method

The ND method is simpler, regarding its applications, than the axiomatic one. The ND system deriving from Jaśkowski was perfected by Jerzy Słupecki and Ludwik Borkowski (see *Borkowski and Słupecki 1958*; *Słupecki and Borkowski 1967*) and their disciples. The more perfect, simplified version of Jaśkowski's system is more practical, since proofs are very intuitive. One can find numerous applications of this version of the ND system in logic and set theory, in books written in Polish (see, e.g. *Słupecki, Halkowska and Piróg-Rzepecka 1976*) and in English (see *Słupecki and Borkowski 1967*). It is used in both teaching logic and proving in different sciences, when the need arises to apply deductive inferences.

The system of Słupecki and Borkowski has yet another great value: it was applied to automatic proving of theorems. The use of computers for this purpose has been made the object of intensive studies in many scientific institutes all over the world for many years now. A suppositional system of logic was the basis for the computer program *MIZAR* designed by Andrzej Trybulec in the 1970s. In the 1980s, Witold Marciszewski initiated a research program dealing with application of *MIZAR* to the construction and formal testing of correctness of proofs of theorems in Słupecki and Borkowski system (proofs in *MIZAR* are constructed similarly as in Słupecki and Borkowski system). The program was successfully implemented in several centers dealing in logic, among others, in Łódź (G. Malinowski, M. Nowak, P. Łukowski) and Opole (U. Wybraniec-Skardowska, E. Bryniarski,

Z. Bonikowski, L. Tendera, M. Chuchro; see *Bryniarski and Wybraniec-Skardowska 1990*; *Wybraniec-Skardowska et al. 1991*). At present, *MIZAR* is widely used by several hundred researchers in the world and the MIZAR Library comprises about 1500 articles (see www.mizar.org and *Indrzejczak 2010*, *2018*).

It was already after the War that Jaśkowski (*1947*) applied the ND method to building different significant logical systems. Later, ND systems were built also by other foreign authors (see *Indrzejczak 2018*, *482*). Some of them make reference, yet rather implicite, to Jaśkowski systems and can be regarded as their simplifications. The systems of Gentzen are better known in the literature on the subject. They are inclined towards theoretical research and apply rather to studies in the sphere of general proof theory. On the other hand, Jaśkowski's systems are more useful when it comes to the practice of proving, and simplifications of this method are used in teaching, mathematical reasoning and more precise non-mathematical sciences.

After the War, as far as studies on formalization of systems are concerned, the generalized natural deduction method (GND method) as a hybrid method – deduction-rejection method – was used on a broad scale. It found its applications as regards many relevant propositional calculi (see *Bryll 1996*). Nowadays, it is a fairly habitually and popularly applied method in formalization of various logical systems. The formal framework of the GND method and its applications can be found in the work of Goranko (*2020*).

It is worth adding that the method of semantic tables, which is fairly broadly applied in logic, does not differ significantly from the GND method and can be regarded as its variation. Moreover, the GND method, upon certain technical modifications and introduction of relations of forming decomposition trees, can be applied with the use of computers.

5. Concluding Remarks

The Lvov-Warsaw School was the most significant Polish philosophical formation in the twentieth century (see *Woleński 1989*). The members of the School were connected with a common method of practicing philosophy, but the interest in logic in its broad scope, understanding of it, was one of the more important attitudes represented by the founders of the LWS – Kazimierz Twardowski and his Lvov disciples. At least several of them undertook to conduct research in the area of logic. Sometime later, those disciples shaped the image and contributed to the success of Polish logic and its international recognition (see *McCall 1967*).

Logic turned out particularly attractive in the peculiar wing of the LWS, that is the Warsaw School of Logic. The founders of the WSL – Jan Łukasiewicz and Stanisław Leśniewski, as well as their famous disciple – Alfred Tarski – invested it with a style that brought about a specific manner of practicing logic. It was based on the theoretical assumption of the autonomous status of formal (mathematical) logic as a discipline which was not part of either philosophy or mathematics (although – as it is well-known – the establishment of the WSL was rooted in both philosophy and mathematics). That resulted in the implementation of a special program of practicing logic, applying special techniques and methods of proving theorems or refuting them, including completely new methods. They also contributed to obtaining significant and outstanding results, also in the global scale.

The aim of the present work was to discuss the proving methods applied in the WSL and more important results obtained with the use of these methods which count into syntactic methods.

Still, the WSL was also famed for using semantic methods and combining them with the syntactic ones. Obviously, issues related to this problem area require a separate study.

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