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Research Article

Impulsive Disturbances on the Dynamical Behavior of Complex-Valued Cohen-Grossberg Neural Networks with Both Time-Varying Delays and Continuously Distributed Delays

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This paper studies the global exponential stability for a class of impulsive disturbance complex-valued Cohen-Grossberg neural networks with both time-varying delays and continuously distributed delays. Firstly, the existence and uniqueness of the equilibrium point of the system are analyzed by using the corresponding property of M-matrix and the theorem of homeomorphism mapping. Secondly, the global exponential stability of the equilibrium point of the system is studied by applying the vector Lyapunov function method and the mathematical induction method. The established sufficient conditions show the effects of both delays and impulsive strength on the exponential convergence rate. The obtained results in this paper are with a lower level of conservatism in comparison with some existing ones. Finally, three numerical examples with simulation results are given to illustrate the correctness of the proposed results.

1. Introduction

Recently, several kinds of complex-valued neural networks have been proposed and attracted researchers' attention due to the broader range of their applications in electromagnetics, quantum waves, optoelectronics, filtering, speech synthesis, remote sensing, signal processing, and so on [1]. Complex-valued neural networks are quite different from real-valued neural networks, and they are not only the simple extension of real-valued systems due to two main aspects. Complex-valued neural networks have more complicated properties than real-valued neural networks because the neuron states, connection matrices, self-feedback functions, and activation functions are all defined in a complex number domain, which is the reason why existing research methods applied for studying the dynamical behavior of the real-valued neural networks cannot be used to study the complex-valued neural networks directly. Besides, complex-valued neural networks can solve some problems that cannot be solved with their real-valued counterparts. For example, the exclusion OR (XOR) problem and the detection of symmetry problem cannot be solved with a single complex-valued neuron with the orthogonal decision boundaries, which reveals the patent computational power of complex-valued neurons [1, 2].

As a very important issue, dynamical behavior analysis including existence, uniqueness, and stability of the equilibrium point of neural networks has attracted growing interest in the past decades (see [3–11] and the references therein). Different stability concepts as an important topic were defined in the existing references, such as exponential stability [2, 7, 12, 13], robust stability [14, 15], complete stability [16], multistability [17, 18], Lagrange stability [19, 20], and asymptotic stability [21].

Due to the finite switching speed of amplifiers, time delay inevitably exists in neural networks. It can cause oscillation and instability behavior of systems. As pointed out in [9], constant fixed time delays in the models of delayed feedback systems serve as a good approximation in simple circuits having a small number of cells. Although we consider that the time delays arise frequently in practical systems, it is difficult to measure them precisely. Up until now, there have been some results about the stability of complex-valued neural networks with time-varying delay (see [6, 7, 13, 19, 22, 23] and the references therein). It is well known that a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths; it is desired to model them by introducing continuously distributed delays over a certain duration of time such that the distant past has less influence compared with the recent behavior of the state [2]. Therefore, it is necessary and accepted to study a neural networks model with both time-varying delays and continuously distributed delays, such as in [2, 9, 10, 15, 24–26]. The existing results on the dynamical behavior analysis for the neural networks models with the above mixed delays were mainly with respect to real-valued neural networks. Xu et al. [9] considered a class of complex-valued Hopfield neural networks with mixed delays and obtained some sufficient conditions for ensuring the existence, uniqueness, and exponential stability of the equilibrium point of the system. Song et al. [2] investigated the stability problem for a class of impulsive complex-valued Hopfield neural networks with the above mixed delays.

The impulsive disturbances are also likely to exist in the system of neural networks and can affect the dynamical behaviors of the system states, the same as the time delays effect. For instance, in the implementation of electronic networks, the states of the neural networks are subject to instantaneous perturbations and experience abrupt changes at certain instants, which may be caused by the switching phenomenon, frequency change, or other sudden noises. This phenomenon of instantaneous perturbations to the system exhibits an impulsive effect [2, 12, 13, 27–29]. The authors in [10] investigated the stability problem for a class of impulsive complex-valued neural networks with both time-varying delays and distributed delays. By applying the vector Lyapunov function method and the mathematical induction method, some sufficient conditions were obtained for judging the exponential stability of the systems and showed the impulsive disturbance on the convergence rate of the system. In [29], the authors considered a class of fractional-order complex-valued neural networks with constant delay and impulsive disturbance. By using the contraction mapping principle, comparison theorem, and inequality scaling skills, some sufficient conditions were obtained for ensuring the existence, uniqueness, and global asymptotic stability of the equilibrium point of the system.

The model of Cohen-Grossberg neural networks was proposed by Cohen and Grossberg in 1983 [30]. It has been widely applied within various engineering and scientific fields such as neurobiology, population biology, and computing technology [31, 32]. Very recently, the authors in [21] considered a class of complex-valued Cohen-Grossberg neural networks with constant delay and studied the global asymptotic stability by separating the model into its real and imaginary parts. As pointed out in [2, 7], this required the existence, continuity, and boundedness of the partial derivatives of the activation functions about the real and imaginary parts of the state variables, which impose restrictions on the applications of the obtained results. Under the assumption that the activation functions only need to satisfy the Lipschitz condition, the authors in [32] removed the mentioned restrictions on the activation functions and studied a class of Cohen-Grossberg complex-valued neural networks with only time-varying delays. As far as we know, there is no result related to the dynamical behavior analysis for complex-valued Cohen-Grossberg neural networks with mixed delays and impulsive disturbances.

Based on the above analysis, in this paper, we will investigate the dynamical behavior for a class of impulsive complex-valued Cohen-Grossberg neural networks with time-varying delays and continuously distributed delays. In this paper, advantages and contributions can be listed as follows. (1) Both impulsive disturbances and mixed delays are considered in the complex-valued Cohen-Grossberg neural networks. The studied model is more universal than the existing ones. (2) The activation functions have not been separated into their real and imaginary parts, and the self-feedback functions are nonlinear functions. (3) Different from the study method in [7], the existence and uniqueness of the equilibrium point of the system are analyzed by using the corresponding property of M-matrix and the theorem of homeomorphism mapping. (4) The sufficient conditions established by the vector Lyapunov function method to ensure the global exponential stability of equilibrium point are expressed in terms of simple forms of inequalities, which are easy to be checked in practice.

2. Model Description and Preliminaries

To make reading easier, the following notations will be used throughout this paper. Let $\mathbf{C}$ denote a complex number set, $\mathbf{N}$ denote a natural number set, and $\mathbf{R}$ denote a real number set. Let $|z| = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}$ be the module of complex number, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real part and the imaginary part of the complex number z, respectively. For complex number vector $\mathbf{z} \in \mathbf{C}^n$, let $|\mathbf{z}| = (|z_1|, |z_2|, \ldots, |z_n|)^T$ be the module of the vector $\mathbf{z}$; here, $(\cdot)^T$ denotes the transpose of vector. Let $\|\mathbf{z}\|_\infty = \max_{1\le i\le n}\{|z_i|\}$ and $\|\mathbf{z}\|_1 = \sum_{i=1}^n |z_i|$ be the ∞-norm and 1-norm of the vector $\mathbf{z}$, respectively.

In this paper, we consider a class of impulsive disturbance complex-valued Cohen-Grossberg neural networks with time-varying delays and continuously distributed delays, which can be described by

$$\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} = h_i(z_i(t)) \left\{ -d_i(z_i(t)) + \sum_{j=1}^n \left[a_{ij} f_j(z_j(t)) \right.\right.$$
$$\left.\left. + b_{ij} f_j(z_j(t-\tau_{ij}(t))) \right.\right.$$

$$+ p_{ij} \int_{-\infty}^{t} \theta_{ij}(t-s) f_j(z_j(s)) \, ds \Big] + J_i(t) \Big\}$$

$$i = 1, 2, \ldots, n$$

$$\Delta z_i(t_k) = z_i(t_k^+) - z_i(t_k^-), \tag{1}$$

where $z_i \in \mathbf{C}$ represents the neuron state, n is the number of neurons, $\Delta z_i(t_k)$ denotes the impulsive jump at discrete moment t_k, the discrete set $\{t_k\}$ satisfies $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $t_k \to \infty$, $k \in N$. It is assumed that $z_i(t_k) = z_i(t_k^+)$ and $z_i(t_k^-) = \lim_{t \to t_k^-} z_i(t)$. $\mathbf{A} = (a_{ij})_{n\times n} \in \mathbf{C}^{n\times n}$, $\mathbf{B} = (b_{ij})_{n\times n} \in \mathbf{C}^{n\times n}$, and $\mathbf{P} = (p_{ij})_{n\times n} \in \mathbf{C}^{n\times n}$ are the connection weight matrices. $\mathbf{J}(t) = (J_1(t), J_2(t), \ldots, J_n(t))^T \in \mathbf{C}^n$ is an external input vector. $h_i(z_i(t))$, $d_i(z_i(t))$, and $f_i(z_i(t))$ represent the amplification function, the self-feedback function, and the activation function, respectively, where $i = 1, 2, \ldots, n$. $\tau_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ are bounded functions and $\tau = \max_{1\le i,j\le n} \sup_{t\ge 0} \tau_{ij}(t)$. Let $\theta_{ij} : [0, +\infty) \to [0, +\infty)$ be piecewise continuous functions that satisfy

$$\int_0^{+\infty} \exp(\beta s) \theta_{ij}(s) \, ds = \mu_{ij}(\beta), \quad i, j = 1, 2, \ldots, n, \tag{2}$$

where $\mu_{ij}(\beta)$ is continuous on $[0, \delta)$ and $\mu_{ij}(0) = 1$, $\delta > 0$.

It is assumed that initial conditions of (1) are $z_i(s) = \varphi_i(s)$; here, $\varphi_i(s)$ are bounded and continuous on $(-\infty, 0]$, $i = 1, 2, \ldots, n$.

Denote by $\mathbf{z}^\# = (z_1^\#, z_2^\#, \ldots, z_n^\#)^T \in \mathbf{C}^n$ the equilibrium point of system (1).

Definition 1. The equilibrium point $\mathbf{z}^\#$ of (1) is globally exponentially stable if there exist positive constants ε and λ such that the inequality $\|\mathbf{z}(t) - \mathbf{z}^\#\| \le \sup_{s\in(-\infty,0]} \|\boldsymbol{\varphi}(s) - \mathbf{z}^\#\| \varepsilon \exp(-\lambda t)$ holds for any external input $\mathbf{J} \in \mathbf{C}^n$ and $t \ge 0$.

Assumption 2. It is assumed that there exist positive numbers ω_i such that the inequalities $(d_i(u_i(t)) - d_i(v_i(t)))/(u_i(t) - v_i(t)) \ge \omega_i$ hold for all $u_i(t), v_i(t) \in \mathbf{C}$, $i = 1, 2, \ldots, n$.

Remark 3. The authors in [21, 32] have studied a class of complex-valued Cohen-Grossberg neural networks and obtained some important stability results. It is supposed that the self-feedback functions are linear functions in [21, 32]. That is to say, the self-feedback functions are supposed to be $d_i z_i(t)$ with $d_i > 0$, $i = 1, 2, \ldots, n$. Obviously, Assumption 2 in this paper is with more generality than that in the mentioned papers.

Assumption 4. Each function $f_i(\cdot)$ is globally Lipschitz with Lipschitz constant $l_i > 0$; that is, the inequality $|f_i(u_i(t)) - f_i(v_i(t))| \le l_i |u_i(t) - v_i(t)|$ holds for all $u_i(t), v_i(t) \in \mathbf{C}$, $i = 1, 2, \ldots, n$. Let $\mathbf{L} = \mathrm{diag}(l_1, l_2, \ldots, l_n)$.

Remark 5. The choice of the activation function is the main challenge in the dynamical behavior analysis of complex-valued neural networks compared to the study of real-valued neural networks. The complex-valued activation functions were supposed to need explicit separation into a real part and an imaginary part in [3, 5, 8–10, 18]. However, this separation is not always expressible in an analytical form. In this paper, the complex-valued activation functions only need to satisfy the Lipschitz condition.

Assumption 6. It is supposed that the amplification function $h_i(z_i(t))$ is with a lower boundary; that is to say, there exists a positive real number σ_i such that the inequality $h_i(z_i(t)) \ge \sigma_i > 0$ holds, $i = 1, 2, \ldots, n$.

Remark 7. The amplification functions of complex-valued Cohen-Grossberg neural networks were supposed to be with upper bounds and lower bounds in [21]. Besides, the authors assumed that the amplification functions needed to be separated into real parts and imaginary parts. Details can be found in Assumption 6 in [21]. In this paper, the complex-valued amplification functions only need to be with lower bounds.

Assumption 8. Let $\Delta z_i(t_k) = z_i(t_k^+) - z_i(t_k^-) = I_{ik}(z_i(t_k^-))$; here, $I_{ik}(\cdot)$ is a complex-valued continuous function. It is assumed that there exists $\eta_{ik} > 0$ such that the inequality $|z_i(t_i^-) + I_{ik}(z_i(t_k^-))| \le \eta_{ik} |z_i(t_k^-)|$ holds, $i = 1, 2, \ldots, n$, $k \in N$.
Let $\eta_k = \max\{1, \eta_{1k}, \eta_{2k}, \ldots, \eta_{nk}\}$, $k \in N$.

Remark 9. The impulsive effect is introduced into (1) as the disturbance to the system, which results in the negative impact on the convergence speed of the equilibrium point of the system. The low bound may not be discussed in this case. If the impulsive disturbance is so weak that $|z_i(t_k^+)| < |z_i(t_k^-)|$, then it will lead to a positive impact on the convergence speed of the equilibrium point of the system. This means the convergence rate of the equilibrium point of the system with impulsive disturbances will be faster than that of the system without them. Therefore, only the upper bound for the impulsive intensity is discussed in this paper.

To proceed with our results, we quote the following lemmas in the proof of the theorems in this paper.

Lemma 10 (see [10]). *Let $\mathbf{A} = (a_{ij})_{n\times n} \in R^{n\times n}$ be a matrix with $a_{ij} \le 0$ $(i, j = 1, 2, \ldots, n,\ i \ne j)$. The following statements are equivalent:*

(i) *$\mathbf{A} = (a_{ij})_{n\times n}$ is an M-matrix.*

(ii) *The real parts of all eigenvalues of $\mathbf{A}$ are positive.*

(iii) *There exists a positive vector $\xi \in \mathbf{R}^n$ such that $\mathbf{A}\xi > 0$.*

Lemma 11 (see [10]). *If $\mathbf{H}(\mathbf{z})$ is a continuous function on $\mathbf{C}^n$ and satisfies the following conditions:*

(i) *$\mathbf{H}(\mathbf{z})$ is univalent injective on $\mathbf{C}^n$,*

(ii) $\lim_{\|\mathbf{z}\|\to\infty} \|\mathbf{H}(\mathbf{z})\| \to \infty$,

then $\mathbf{H}(\mathbf{z})$ is a homeomorphism of $\mathbf{C}^n$ into itself.

3. Main Results

Theorem 12. *It is supposed that Assumptions 2~8 are satisfied. If there exist constants* $\lambda > 0$, $\eta > 0$ *and a series of positive constants* ξ_i *such that the following inequalities hold:*

$$\xi_i\left(-2\omega_i + \frac{\lambda}{\sigma_i}\right) + 2\sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \exp(0.5\lambda\tau)\left|b_{ij}\right| + \mu_{ij}(0.5\lambda)\left|p_{ij}\right|\right) < 0, \quad (3)$$

where $\eta = \lim_{k\to\infty}\sup(2\ln\eta_k/(t_k - t_{k-1}))$ *with* $\lambda > \eta$, $i = 1, 2, \ldots, n$, $k \in N$, *then for arbitrary input* $\mathbf{J} \in \mathbf{C}^n$, *(1) has a unique equilibrium point* $\mathbf{z}^{\#}$, *and* $\mathbf{z}^{\#}$ *is globally exponentially stable with convergence rate* $0.5(\lambda - \eta)$.

Proof. The proof of the theorem is separated into two steps.

Step 1. Firstly, the existence and uniqueness of the equilibrium point $\mathbf{z}^{\#}$ of (1) will be proved by using the corresponding properties of homeomorphism and M-matrix.

Define a map $\mathbf{H}(\mathbf{z}) = [H_1(\mathbf{z}), H_2(\mathbf{z}), \ldots, H_n(\mathbf{z})]^T$ associated with (1) with the following forms:

$$H_i(\mathbf{z}) = -d_i(z_i) + \sum_{j=1}^{n}\left(a_{ij} + b_{ij} + p_{ij}\right) f_j\left(z_j\right) + J_i, \quad i = 1, 2, \ldots, n. \quad (4)$$

It is well known that if $\mathbf{H}(\mathbf{z})$ is a homeomorphism on $\mathbf{C}^n$, then (1) has a unique equilibrium point $\mathbf{z}^{\#}$ obviously.

① We prove that the map $\mathbf{H}(\mathbf{z})$ is univalent injective on $\mathbf{C}^n$ under Assumptions 2 and 4.

From inequalities (3), it can be concluded that the following inequalities hold:

$$-\omega_i\xi_i + \sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \left|b_{ij}\right| + \left|p_{ij}\right|\right) < 0, \quad i = 1, 2, \ldots, n. \quad (5)$$

According to Lemma 10, we can obtain that the matrix $\mathbf{Q}$ is M-matrix, where $q_{ii} = \omega_i$, $i = j$; $q_{ij} = -\sum_{j=1}^{n} l_j(|a_{ij}| + |b_{ij}| + |p_{ij}|)$, $i \neq j$, $i = 1, 2, \ldots, n$.

Moreover, because inequalities (5) hold, we know that there exists a sufficient small positive number $\delta > 0$ such that the following inequalities hold:

$$\omega_i\xi_i - \sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \left|b_{ij}\right| + \left|p_{ij}\right|\right) \geq \delta > 0, \quad i = 1, 2, \ldots, n. \quad (6)$$

It is assumed that there exist $\mathbf{u}, \mathbf{v} \in \mathbf{C}^n$ with $\mathbf{u} \neq \mathbf{v}$, such that $H_i(\mathbf{u}) = H_i(\mathbf{v})$, $i = 1, 2, \ldots, n$; that is,

$$-d_i(u_i) + \sum_{j=1}^{n}\left(a_{ij} + b_{ij} + p_{ij}\right) f_j\left(u_j\right) + J_i = -d_i(v_i) + \sum_{j=1}^{n}\left(a_{ij} + b_{ij} + p_{ij}\right) f_j\left(v_j\right) + J_i. \quad (7)$$

Taking absolute value on both sides of (7), we get

$$\left|d_i(u_i) - d_i(v_i)\right| = \left|\sum_{j=1}^{n}\left(a_{ij} + b_{ij} + p_{ij}\right)\left[f_j\left(u_j\right) - f_j\left(v_j\right)\right]\right|, \quad i = 1, 2, \ldots, n. \quad (8)$$

Considering Assumptions 2 and 4, we get

$$\omega_i\left|u_i - v_i\right| \leq \sum_{j=1}^{n}\left(\left|a_{ij}\right| + \left|b_{ij}\right| + \left|p_{ij}\right|\right) l_j\left|u_j - v_j\right|, \quad i = 1, 2, \ldots, n. \quad (9)$$

Furthermore, inequalities (9) can be rewritten as follows:

$$\omega_i\left|u_i - v_i\right| - \sum_{j=1}^{n}\left(\left|a_{ij}\right| + \left|b_{ij}\right| + \left|p_{ij}\right|\right) l_j\left|u_j - v_j\right| \leq 0, \quad i = 1, 2, \ldots, n. \quad (10)$$

Obviously, $\mathbf{Q}|\mathbf{u}-\mathbf{v}| \leq 0$. Because $\mathbf{Q}$ is an M-matrix, we get $\det\mathbf{Q} > 0$ and $\mathbf{Q}^{-1}$ exists. Furthermore, it can be concluded that $|\mathbf{u} - \mathbf{v}| = 0$ (i.e., $\mathbf{u} = \mathbf{v}$). To sum up, the map $\mathbf{H}(\mathbf{z})$ is univalent injective on $\mathbf{C}^n$, $i = 1, 2, \ldots, n$.

② In what follows, we will prove that $\lim_{\|\mathbf{z}\|\to\infty}\|\mathbf{H}(\mathbf{z})\| \to \infty$.

Let $\widetilde{H}_i(\mathbf{z}) = H_i(\mathbf{z}) - H_i(0)$, where $H_i(0) = -d_i(0) + \sum_{j=1}^{n}(a_{ij} + b_{ij} + p_{ij}) f_j(0) + J_i$; that is,

$$\widetilde{H}_i(\mathbf{z}) = -\left[d_i(z_i) - d_i(0)\right] + \sum_{j=1}^{n}\left(a_{ij} + b_{ij} + p_{ij}\right)\left[f_j\left(z_j\right) - f_j(0)\right]. \quad (11)$$

Multiplying by the conjugate complex number of z_i on both sides of (11), we get

$$\widetilde{H}_i(\mathbf{z})\,\overline{z}_i = -\left[d_i(z_i) - d_i(0)\right]\overline{z}_i + \overline{z}_i\sum_{j=1}^{n}\left(a_{ij} + b_{ij} + p_{ij}\right)\left[f_j\left(z_j\right) - f_j(0)\right]. \quad (12)$$

Taking the conjugate operation on both sides of (12), we have

$$\overline{\widetilde{H}}_i(\mathbf{z})\, z_i = -\left[\overline{d}_i(z_i) - \overline{d}_i(0)\right] z_i + z_i\sum_{j=1}^{n}\left(\overline{a}_{ij} + \overline{b}_{ij} + \overline{p}_{ij}\right)\left[\overline{f}_j\left(z_j\right) - \overline{f}_j(0)\right]. \quad (13)$$

Taking the summation operation on both sides of (12) and (13) and considering Assumptions 2 and 4, we obtain

$$\begin{aligned}\operatorname{Re}\left[\widetilde{H}_i(\mathbf{z})\,\overline{z}_i\right] &= -\operatorname{Re}\left\{\left[d_i(z_i)-d_i(0)\right]\overline{z}_i\right\} \\ &\quad + \operatorname{Re}\left\{\overline{z}_i\sum_{j=1}^{n}\left(a_{ij}+b_{ij}+p_{ij}\right)\left[f_j(z_j)-f_j(0)\right]\right\} \\ &\le -\omega_i\left|z_i\right|^2+\left|\overline{z}_i\right|\sum_{j=1}^{n}l_j\left(\left|a_{ij}\right|+\left|b_{ij}\right|+\left|p_{ij}\right|\right)\left|z_j\right|, \\ &\qquad i=1,2,\ldots,n.\end{aligned} \tag{14}$$

Multiplying by ξ_i on both sides of (14), $i=1,2,\ldots,n$, we get

$$\begin{aligned}&\sum_{i=1}^{n}\xi_i\operatorname{Re}\left[\widetilde{H}_i(\mathbf{z})\,\overline{z}_i\right] \\ &\quad\le\sum_{i=1}^{n}\xi_i\left|z_i\right|\left[-\omega_i+\sum_{j=1}^{n}l_j\left(\left|a_{ij}\right|+\left|b_{ij}\right|+\left|p_{ij}\right|\right)\right]\left|z_j\right|.\end{aligned} \tag{15}$$

Considering inequalities (6), we have $\sum_{i=1}^{n}\xi_i\operatorname{Re}[\widetilde{H}_i(\mathbf{z})\overline{z}_i]\le-\delta\sum_{i=1}^{n}|z_i|\cdot\sum_{j=1}^{n}|z_j|$. Furthermore, it can be concluded that

$$\begin{aligned}\sum_{i=1}^{n}\left|z_i\right|\cdot\sum_{j=1}^{n}\left|z_j\right| &\le -\sum_{i=1}^{n}\xi_i\operatorname{Re}\left[\widetilde{H}_i(\mathbf{z})\,\overline{z}_i\right] \\ &\le \max_{1\le i\le n}\{\xi_i\}\sum_{i=1}^{n}\left|\widetilde{H}_i(\mathbf{z})\right|\sum_{i=1}^{n}\left|z_i\right|.\end{aligned} \tag{16}$$

Namely, $\delta\|\mathbf{z}\|_1\le\max_{1\le i\le n}\{\xi_i\}\|\widetilde{H}(\mathbf{z})\|_1$. That is to say, $\|\mathbf{z}\|_1\le\delta^{-1}\max_{1\le i\le n}\{\xi_i\}\|\widetilde{H}(\mathbf{z})\|_1$ holds. Obviously, we have $\|\widetilde{H}(\mathbf{z})\|\to\infty$ as $\|\mathbf{z}\|\to\infty$. This means that $\|H(\mathbf{z})\|\to\infty$ as $\|\mathbf{z}\|\to\infty$.

Combining ① and ② above, we know that $\mathbf{H}(\mathbf{z})$ is a homeomorphism on $\mathbf{C}^n$. So, (1) has a unique equilibrium point $\mathbf{z}^{\#}$.

Step 2. In this section, the global exponential stability of the equilibrium point $\mathbf{z}^{\#}$ under impulsive disturbances will be proved by applying the vector Lyapunov function method and the mathematical induction method.

For analysis convenience, we translate the coordinate of (1). Let $\widetilde{\mathbf{z}}(t)=\mathbf{z}(t)-\mathbf{z}^{\#}$. By translation, we change (1) into the following forms:

$$\begin{aligned}\frac{\mathrm{d}\widetilde{z}_i(t)}{\mathrm{d}t} &= \widetilde{h}_i(\widetilde{z}_i(t))\left\{-r_i(\widetilde{z}_i(t))+\sum_{j=1}^{n}\left[a_{ij}g_j(\widetilde{z}_j(t))\right.\right. \\ &\quad + b_{ij}g_j\left(\widetilde{z}_j\left(t-\tau_{ij}(t)\right)\right) \\ &\quad \left.\left.+ p_{ij}\int_{-\infty}^{t}\theta_{ij}(t-s)\,g_j(\widetilde{z}_j(s))\,\mathrm{d}s\right]\right\} \\ \Delta\widetilde{z}_i(t_k) &= \widetilde{z}_i(t_k^+)-\widetilde{z}_i(t_k^-),\end{aligned} \tag{17}$$

where $\widetilde{h}_i(\widetilde{z}_i(t))=h_i(\widetilde{z}_i(t)+z_i^{\#})$, $r_i(\widetilde{z}_i(t))=d_i(\widetilde{z}_i(t)+z_i^{\#})-d_i(z_i^{\#})$, $g_j(\widetilde{z}_j(t))=f_j(\widetilde{z}_j(t)+z_j^{\#})-f_j(z_j^{\#})$, $i,j=1,2,\ldots,n$.

Let the initial conditions of (17) be with the forms $\psi_i(s)=\varphi_i(s)-z_i^{\#}$, $i=1,2,\ldots,n$, $-\infty<s\le0$.

Obviously, if the zero solution of (17) is globally exponentially stable, the equilibrium point of (1) is also globally exponentially stable.

Choose the vector Lyapunov function as follows:

$$\begin{aligned}V_i(\widetilde{z}_i(t),t) &= \frac{1}{2}\exp(\lambda t)\left|\widetilde{z}_i(t)\right|^2 \\ &= \frac{1}{2}\exp(\lambda t)\,\overline{\widetilde{z}}_i(t)\,\widetilde{z}_i(t),\quad i=1,2,\ldots,n.\end{aligned} \tag{18}$$

Let $V_i(\widetilde{z}_i(t),t)$ be $V_i(t)$ if there is no confusion, $i=1,2,\ldots,n$.

When $0<t<t_1$, calculating the upper right derivative of $V_i(t)$ along (17) and considering Assumptions 2~6, we have

$$\begin{aligned}D^+V_i(t) &= \frac{1}{2}\exp(\lambda t)\left[\lambda\left|\widetilde{z}_i(t)\right|^2+2\overline{\widetilde{z}}_i(t)\,\dot{\widetilde{z}}_i(t)\right]\le\frac{1}{2}\lambda \\ &\quad\cdot\exp(\lambda t)\left|\widetilde{z}_i(t)\right|^2+\exp(\lambda t)\,\overline{\widetilde{z}}_i(t)\,\widetilde{h}_i(\widetilde{z}_i(t)) \\ &\quad\cdot\left\{-r_i(\widetilde{z}_i(t))+\sum_{j=1}^{n}\left[a_{ij}g_j(\widetilde{z}_j(t))\right.\right. \\ &\quad+b_{ij}g_j\left(\widetilde{z}_j\left(t-\tau_{ij}(t)\right)\right) \\ &\quad\left.\left.+p_{ij}\int_{-\infty}^{t}\theta_{ij}(t-s)\,g_j(\widetilde{z}_j(s))\,\mathrm{d}s\right]\right\}\le\frac{1}{2}\lambda\exp(\lambda t) \\ &\quad\cdot\left|\widetilde{z}_i(t)\right|^2+\exp(\lambda t)\,\widetilde{h}_i(\widetilde{z}_i(t))\left\{-\omega_i\left|\widetilde{z}_i(t)\right|^2\right. \\ &\quad+\left|\overline{\widetilde{z}}_i(t)\right|\sum_{j=1}^{n}l_j\left[\left|a_{ij}\right|\left|\widetilde{z}_j(t)\right|+\left|b_{ij}\right|\left|\widetilde{z}_j\left(t-\tau_{ij}(t)\right)\right|\right. \\ &\quad\left.\left.+\left|p_{ij}\right|\int_{-\infty}^{t}\theta_{ij}(t-s)\left|\widetilde{z}_j(s)\right|\mathrm{d}s\right]\right\}\le\frac{\sqrt{2}}{2} \\ &\quad\cdot\exp(0.5\lambda t)\left|\widetilde{z}_i(t)\right|\widetilde{h}_i(\widetilde{z}_i(t))\left\{\frac{\sqrt{2}}{2}\exp(0.5\lambda t)\left(\frac{\lambda}{\sigma_i}\right.\right.\end{aligned}$$

$$- 2\omega_i\Big) \left|\tilde{z}_i(t)\right| + \sqrt{2}\exp(0.5\lambda t) \times \sum_{j=1}^{n} l_j \Big[\left|a_{ij}\right| \left|\tilde{z}_j(t)\right| + \left|b_{ij}\right| \left|\tilde{z}_j\left(t-\tau_{ij}(t)\right)\right| + \left|p_{ij}\right| \cdot \int_{-\infty}^{t} \theta_{ij}(t-s) \left|\tilde{z}_j(s)\right| \mathrm{d}s \Big] \Big\} \le \sqrt{V_i(t)}\tilde{h}_i(\tilde{z}_i(t)) \cdot \Big\{ \left(\frac{\lambda}{\sigma_i} - 2\omega_i\right)\sqrt{V_i(t)} + 2\sum_{j=1}^{n} l_j \Big[\left|a_{ij}\right| \sqrt{V_j(t)} + \exp(0.5\lambda\tau)\left|b_{ij}\right| \times \sqrt{V_j\left(t-\tau_{ij}(t)\right)} + \left|p_{ij}\right| \cdot \int_{-\infty}^{t} \theta_{ij}(t-s)\exp(0.5\lambda(t-s))\sqrt{V_j(s)}\mathrm{d}s \Big] \Big\}. \tag{19}$$

Let $U_i(t) = \sqrt{V_i(t)} = (\sqrt{2}/2)\exp(0.5\lambda t)|\tilde{z}_i(t)|$, $i = 1, 2, \dots, n$. It is easy to obtain $D^+V_i(t) = 2U_i(t)D^+U_i(t)$, $i = 1, 2, \dots, n$. Substituting them into inequalities (19), we obtain

$$D^+U_i(t) \le 0.5\tilde{h}_i(\tilde{z}_i(t)) \Big\{ \left(\frac{\lambda}{\sigma_i} - 2\omega_i\right) U_i(t) + 2\sum_{j=1}^{n} l_j \Big[\left|a_{ij}\right| U_j(t) + \exp(0.5\lambda\tau)\left|b_{ij}\right| \times U_j\left(t-\tau_{ij}(t)\right) + \left|p_{ij}\right| \cdot \int_{-\infty}^{t} \theta_{ij}(t-s)\exp(0.5\lambda(t-s)) U_j(s)\,\mathrm{d}s \Big] \Big\}. \tag{20}$$

Define the curve $\boldsymbol{\zeta} = \{\gamma(\chi) : \gamma_i = \xi_i\chi,\ \chi > 0,\ i = 1, 2, \dots, n\}$ and the set $\boldsymbol{\Omega}(\gamma) = \{y : 0 \le y \le \gamma,\ \gamma \in \zeta\}$. When $\chi > \chi'$, it is obvious that $\boldsymbol{\Omega}(\gamma(\chi)) \supset \boldsymbol{\Omega}(\gamma(\chi'))$. Let $\xi_{\max} = \max_{1\le i\le n}\{\xi_i\}$, $\xi_{\min} = \min_{1\le i\le n}\{\xi_i\}$, $\chi_0 = \varepsilon\|\boldsymbol{\psi}\|^2/\xi_{\min}$, where $\varepsilon > 1$ is a constant. Then, $\{U : U = (\sqrt{2}/2)\exp(0.5\lambda s)|\boldsymbol{\psi}(s)|^2,\ -\infty < s \le 0\} \subset \boldsymbol{\Omega}(\gamma_0(\chi_0))$; that is,

$$U_i(s) = \frac{\sqrt{2}}{2}\exp(0.5\lambda s)\left|\psi_i(s)\right|^2 < \xi_i\chi_0, \quad -\infty < s \le 0,\ i = 1, 2, \dots, n. \tag{21}$$

Furthermore, we can claim that $U_i(t) < \xi_i\chi_0$, $i = 1, 2, \dots, n$, $0 < t < t_1$. If it is not true, then there exist some $m \in \{1, 2, \dots, n\}$ and t^* $(0 < t^* < t_1)$ such that $U_m(t^*) = \xi_m\chi_0$, $D^+U_m(t^*) \ge 0$, $U_j(t^*) \le \xi_j\chi_0$, $j = 1, 2, \dots, n$. Substituting them into inequalities (20) and considering inequalities (3), we get

$$D^+U_m(t^*) \le 0.5\tilde{h}_m(\tilde{z}_m(t^*)) \Big\{ \left(\frac{\lambda}{\sigma_m} - 2\omega_m\right)\xi_m\chi_0 + 2\sum_{j=1}^{n} \xi_j\chi_0 l_j \Big[\left|a_{mj}\right| + \exp(0.5\lambda\tau)\left|b_{mj}\right| + \left|p_{mj}\right| \mu_{mj}(0.5\lambda) \Big] \Big\} < 0. \tag{22}$$

This is a contradiction with the above assumption $D^+U_m(t^*) \ge 0$. So, we have $U_i(t) < \xi_i\chi_0$, $i = 1, 2, \dots, n$. That is to say, $|\tilde{z}_i(t)| < \sqrt{2}\exp(-0.5\lambda t)\xi_i\chi_0$, $i = 1, 2, \dots, n$, $0 < t < t_1$.

Next, the mathematical induction method will be applied to prove that the following inequalities hold:

$$\left|\tilde{z}_i(t)\right| < \sqrt{2}\exp(-0.5\lambda t)\eta_0\eta_1\eta_2\cdots\eta_{k-1}\xi_i\chi_0, \quad i = 1, 2, \dots, n,\ t_{k-1} \le t < t_k,\ k \in N. \tag{23}$$

When $k = 1$, $|\tilde{z}_i(t)| < \sqrt{2}\eta_0\exp(-0.5\lambda t)\xi_i\chi_0$, $i = 1, 2, \dots, n$, $t_0 \le t < t_1$; here, $\eta_0 = 1$. According to the preceding analysis, the inequalities are satisfied obviously.

It is assumed that the following inequalities hold:

$$\left|\tilde{z}_i(t)\right| < \sqrt{2}\exp(-0.5\lambda t)\eta_0\eta_1\eta_2\cdots\eta_{w-1}\xi_i\chi_0, \quad i = 1, 2, \dots, n,\ t_{w-1} \le t < t_w,\ w = 1, 2, \dots, k. \tag{24}$$

When $t = t_w$, according to Assumption 8, we get

$$\left|\tilde{z}_i(t_w^+)\right| = \left|\tilde{z}_i(t_w^-) + I_{iw}(\tilde{z}_i(t_w^-))\right| \le \eta_{iw}\left|\tilde{z}_i(t_w^-)\right| \le \eta_w\left|\tilde{z}_i(t_w^-)\right|, \quad i = 1, 2, \dots, n,\ w = 1, 2, \dots, k. \tag{25}$$

Due to $\eta_w \ge 1$, inequalities (24) can be changed into the following forms:

$$\left|\tilde{z}_i(t)\right| < \sqrt{2}\exp(-0.5\lambda t)\eta_0\eta_1\eta_2\cdots\eta_{w-1}\eta_w\xi_i\chi_0, \quad i = 1, 2, \dots, n,\ t_w - \tau \le t \le t_w,\ w \in N. \tag{26}$$

Furthermore, we can conclude that the following inequalities hold:

$$\left|\tilde{z}_i(t)\right| < \sqrt{2}\exp(-0.5\lambda t)\eta_0\eta_1\eta_2\cdots\eta_{w-1}\eta_w\xi_i\chi_0, \quad i = 1, 2, \dots, n,\ t_w \le t < t_{w+1},\ w \in N. \tag{27}$$

If inequalities (27) are not true, there exist some m' and t' such that $D^+U_{m'}(t') \ge 0$ and

$$\begin{aligned} \left|\tilde{z}_{m'}(t')\right| &= \sqrt{2}\exp(-0.5\lambda t')\eta_0\eta_1\eta_2\cdots\eta_{w-1}\eta_w\xi_{m'}\chi_0, \quad t_w \le t' < t_{w+1} \\ \left|\tilde{z}_j(t')\right| &\le \sqrt{2}\exp(-0.5\lambda t')\eta_0\eta_1\eta_2\cdots\eta_{w-1}\eta_w\xi_j\chi_0, \quad t_w - \tau < t \le t',\ j = 1, 2, \dots, n. \end{aligned} \tag{28}$$

Substituting (28) into inequalities (20) and considering inequalities (3), we have

$$D^+U_{m'}\left(t'\right) \le 0.5\widetilde{h}_{m'}\left(\widetilde{z}_{m'}(t)\right)\left\{\left(\frac{\lambda}{\sigma_{m'}} - 2\omega_{m'}\right)\xi_{m'} + 2\sum_{j=1}^{n} l_j\xi_j\left[\left|a_{m'j}\right| + \exp(0.5\lambda\tau)\left|b_{m'j}\right| + \left|p_{m'j}\right|\mu_{m'j}(0.5\lambda)\right]\right\}\eta_0\eta_1\eta_2\cdots\eta_{w-1}\eta_w\chi_0 < 0. \quad (29)$$

This is a contradiction with the assumption $D^+U_{m'}(t') \ge 0$. So, inequalities (24) hold.

Based on the idea of the mathematical induction method, we have

$$\left|\widetilde{z}_i(t)\right| < \sqrt{2}\exp(-0.5\lambda t)\,\eta_0\eta_1\eta_2\cdots\eta_{k-1}\xi_i\chi_0, \quad i = 1,2,\ldots,n,\ t_{k-1} \le t < t_k,\ k \in N. \quad (30)$$

It follows from the condition of the theorem $\eta = \lim_{k\to\infty}\sup(2\ln\eta_k/(t_k - t_{k-1}))$ that $\eta_k \le \exp(0.5\eta(t_k - t_{k-1}))$, $k \in N$. Substituting it into inequalities (30), we obtain

$$\begin{aligned}\left|\widetilde{z}_i(t)\right| &< \sqrt{2}\exp\left(0.5\eta\left(t_1 - t_0\right)\right) \cdot \exp\left(0.5\eta\left(t_2 - t_1\right)\right)\cdots\exp\left(0.5\eta\left(t_{k-1} - t_{k-2}\right)\right)\xi_i\chi_0 \\ &\cdot \exp(-0.5\lambda t) < \sqrt{2}\xi_i\chi_0 \cdot \exp\left(-0.5\left(\lambda - \eta\right)\left(t - t_0\right)\right), \\ &t_{k-1} \le t < t_k,\ k \in N.\end{aligned} \quad (31)$$

Furthermore, we have

$$\begin{aligned}\|\widetilde{\mathbf{z}}(t)\| &< \frac{\sqrt{2}\varepsilon\|\psi\|^2\xi_{\max}}{\xi_{\min}\exp\left(-0.5\left(\lambda-\eta\right)\left(t-t_0\right)\right)} \\ &= \Gamma\|\psi\|\exp\left(-0.5\left(\lambda-\eta\right)\left(t-t_0\right)\right),\end{aligned} \quad (32)$$

where $\Gamma = \sqrt{2}\varepsilon\xi_{\max}/\xi_{\min}$.

According to Definition 1, the zero solution of system (17) is globally exponentially stable. That is to say, the equilibrium point $\mathbf{z}^{\#}$ of system (1) is also globally exponentially stable.

To sum up, it can be concluded from Steps 1 and 2 that system (1) has a unique equilibrium point $\mathbf{z}^{\#}$, and the equilibrium point is globally exponentially stable with exponential converge rate $0.5(\lambda - \eta)$. The proof is completed. □

Remark 13. Although there have been various methods for studying the diverse complex-valued neural networks, the scalar Lyapunov function method combined with the LMI method is nearly the most popular method to research the stability problem and synchronization problem (see [3, 4, 11, 13, 20, 29, 33]). The continuously distributed delays were not considered in the mentioned references. Mixed time delays in the model of complex-valued neural networks make the system become an infinite-dimensional interconnected system. Using the vector Lyapunov function method used in this paper can avoid discussing the convergence of the candidate scalar Lyapunov function, which is extremely difficult to prove in most cases.

From Theorem 12, we can directly obtain corresponding corollaries for guaranteeing the existence, uniqueness, and global exponential stability of the equilibrium point of system (33) and system (35) described as follows.

If there are no continuously distributed delays in (1), that is, when $\mathbf{P} = 0$, the corresponding system is with the following forms:

$$\begin{aligned}\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} &= h_i\left(z_i(t)\right)\left\{-d_i\left(z_i(t)\right) + \sum_{j=1}^{n}\left[a_{ij}f_j\left(z_j(t)\right) + b_{ij}f_j\left(z_j\left(t-\tau_{ij}(t)\right)\right)\right] + J_i(t)\right\} \\ \Delta z_i\left(t_k\right) &= z_i\left(t_k^+\right) - z_i\left(t_k^-\right).\end{aligned} \quad (33)$$

Corollary 14. *Suppose that Assumptions 2~8 are satisfied. If there exist positive constants $\lambda > \eta > 0$ and a series of positive constants ξ_i such that the following inequalities hold, here $\eta = \lim_{k\to\infty}\sup(2\ln\eta_k/(t_k - t_{k-1}))$, $k \in N$,*

$$\xi_i\left(-2\omega_i + \frac{\lambda}{\sigma_i}\right) + 2\sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \exp(0.5\lambda\tau)\left|b_{ij}\right|\right) < 0, \quad i = 1,2,\ldots,n. \quad (34)$$

Then, (33) has a unique equilibrium point $\mathbf{z}^{\#}$ for arbitrary external input $\mathbf{J} \in \mathbf{C}^n$, and the equilibrium point $\mathbf{z}^{\#}$ is globally exponentially stable with the convergence rate $0.5(\lambda - \eta)$.

Similarly, if there are no time-varying delays in (1), that is, when $\mathbf{B} = 0$, the corresponding system is with the following forms:

$$\begin{aligned}\frac{\mathrm{d}z_i(t)}{\mathrm{d}t} &= h_i\left(z_i(t)\right)\left\{-d_i\left(z_i(t)\right) + \sum_{j=1}^{n}\left[a_{ij}f_j\left(z_j(t)\right) + p_{ij}\int_{-\infty}^{t}\theta_{ij}(t-s)f_j\left(z_j(s)\right)\mathrm{d}s\right] + J_i(t)\right\} \\ \Delta z_i\left(t_k\right) &= z_i\left(t_k^+\right) - z_i\left(t_k^-\right).\end{aligned} \quad (35)$$

Corollary 15. *Suppose that Assumptions 2~8 are satisfied. If there exist positive constants $\lambda > \eta > 0$ and a series of positive* *constants ξ_i such that the following inequalities hold, here* $\eta = \lim_{k\to\infty}\sup(2\ln\eta_k/(t_k - t_{k-1}))$, $k \in N$,

$$\xi_i\left(-2\omega_i + \frac{\lambda}{\sigma_i}\right) + 2\sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \mu_{ij}(0.5\lambda)\left|p_{ij}\right|\right) < 0, \quad i = 1, 2, \ldots, n. \tag{36}$$

Then, (35) has a unique equilibrium point $\mathbf{z}^{\#}$ *for arbitrary external input* $\mathbf{J} \in \mathbf{C}^n$, *and the equilibrium point* $\mathbf{z}^{\#}$ *is globally exponentially stable with the convergence rate* $0.5(\lambda - \eta)$.

When $h_i(z_i(t)) = 1$ in system (1), model (1) is changed into impulsive complex-valued Hopfield neural networks with time-varying delays and continuously distributed delays, which can be described as follows:

$$\begin{aligned} \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} &= -d_i(z_i(t)) + \sum_{j=1}^{n}\Big[a_{ij}f_j(z_j(t)) \\ &\quad + b_{ij}f_j(z_j(t-\tau_{ij}(t))) \\ &\quad + p_{ij}\int_{-\infty}^{t}\theta_{ij}(t-s)f_j(z_j(s))\,\mathrm{d}s\Big] + J_i(t) \\ \Delta z_i(t_k) &= z_i(t_k^+) - z_i(t_k^-). \end{aligned} \tag{37}$$

It is easy to obtain sufficient conditions for ensuring the existence, uniqueness, and global exponential stability of the equilibrium point of system (37).

Corollary 16. *Suppose that Assumptions 2, 4, and 8 are satisfied. If there exist positive constants $\lambda > \eta > 0$ and a series of positive constants ξ_i such that the following inequalities hold, here* $\eta = \lim_{k\to\infty}\sup(2\ln\eta_k/(t_k - t_{k-1}))$, $k \in N$,

$$\begin{aligned} &\xi_i(-2\omega_i + \lambda) \\ &+ 2\sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \exp(0.5\lambda\tau)\left|b_{ij}\right| + \mu_{ij}(0.5\lambda)\left|p_{ij}\right|\right) \\ &< 0, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{38}$$

Then, (35) has a unique equilibrium point $\mathbf{z}^{\#}$ *for arbitrary external input* $\mathbf{J} \in \mathbf{C}^n$, *and the equilibrium point* $\mathbf{z}^{\#}$ *is globally exponentially stable with the convergence rate* $0.5(\lambda - \eta)$.

The same as the preceding analysis, we can easily obtain the corresponding criteria for guaranteeing the stability of the equilibrium point of impulsive complex-valued Hopfield neural networks with only time-varying delays or continuously distributed delays. Therefore, we omit the similar works here.

When there is no impulsive disturbance in model (1), the complex-valued Cohen-Grossberg neural networks with time-varying delays and continuously distributed delays can be described as follows:

$$\begin{aligned} \frac{\mathrm{d}z_i(t)}{\mathrm{d}t} &= h_i(z_i(t))\Big\{-d_i(z_i(t)) + \sum_{j=1}^{n}\Big[a_{ij}f_j(z_j(t)) \\ &\quad + b_{ij}f_j(z_j(t-\tau_{ij}(t))) \\ &\quad + p_{ij}\int_{-\infty}^{t}\theta_{ij}(t-s)f_j(z_j(s))\,\mathrm{d}s\Big] + J_i(t)\Big\}, \\ &\qquad i = 1, 2, \ldots, n. \end{aligned} \tag{39}$$

All variables and functions in model (39) are the same as in system (1). Next, we will establish some sufficient conditions for judging the dynamical behavior of the equilibrium point $\mathbf{z}^{\#}$ of system (39).

Theorem 17. *It is supposed that Assumptions 2~6 are satisfied. If the matrix $\mathbf{Q}$ is an M-matrix, where $q_{ii} = \omega_i$, $i = j$; $q_{ij} = -\sum_{j=1}^{n} l_j(|a_{ij}|+|b_{ij}|+|p_{ij}|)$, $i \neq j$, $i, j = 1, 2, \ldots, n$, then system (39) has a unique equilibrium point $\mathbf{z}^{\#}$ for arbitrary external input $\mathbf{J} \in \mathbf{C}^n$, which is globally exponentially stable.*

Proof. According to Step 1 in the analysis of Theorem 12, we can directly conclude that system (39) has a unique equilibrium point $\mathbf{z}^{\#}$. In what follows, we will prove that the equilibrium point $\mathbf{z}^{\#}$ is globally exponentially stable. Because matrix $\mathbf{Q}$ is an M-matrix, it follows from Lemma 10 that there exists a vector $\boldsymbol{\xi} \in R^n > 0$ such that the following inequalities hold:

$$-\xi_i\omega_i + \sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \left|b_{ij}\right| + \left|p_{ij}\right|\right) < 0, \quad i = 1, 2, \ldots, n. \tag{40}$$

We construct functions as follows:

$$\begin{aligned} F_i(\beta) &= \xi_i\left(-2\omega_i + \frac{\beta}{\sigma_i}\right) + 2\sum_{j=1}^{n}\xi_j \\ &\quad \cdot l_j\left(\left|a_{ij}\right| + \exp(0.5\beta\tau)\left|b_{ij}\right| + \mu_{ij}(0.5\beta)\left|p_{ij}\right|\right), \\ &\qquad i = 1, 2, \ldots, n. \end{aligned} \tag{41}$$

From inequalities (40), it is easy to get

$$F_i(0) = -\xi_i\omega_i + \sum_{j=1}^{n}\xi_j \cdot l_j\left(\left|a_{ij}\right| + \left|b_{ij}\right| + \left|p_{ij}\right|\right) < 0, \quad i = 1, 2, \ldots, n. \tag{42}$$

Because $F_i(\cdot)$ is a continuous function, there exists a constant $\lambda > 0$ such that $F_i(\lambda) < 0$, $i = 1, 2, \ldots, n$. That is to say, the following inequalities hold:

$$F_i(\lambda) = \xi_i \left(-2\omega_i + \frac{\lambda}{\sigma_i} \right) + 2\sum_{j=1}^{n} \xi_j \cdot l_j \left(\left| a_{ij} \right| + \exp(0.5\lambda\tau) \left| b_{ij} \right| + \mu_{ij}(0.5\lambda) \left| p_{ij} \right| \right) < 0. \quad (43)$$

We choose the vector Lyapunov function as follows:

$$V_i(\tilde{z}_i(t), t) = \frac{1}{2} \exp(\lambda t) \left| \tilde{z}_i(t) \right|^2 = \frac{1}{2} \exp(\lambda t) \overline{\tilde{z}}_i(t) \tilde{z}_i(t), \quad i = 1, 2, \ldots, n. \quad (44)$$

According to the proof of Step 2 in Theorem 12, we can conclude that the equilibrium point of system (39) is globally exponentially stable, and the convergence rate is 0.5λ. The proof is completed. □

When there are only time-varying delays or continuously distributed delays in system (39), it is easy to obtain the corresponding criteria for guaranteeing the global exponential stability of the equilibrium point of the complex-valued Cohen-Grossberg neural networks. We omit the similar works here.

Remark 18. Separating the model of complex-valued neural networks into its real and imaginary parts is a routine method (e.g., [3, 8–10, 18]). The complex-valued activation functions were supposed to be with existence, continuity, and boundedness of the partial derivatives of the activation functions about the real and imaginary parts of the state variables, which impose restrictions on the applications of the obtained results. Assumptions 1 and 2 in [34] deviate from the boundedness and differentiability assumption for activation functions. In future works, we will furthermore study the dynamical behavior of complex-valued neural networks with a lower level of conservation of assumption conditions, including activation function, self-feedback function, and amplification function.

4. Numerical Examples

In this section, we will give three examples with numerical simulations to demonstrate the correctness of the above results.

Example 1. Consider a class of two-order system described by (1) with the following assumptions.

The interconnected matrices are given as $\mathbf{A} = \begin{bmatrix} 2.5-2i & 3+1.5i \\ -2-3i & -4i \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1.7+i & -1.0+1i \\ 1.5-0.8i & -2 \end{bmatrix}$, and $\mathbf{P} = \begin{bmatrix} 1 & 1.5i \\ 1-i & -1 \end{bmatrix}$, respectively, where i denotes the imaginary unit. Let $h_1(z_1(t)) = 2 + \sin(|z_1(t)|)$, $h_2(z_2(t)) = 3 + \cos(|z_2(t)|)$; $d_1(z_1(t)) = 15z_1(t)$, $d_2(z_2(t)) = 14z_2(t)$; $f_1(z_1(t)) = 0.5(|x_1| + i|y_1|)$, $f_2(z_2(t)) = (|x_2| + i|y_2|)$, where $z_i = x_i + iy_i$, $i = 1, 2$. It is supposed that $z_1(t_k^+) = 1.4z_1(t_k^-)$ and $z_2(t_k^+) = 1.2z_2(t_k^-)$, where $t_k \in \{3s, 6s, 9s, \ldots\}$. Let $\lambda = 8$; $\boldsymbol{\xi} = [1, 1]^T$; $\theta_{ij}(t - s) = \exp(-5(t - s))$, $i, j = 1, 2$. Let $\tau_{1j} = 0.02 + 0.01 \sin t$, $\tau_{2j} = 0.03 - 0.01 \cos t$, $j = 1, 2$, $t \geq 0$. It is assumed that the initial conditions are $z_1(s) = 1 - 0.9i$, $z_2(s) = -0.6 + 0.8i$, $s \in (-\infty, 0]$. Let external inputs be $J_1(t) = 1 + i$, $J_2(t) = 5 + 5i$.

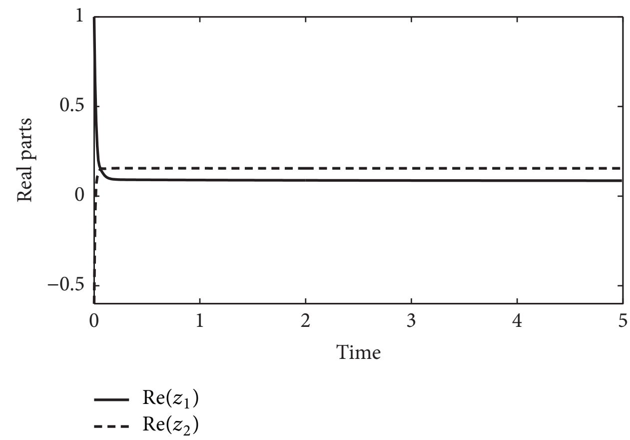

Figure 1: State curves of Re(z_1) and Re(z_2) without impulsive disturbances.

By calculation, we obtain $\omega_1 = 15$, $\omega_2 = 14$, $\sigma_1 = 1$, $\sigma_2 = 2$, $l_1 = 0.5$, $l_2 = 1$, $\eta = 0.22$, $\tau = 0.04$. Furthermore, we get

$$\begin{aligned} &\xi_1 \left(-2\omega_1 + \frac{\lambda}{\sigma_1} \right) + 2\sum_{j=1}^{n} \xi_j \cdot l_j \left(\left| a_{1j} \right| + \exp(0.5\lambda\tau) \left| b_{1j} \right| + \mu_{1j}(0.5\lambda) \left| p_{1j} \right| \right) = -2.21 < 0 \\ &\xi_2 \left(-2\omega_2 + \frac{\lambda}{\sigma_2} \right) + 2\sum_{j=1}^{n} \xi_j \cdot l_j \left(\left| a_{2j} \right| + \exp(0.5\lambda\tau) \left| b_{2j} \right| + \mu_{2j}(0.5\lambda) \left| p_{2j} \right| \right) = -2.31 < 0. \end{aligned} \quad (45)$$

From the above computing analysis, the assumption conditions in Theorem 12 are satisfied. According to Theorem 12, it can be concluded that the equilibrium point of (1) under the above assumptions is existent, unique, and globally exponentially stable, and the exponential convergence rate is 3.89.

The numerical simulations of the above system are shown in Figures 1–4. Figures 1 and 2 show the state curves of the real parts and imaginary parts of neuron states, respectively. Figure 3 shows the modulus of state curves under the condition that there is no impulsive disturbance in the system. Figure 4 shows the modulus of state curves under the condition that there exist impulsive disturbances in the system. From the simulation results, it can be seen that the equilibrium point of the system is existent, unique, and stable.

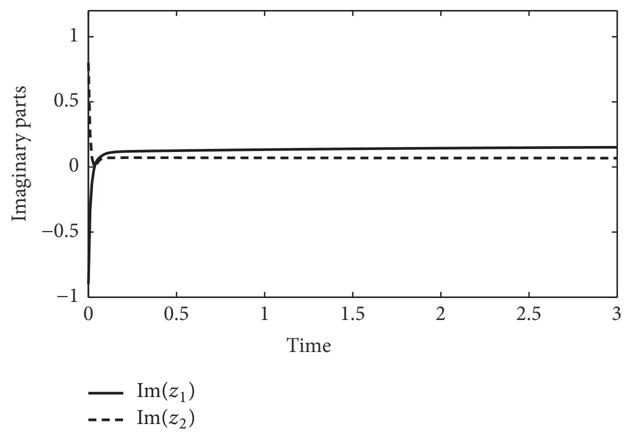

Figure 2: State curves of Im(z_1) and Im(z_2) with impulsive disturbances.

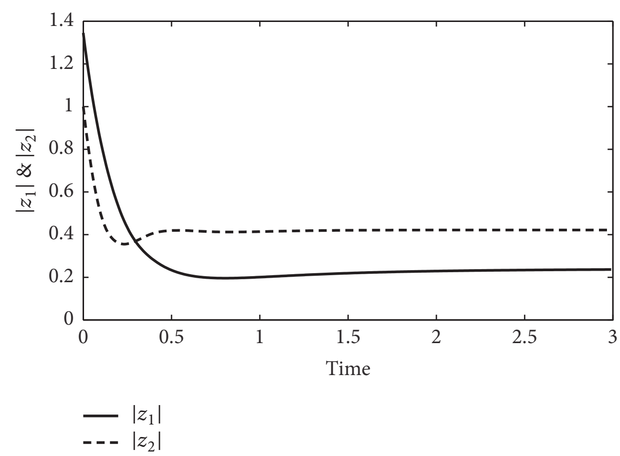

Figure 3: State curves of $|z_1|$ and $|z_2|$ without impulsive disturbances.

Remark 19. In [3, 10], under the assumption that the activation functions satisfied boundedness and analyticity, some sufficient conditions were obtained to guarantee the stability of the equilibrium point. As pointed out in [7], if the activation function only needs to satisfy the condition of global Lipschitz, the restriction for the activation function in this paper is weaker than Assumption 1 in [3, 10]. Besides, model (1) in this paper includes the models studied in there.

Example 2. Consider a class of two-order system described by (1) with the following assumptions.

Let $\theta_{ij}(t-s) = \exp(-2.5(t-s))$, $i, j = 1, 2$. It is assumed that $h_1(z_1(t)) = 0.6 + 0.1\sin(|z_1(t)|)$, $h_2(z_2(t)) = 0.7 + 0.1\cos(|z_2(t)|)$, $\lambda = 3$. Let the time-varying delays in system (1) be $\tau_{1j} = 0.2 + 0.1\sin t$, $\tau_{2j} = 0.3 - 0.1\cos t$, $j = 1, 2$, $t \geq 0$. Suppose that $z_1(t_k^+) = 1.4z_1(t_k^-)$, $t_k \in \{5s, 10s, 15s, \ldots\}$, $z_2(t_k^+) = 0.9z_2(t_k^-)$, $t_k \in \{6s, 12s, 18s, \ldots\}$. The rest of the assumptions are the same as in Example 1.

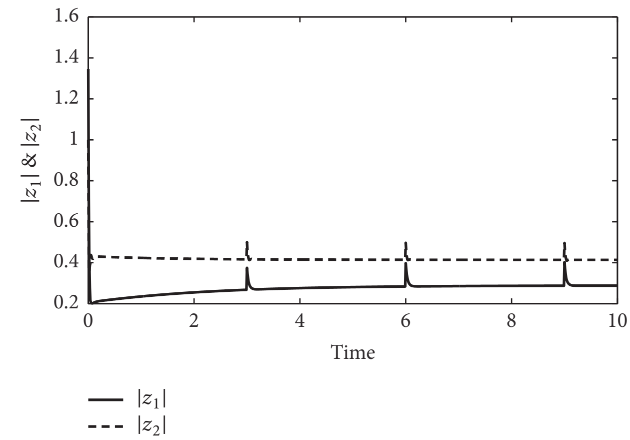

Figure 4: State curves of $|z_1|$ and $|z_2|$ with impulsive disturbances.

By calculation, we have $\eta = 0.14$, $\omega_1 = 15$, $\omega_2 = 16$, $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $l_1 = 0.5$, $l_2 = 1$, $\tau = 0.4$. Taking the above parameters into inequalities (3), we obtain

$$\begin{aligned}
&\xi_1\left(-2\omega_1 + \frac{\lambda}{\sigma_1}\right) + 2\sum_{j=1}^{n}\xi_j \\
&\quad \cdot l_j\left(\left|a_{1j}\right| + \exp(0.5\lambda\tau)\left|b_{1j}\right| + \mu_{1j}(0.5\lambda)\left|p_{1j}\right|\right) \\
&\quad = -0.94 < 0 \\
&\xi_1\left(-2\omega_1 + \frac{\lambda}{\sigma_1}\right) + 2\sum_{j=1}^{n}\xi_j \\
&\quad \cdot l_j\left(\left|a_{1j}\right| + \exp(0.5\lambda\tau)\left|b_{1j}\right| + \mu_{1j}(0.5\lambda)\left|p_{1j}\right|\right) \\
&\quad = -1.6 < 0.
\end{aligned} \tag{46}$$

According to Theorem 12, it can be concluded that the equilibrium point of (1) under the above assumptions is existent, unique, and globally exponentially stable, and the exponential convergence rate is 1.43.

The numerical simulation of the system is shown in Figure 5. It can be seen from the simulation result that the equilibrium point of the system is existent, unique, and stable.

Remark 20. According to results of calculation and simulation, we find that the convergence rate in Example 2 is slower than that in Example 1 due to the larger amplification function and time delays. The correctness of Theorem 12 is verified by the comparison between Examples 1 and 2.

Example 3. Consider a class of two-order system described by (39) with the following assumptions.

It is assumed that activation functions are $f_1(z_1(t)) = 0.5(1-\exp(-\overline{z}_1(t)))/(1+\exp(-\overline{z}_1(t)))$, $f_2(z_2(t)) = 1.5/(1+\exp(-\overline{z}_2(t)))$, and amplification functions are $h_1(z_1(t)) = 0.2 + 0.1\sin(|z_1(t)|)$, $h_2(z_2(t)) = 0.3 + 0.1\cos(|z_2(t)|)$. Let self-feedback functions be $d_1(z_1(t)) = 10z_1(t)$, $d_2(z_2(t)) = 8z_2(t)$. Suppose that external inputs are $J_1(t) = 5i$, $J_2(t) = 4$. Let

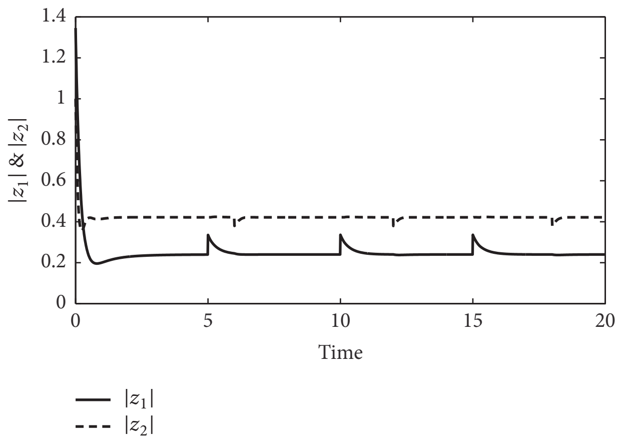

Figure 5: State curves of $|z_1|$ and $|z_2|$ with impulsive disturbances.

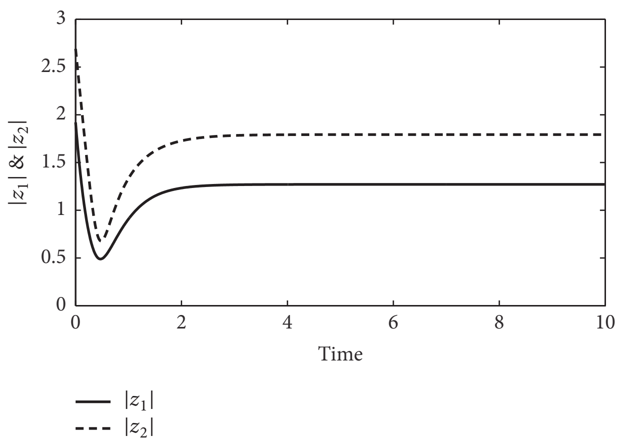

Figure 6: State curves of $|z_1|$ and $|z_2|$.

initial conditions be $z_1(s) = 1.5 - 1.2i$, $z_2(s) = -2 - 1.8i$, $s \in (-\infty, 0]$. Let time-varying delays be $\tau_{1j} = 0.5|\sin t|$, $\tau_{2j} = 0.6|\cos t|$, $j = 1, 2$, $t \geq 0$. Let $\theta_{ij}(t - s) = \exp(-3(t - s))$, $i, j = 1, 2$.

By calculation, we have $\omega_1 = 10$, $\omega_2 = 8$, $l_1 = 0.25$, $l_2 = 0.375$, $|\mathbf{A}| = \begin{bmatrix} 3.20 & 3.35 \\ 3.61 & 4.00 \end{bmatrix}$, $|\mathbf{B}| = \begin{bmatrix} 2.21 & 1.41 \\ 1.70 & 2.00 \end{bmatrix}$, $|\mathbf{P}| = \begin{bmatrix} 1.00 & 1.50 \\ 1.41 & 1.00 \end{bmatrix}$. Furthermore, we get $\mathbf{Q} = \begin{bmatrix} 8.397 & -2.348 \\ -1.680 & 7.375 \end{bmatrix}$.

Obviously, it is known from Lemma 10 that the matrix $\mathbf{Q}$ is an M-matrix. According to Theorem 17, the equilibrium point of system (39) is with existence, uniqueness, and global exponential stability under the assumption conditions above.

The numerical simulation of the system is shown in Figure 6. It can be seen from the simulation result that the equilibrium point of the system is existent, unique, and stable, which verifies the correctness of Theorem 17.

5. Conclusions and Future Directions

This paper has studied the dynamical behavior for a class of impulsive disturbance complex-valued Cohen-Grossberg neural networks with both time-varying delays and continuously distributed delays. Based on the idea of the vector Lyapunov function method, some sufficient conditions have been established for ensuring the existence, uniqueness, and global exponential stability of the equilibrium point of the system by using the corresponding properties of M-matrix and homeomorphism mapping. Not only are the established criteria easy to be verified, but also they improve existing results. Three numerical examples with simulation results have been given to illustrate the effectiveness of the obtained results in this paper.

It is well known that the synchronization problem of chaotic neural networks can be translated into the stability problem of the corresponding error system of driving system and driven system. There have been some literatures concerning the analysis of synchronization control for some complex-valued neural networks with time delays by using the idea of adaptive control [33, 35] and sliding mode control [36]. In future works, we will furthermore investigate the synchronization problems of complex-valued chaotic neural networks with a lower level of conservation of assumption conditions.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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