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Research Article

Robust Exponential Stability Analysis of Switched Neural Networks with Interval Parameter Uncertainties and Time Delays

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In this paper, the stability of switched neural networks (SNNs) with interval parameter uncertainties and time delays is investigated. First, the conditions for the existence and uniqueness of the equilibrium point of the system are discussed. Second, the average dwell time approach and M-matrix property are employed to obtain conditions to ensure the globally exponential stability of the delayed SNNs under constrained switching. Third, by resorting to inequality technique and the idea of vector Lyapunov function, sufficient condition to ensure the robust exponential stability of the delayed SNNs under arbitrary switching is derived. The form of the constructed Lyapunov functions is simple, which has certain commonality in studying delayed SNNs, and the proposed results not only are explicit but also reveal the relationship between the constrained switching and the arbitrary switching of the SNNs. Finally, two numerical examples are presented to illustrate the effectiveness and less conservativeness of the main results compared with the existing literature.

1. Introduction

In the past years, neural networks have been widely studied and successfully applied to various realms such as dynamic optimization, associative memory, and pattern recognition and to solve nonlinear algebraic equations and so on [1–5]. In the real world, the connections among different nodes of the networks are not always fixed or consistent, which frequently result in link failure and new link creation. Therefore, the abrupt changes in the structures and parameters of the neural networks often occur, which bring about switchings among certain different topologies and the instability of the networks [6]. In application's point of view, a fundamental problem of applying neural networks is stability. This is a prerequisite for ensuring that the developed networks can work normally [7–10]. Thus, a popular topic about the stability analysis and stabilization of SNNs has been considered in [11–24].

A switched neural network is a hybrid system, which is essentially composed of a family of subnetworks and a switching signal which defines a specially designated subnetwork being activated at each instant of time. SNNs have attracted significant attention and have been successfully applied to many fields such as artificial intelligence, high-speed signal processing, and gene selection in DNA microarray analysis [25–28]. Generally, a switching system can be described by the following differential equation:

$$\dot{\mathbf{x}} = \mathbf{f}_{\sigma}(\mathbf{x}), \tag{1}$$

where $\{\mathbf{f}_p : p \in \mathscr{P}\}$ is a family of functions parameterized by some index set $\mathscr{P}$ and switching signal σ is a piecewise constant and right continuous function of time mapping from $[0, +\infty)$ to $\mathscr{P}$. The original motivation for the study on switched systems comes partly from that switching among different systems may cause many nonlinear system behaviors such as chaos and multiple limit cycles [29]. In recent years, switched systems have gained increasing attention because many practical systems (for example, constrained robotics, computer-controlled systems, and automated highway systems) can be modeled as switched systems. Furthermore, from the point of view of control, multicontroller switching is an effective way to deal with complex systems. It is well-known that time delays are inevitable in a practical control design which usually leads to unsatisfactory performances and the stability of the dynamic systems may even be destroyed with the increase of delays [30–35]. Attributing to the interaction among the discrete dynamics, continuous dynamics, and time delays, the behaviors of delayed SNNs are very complicated. Besides, due to many inevitable factors such as modelling errors and external perturbations, the models certainly contain uncertainties which can have a serious effect on the dynamical behavior of the systems. To analyze the robustness of the SNNs, one feasible method is to assume that the parameters are included in certain intervals [36]. Therefore, the robust stability analysis of SNNs with interval parameter uncertainties and time delays is of practical and theoretical importance.

For switched dynamical systems, the unpredictable change of system dynamics, such as abrupt perturbation of external environment or sudden change of the system structure due to the failure of a component, may cause the sudden change of the switching signal. In these cases, in order to keep the system working, the system should be stable under arbitrary switching. A typical approach for the stability analysis of switched dynamical systems with arbitrary switching signal is to search for a suitable common Lyapunov function (CLF) $V(\mathbf{x})$ such that the rate of the decrease of $V(\mathbf{x})$ along the trajectories of systems is not affected by switching (see, e.g., [37–40] and the references therein). If the CLF for the systems does not exist or is not known, in this case, we can study the stability of the system by using multiple Lyapunov functions (MLFs) $V_p(\mathbf{x}), p \in \mathscr{P}$, (see [37, 41, 42]). However, it is worth noting that to apply this MLF method, one needs to know some information of the state at each switching time. This is to be contrasted with the Lyapunov second method, which do not need to know the knowledge of the solutions. For example, Wu et al. studied the exponential stability of delayed SNNs by using a linear matrix inequality approach and an average dwell time method [12]; based on the piecewise Lyapunov function technique and average dwell time approach, the problem of the exponential stability of SNNs with constant and time-varying delays was investigated, respectively, in [43] and in [44]; by resorting to a novel delay division method, the stability analysis for uncertain SNNs with mixed time-varying delays was addressed. A common feature in these articles is that they all resort to scalar Lyapunov function (or functional). In this paper, the stability of the delayed SNNs with switching signal will be studied by using the idea of vector Lyapunov function with simple forms, which have certain commonality in studying SNNs, and this is the main reason why the obtained results in this paper have less conservativeness. By using the M-matrix property and average dwell time approach, the differential inequalities with time delays will be constructed. By the stability analysis of the differential inequalities, the sufficient conditions to ensure the robust exponential stability of the SNNs under arbitrary switching and constrained switching will be obtained.

Compared with the existing results on SNNs, the contributions of this paper are listed as follows: (a) the forms of the constructed Lyapunov functions are simple, which have certain commonality in studying delayed SNNs under arbitrary switching; (b) unlike asymptotic stability, we analyze the exponential stability of SNNs which include uncertainty and time delays, and the exponential convergence rate can also be obtained; (c) the obtained results not only have less conservativeness but also reveal the relationship between the constrained switching and the arbitrary switching of the delayed SNNs; and (d) comparing with most of the previous results obtained by linear matrix inequalities approach (to apply LMIs approach, one has to determine too many unknown parameters), the proposed criteria are straightforward, which are conducive to practical applications.

Notation. Let $\mathbf{x} = (x_1, \cdots, x_n)^{\mathrm{T}}$ denote a column vector of $\mathbb{R}^n$ (the symbol "T" denotes transpose), $|\mathbf{x}|$ denote $|\mathbf{x}| = (|x_1|, |x_2|, \cdots, |x_n|)^{\mathrm{T}}$, and $\|\mathbf{x}\|$ denote a vector norm defined by $\|\mathbf{x}\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} > \mathbf{y}$ means that each pair of the corresponding elements of $\mathbf{x}$ and $\mathbf{y}$ satisfies the inequality ">." For matrix $\mathbf{A} = (a_{ij})_{n\times n}$, $|\mathbf{A}|$ denote $|\mathbf{A}| = (|a_{ij}|)_{n\times n}$. $C([-\tau, t_0]\,; \mathbb{R}^n)$ denotes the set of continuous functions mapping from $[-\tau, t_0]$ to $\mathbb{R}^n$.

2. Preliminaries

The model of a delayed SNNs can be described by the delayed differential equations as follows:

$$\begin{cases} \dfrac{\mathrm{d}w_i(t)}{dt} = -e_i^{\sigma(t)} w_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij}^{\sigma(t)} g_j^{\sigma(t)}\left(w_j(t)\right) + \sum_{j=1}^{n} b_{ij}^{\sigma(t)} g_j^{\sigma(t)}\left(w_j\left(t - \tau_{ij}^{\sigma(t)}\right)\right) + J_i^{\sigma(t)}, \\ w_i(s + t_0) = \phi_i(s), s \in [-\tau, 0], \end{cases} \tag{2}$$

where $i = 1, 2, \ldots, n$, n is the number of neurons, $w_i(t)$ is the state of neuron i at time t, $\sigma(t)$: $[0, +\infty] \to \Sigma = \{1, 2, \cdots, m\}$ is the switching signal, which is a piecewise constant and right continuous function of time, and $\sigma(t) = k \in \Sigma$ means that the kth subnetwork is activated. $\mathbf{E}_k = \operatorname{diag}\{e_1^k, e_2^k, \cdots, e_n^k\}$ denotes the neuron self-feedback coefficient matrix of the k subnetwork, and $e_i^k > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; $\mathbf{g}^k(\mathbf{w}(t)) = (g_1^k(w_1(t)), g_2^k(w_2(t)), \cdots, g_n^k(w_n(t)))^{\mathrm{T}}$ is the activation functions of neurons at time t; $\mathbf{A}_k = (a_{ij}^k)_{n\times n}$ and $\mathbf{B}_k = (b_{ij}^k)_{n\times n}$ are the connection weight matrices of the kth subnetwork, and a_{ij}^k and b_{ij}^k denote the connection strengths of the jth neuron on the ith neuron at time t and $t - \tau_{ij}^k$, respectively; the delay $\tau_{ij}^k \geq 0$ is the bounded function with $\tau^k = \max_{1\leq i,j\leq n}\{\tau_{ij}^k\} \geq 0$ and $\tau = \max_{1\leq k\leq m}\{\tau^k\}$; $\mathbf{J}_k = (J_1^k, J_2^k, \cdots, J_n^k)^{\mathrm{T}}$ is the constant external input vector of the k th subnetwork. $w_i(s + t_0) = \phi_i(s)$ is the initial condition of the system, where $\phi_i \in C([-\tau, t_0], \mathbb{R})$, $i = 1, 2, \ldots, n$.

We assume that the switching signal $\sigma(t)$ is unknown a priori. Corresponding to the switching signal $\sigma(t)$, we have a switching sequence $\{(t_0, i_0) \cdots (t_k, i_k) \cdots |_{i_k\in\Sigma, k=0,1,\ldots}\}$, which means that the i_kth subsystem is activated when $t \in [t_k, t_{k+1})$. We also assume that there is only finite switching in any finite interval and satisfy the following conditions.

Assumption 1. Each activation function $g_i^k(\cdot)$ in the delayed SNNs (2) is assumed to satisfy

$$\underline{L}_i^k \leq \frac{g_i^k(u) - g_i^k(v)}{u - v} \leq \bar{L}_i^k, \tag{3}$$

for any $u, v \in \mathbb{R}$, $u \neq v$, $i = 1, 2, \ldots, n$, $k \in \Sigma$, where $\underline{L}_i^k$ and $\bar{L}_i^k$ are known constant scalars and $\underline{L}_i^k < \bar{L}_i^k$.

Remark 1. Assumption 1 was first proposed in [45]. The constants $\underline{L}_i^k$ and $\bar{L}_i^k$ in this assumption are allowed to be any real number (positive, negative, or zero). Therefore, the activation functions can be nonmonotonic, which are more general than commonly used Lipschitz conditions and sigmoid functions. Such assumption is very useful to obtain less conservative results.

To facilitate the following analysis, let $\mathbf{L}_k = \operatorname{diag}\{L_1^k, L_2^k, \cdots, L_n^k\}$ with $L_i^k = \max\{|\underline{L}_i^k|, |\bar{L}_i^k|\}$. In order to study the stability of SNNs under parameter uncertainties, for $k \in \Sigma$, the matrices are intervalized as follows:

$$\begin{aligned}
\mathbf{E}_k^{\mathrm{I}} &= \left\{\mathbf{E}_k = \operatorname{diag}\left(e_i^k\right)_{n\times n} : \underline{\mathbf{E}}_k \leq \mathbf{E}_k \leq \bar{\mathbf{E}}_k, \quad \text{i.e., } 0 < \underline{e}_i^k \leq e_i^k \leq \bar{e}_i^k\right\},\\
\mathbf{A}_k^{\mathrm{I}} &= \left\{\mathbf{A}_k = \left(a_{ij}^k\right)_{n\times n} : \underline{\mathbf{A}}_k \leq \mathbf{A}_k \leq \bar{\mathbf{A}}_k, \quad \text{i.e., } \underline{a}_{ij}^k \leq a_{ij}^k \leq \bar{a}_{ij}^k\right\},\\
\mathbf{B}_k^{\mathrm{I}} &= \left\{\mathbf{B}_k = \left(b_{ij}^k\right)_{n\times n} : \underline{\mathbf{B}}_k \leq \mathbf{B}_k \leq \bar{\mathbf{B}}_k, \quad \text{i.e., } \underline{b}_{ij}^k \leq b_{ij}^k \leq \bar{b}_{ij}^k\right\}.
\end{aligned} \tag{4}$$

Define

$$\begin{aligned}
\mathbf{E}_k^* &= \operatorname{diag}\left\{\underline{e}_1^k, \underline{e}_2^k, \cdots, \underline{e}_n^k\right\},\\
\mathbf{A}_k^* &= \left(a_{ij}^{k*}\right)_{n\times n} \text{with } a_{ij}^{k*} = \max\left\{\left|\underline{a}_{ij}^k\right|, \left|\bar{a}_{ij}^k\right|\right\},\\
\mathbf{B}_k^* &= \left(b_{ij}^{k*}\right)_{n\times n} \text{with } b_{ij}^{k*} = \max\left\{\left|\underline{b}_{ij}^k\right|, \left|\bar{b}_{ij}^k\right|\right\},\\
\mathbf{B}_k^\star &= \left(b_{ij}^{k\star}\right)_{n\times n} \text{with } b_{ij}^{k\star} = \min\left\{\left|\underline{b}_{ij}^k\right|, \left|\bar{b}_{ij}^k\right|\right\}.
\end{aligned} \tag{5}$$

Definition 1. For the delayed SNNs (2), the equilibrium point $\mathbf{w}^* = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is said to be robustly exponentially stable if for each $\mathbf{E} \in \mathbf{E}_k^{\mathrm{I}}$, $\mathbf{A} \in \mathbf{A}_k^{\mathrm{I}}$, and $\mathbf{B} \in \mathbf{B}_k^{\mathrm{I}}$, there exist constants $\lambda > 0$ and $M > 1$ such that

$$\|\mathbf{w}(t) - \mathbf{w}^*\| \leq M\|\phi - \mathbf{w}^*\|_{t_0} \exp\left(-\lambda(t - t_0)\right), \quad t \geq t_0, \tag{6}$$

where $\|\phi - \mathbf{w}^*\|_{t_0} = \sum_{i=1}^n \left(\sup_{s\in[-\tau,t_0]}(\phi_i(s) - w_i^*)^2\right)^{1/2}$.

3. Existence and Uniqueness of the Equilibrium Point

The purpose of the present section is to give a sufficient condition which ensures that the equilibrium point of each subsystem satisfies the existence and uniqueness, which implies that for any initial condition $\phi \in C([-\tau, t_0]; \mathbb{R}^n)$, system (2) admits a solution $\mathbf{w}(t, t_0, \phi)$ which exists in a maximal interval $[-\tau, t_0 + \mathscr{K})$, where $0 < \mathscr{K} \leq \infty$.

Definition 2. A real $n \times n$ matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is said to be an M-matrix if $a_{ij} \leq 0$, $i, j = 1, 2, \ldots, n$, $i \neq j$, and all successive principal minors of $\mathbf{A}$ are positive.

Lemma 1 *([46]).*
Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix with nonpositive off-diagonal elements. Then the following statements are equivalent:

(i) $\mathbf{A}$ is an M-matrix

(ii) There exists a vector $\boldsymbol{\xi} > 0$ such that $\mathbf{A}\boldsymbol{\xi} > 0$.

Definition 3. A mapping $\mathscr{H} : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism of $\mathbb{R}^n$ onto itself if $\mathscr{H} \in C^0$, $\mathscr{H}$ is one to one, $\mathscr{H}$ is onto, and the inverse mapping $\mathscr{H}^{-1} \in C^0$, where C^0 denotes the set of continuous functions.

Lemma 2 *([47]).*
If $\mathscr{H}(\mathbf{u}) \in C^0$ satisfies the following conditions:

(i) $\mathscr{H}(\mathbf{u})$ is injective on $\mathbb{R}^n$

(ii) $\|\mathscr{H}(\mathbf{u})\| \to \infty$ as $\|\mathbf{u}\| \to \infty$

Then $\mathscr{H}(\mathbf{u})$ is a homeomorphism of $\mathbb{R}^n$.

Theorem 1. *Under Assumption 1, if for all $k \in \Sigma$, $\mathbf{C}_k^* = \mathbf{E}_k^* - (\mathbf{A}_k^* + \mathbf{B}_k^*)\mathbf{L}_k$ are nonsingular M-matrices, then for each specified switching signal $\sigma(t)$, system (2) has a unique equilibrium point.*

Proof 1. Because the equilibrium point of subsystems,

$$\frac{\mathrm{d}w_i(t)}{\mathrm{d}t} = -e_i^k w_i(t) + \sum_{j=1}^{n} a_{ij}^k g_j^k(w_j(t)) + \sum_{j=1}^{n} b_{ij}^k g_j^k\left(w_j\left(t - \tau_{ij}^k\right)\right) + J_i^k \tag{7}$$

satisfies the following equation:

$$-e_i^k w_i(t) + \sum_{j=1}^{n}\left(a_{ij}^k + b_{ij}^k\right) g_j^k(w_j(t)) + J_i^k = 0, \tag{8}$$

for $i = 1, 2, \ldots, n$ and $k \in \Sigma$. Let

$$\mathscr{H}^k(\mathbf{w}(t)) = \left(\mathscr{H}_1^k(\mathbf{w}(t)), \mathscr{H}_2^k(\mathbf{w}(t)), \cdots, \mathscr{H}_n^k(\mathbf{w}(t))\right)^{\mathrm{T}}, \tag{9}$$

where

$$\mathscr{H}_i^k(\mathbf{w}(t)) = -e_i^k w_i(t) + \sum_{j=1}^{n}\left(a_{ij}^k + b_{ij}^k\right) g_j^k(w_j(t)) + J_i^k, \tag{10}$$

for $i = 1, 2, \ldots, n$. In the following, we will give a proof that $\mathscr{H}^k(\mathbf{w}(t))$ are homeomorphisms of $\mathbb{R}^n$ onto itself.

First, we prove that $\mathscr{H}^k(\mathbf{w}(t))$ are injective mappings on $\mathbb{R}^n$. Actually, if there exist vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, $\mathbf{y} = (y_1, y_2, \cdots y_n)^{\mathrm{T}}$, and $\mathbf{x} \neq \mathbf{y}$ such that $\mathscr{H}^k(\mathbf{x}) = \mathscr{H}^k(\mathbf{y})$; then

$$-e_i^k(x_i - y_i) + \sum_{j=1}^{n}\left(a_{ij}^k + b_{ij}^k\right)\left[g_j^k(x_j) - g_j^k\left(y_j\right)\right] = 0, \tag{11}$$

for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. From Assumption 1, it can be derived that

$$-e_i^k|x_i - y_i| + \sum_{j=1}^{n}\left(\left|a_{ij}^k\right| + \left|b_{ij}^k\right|\right) L_j^k\left|x_j - y_j\right| \geq 0, \tag{12}$$

for $i = 1, 2, \ldots, n$. That is,

$$[\mathbf{E}_k - (|\mathbf{A}_k| + |\mathbf{B}_k|)\mathbf{L}_k]|x - y| \leq 0. \tag{13}$$

Let $\mathbf{C}_k = \mathbf{E}_k - (|\mathbf{A}_k| + |\mathbf{B}_k|)\mathbf{L}_k$. Obviously, $\mathbf{C}_k$ have nonpositive off-diagonal entries and $\mathbf{C}_k \geq \mathbf{C}_k^*$ which implies that $\mathbf{C}_k$ are nonsingular M-matrices. From Theorem 2.3 of [48], we can get $\mathbf{x} = \mathbf{y}$. That is,

$$x_i = y_i, \quad i = 1, 2, \ldots, n, \tag{14}$$

which is a contradiction. As a result, $\mathscr{H}^k(\mathbf{w}(t))$ are injective mappings on $\mathbb{R}^n$.

Next, we prove that $\|\mathscr{H}^k(\mathbf{w}(t))\| \to \infty$ as $\|\mathbf{w}(t)\| \to \infty$.

Because $\mathbf{C}_k$ are nonsingular M-matrices, we know that there exist positive diagonal matrices $\mathbf{D}_k = \mathrm{diag}\,(d_1^k, d_2^k, \cdots, d_n^k)$, which make matrices $\mathbf{D}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{D}_k$ positively definite. Let

$$\tilde{\mathscr{H}}^k(\mathbf{w}(t)) = \left(\tilde{\mathscr{H}}_1^k(\mathbf{w}(t)), \tilde{\mathscr{H}}_2^k(\mathbf{w}(t)), \cdots, \tilde{\mathscr{H}}_n^k(\mathbf{w}(t))\right)^{\mathrm{T}}, \tag{15}$$

where

$$\tilde{\mathscr{H}}_i^k(\mathbf{w}(t)) = -e_i^k w_i(t) + \sum_{j=1}^{n}\left(a_{ij}^k + b_{ij}^k\right)\left(g_j^k(w_j(t)) - g_j^k(0)\right), \tag{16}$$

for $i = 1, 2, \ldots, n$. Calculate

$$\begin{aligned}
&(w_1, w_2, \cdots, w_n)\mathbf{D}_k\tilde{\mathscr{H}}^k(\mathbf{w}(t)) \\
&\quad = \sum_{i=1}^{n} w_i d_i^k \tilde{\mathscr{H}}_i^k(\mathbf{w}(t)) = \sum_{i=1}^{n}\Big[-e_i^k d_i^k w_i^2(t) \\
&\qquad + \sum_{j=1}^{n}\left(a_{ij}^k + b_{ij}^k\right) d_i^k w_i(t)\left(g_j^k(w_j(t)) - g_j^k(0)\right)\Big] \\
&\quad \leq \sum_{i=1}^{n}\left[-e_i^k d_i^k w_i^2(t) + \sum_{j=1}^{n}\left(\left|a_{ij}^k\right| + \left|b_{ij}^k\right|\right) d_i^k L_j^k |w_i(t)||w_j(t)|\right] \\
&\quad = -(|w_1(t)|, |w_2(t)|, \cdots, |w_n(t)|)\mathbf{D}_k\mathbf{C}_k(|w_1(t)|, |w_2 \\
&\qquad \cdot (t)|, \cdots, |w_n(t)|)^{\mathrm{T}} = -\frac{1}{2}|\mathbf{w}(t)|^{\mathrm{T}}\left(\mathbf{D}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{D}_k\right)|\mathbf{w}(t)| \\
&\quad \leq -\frac{1}{2}\lambda_{\min}\left(\mathbf{D}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{D}_k\right)\|\mathbf{w}(t)\|^2.
\end{aligned} \tag{17}$$

Using Schwartz inequality, we have

$$\|\mathbf{w}(t)\| \cdot \|\mathbf{D}_k\| \cdot \left\|\tilde{\mathscr{H}}^k(\mathbf{w}(t))\right\| \geq \frac{1}{2}\lambda_{\min}\left(\mathbf{D}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{D}_k\right)\|\mathbf{w}(t)\|^2. \tag{18}$$

When $\|\mathbf{w}(t)\| \neq 0$, we get

$$\left\|\tilde{\mathscr{H}}^k(\mathbf{w}(t))\right\| \geq \frac{1}{2}\lambda_{\min}\left(\mathbf{D}_k\mathbf{C}_k + \mathbf{C}_k^{\mathrm{T}}\mathbf{D}_k\right)\frac{\|\mathbf{w}(t)\|}{\|\mathbf{D}_k\|}, \tag{19}$$

which implies $\|\tilde{\mathscr{H}}^k(\mathbf{w}(t))\| \to \infty$ as $\|\mathbf{w}(t)\| \to \infty$.

Since $\|\tilde{\mathscr{H}}^k(\mathbf{w}(t))\| \to \infty$ implies $\|\mathscr{H}^k(\mathbf{w}(t))\| \to \infty$, by Lemma 2, we know that $\mathscr{H}^k(\mathbf{w}(t))$ are homeomorphisms of $\mathbb{R}^n$. So each subnetwork has a unique equilibrium point. Therefore, for each specified switching signal $\sigma(t)$, system (2) has a unique equilibrium point. The proof is completed.

4. Exponential Stability of the Delayed SNNs

4.1. Exponential Stability under Constrained Switching. In this section, we will give a sufficient condition ensuring the global exponential stability of delayed SNNs (2) by using the average dwell time method. Let $\mathbf{Q}$ be an M-matrix; we denote

$$\mathbf{\Lambda}(\mathbf{Q}) \triangleq \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \mathbf{Q}\boldsymbol{\xi} > 0, \boldsymbol{\xi} > 0\}. \tag{20}$$

Definition 4 (see [37]).
Let $N_\sigma(t_1, t_2)$ denote the number of discontinuities of a switching signal σ on an interval (t_1, t_2). $\mathcal{T} > 0$ is called the average dwell time, if for any $t_2 \geq t_1 \geq 0$ and $N_0 \geq 0$,

$$N_\sigma(t_1, t_2) \leq N_0 + \frac{t_2 - t_1}{\mathcal{T}} \tag{21}$$

hold.

Theorem 2. *Under Assumption 1, if for all $k \in \Sigma$, $\mathbf{C}_k^* = \mathbf{E}_k^* - (\mathbf{A}_k^* + \mathbf{B}_k^*)\mathbf{L}_k$ are nonsingular M-matrices, then for all $\mathbf{E}_k \in \mathbf{E}_k^I$, $\mathbf{A}_k \in \mathbf{A}_k^I$, $\mathbf{B}_k \in \mathbf{B}_k^I$ and any external input $\mathbf{J}_k$, the delayed SNN (2) is robustly exponentially stable for any switching signal with the average dwell time satisfying*

$$\mathcal{T} > \mathcal{T}^* = \frac{\ln \vartheta}{\varepsilon}, \tag{22}$$

where $\varepsilon > 0$ is determined by inequalities

$$\frac{-\underline{e}_i^k + \varepsilon}{\exp(\varepsilon\tau^k)} \xi_i^k + \sum_{j=1}^n \xi_j^k L_j^k \left(a_{ji}^{k*} + b_{ji}^{k*}\right) < 0, \tag{23}$$

for some given $\xi_k = (\xi_1^k, \xi_2^k, \cdots, \xi_n^k)^{\mathrm{T}} \in \mathbf{\Lambda}(\mathbf{C}_k^*)$ and $\vartheta = \max_{1\leq i\leq n, 1\leq i_{k-1}\leq m} \{\eta_i^{i_{k-1}}, \eta_i^{i_{k-1}}\beta_i^{i_{k-1}}\} \geq 1$ with

$$\eta_i^{i_{k-1}} = \frac{\max_{1\leq i\leq n}\left\{\xi_i^{i_k}\right\}}{\min_{1\leq i\leq n}\left\{\xi_i^{i_{k-1}}\right\}},$$
$$\beta_i^{i_{k-1}} = \frac{\max_{1\leq j\leq n}\left\{\exp\left(\varepsilon\tau_{ij}^{i_k}\right)L_j^{i_k} b_{ij}^{i_k *}\right\}}{\min_{1\leq j\leq n}\left\{\exp\left(\varepsilon\tau_{ij}^{i_{k-1}}\right)L_j^{i_{k-1}} b_{ij}^{i_{k-1} *}\right\}}. \tag{24}$$

Proof 2. According to Theorem 1, we know that if $\mathbf{C}_k^* = \mathbf{E}_k^* - (\mathbf{A}_k^* + \mathbf{B}_k^*)\mathbf{L}_k$ are M-matrices, then the system has a unique equilibrium point for each specified switching signal. Let $\mathbf{w}^* = (w_1^*, w_2^*, \cdots, w_n^*)^{\mathrm{T}}$ be an equilibrium point of system (2) and $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^{\mathrm{T}}$ be any solution of system (2). Denote $x_i(t) = w_i(t) - w_i^*$, $f_j^k(x_j(t)) = g_j^k(x_j(t) + w_j^*) - g_j^k(w_j^*)$, and $\psi_i(s) = \phi_i(s) - w_i^*$; then system (2) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{dt} = -e_i^{\sigma(t)} x_i(t) + \displaystyle\sum_{j=1}^n a_{ij}^{\sigma(t)} f_j^{\sigma(t)}(x_j(t)) + \sum_{j=1}^n b_{ij}^{\sigma(t)} f_j^{\sigma(t)}\left(x_j\left(t - \tau_{ij}^{\sigma(t)}\right)\right), \\ x_i(s + t_0) = \psi_i(s), \quad s \in [-\tau, 0], \end{cases} \tag{25}$$

with $i = 1, 2, \ldots, n$.

Due to $\mathbf{C}_k^*$ being M-matrices, by Lemma 1 (ii), we know that there exist $\xi_i^k > 0$ and $\delta_i^k > 0$, $(k \in \Sigma, i = 1, 2, \cdots, n)$ such that

$$-\underline{e}_i^k \xi_i^k + \sum_{j=1}^n \xi_j^k L_j^k \left(a_{ji}^{k*} + b_{ji}^{k*}\right) = -\delta_i^k < 0. \tag{26}$$

Consider a Lyapunov functional candidate

$$V(\mathbf{x}, t) = \sum_{i=1}^n \xi_i^{\sigma(t)} \left\{ \exp(\varepsilon t)|x_i| + \sum_{j=1}^n L_j^{\sigma(t)} \left|b_{ij}^{\sigma(t)}\right| \left|\int_{t-\tau_{ij}^{\sigma(t)}}^t \exp\left(\varepsilon\left(s + \tau_{ij}^{\sigma(t)}\right)\right)|x_j(s)|\mathrm{d}s\right\}. \tag{27}$$

Calculating the upper right derivative $\mathrm{D}^+ V$ of V along the solutions of (25), we get

$$\begin{aligned} \mathrm{D}^+ V(\mathbf{x}, t) = {} & \sum_{i=1}^n \xi_i^{\sigma(t)} \left\{ \exp(\varepsilon t) \operatorname{sgn} x_i \frac{\mathrm{d}x_i}{\mathrm{d}t} + \varepsilon \exp(\varepsilon t)|x_i| \right. \\ & + \sum_{j=1}^n L_j^{\sigma(t)} \left|b_{ij}^{\sigma(t)}\right| \left[\exp\left(\varepsilon\left(t + \tau_{ij}^{\sigma(t)}\right)\right)|x_j(t)| \right. \\ & \left.\left. - \exp(\varepsilon t)\left|x_j\left(t - \tau_{ij}^{\sigma(t)}\right)\right|\right]\right\} \\ = {} & \sum_{i=1}^n \xi_i^{\sigma(t)} \left\{ \exp(\varepsilon t) \operatorname{sgn} x_i \left[-e_i^{\sigma(t)} x_i(t) \right.\right. \\ & + \sum_{j=1}^n a_{ij}^{\sigma(t)} f_j^{\sigma(t)}(x_j(t)) \\ & \left. + \sum_{j=1}^n b_{ij}^{\sigma(t)} f_j^{\sigma(t)}\left(x_j\left(t - \tau_{ij}^{\sigma(t)}\right)\right)\right] + \varepsilon \exp(\varepsilon t)|x_i| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} L_j^{\sigma(t)}\left|b_{ij}^{\sigma(t)}\right|\Big[\exp\left(\varepsilon\left(t+\tau_{ij}^{\sigma(t)}\right)\right)|x_j(t)| \\
&-\exp(\varepsilon t)\left|x_j\left(t-\tau_{ij}^{\sigma(t)}\right)\right|\Big]\Big\} \le \sum_{i=1}^{n}\xi_i^{\sigma(t)}\Big\{\exp(\varepsilon t)\Big[-e_i^{\sigma(t)}|x_i(t)| \\
&+\sum_{j=1}^{n}\left|a_{ij}^{\sigma(t)}\right|\left|f_j^{\sigma(t)}(x_j(t))\right| + \sum_{j=1}^{n}\left|b_{ij}^{\sigma(t)}\right|\left|f_j^{\sigma(t)}\left(x_j\left(t-\tau_{ij}^{\sigma(t)}\right)\right)\right|\Big] \\
&+\varepsilon\exp(\varepsilon t)|x_i| + \sum_{j=1}^{n} L_j^{\sigma(t)}\left|b_{ij}^{\sigma(t)}\right|\Big[\exp\left(\varepsilon\left(t+\tau_{ij}^{\sigma(t)}\right)\right)|x_j(t)| \\
&-\exp(\varepsilon t)\left|x_j\left(t-\tau_{ij}^{\sigma(t)}\right)\right|\Big]\Big\} \le \sum_{i=1}^{n}\xi_i^{\sigma(t)}\Big\{\exp(\varepsilon t)\Big[-e_i^{\sigma(t)}|x_i(t)| \\
&+\sum_{j=1}^{n} L_j^{\sigma(t)}\left|a_{ij}^{\sigma(t)}\right||x_j(t)| + \sum_{j=1}^{n} L_j^{\sigma(t)}\left|b_{ij}^{\sigma(t)}\right|\left|x_j\left(t-\tau_{ij}^{\sigma(t)}\right)\right|\Big] \\
&+\varepsilon\exp(\varepsilon t)|x_i| + \sum_{j=1}^{n} L_j^{\sigma(t)}\left|b_{ij}^{\sigma(t)}\right|\Big[\exp\left(\varepsilon\left(t+\tau_{ij}^{\sigma(t)}\right)\right)|x_j(t)| \\
&-\exp(\varepsilon t)\left|x_j\left(t-\tau_{ij}^{\sigma(t)}\right)\right|\Big]\Big\} = \sum_{i=1}^{n}\exp(\varepsilon t)\xi_i^{\sigma(t)}\Big[\left(-e_i^{\sigma(t)}+\varepsilon\right)|x_i(t)| \\
&+\sum_{j=1}^{n}\left(\left|a_{ij}^{\sigma(t)}\right| + \exp\left(\varepsilon\tau_{ij}^{\sigma(t)}\right)\left|b_{ij}^{\sigma(t)}\right|\right)L_j^{\sigma(t)}|x_j(t)|\Big] \\
&\le \exp(\varepsilon t)\sum_{i=1}^{n}\Big[\left(-e_i^{\sigma(t)}+\varepsilon\right)\xi_i^{\sigma(t)} + \exp\left(\varepsilon\tau^{\sigma(t)}\right) \\
&\cdot\sum_{j=1}^{n}\xi_j^{\sigma(t)}L_j^{\sigma(t)}\left(\left|a_{ji}^{\sigma(t)}\right| + \left|b_{ji}^{\sigma(t)}\right|\right)\Big]|x_i(t)| \le \exp\left(\varepsilon\left(t+\tau^{\sigma(t)}\right)\right) \\
&\cdot\sum_{i=1}^{n}\left[\frac{-\underline{e}_i^{\sigma(t)}+\varepsilon}{\exp\left(\varepsilon\tau^{\sigma(t)}\right)}\xi_i^{\sigma(t)} + \sum_{j=1}^{n}\xi_j^{\sigma(t)}L_j^{\sigma(t)}\left(a_{ji}^{\sigma(t)*}+b_{ji}^{\sigma(t)*}\right)\right]|x_i(t)| \\
&= \exp\left(\varepsilon\left(t+\tau^{\sigma(t)}\right)\right)\sum_{i=1}^{n}\left[-\delta_i^{\sigma(t)} + \left(\frac{-\underline{e}_i^{\sigma(t)}+\varepsilon}{\exp\left(\varepsilon\tau^{\sigma(t)}\right)} + \underline{e}_i^{\sigma(t)}\right)\xi_i^{\sigma(t)}\right] \\
&\cdot|x_i(t)|.
\end{aligned}
\tag{28}
$$

Defining functions,

$$
\mathscr{F}_i^k(z) = -\delta_i^k + \left(\frac{-\underline{e}_i^k+z}{\exp\left(z\tau^k\right)} + \underline{e}_i^k\right)\xi_i^k, \quad (i=1,2,\cdots,n, \quad k\in\Sigma). \tag{29}
$$

Obviously, $\mathscr{F}_i^k(0) = -\delta_i^k < 0$. Since $\mathscr{F}_i^k(z)$ are continuous functions, there exist $\varepsilon_i^k > 0, (i=1,2,\cdots,n)$, such that $\mathscr{F}_i^k(\varepsilon_i^k) < 0$. Let $\varepsilon = \min_{1\le k\le m, 1\le i\le n}\{\varepsilon_i^k\}$; we can get $\mathscr{F}_i^k(\varepsilon) < 0, (i=1,2,\cdots,n, k\in\Sigma)$. Combining it with inequality (28), we get

$$
D^+V(\mathbf{x},t) \le \exp\left(\varepsilon\left(t+\tau^{\sigma(t)}\right)\right)\sum_{i=1}^{n}\mathscr{F}_i^{\sigma(t)}(\varepsilon)|x_i(t)| \le 0. \tag{30}
$$

So for $t\in[t_k, t_{k+1})$,

$$
V(\mathbf{x},t) \le V(\mathbf{x},t_k). \tag{31}
$$

For convenience, we denote $\sigma(t) = i_k$ when $t\in[t_k, t_{k+1})$, $k = 0,1,\dots,n$. That is, the i_kth subnetwork is activated for $t\in[t_k, t_{k+1})$; then

$$
\begin{aligned}
V(\mathbf{x},t_k) &= \sum_{i=1}^{n}\xi_i^{i_k}\Big\{\exp(\varepsilon t_k)|x_i(t_k)| \\
&\quad + \sum_{j=1}^{n}L_j^{i_k}\left|b_{ij}^{i_k}\right|\int_{t_k-\tau_{ij}^{i_k}}^{t_k}\exp\left(\varepsilon\left(s+\tau_{ij}^{i_k}\right)\right)|x_j(s)|\mathrm{d}s\Big\} \\
&= \sum_{i=1}^{n}\xi_i^{i_k}\Big\{\exp(\varepsilon t_k^-)|x_i(t_k^-)| \\
&\quad + \sum_{j=1}^{n}L_j^{i_k}\left|b_{ij}^{i_k}\right|\int_{t_k^- -\tau_{ij}^{i_k}}^{t_k^-}\exp\left(\varepsilon\left(s+\tau_{ij}^{i_k}\right)\right)|x_j(s)|\mathrm{d}s\Big\} \\
&\le \sum_{i=1}^{n}\eta_i^{i_{k-1}}\xi_i^{i_{k-1}}\Big\{\exp(\varepsilon t_k^-)|x_i(t_k^-)| \\
&\quad + \beta_i^{i_{k-1}}\sum_{j=1}^{n}L_j^{i_{k-1}}\left|b_{ij}^{i_{k-1}}\right|\int_{t_k^- -\tau_{ij}^{i_{k-1}}}^{t_k^-}\exp(\varepsilon(s \\
&\quad + \tau_{ij}^{i_{k-1}}))|x_j(s)|\mathrm{d}s\Big\},
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
\eta_i^{i_{k-1}} &= \frac{\max_{1\le i\le n}\left\{\xi_i^{i_k}\right\}}{\min_{1\le i\le n}\left\{\xi_i^{i_{k-1}}\right\}}, \\
\beta_i^{i_{k-1}} &= \frac{\max_{1\le j\le n}\left\{\exp\left(\varepsilon\tau_{ij}^{i_k}\right)L_j^{i_k}b_{ij}^{i_k *}\right\}}{\min_{1\le j\le n}\left\{\exp\left(\varepsilon\tau_{ij}^{i_{k-1}}\right)L_j^{i_{k-1}}b_{ij}^{i_{k-1}*}\right\}}.
\end{aligned}
\tag{33}
$$

Let $\vartheta = \max_{1\le i\le n, 1\le i_{k-1}\le m}\{\eta_i^{i_{k-1}}, \eta_i^{i_{k-1}}\beta_i^{i_{k-1}}\} \ge 1$; we can get

$$
\begin{aligned}
V(\mathbf{x},t_k) &\le \vartheta\sum_{i=1}^{n}\xi_i^{i_{k-1}}\Big\{\exp(\varepsilon t_k^-)|x_i(t_k^-)| + \sum_{j=1}^{n}L_j^{i_{k-1}}\left|b_{ij}^{i_{k-1}}\right| \\
&\quad \cdot\int_{t_k^- -\tau_{ij}^{i_{k-1}}}^{t_k^-}\exp\left(\varepsilon\left(s+\tau_{ij}^{i_{k-1}}\right)\right)|x_j(s)|\mathrm{d}s\Big\} = \vartheta V(\mathbf{x},t_k^-).
\end{aligned}
\tag{34}
$$

Combining (31) and (34) yields

$$
\begin{aligned}
\mathrm{D}^+V(\mathbf{x},t) &\le V(\mathbf{x},t_k) \le \vartheta V(\mathbf{x},t_k^-) \\
&\le \vartheta V(\mathbf{x},t_{k-1}) \le \cdots \le \vartheta^{N_\sigma(t_0,t)}V(\mathbf{x},t_0).
\end{aligned}
\tag{35}
$$

When $t = t_0$, the i_0th subnetwork is activated; then

$$\begin{aligned}
V(\mathbf{x}, t_0) &= \sum_{i=1}^{n} \xi_i^{i_0} \Bigg\{ \exp(\varepsilon t_0)|x_i(t_0)| + \sum_{j=1}^{n} L_j^{i_0} \left| b_{ij}^{i_0} \right| \\
&\quad \cdot \int_{t_0-\tau_{ij}^{i_0}}^{t_0} \exp\left(\varepsilon\left(s+\tau_{ij}^{i_0}\right)\right) |x_j(s)| \mathrm{d}s \Bigg\} \\
&= \sum_{i=1}^{n} \xi_i^{i_0} \Bigg\{ \exp(\varepsilon t_0)|w_i(t_0) - w_i^*| \\
&\quad + \sum_{j=1}^{n} L_j^{i_0} \left| b_{ij}^{i_0} \right| \int_{t_0-\tau_{ij}^{i_0}}^{t_0} \exp\left(\varepsilon\left(s+\tau_{ij}^{i_0}\right)\right) \left| w_j(s) - w_j^* \right| \mathrm{d}s \Bigg\} \\
&\le \sum_{i=1}^{n} \xi_i^{i_0} \Bigg\{ \exp(\varepsilon t_0) \sup_{s\in[t_0-\tau,t_0]} |\phi_i(s) - w_i^*| \\
&\quad + \tau^{i_0} \sum_{j=1}^{n} L_j^{i_0} \left| b_{ij}^{i_0} \right| \exp\left(\varepsilon\left(t_0 + \tau^{i_0}\right)\right) \sup_{s\in[t_0-\tau,t_0]} |\phi_i(s) \\
&\quad - w_j^*| \Bigg\} = \exp(\varepsilon t_0) \sum_{i=1}^{n} \xi_i^{i_0} \sup_{s\in[t_0-\tau,t_0]} |\phi_i(s) - w_i^*| \\
&\quad + \tau^{i_0} \exp(\varepsilon t_0) \sum_{i=1}^{n} \xi_i^{i_0} \sum_{j=1}^{n} L_j^{i_0} \left| b_{ij}^{i_0} \right| \exp\left(\varepsilon \tau^{i_0}\right) \\
&\quad \cdot \sup_{s\in[t_0-\tau,t_0]} \left| \phi_i(s) - w_j^* \right| \\
&\le \exp(\varepsilon t_0) \sum_{i=1}^{n} \xi_i^{i_0} \sup_{s\in[t_0-\tau,t_0]} |\phi_i(s) - w_i^*| \\
&\quad + \tau^{i_0} \exp(\varepsilon t_0) \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \xi_j^{i_0} L_i^{i_0} \left| b_{ji}^{i_0} \right| \exp\left(\varepsilon \tau^{i_0}\right) \right) \\
&\quad \cdot \sup_{s\in[t_0-\tau,t_0]} |\phi_i(s) - w_i^*| \\
&= \exp(\varepsilon t_0) \sum_{i=1}^{n} \left[\xi_i^{i_0} + \tau^{i_0} \left(\sum_{j=1}^{n} \xi_j^{i_0} L_i^{i_0} \left| b_{ji}^{i_0} \right| \exp\left(\varepsilon \tau^{i_0}\right) \right) \right] \\
&\quad \cdot \sup_{s\in[t_0-\tau,t_0]} |\phi_i(s) - w_i^*| \\
&\le \exp(\varepsilon t_0) M^{i_0} \left(\sum_{i=1}^{n} \sup_{s\in[t_0-\tau,t_0]} |\phi_i(s) - w_i^*|^2 \right)^{1/2} \\
&= \exp(\varepsilon t_0) M^{i_0} \|\phi - \mathbf{w}\|_{t_0},
\end{aligned} \tag{36}$$

where

$$M^{i_0} = \sqrt{n} \max_{1\le i\le n} \left\{ \xi_i^{i_0} + \tau^{i_0} \left(\sum_{j=1}^{n} \xi_j^{i_0} L_i^{i_0} \left| b_{ji}^{i_0} \right| \exp\left(\varepsilon \tau^{i_0}\right) \right) \right\}. \tag{37}$$

Combining (27), (35), and (36) yields

$$\exp(\varepsilon t) \sum_{i=1}^{n} \xi_i^{i_k} |x_i(t)| \le \vartheta^{N_\sigma(t_0,t)} \exp(\varepsilon t_0) M^{i_0} \|\phi - \mathbf{w}^*\|_{t_0}. \tag{38}$$

Let $\xi = \min_{1\le i\le n, 1\le k\le m}\{\xi_i^k\}$; (38) becomes

$$\begin{aligned}
\|\mathbf{w}(t) - \mathbf{w}^*\| &\le \frac{\vartheta^{N_\sigma(t_0,t)} M^{i_0}}{\xi} \|\phi - \mathbf{w}^*\|_{t_0} \exp(-\varepsilon(t - t_0)) \\
&= \vartheta^{N_\sigma(t_0,t)} \exp(-\varepsilon(t - t_0)) \frac{M^{i_0}}{\xi} \|\phi - \mathbf{w}^*\|_{t_0} \\
&= \exp(N_\sigma(t_0, t) \ln \vartheta - \varepsilon(t - t_0)) \frac{M^{i_0}}{\xi} \|\phi - \mathbf{w}^*\|_{t_0} \\
&\le \exp\left(N_0 \ln \vartheta + \frac{t - t_0}{\mathscr{T}} \ln \vartheta - \varepsilon(t - t_0) \right) \frac{M^{i_0}}{\xi} \|\phi - \mathbf{w}^*\|_{t_0} \\
&= \exp\left(\frac{t - t_0}{T} \ln \vartheta - \varepsilon(t - t_0) \right) \vartheta^{N_0} \frac{M^{i_0}}{\xi} \|\phi - \mathbf{w}^*\|_{t_0} \\
&= \exp\left(-(t - t_0) \left(\frac{\varepsilon - \ln \vartheta}{\mathscr{T}} \right) \right) \vartheta^{N_0} \frac{M^{i_0}}{\xi} \|\phi - \mathbf{w}^*\|_{t_0}.
\end{aligned} \tag{39}$$

Let $M = \vartheta^{N_0} M^{i_0}/\xi$ and $\lambda = \varepsilon - \ln\vartheta/\mathscr{T}$; we have

$$\|\mathbf{w}(t) - \mathbf{w}^*\| \le M \|\phi - \mathbf{w}^*\|_{t_0} \exp(-\lambda(t - t_0)), \tag{40}$$

when $\mathscr{T} > \ln \vartheta/\varepsilon$ and $\lambda > 0$. According to Definition 1, equilibrium point system (2) $\mathbf{w}^*$ is robustly exponentially stable. The proof is completed.

Remark 2. For all $k \in \Sigma$, $\mathbf{C}_k^*$ are M-matrices which mean that delayed SNN (2) is globally exponentially stable under constrained switching. From the definitions of $\mathscr{F}_i^k(z)$ functions, we know that the value of ε relies on vector $\boldsymbol{\xi}_k = (\xi_1^k, \xi_2^k, \cdots, \xi_n^k)^{\mathrm{T}} \in \boldsymbol{\Lambda}(\mathbf{C}_k^*)$. So, for obtaining the maximum convergence rate λ^* or the minimum average dwell time $\mathscr{T}^*$, one can solve the optimization problem under constraint conditions $\mathscr{F}_i^k(\lambda, \boldsymbol{\xi}_k) < 0, \boldsymbol{\xi}_k \in \boldsymbol{\Lambda}(\mathbf{C}_k^*)$, $(i = 1, 2, \cdots, n, k \in \Sigma)$.

4.2. Exponential Stability under Arbitrary Switching. Define the indicator function

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \cdots, \gamma_m(t))^{\mathrm{T}}, \tag{41}$$

where

$$\gamma_k(t) = \begin{cases} 1, & \text{when the } k\text{th subnetwork is activated,} \\ 0, & \text{otherwise,} \end{cases} \tag{42}$$

with $k = 1, 2, \ldots, m$. Therefore, delayed SNN system (25) can be described as follows:

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \displaystyle\sum_{k=1}^{m} \gamma_k(t) \left[-e_i^k x_i(t) + \sum_{j=1}^{n} a_{ij}^k f_j^k(x_j(t)) + \sum_{j=1}^{n} b_{ij}^k f_j^k\left(x_j\left(t - \tau_{ij}^k\right)\right) \right], \\ x_i(s + t_0) = \psi_i(s), \quad s \in [-\tau, 0], \quad i = 1, \ldots, n. \end{cases} \tag{43}$$

For any switching signal, only one subnetwork is activated at any time, so it follows that $\sum_{k=1}^{m}\gamma_k(t)=1$.

Theorem 3. *Under Assumption 1, the equilibrium point of delayed SNNs (2) is robustly exponentially stable for all* $\mathbf{E}_k \in \mathbf{E}_k^{\mathrm{I}}$, $\mathbf{A}_k \in \mathbf{A}_k^{\mathrm{I}}$, $\mathbf{B}_k$ *in* $\mathbf{B}_k^{\mathrm{I}}$, *and any switching signal if the following conditions are satisfied:*

(i) $\mathbf{C}_k^* = \mathbf{E}_k^* - (\mathbf{A}_k^* + \mathbf{B}_k^*)\mathbf{L}_k, k \in \Sigma$, *are nonsingular M-matrices*

(ii) $\boldsymbol{\chi} = \cap_{k=1}^{m}\boldsymbol{\Lambda}(\mathbf{C}_k^*)$ *is nonempty.*

Moreover, the exponential convergence rate of system (2) is equal to λ, *which is determined by*

$$-\xi_i\left(\underline{e}_i^k-\lambda\right)+\sum_{j=1}^{n}\xi_j L_j^k\left(a_{ij}^{k*}+e^{\lambda\tau^k}b_{ij}^{k*}\right)<0, \quad (44)$$

for a given vector $\boldsymbol{\xi}\in\boldsymbol{\chi}$.

Proof 3. Consider Lyapunov function candidates $v_i(t)=|x_i(t)|\exp(\lambda(t-t_0))$. Calculating the upper right derivative D^+v_i of v_i along the solutions of (43), we get

$$\begin{aligned}
D^+v_i(t) &= \exp(\lambda(t-t_0))\,\mathrm{sgn}\,x_i\sum_{k=1}^{m}\gamma_k(t)\left[-e_i^k x_i(t)\right.\\
&\quad\left.+\sum_{j=1}^{n}a_{ij}^k f_j^k(x_j(t))+\sum_{j=1}^{n}b_{ij}^k f_j^k\left(x_j\left(t-\tau_{ij}^k\right)\right)\right]\\
&\quad+\lambda\exp(\lambda(t-t_0))|x_i(t)|\\
&\le \exp(\lambda(t-t_0))\sum_{k=1}^{m}\gamma_k(t)\left[\left(-e_i^k+\lambda\right)|x_i(t)|\right.\\
&\quad\left.+\sum_{j=1}^{n}\left|a_{ij}^k\right|\left|f_j^k(x_j(t))\right|+\sum_{j=1}^{n}\left|b_{ij}^k\right|\left|f_j^k\left(x_j\left(t-\tau_{ij}^k\right)\right)\right|\right]\\
&\le \exp(\lambda(t-t_0))\sum_{k=1}^{m}\gamma_k(t)\left[\left(-e_i^k+\lambda\right)|x_i(t)|\right.\\
&\quad\left.+\sum_{j=1}^{n}L_j^k\left|a_{ij}^k\right||x_j(t)|+\sum_{j=1}^{n}L_j^k\left|b_{ij}^k\right|\left|x_j\left(t-\tau_{ij}^k\right)\right|\right]\\
&\le \exp(\lambda(t-t_0))\sum_{k=1}^{m}\gamma_k(t)\left[\left(-\underline{e}_i^k+\lambda\right)|x_i(t)|\right.\\
&\quad\left.+\sum_{j=1}^{n}L_j^k a_{ij}^{k*}|x_j(t)|+\sum_{j=1}^{n}L_j^k b_{ij}^{k*}\left|x_j\left(t-\tau_{ij}^k\right)\right|\right]\\
&\le \sum_{k=1}^{m}\gamma_k(t)\left[\left(-\underline{e}_i^k+\lambda\right)|\exp(\lambda(t-t_0))x_i(t)|\right.\\
&\quad+\sum_{j=1}^{n}L_j^k a_{ij}^{k*}|\exp(\lambda(t-t_0))x_j(t)|\\
&\quad\left.+\exp\left(\lambda\tau^k\right)\sum_{j=1}^{n}L_j^k b_{ij}^{k*}\left|\exp\left(\lambda\left(t-\tau_{ij}^k-t_0\right)\right)x_j\left(t-\tau_{ij}^k\right)\right|\right].
\end{aligned} \quad (45)$$

Substituting $v_i(t)=\exp(\lambda(t-t_0))|x_i(t)|$ into the above inequality, we can get

$$\begin{aligned}
D^+v_i(t) &\le \sum_{k=1}^{m}\gamma_k(t)\left[\left(-\underline{e}_i^k+\lambda\right)|v_i(t)|+\sum_{j=1}^{n}L_j^k a_{ij}^{k*}|v_j(t)|\right.\\
&\quad\left.+\exp\left(\lambda\tau^k\right)\sum_{j=1}^{n}L_j^k b_{ij}^{k*}\sup_{t-\tau^k\le s\le t}v_j(s)\right],
\end{aligned} \quad (46)$$

for $i=1,2,\ldots,n$.

Since $\mathbf{C}_k^*$ are nonsingular M-matrices and $\boldsymbol{\chi}$ is nonempty, from Lemma 1, we know that there exists at least one vector $\boldsymbol{\xi}=(\xi_1,\xi_2,\cdots,\xi_n)^{\mathrm{T}}\in\boldsymbol{\chi}\subseteq\boldsymbol{\Lambda}(\mathbf{C}_k^*)$ such that

$$-\underline{e}_i^k\xi_i+\sum_{j=1}^{n}\left(\left|a_{ij}^{k*}\right|+\left|b_{ij}^{k*}\right|\right)L_j^k\xi_j<0, \quad (47)$$

for $i=1,2,\ldots,n, k\in\Sigma$.

Consider functions

$$\mathcal{G}_i^k\left(z_i^k\right)=-\xi_i\left(\underline{e}_i^k-z_i^k\right)+\sum_{j=1}^{n}\left(a_{ij}^{k*}+\exp\left(z_i^k\tau^k\right)b_{ij}^{k*}\right)L_j^k\xi_j, \quad (48)$$

with $i=1,2,\ldots,n$ and $k=1,2,\ldots,m$.

By inequality (47) and the definition of functions $\mathcal{G}_i^k$, it is clear that $\mathcal{G}_i^k(z_i^k)\in C^0$ and $\mathcal{G}_i^k(0)<0$. Because $\mathrm{d}\mathcal{G}_i^k(z_i^k)/\mathrm{d}z_i^k>0$, there are constants $\lambda_i^k>0$ such that

$$\mathcal{G}_i^k\left(\lambda_i^k\right)=-\xi_i\left(\underline{e}_i^k-\lambda_i^k\right)+\sum_{j=1}^{n}\left(a_{ij}^{k*}+\exp\left(\lambda_i^k\tau^k\right)b_{ij}^{k*}\right)L_j^k\xi_j=0. \quad (49)$$

Let $0<\lambda<\min_{1\le k\le m,1\le i\le n}\{\lambda_i^k\}$; then

$$\mathcal{G}_i^k(\lambda)=-\xi_i\left(\underline{e}_i^k-\lambda\right)+\sum_{j=1}^{n}\left(a_{ij}^{k*}+\exp\left(\lambda\tau^k\right)b_{ij}^{k*}\right)L_j^k\xi_j<0. \quad (50)$$

for $i=1,2,\ldots,n$ and $k\in\Sigma$.

Let $l_0=\|\phi-\mathbf{w}^*\|_{t_0}/\xi_{\min}$, where $\xi_{\min}=\min_{1\le i\le n}\{\xi_i\}$. So

$$v_i(s)=\exp(\lambda(s-t_0))|\phi_i(s)-w_i^*|<\xi_i l_0, t_0-\tau\le s<t_0, \quad i=1,2,\ldots,n. \quad (51)$$

For $t\ge t_0$, we claim that $v_i(t)<\xi_i l_0, i=1,2,\ldots,n$. If this is not true, there exist some i and corresponding $t'>0$, which make $v_i(t')=\xi_i l_0$, $\mathrm{D}^+v_i(t')\ge 0$, and $v_j(t)<\xi_j l_0$ for $t_0\le t\le t'$, $j=1,2,\ldots,n, j\ne i$. However, applying (44) and (46) leads to

$$D^+v_i(t') \le \sum_{k=1}^{m}\gamma_k(t')\Bigg[\left(-\underline{e}_i^k+\lambda\right)\left|v_i(t')\right| + \sum_{j=1}^{n}L_j^k a_{ij}^{k*}\left|v_j(t')\right| + \exp\left(\lambda\tau^k\right)\sum_{j=1}^{n}L_j^k b_{ij}^{k*}\sup_{t'-\tau^k\le s\le t'}v_j(s)\Bigg] \le \sum_{k=1}^{m}\gamma_k(t')\Bigg[\left(-\underline{e}_i^k+\lambda\right)\xi_i l_0 + \sum_{j=1}^{n}L_j^k a_{ij}^{k*}\xi_j l_0 + \exp\left(\lambda\tau^k\right)\sum_{j=1}^{n}L_j^k b_{ij}^{k*}\xi_j l_0\Bigg] < 0. \tag{52}$$

This is a contradiction. So $v_i(t) < \xi_i l_0$, $i = 1, 2, \ldots, n$, for $t \ge t_0$. That is, for $t \ge t_0$,

$$|x_i| < \xi_i l_0 \exp(-\lambda(t-t_0)) = \frac{\xi_i}{\xi_{\min}}\|\phi - \mathbf{w}^*\|_{t_0}\exp(-\lambda(t-t_0)), \quad i = 1, 2, \ldots, n. \tag{53}$$

Let $M = \sqrt{n}\cdot\max_{1\le i\le n}\{\xi_i\}/\xi_{\min}$; then we can get

$$\|\mathbf{w} - \mathbf{w}^*\| < M\|\phi - \mathbf{w}^*\|_{t_0}\exp(-\lambda(t-t_0)), \tag{54}$$

for $t \ge t_0$. From Definition 1, the equilibrium point of system (2) is robustly exponentially stable. Moreover, the exponential convergence rate is λ. The proof is completed.

Remark 3. The existence of exponential convergence rate λ has been proved, and from the definitions of functions $\mathcal{G}_i^k(z)$ and (44), we know that the value of λ relies on vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)^{\mathrm{T}} \in \boldsymbol{\chi}$. So, for obtaining maximum convergence rate λ^*, one can solve the optimization problem under constraint conditions $\mathcal{G}_i^k(\lambda, \xi) < 0, \xi \in \chi(i = 1, 2, \cdots, n, k \in \Sigma)$.
By virtue of Theorem 3, it is easy to get the following result.

Corollary 1. *Under Assumption 1, if for $k \in \Sigma$, $\mathbf{C}_k^* = \mathbf{E}_k^* - (\mathbf{A}_k^* + \mathbf{B}_k^*)\mathbf{L}_k$ are nonsingular M-matrices, then for all $\boldsymbol{E}_k \in \boldsymbol{E}_k^{\mathrm{I}}$, $\boldsymbol{A}_k \in \boldsymbol{A}_k^{\mathrm{I}}$, and $\boldsymbol{B}_k \in \boldsymbol{B}_k^{\mathrm{I}}$, the equilibrium point of system (2) is robustly exponentially stable for any switching signal with the average dwell time satisfying*

$$\mathcal{T} > \mathcal{T}^* = \frac{\ln\eta_{\max}}{\lambda}, \tag{55}$$

where $\eta_{\max} = \max_{1\le i\le n, 1\le k\le m}\{\xi_i^k\}/\min_{1\le i\le n,1\le k\le m}\{\xi_i^k\}$ and $\lambda > 0$ is determined by inequality

$$\xi_i^k\left(-\underline{e}_i^k + \lambda\right) + \sum_{j=1}^{n}\left(a_{ij}^{k*} + \exp\left(\lambda\tau^k\right)b_{ij}^{k*}\right)L_j^k\xi_j^k < 0, \tag{56}$$

for some given $\boldsymbol{\xi}_k = (\xi_1^k, \xi_2^k, \cdots, \xi_n^k)^{\mathrm{T}} \in \boldsymbol{\chi}(\mathbf{C}_k^*)$.

Proof 4. Consider Lyapunov function candidates $v_i(t) = |x_i(t)|\exp(\lambda(t-t_k))$. For convenience, we assume that the j_kth subsystem is activated when $t \in [t_k, t_{k+1})$. Calculating the upper right derivative $D^+v_i(t)$ of $v_i(t)$ along the solutions of (25), we get

$$D^+v_i(t) = \exp(\lambda(t-t_k))\,\mathrm{sgn}\,x_i\Bigg[-e_i^{j_k}x_i(t) + \sum_{j=1}^{n}a_{ij}^{j_k}f_j^{j_k}(x_j(t)) + \sum_{j=1}^{n}b_{ij}^{j_k}f_j^{j_k}\left(x_j\left(t-\tau_{ij}^{j_k}(t)\right)\right)\Bigg] + \lambda\exp(\lambda(t-t_k))|x_i(t)|. \tag{57}$$

It can be known from Theorem 3 that there exists $\lambda_i^{j_k} > 0$ such that

$$\mathcal{G}_i^{j_k}\left(\lambda_i^{j_k}\right) = -\xi_i^{j_k}\left(\underline{e}_i^{j_k} - \lambda_i^{j_k}\right) + \sum_{j=1}^{n}\left(a_{ij}^{j_k*} + \exp\left(\lambda_i^{j_k}\tau^{j_k}\right)b_{ij}^{j_k*}\right)L_j^{j_k}\xi_j^{j_k} = 0. \tag{58}$$

Let $0 < \lambda < \min_{1\le i\le n}\{\lambda_i^{j_k}\}$; then

$$\mathcal{G}_i^{j_k}(\lambda) = -\xi_i^{j_k}\left(\underline{e}_i^{j_k} - \lambda\right) + \sum_{j=1}^{n}\left(b_{ij}^{j_k*} + \exp\left(\lambda\tau^{j_k}\right)c_{ij}^{j_k*}\right)L_j^{j_k}\xi_j^{j_k} < 0. \tag{59}$$

for $i = 1, 2, \ldots, n$, $j_k \in \Sigma$.
By the proof of Theorem 3, we know that

$$|x_i(t)| < \frac{\xi_i^{j_k}}{\min_{1\le i\le n}\left\{\xi_i^{j_k}\right\}}|x(t_k)|\exp(-\lambda(t-t_k)), \tag{60}$$

for $t \in [t_k, t_{k+1})$. Since the system state is continuous, it follows from (60) that

$$\begin{aligned}|x_i(t)| &< \frac{\xi_i^{j_k}}{\min_{1\le i\le n}\left\{\xi_i^{j_k}\right\}}|x_i(t_k)|\exp(-\lambda(t-t_k)) \\ &< \cdots < \exp\left(\sum_{l=0}^{k}\ln\eta_{j_l} - \lambda(t-t_0)\right)|x_i(t_0)| \\ &< \exp((k+1)\ln\eta_{\max} - \lambda(t-t_0))|x_i(t_0)| \\ &= \eta_{\max}\exp(N_\sigma(t_0,t)\ln\eta_{\max} - \lambda(t-t_0))|x_i(t_0)| \\ &\le \eta_{\max}{}^{(N_0+1)}\exp\left(-\left(\lambda - \frac{\ln\eta_{\max}}{\mathcal{T}}\right)(t-t_0)\right)|x_i(t_0)|,\end{aligned} \tag{61}$$

where $\eta_{j_v} = \xi_i^{j_v}/\min_{1\le i\le n}\{\xi_i^{j_v}\}$ and $\eta_{\max} = \max_{1\le j_v\le m}\{\eta_{j_v}\}$. Let $M = \sqrt{n}(\eta_{\max})^{N_0+1}$ and $\varepsilon = \lambda - \ln\eta_{\max}/\mathcal{T}$ yield

$$\|\mathbf{w}(t)-\mathbf{w}^*\| \le M\|\phi-\mathbf{w}^*\|_{t_0} \exp\left(-\varepsilon(t-t_0)\right). \quad (62)$$

When $\mathcal{T} > \mathcal{T}^* = \ln \eta_{\max}/\lambda$, $\varepsilon > 0$. According to Definition 1, system (2) is robustly exponentially stable, and the exponential convergence rate is ε. The proof is completed.

Remark 4. Stability conditions in Theorems 2 and 3 and Corollary 1 are explicit for SNNs, which are convenient to verify in practice. However, they have the disadvantage of neglecting the signs of entries in the connection weight matrices $\mathbf{A}_k$ and $\mathbf{B}_k$, and thus, differences between excitatory and inhibitory effects might be ignored.

Remark 5. Theorems 2 and 3 and Corollary 1 reflect the relationship between arbitrary switching and constrained switching of system (2). If $\mathbf{C}_k^*$ are M-matrices for all $k \in \Sigma$, then system (2) would be exponentially stable at least under constrained switching. If $\boldsymbol{\chi} = \cap_{k=1}^{m} \boldsymbol{\Lambda}(\mathbf{C}_k^*)$ is nonempty, then system (2) is exponentially stable for any switching signal.

5. Numerical Examples

We present two examples to illustrate the main results.

Example 1. Consider a delayed SNNs with two subnetworks, and the relevant parameters of system (25) are given as follows [18, 21]:

$$\mathbf{E}_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} 1.0 & -0.4 & 0.4 \\ -0.4 & 0.2 & 0.2 \\ 0.2 & 0.4 & -0.4 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 0.3 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.2 \\ 0.3 & 0.2 & 0.2 \end{pmatrix},$$
$$\mathbf{E}_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2.8 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 1.5 & -0.3 & 0.4 \\ -0.4 & 0.3 & 0.4 \\ 0.2 & 0.3 & -0.5 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0.2 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.2 \end{pmatrix}. \quad (63)$$

Take the activation functions as $\mathbf{f}_1(\mathbf{x}) = ((1/2)(x_1 + \sin x_1), \tanh(x_2), (1/2)(|x_3+1| - |x_3-1|))^{\mathrm{T}}$ and $\mathbf{f}_2(\mathbf{x}) = ((1/2)(x_1 + \sin x_1), (1/2)(x_2 + \sin x_2), (1/2)(x_3 + \sin x_3))^{\mathrm{T}}$.
Obviously, $\mathbf{f}_1(\mathbf{x}) = \mathbf{f}_2(\mathbf{x})$ satisfy Assumption 1 and $\mathbf{L}_1 = \mathbf{L}_2 = \mathrm{diag}(1,1,1)$.

Step 1. Determine whether $\mathbf{C}_k^* = \mathbf{E}_k^* - (\mathbf{A}_k^* + \mathbf{B}_k^*)\mathbf{L}_k$ are M-matrices.

$$\mathbf{C}_1^* = \begin{pmatrix} 0.7 & -0.6 & -0.7 \\ -0.6 & 2.6 & -0.4 \\ -0.5 & -0.6 & 1.4 \end{pmatrix}, \quad \mathbf{C}_2^* = \begin{pmatrix} 1.3 & -0.6 & -0.7 \\ -0.7 & 2.3 & -0.6 \\ -0.5 & -0.6 & 1.3 \end{pmatrix} \quad (64)$$

are both M-matrices, which imply that the considered SNN is at least exponentially stable under constrained switching. If $\boldsymbol{\chi}$ is nonempty, then the system is exponentially stable under arbitrary switching.

Step 2. Determine whether $\boldsymbol{\chi} = \cap_{k=1}^{2} \boldsymbol{\Lambda}(\mathbf{C}_k^*)$ is nonempty.
Let $\boldsymbol{\xi} = (0.2146, 0.1000, 0.1288)^{\mathrm{T}}$; then we can get $\mathbf{C}_1^*\xi > 0$ and $\mathbf{C}_2^*\xi > 0$. That is, $\cap_{k=1}^{2} \boldsymbol{\Lambda}(\mathbf{C}_k^*) \neq \varnothing$. Therefore, the considered SNN is globally exponentially stable for any switching signal.

Step 3. Calculate the maximum exponential convergence rate λ.
By using LINGO solver, we can get the maximum convergence rate $\lambda = 0.6999$ under the constraint conditions $\mathcal{G}_i^k(\lambda, \boldsymbol{\xi}) < 0$, $\boldsymbol{\xi} \in \boldsymbol{\chi} = \cap_{k=1}^{2} \boldsymbol{\Lambda}(\mathbf{C}_k^*)$, $k, i = 1, 2$, and the corresponding vector $\boldsymbol{\xi} = (0.2605, 0.1205, 0.1562)^{\mathrm{T}}$.

The numerical simulations are given in Figures 1–5. We can see that the state trajectories converge to the equilibrium point of the system, which is consistent with the conclusion of Theorem 3. On the other hand, from [18, 21], we know that when the average dwell time of switched signal is greater than or equal to 9.1936*s* and 0.8396*s*; then the considered neural network is exponentially stable. Table 1 shows that the stability criteria obtained in this paper are less conservative than those in [18, 21].

Example 2. Consider the second-order delayed SNNs in system (25) described by [49]: $\sigma(t)\colon [0, +\infty) \to \sum = \{1, 2\}$, $\mathbf{f}_1(\mathbf{x}) = \mathbf{f}_2(\mathbf{x}) = (0.5x_1 + 0.5 \sin x_1, 0.5x_2 + 0.5 \sin x_2)^{\mathrm{T}}$, $\tau_{ij}^k(t) = |0.5 + 0.5 \sin(t)|$, $i, j, k = 1, 2$, and

$$
\begin{aligned}
\underline{\mathbf{E}}_1 &= \begin{pmatrix} 3.99 & 0 \\ 0 & 2.99 \end{pmatrix}, \\
\bar{\mathbf{E}}_1 &= \begin{pmatrix} 4.01 & 0 \\ 0 & 3.02 \end{pmatrix}, \\
\underline{\mathbf{E}}_2 &= \begin{pmatrix} 2.81 & 0 \\ 0 & 3.60 \end{pmatrix}, \\
\bar{\mathbf{E}}_2 &= \begin{pmatrix} 2.95 & 0 \\ 0 & 3.72 \end{pmatrix}, \\
\underline{\mathbf{A}}_1 &= \begin{pmatrix} 1.19 & 2.35 \\ 0.05 & 0.03 \end{pmatrix}, \\
\bar{\mathbf{A}}_1 &= \begin{pmatrix} 1.21 & 2.41 \\ 0.06 & 0.04 \end{pmatrix}, \\
\underline{\mathbf{A}}_2 &= \begin{pmatrix} 0.87 & -0.03 \\ 2.07 & 0.68 \end{pmatrix}, \\
\bar{\mathbf{A}}_2 &= \begin{pmatrix} 1.01 & 0.10 \\ 2.28 & 0.80 \end{pmatrix}, \\
\underline{\mathbf{B}}_1 &= \begin{pmatrix} 0.09 & 3.14 \\ -0.05 & 0.43 \end{pmatrix}, \\
\bar{\mathbf{B}}_1 &= \begin{pmatrix} 0.11 & 3.32 \\ 0.13 & 0.54 \end{pmatrix}, \\
\underline{\mathbf{B}}_2 &= \begin{pmatrix} 0.11 & -0.02 \\ 2.87 & 0.05 \end{pmatrix}, \\
\bar{\mathbf{B}}_2 &= \begin{pmatrix} 0.35 & 0.10 \\ 3.00 & 0.13 \end{pmatrix}.
\end{aligned} \tag{65}
$$

Obviously, $\mathbf{f}_1(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$ satisfy Assumption 1 and $\mathbf{L}_1 = \mathbf{L}_2 = \text{diag}\,(1, 1)$,

$$
\begin{aligned}
\mathbf{A}_1^* &= \begin{pmatrix} 3.99 & 0 \\ 0 & 2.99 \end{pmatrix}, \\
\mathbf{B}_1^* &= \begin{pmatrix} 1.21 & 2.41 \\ 0.06 & 0.04 \end{pmatrix}, \\
\mathbf{C}_1^* &= \begin{pmatrix} 0.11 & 3.32 \\ 0.13 & 0.54 \end{pmatrix}, \\
\mathbf{A}_2^* &= \begin{pmatrix} 2.81 & 0 \\ 0 & 3.60 \end{pmatrix}, \\
\mathbf{B}_2^* &= \begin{pmatrix} 1.01 & 0.10 \\ 2.28 & 0.80 \end{pmatrix},
\end{aligned}
$$

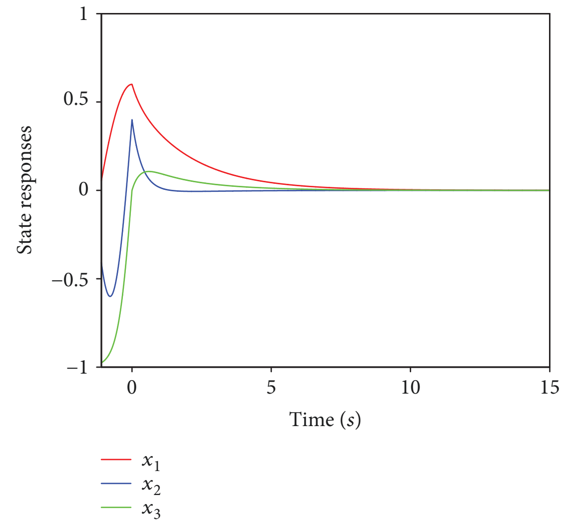

Figure 1: State responses of subnetwork 1 of the considered SNNs for Example 1 with the initial condition $\psi(s) = [\cos(s) - 0.4;\ \sin(2s) + 0.4;\ \tanh(2s)]^{\mathrm{T}}$.

$$
\mathbf{C}_2^* = \begin{pmatrix} 0.35 & 0.10 \\ 3.00 & 0.13 \end{pmatrix}. \tag{66}
$$

Step 1. Determine whether $\mathbf{C}_k^* = \mathbf{E}_k^* - (\mathbf{A}_k^* + \mathbf{B}_k^*)\mathbf{L}_k$, $k = 1, 2$, are M-matrices.

$$
\begin{aligned}
\mathbf{C}_1^* &= \begin{pmatrix} 2.67 & -5.73 \\ -0.19 & 2.41 \end{pmatrix}, \\
\mathbf{C}_2^* &= \begin{pmatrix} 1.45 & -0.20 \\ -5.28 & 2.67 \end{pmatrix}
\end{aligned} \tag{67}
$$

are both M-matrices, which mean that the considered system is at least globally exponentially stable under constrained switching.

Step 2. Determine whether $\chi = \cap_{k=1}^{2} \mathbf{\Lambda}(\mathbf{C}_k^*)$, $k = 1, 2$, is nonempty.
As shown in Figure 6, $\cap_{k=1}^{2} \mathbf{\Lambda}(\mathbf{C}_k^*) = \varnothing$. Therefore, we can not claim that the considered system is stable under arbitrary switching.

Step 3. Calculate the average dwell time $\mathcal{T}^*$.
By using LINGO solver, we can get the maximum convergence rate $\lambda = 0.4387$ under the constraint conditions $\mathcal{G}_i^k(\lambda, \xi_k) < 0$, $\xi_k \in \mathbf{\Lambda}(\mathbf{C}_k^*)$, $k, i = 1, 2$, and the corresponding vectors $\boldsymbol{\xi}_1 = (5.5719, 1.4679)^{\mathrm{T}}$, $\boldsymbol{\xi}_2 = (1.2981, 4.1662)^{\mathrm{T}}$, and $\eta_{\max} = 4.2923$. So we can get the average dwell time $\mathcal{T}^* = \ln \eta_{\max} / \lambda = 3.3208s$.

For numerical simulation, let $\mathbf{E}_k = \mathbf{E}_k^*$, $\mathbf{A}_k = \mathbf{A}_k^*$, and $\mathbf{B}_k = \mathbf{B}_k^*$, where $k = 1, 2$, and choose the initial value $(\psi_1, \psi_2)^{\mathrm{T}} = (\cos(2s) - 0.4, \sin(2s) + 0.4)^{\mathrm{T}}$, $s \in [-1, 0]$. Figures 7–9 display the state responses and state norm responses of

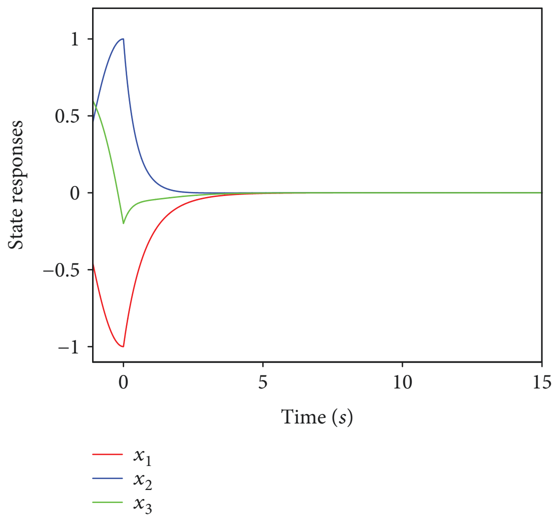

Figure 2: State responses of subnetwork 2 of the considered SNNs for Example 1 with the initial condition $\psi(s) = [-\cos(s); \ \cos(s); -\tanh(s) - 0.2]^{\mathrm{T}}$.

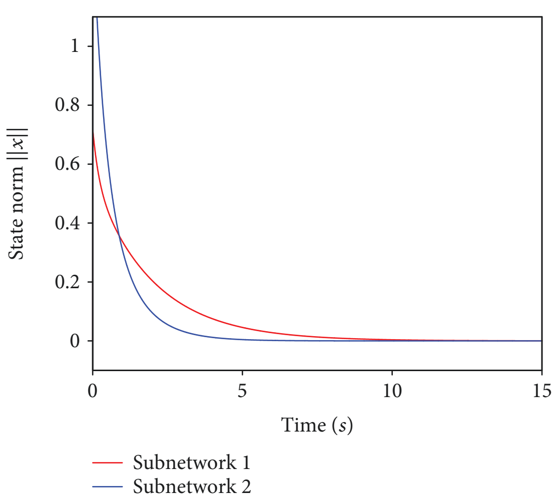

Figure 3: State norm responses of the subnetworks of the considered SNNs for Example 1.

these two subnetworks. Figures 10–13 display the state responses and state norm responses of the delayed SNNs under two different switching signals. From Figures 10 and 11, we can see that with the dwell time $\mathcal{T}_1 = 1s$ that is less than $\mathcal{T}^*$, the trajectories can not converge to the equilibrium point of the system; Figures 12 and 13 show that with the dwell time $\mathcal{T}_2 = 4s$ that is larger than $\mathcal{T}^*$, the trajectories converge to the equilibrium point of the system. This is consistent with the conclusion of Corollary 1.

These two examples indicate the correctness and effectiveness of the results proposed in this paper.

6. Conclusion

The existence, uniqueness, and robust exponential stability of the equilibrium point of SNNs with time delays were

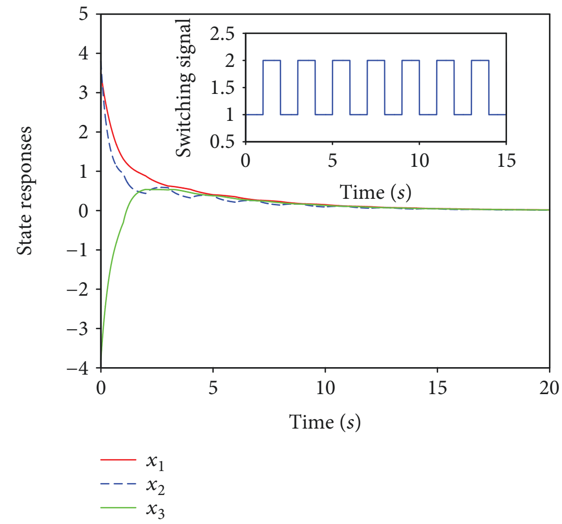

Figure 4: State responses of the considered SNNs for Example 1 with the initial condition $\psi(s) = [3.5; 3.8; -3.8]^{\mathrm{T}}$.

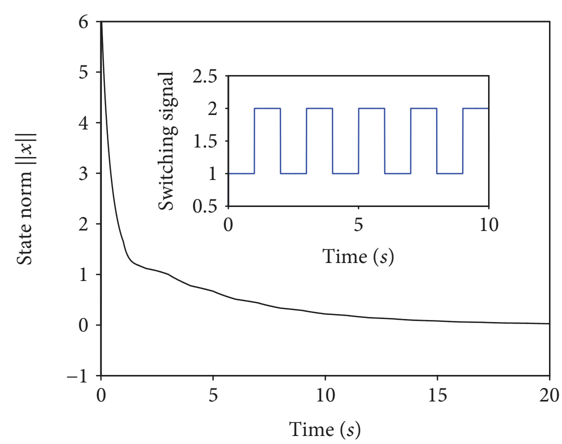

Figure 5: State norm response of the considered SNNs for Example 1 with switching signal.

Table 1: Stability conditions are derived by different methods.

Methods	Switching signal	Average dwell time
[18]	Constrained	9.1936
[21]	Constrained	0.8396
Theorem 3	Arbitrary	—

investigated in this paper. For each specified switching signal $\sigma(t)$, conditions for guaranteeing the existence and uniqueness of the delayed SNNs were obtained by resorting to the homomorphism mapping theorem and M-matrix theory. By using average dwell time approach, sufficient conditions to ensure the robust exponential stability of the delayed SNNs under constrained switching were derived, and inequality technique and the idea of vector Lyapunov function were employed to obtain conditions for ensuring the globally exponential stability of the delayed SNNs under

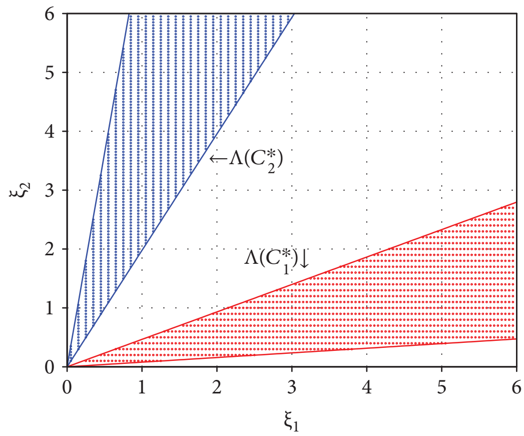

FIGURE 6: $\mathbf{\Lambda}(\mathbf{C}_k^*)$ of the considered SNNs for Example 2.

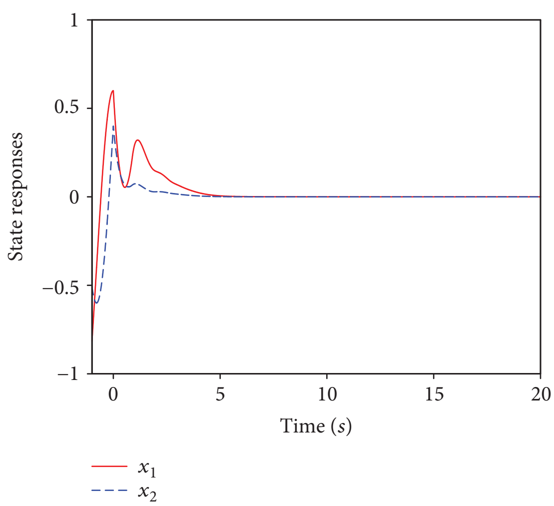

FIGURE 7: State responses of subnetwork 1 of the considered SNNs for Example 2.

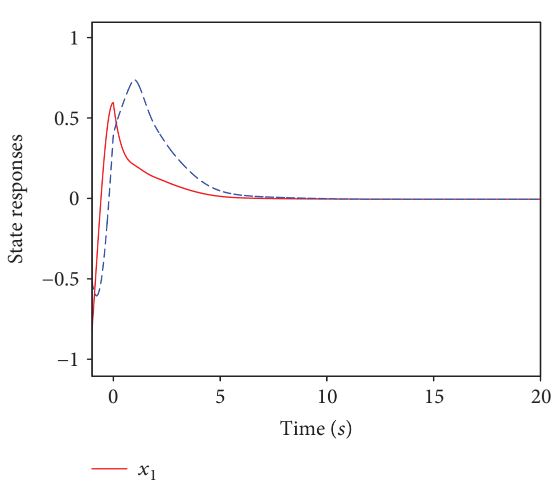

FIGURE 8: State responses of subnetwork 2 of the considered SNNs for Example 2.

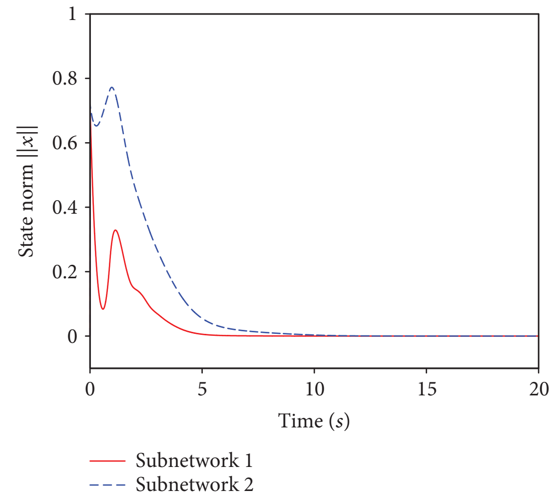

FIGURE 9: State norm responses of the subnetworks of the considered SNNs for Example 2.

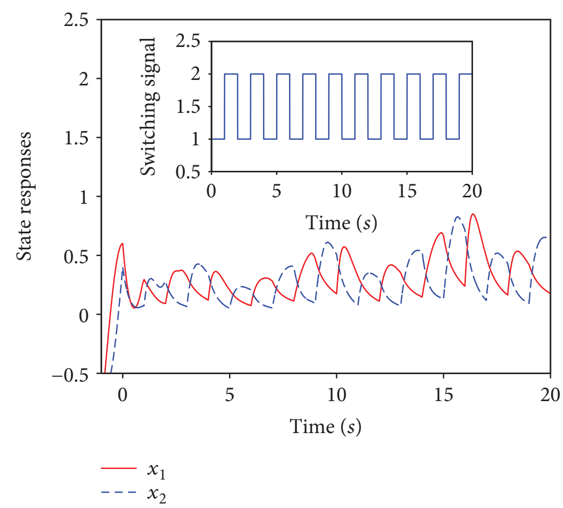

FIGURE 10: State responses of the considered SNNs for Example 2 with average dwell time $\mathcal{T}_1 = 1s < \mathcal{T}^*$.

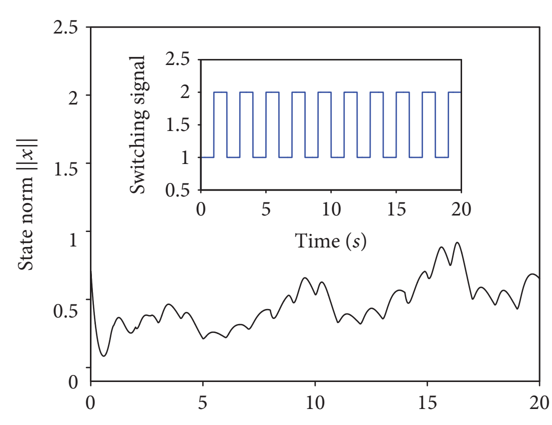

FIGURE 11: State norm response of the considered SNNs for Example 2 with average dwell time $\mathcal{T}_1 = 1s < \mathcal{T}^*$.

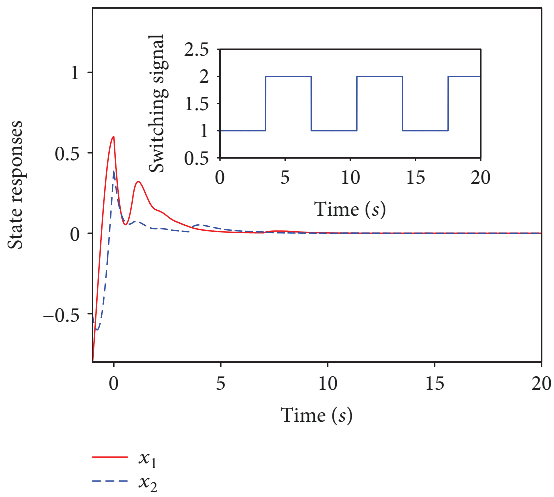

FIGURE 12: State responses of the considered SNNs for Example 2 with average dwell time $\mathcal{T}_2 = 4s > \mathcal{T}^*$.

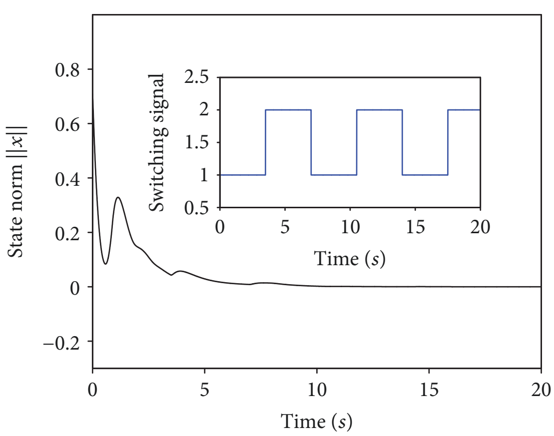

FIGURE 13: State norm response of the considered SNNs for Example 2 with average dwell time $\mathcal{T}_2 = 4s > \mathcal{T}^*$.

arbitrary switching. The obtained results not only have less conservativeness but also reveal the relationship between the constrained switching and the arbitrary switching of the delayed SNNs. Finally, two numerical examples were presented to demonstrate the effectiveness and less conservativeness of the main results over existing literature.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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