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# The Geometrical Meaning of Time

**Asher Yahalom**

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**Abstract** It is stated in many text books that the any metric appearing in general relativity should be locally Lorentzian i.e. of the type $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ this is usually presented as an independent axiom of the theory, which can not be deduced from other assumptions. The meaning of this assertion is that a specific coordinate (the temporal coordinate) is given a unique significance with respect to the other spatial coordinates. In this work it is shown that the above assertion is a consequence of requirement that the metric of empty space should be linearly stable and need not be assumed.

**Keywords** General relativity · Stability of solutions

## 1 Introduction

It is well known that our daily space-time is approximately of Lorentz (Minkowski) type that is, it possesses the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The above statement is taken as one of the central assumptions of the theory of special relativity and has been supported by numerous experiments. Some may be satisfied by the overwhelming evidence that space-time is Lorentzian and see no need to investigate this issue any further, others including the author of this paper see it as a profound mystery of nature and ask why should it be so?

For instance the mass of the electron is known empirically to a reasonably high accuracy but this does not explain why electrons have such a mass nor does it ex-

A. Yahalom
Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge CB3 0HA, UK

A. Yahalom (✉)
Ariel University Center of Samaria, Ariel 40700, Israel
e-mail: asya@ariel.ac.il

plain why electrons exist at all. To explain such phenomena researchers try to construct theories such as string theory which hopefully will yield an explanation. Thus well established empirical facts demand theories to explain them. To put this in other words, science is not just a collection of well established facts, rather it is a struggle to explain those well established facts using a *minimal* number of assumptions. The emphasis is on the word *minimal* because explaining the well established empirical facts with an arbitrary number of assumptions is not a challenge at all.

Furthermore it is assumed in the general theory of relativity that any space-time is locally of the type $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, although it can not be presented so globally due to the effect of matter. This is a part of the demands dictated by the well known equivalence principle. The above principle is taken to be one of the assumptions of general relativity. Other assumptions such as diffeomorphism invariance and the requirement that the theory reduces to Newtonian gravity in the proper regime leads to the Einstein equations:

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

in which $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor, $G$ is the gravitational constant and $c$ is the velocity of light. The Principle of Equivalence rests on the equality of gravitational and inertial mass, demonstrated by Galileo, Huygens, Newton, Bessel, and Eötvös. Einstein reflected that, as a consequence, no external static homogeneous gravitational field could be detected in a freely falling elevator, for the observers, their test bodies, and the elevator itself would respond to the field with the same acceleration [1]. This means that the observer will experience himself as free, not feeling the effect of any force at all. Mathematically speaking for the observer space time is locally (but not globally) flat and Minkowskian. Assuming that the metric is Minkowskian apart from a small linear correction to the metric, Einstein's field equations reduce in the case of slow particles to Newton's field equation:

$$\nabla^2 \phi = 4\pi G \rho \tag{2}$$

were $\phi$ is the gravitational potential and $\rho$ is the density. The point of this paper is that one need not assume that space-time is locally Minkowskian based on an empirical (unexplained) fact, rather one can derive this property from the field equations themselves. Once this fact is established one can proceed with the usual linear analysis to derive Newton's field equation.

In what follows we will show that the assumption about space-time being locally of the type $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is not necessary, (contrary to what is argued in so many text books, see for example [2]) rather we will argue that this metric is the only possible linearly stable solution to the Einstein equation (1) in vacuum, that is for the case $T_{\mu\nu} = 0$. And thus reduce the number of assumptions needed to obtain the celebrated results of general relativity. By making the theory more compact we enhance its predictive strength. Moreover, it is shown that the existence of a temporal coordinate is a necessary consequence of the geometrical structure of the four dimensional space and not a separate ad-hoc assumption.

Eddington [3, p. 25] has considered the possibility that the universe contains different domains in which some domains are locally Lorentzian and others have some other local metric of the type $\eta_{\mu\nu} = \text{diag}(-1, -1, -1, -1)$ or the type $\eta_{\mu\nu} = \text{diag}(+1, +1, -1, -1)$. For the first case he concluded that the transition will not be possible since one will have to go through a static hypersurface with a metric $\eta_{\mu\nu} = \text{diag}(0, -1, -1, -1)$.[1] Going to the domain in which $\eta_{\mu\nu} = \text{diag}(+1, +1, -1, -1)$ means that one will have to pass through $\eta_{\mu\nu} = \text{diag}(+1, 0, -1, -1)$ in which space becomes two dimensional.[2] The stability of those domains was not discussed by Eddington.

Greensite [4] and Carlini and Greensite [5, 6] have studied the metric $\eta_{\mu\nu} = \text{diag}(e^{i\theta}, -1, -1, -1)$ in which $\theta$ the "wick angle" was treated as a quantum field dynamical variable. They have shown that the real part of the quantum field effective potential is minimized for the Lorentzian metric $\theta = 0$ and for the same case the imaginary part of the quantum field effective potential is stationary. Furthermore they have calculated the fluctuations around this minimal value and have shown them to be of the order $(\frac{l_p}{R})^3$ in which $l_p$ is the Planck length and $R$ is the scale of the universe. Elizalde and collaborators [7] have shown that the same arguments apply to a five dimensional Kaluza-Klein universe of the type $R^4 \times T^1$.

Itin and Hehl [8] have deduced that space time must have a Lorentzian metric in order to support classical electric/magnetic reciprocity.

H. van Dam and Y. Jack Ng [9] have argued that in the absence of a Lorentzian metric one can not obtain an appropriate finite representation of the relevant groups and hence the various quantum wave equations can not be written.

What is common to the above approaches is that additional theoretical structures and assumptions are needed in order to justify what appears to be a fundamental property of space-time. In this paper we claim otherwise. We will show that General relativistic equations and some "old fashioned" linear stability analysis will lead to a unique choice of the Lorentzian metric being the only one which is linearly stable.

H. Nikolić [10] has argued that space time must have a Lorentzian metric in order that various field equations (including the equations for linear gravitational metrics) will have a "Cauchy problem" form. The author of this paper does not agree. Rather it seems that the fact that various field equations can be presented as a "Cauchy problem" is a *consequence* of space time having a Lorentzian metric but does not explain this fact.

This paper assumes that space time must have four dimensions, it does not explain why this is so. For a possible explanation derived from string theory one can consult a paper by S.K. Rama [11].

The plan of this paper is as follows: in the first section we describe the possible constant metrics which are not equivalent to each other by trivial manipulations. In the second section we use the linearized theory to study the linearized stability of the possible constant metrics and thus divide them into two classes: linearly stable and linearly unstable. The last section will discuss some possible implications of our results.

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[1] Prof. Lynden Bell has noticed that there may be another way going through the metric $\eta_{\mu\nu} = \text{diag}(\infty, -1, -1, -1)$, the author thanks him for his remark.

[2] Again there may be another way going through the metric $\eta_{\mu\nu} = \text{diag}(+1, \infty, -1, -1)$.

## 2 Possible Constant Metrics

In this section we study what are the possible constant metrics available in the general theory of relativity which are not equivalent to one another by a trivial transformation, that amounts to a simple change of coordinates.

Let us thus study the four-dimensional interval:

$$d\tau^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \tag{3}$$

in what follows Greek letters take the traditional values of 0–3, and summation convention is assumed. $\eta_{\mu\nu}$ is any real constant matrix.

Since $\eta_{\mu\nu}$ is symmetric we can diagonalize it using a unitary transformation in which both the transformation matrix and the eigenvalues obtained are real. Thus without loss of generality we can assume that in a proper coordinate basis:

$$\eta = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3). \tag{4}$$

Next, by changing the units of the coordinates, we can always obtain:

$$\eta = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1) \tag{5}$$

notice that a zero eigen-value is not possible due to our assumption that the space is four dimensional.

We conclude that the metrics $\eta$ given in (5) are the most general constant metrics possible. In what follows we will study the stability of those solutions.

## 3 Linear Stability Analysis

In the lack of matter Einstein equation (1) becomes $G_{\mu\nu} = 0$. For small perturbations to flat space time, the Einstein equation takes the form:

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0 \tag{6}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the Lorentz gauge has been assumed and $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ (see Misner, Thorne and Wheeler [2] for further details).[3] In addition the $\bar{h}$ tensor must satisfy the gauge condition:

$$\bar{h}_{\mu\alpha,}{}^{\alpha} = 0. \tag{7}$$

Notice that the general theory of relativity does not dictate any type of boundary conditions to (6). Neither initial conditions of Cauchy form are specified, nor Dirichlet or Neuman conditions on a closed boundary are given. The question one

[3] In the above equation standard notations of general relativity are assumed. Those include the summation convention, the lowering and raising of indices using the metric and the symbol "," representing partial derivation.

should ask is in the absence of any type of boundary conditions is: can one construct a *general* solution to (6) which is finite in every point in space time.

To answer this question we introduce the Fourier decomposition of $\bar{h}_{\mu\nu}$:

$$\bar{h}_{\mu\nu} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} A_{\mu\nu}(x_0, \vec{k}) e^{i\vec{k}\cdot\vec{x}} d^3k, \quad \vec{k} = (k^1, k^2, k^3), \ \vec{x} = (x^1, x^2, x^3). \tag{8}$$

Although it seems that $x_0$ is granted a unique status in this representation this is not so, since $x_0$ is just any arbitrary coordinate and the same coordinate could be denoted as $x_1$ or by any other name which the reader would care to embrace.

Notice, however, that Fourier decomposing a function in some manifold $x_0 =$ *const* means that the function is well behaved at least in that slice. This could be of course otherwise but then our result would be contradicted by our assumptions.

Introducing the decomposition equation (8) into (6) leads to:

$$\eta^{00}\partial_0^2 A_{\mu\nu} - \eta^{mn} k_m k_n A_{\mu\nu} = 0 \tag{9}$$

in which $m, n$ are integers between 1–3. Choosing $\eta^{00} = 1$ we see that the only way to avoid exploding solutions is to choose $\eta^{mn} = \text{diag}(-1, -1, -1)$[4] thus one stable metric would be:

$$\eta^{(1)} = \text{diag}(1, -1, -1, -1) \tag{10}$$

alternatively we can choose $\eta^{00} = -1$ in this case the only way to avoid exploding solutions is to choose $\eta^{mn} = \text{diag}(1, 1, 1)$, thus a second stable metric would be:

$$\eta^{(2)} = \text{diag}(-1, 1, 1, 1) \tag{11}$$

that is $\eta^{(1)} = -\eta^{(2)}$.

A famous example due to Hadamard shows that exploding solutions appear when the Cauchy conditions are applied to the Laplace equation. This example is in fact the reason behind part of the author's results. When one can not prevent nature from enforcing arbitrary Cauchy type conditions one is forced to give up Euclidean (and other types) of flat metrics to avoid Laplace like equations and hence exploding solutions.

One can claim that even in the Lorentzian case we do have an exploding solution $A_{\mu\nu} \sim x_0$ in the degenerate case $|\vec{k}| = 0$ in which the perturbation is uniform over the entire manifold $x_0 =$ *const*. Notice, however, this solution is only linearly exploding and not exponentially exploding as is the generic case of the other flat metrics considered. A way to avoid this type of solution is to assume that space time is topologically a 4D Torus. In this case, this type of solution may be rejected on the grounds that the metric must be a single valued function of the space time coordinates.

The reader should notice that only the "spatial" functions $A_{mn}$ should be thought of as independent. The "temporal" function $A_{0\mu}$ are dependent on $A_{mn}$ through (7),

[4] The inability to construct a *general* finite solution for a metric of the type $\eta = \text{diag}(1, 1, 1, 1)$, is a generalization of the argument leading to the well known result (in 2D) that one can not construct a function which is analytic over the entire complex plane unless this function is constant.

which takes the form:

$$\partial_0 A_{0n} + i\frac{\eta^{rs}}{\eta^{00}}k_r A_{ns} = 0, \qquad \partial_0 A_{00} + i\frac{\eta^{rs}}{\eta^{00}}k_r A_{0s} = 0. \tag{12}$$

It is easy to show that (12) can be solved simultaneously with (9) for the components $A_{0\mu}$.

Notice that when $\bar{h}_{\mu\nu}$ becomes large, the linearized Einstein equations are no longer appropriate. It is then necessary to study the full, nonlinear Einstein equations to determine if the solution in question continues diverging. It is possible to imagine that a solution that appears to diverge is actually well-behaved when studied in full. Although linear theory seems most appropriate for most astronomical phenomena excluding extreme cases such as black holes one can not neglect the mathematical possibility that non-linear terms are important. However, this paper deals with linear stability and the study of non-linear stability is left for future endeavors probably using numerical methods.

The method of linear stability analysis for a solution of nonlinear equations is quite common in the physical literature and its merits and limitation are well known. For example in fluid dynamics which is a non linear theory stability criteria obtained through linear analysis are ubiquitous. The reader may wish to consult [12–14]. Thus although linear stability analysis will not give one the entire stability information for the full Einstein equation, the information obtained is not null.

It must be stressed that "linear stability" is not an assumption. Solutions that are not linearly stable must develop under the equations due to the slightest perturbation to something else. Thus they cannot be observed in reality. The formal meaning of linear stability which is ubiquitous in the literature is as follows. Linear stability means that a small perturbation will not grow under the linear equations and will therefore remain bound and small. An instability means that the equations will cause an initially small perturbation to become unbound and eventually not small. An instability can be either exponentially fast growing as is the case in non-Lorentzian metrics or linearly slow growing as in the case of a Lorentzian metric in either case it will not remain small. The question is, since the Lorentzian metric is not stable why is it still here. The answer is two fold: one possibility is that it will not stay here for long but the instability is slow so we may observe it for a while, the second possibility is more intriguing since it contains information not just on the local geometry of space time but on the topology of space time as well. The question of the possible topology of space-time is not settled at this time. And the current analysis leads to a new angle regarding this problem. On a topological torus the Lorentzian metric will remain constant for ever.

In the case that the universe has a spatial cyclic topology in one or more directions the Fourier integral in this direction can be replaced by a Fourier series such that we only have $k_i$'s of the type:

$$k_i = \frac{2\pi n_i}{L_i} \tag{13}$$

in which $n_i$ is an integer and $L_i$ is the dimension of the spatially cyclic universe in the $i$ direction. In this case, the linear type of solution mentioned above may be rejected on the grounds that the metric must be a single valued function of the space time coordinates.

Notice that the instabilities discussed in this section are not mere coordinate instabilities as they lead to growth in gravitational waves and components of the Riemann tensor.

## 4 Conclusions

The historical development of the general theory of relativity can be succinctly descried as follows. First, Maxwell established his set of equations based on well established empirical facts. Then it was noticed that the equations do not respect the Galileo group of transformations but rather the Lorentz group of transformations. The new physics, consisting of Maxwell's electrodynamics and Einstein's mechanics, then satisfied a new principle of relativity, the Principle of Special Relativity, which says that all physical equations must be invariant under Lorentz transformations [1]. The development of General relativity followed later when Einstein incorporated gravity into the relativity frame-work, still demanding that locally the space-time metric should be of Lorentz type.

From a fundamental point of view the derivation should be somewhat different. That is, one may imagine that space-time is a geometrical manifold and ask how should such a manifold arrange itself. The behavior of such a manifold could be contemplated to emerge from a variational principle in which the action is

$$\int \sqrt{g} R d^4 x, \tag{14}$$

in which $g$ is the metric Jacobean and $R$ is the curvature scalar (the manifold will try to be as flat as possible). This variational principle will yield the Einstein vacuum field equations. Having derived the equations one may ask what are the stable solutions of such equations. Taking into account that a manifold must have a tangent space and hence locally must have a constant metric, one may ask what type of constant metrics can exist. The arguments outlined in the paper suggests that such a metric should be Lorentz. Only after specifying the properties of an empty space-time one is allowed to contemplate what type of entities such as matter and fields can inhabit such a manifold, leading perhaps to Maxwell's electrodynamics and Einstein's mechanics. Thus the fundamental derivation takes an opposite route than the historical one.

We conclude from (10, 11) that the only constant stable solution is of a Lorentz (Minkowski) type.

For other constant solutions we expect instabilities for $k_i \to \infty$ where $i$ depends on the unstable solution chosen. Thus the instabilities vary on very small length scale of which $\lambda = \frac{2\pi}{k} \to 0$, this length can be the smallest for which the general theory of relativity is applicable, perhaps the Planck scale $\lambda = l_p = 1.61610^{-35}$ m, in that case an unstable solution will last for about $t = \frac{\lambda}{c} = 5.3910^{-44}$ s. However, in the presence of matter this may take longer. This may explain why in QED an unstable Euclidean metric is used such that $\eta = \mathrm{diag}(1, 1, 1, 1)$, this is referred to as "Wick's rotation" [15].

This work gives a plausible explanation why the flat metric of the 4 dimensional space of general relativity is Lorentzian. And hence explains the existence of the intuitive partition of this 4 dimensional space into "spatial" space and "temporal" time. The existence of a such a partition in an almost empty space-time does not contradict the fact that such a partition can not be demonstrated in general solutions of (1) such as the one discovered by Gödel [16]. The main difference between the solutions studied in this paper and the solutions studied by Gödel, is that in this paper we study solutions of a vacuum theory while in Gödel's analysis a distribution of matter is assumed. What is claimed is that the concept of time is meaningful in moderate mass distributions scenarios in which the gravitational field is weak which is the prevalent case in astrophysical scenarios. However, there is no guarantee the one can meaningfully discuss time under extreme gravitational fields (and extreme mass distributions) as in the cases discussed by Gödel.

The reader may inquire why can't 3D theories such as geometrostatics and electrostatics be subject to the same analysis presented in this paper? The reason is the following: Geometrostatics and electrostatics are differential presentations of Newton's inverse square law and Coulomb's inverse square law and hence they are empirical in nature. Those theories contain in addition to Poisson's equation also the boundary conditions of this equation at infinity (that is the fields should vanish far away from the source). Under those conditions it is well known that well behaved solutions exist.

On the other hand the equations of the general theory of relativity which is considered to be a fundamental theory derived from first principles should also have "boundary conditions" derived from first principles. Since the theory is local in nature it is only reasonable to assume that the "boundary conditions" should be also local and not "global" dictating a certain behavior at "infinity". This of course does not dictate what type of values the linear metric should take locally and thus we have a freedom to choose arbitrary values which lead to stability analysis described in this paper and to the result that only the Lorentz metric can be considered stable.

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## References

1. Weinberg, S.: Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Wiley, New York (1972)
2. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Freeman, New York (1973)
3. Eddington, A.S.: The Mathematical Theory of Relativity. Cambridge University Press, Cambridge (1923)
4. Greensite, J.: Los Alamos Archive gr-qc/9210008 (14 Oct. 1992)
5. Carlini, A., Greensite, J.: Los Alamos Archive gr-qc/9308012 (12 Aug. 1993)
6. Carlini, A., Greensite, J.: Phys. Rev. D **49**(2), 866 (1994)
7. Elizalde, E., Odintsov, S.D., Romeo, A.: Class. Quantum Gravity **11**, L61–L67 (1994)
8. Itin, Y., Hehl, F.W.: Los Alamos Archive gr-qc/0401016 (6 Jan. 2004)
9. van Dam, H., Ng, Y.J.: Los Alamos Archive hep-th/0108067 (10 Aug. 2001)
10. Nikoliȅ, H.: Los Alamos Archive gr-qc/9901045 v1 (15 Jan. 1999)
11. Kalyana Rama, S.: Los Alamos Archive hep-th/0610071 v2 (18 Oct. 2006)

12. Rayleigh, J.W.S.: Proc. Lond. Math. Soc. **10**, 4 (1880)
13. Binney, J., Tremaine, S.: Galactic Dynamics. Princeton University Press, Princeton (1987), Chap. 5
14. Yahalom, A., Katz, J., Inagaki, K.: Mon. Not. R. Astron. Soc. **268**, 506–516 (1994)
15. Weinberg, S.: The Quantum Theory of Fields. Cambridge University Press, Cambridge (1995)
16. Yourgrau, P.: A World Without Time. Basic Books, New York (2006)