Low c.e. and low 2-c.e. degrees are not elementarily equivalent.

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Basic definitions.

If a set $A \subseteq \omega$ is Turing reducible to $B \subseteq \omega$ then we denote $A \leq_T B$.

$A \equiv_T B$ iff $A \leq_T B$ and $B \leq_T A$.

$\mathbf{a} = deg(A) = \{B \mid B \equiv_T A\}$.

The degrees with "$\leq$" and "$\cup$" form an upper semilattice, where $\mathbf{a} \cup \mathbf{b} = deg(A \oplus B)$ and $A \oplus B = \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$.

Also in this structure a jump operator is defined such that $\mathbf{b} \leq \mathbf{a} \rightarrow \mathbf{b}' \leq \mathbf{a}'$.

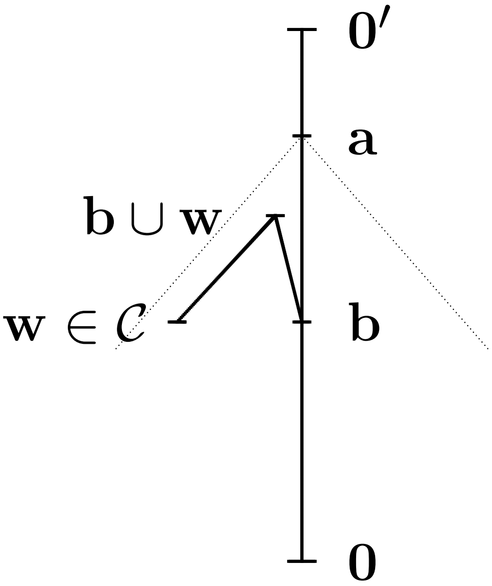
$\mathbf{0}'$ is known as the degree of the halting problem.

Given Turing degrees $\mathbf{0} < \mathbf{b} < \mathbf{a}$ and a class of Turing degrees $\mathcal{C}$.

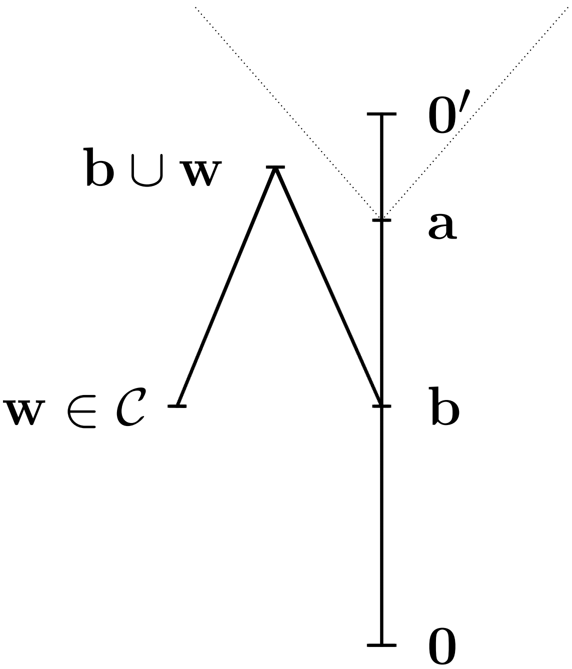
Definition. We say that $\mathbf{b}$ *is noncuppable to* $\mathbf{a}$ *in the class* $\mathcal{C}$ if there is no degree $\mathbf{w} \in \mathcal{C}$ such that $\mathbf{w} < \mathbf{a}$ and $\mathbf{a} = \mathbf{b} \cup \mathbf{w}$

Definition. We say that $\mathbf{b}$ *is strongly noncuppable to* $\mathbf{a}$ *in the class* $\mathcal{C}$ if there is no degree $\mathbf{w} \in \mathcal{C}$ such that $\mathbf{a} \not\leq \mathbf{w}$ and $\mathbf{a} \leq \mathbf{b} \cup \mathbf{w}$.

Noncuppability.

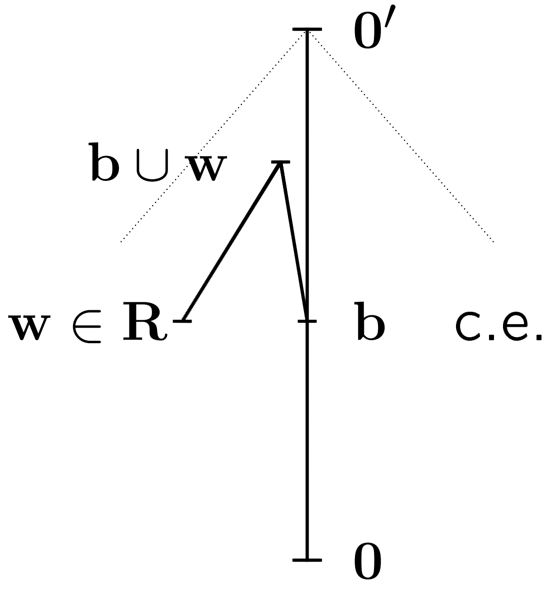


Strongly noncuppability.



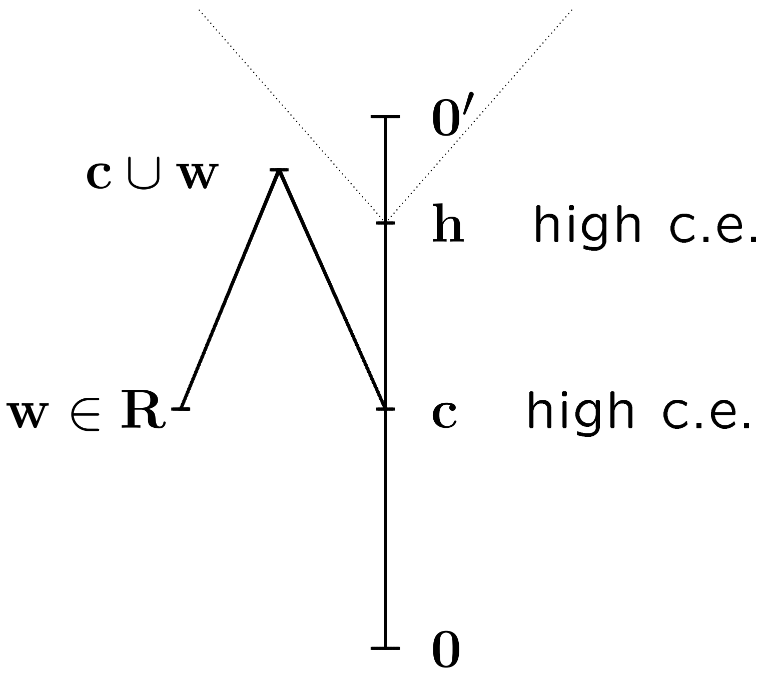
A REVIEW AND THE RESULTS

Theorem (Cooper; Yates; 1974г.). *There exists noncomputable c.e. degree* $\mathbf{b}$ *such that it is noncuppable to* $\mathbf{0}'$ *in the class of computably enumerable (c.e.) degrees* **R**.

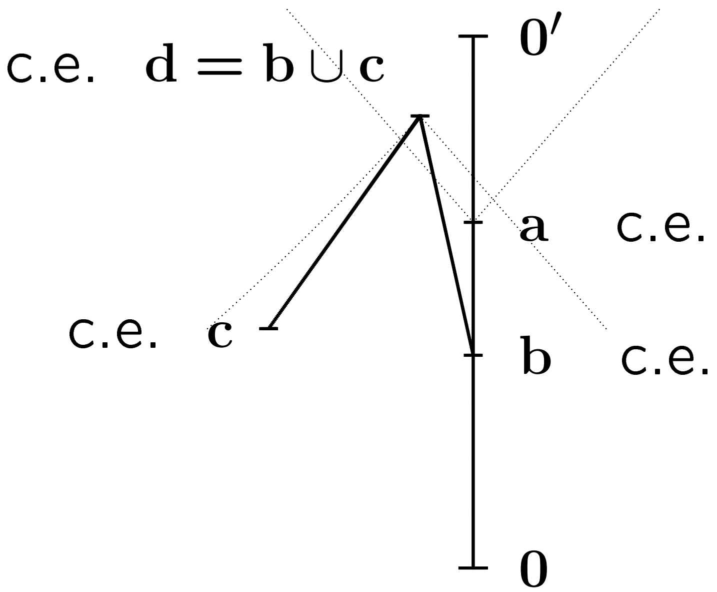


Remind that a degree $\mathbf{h} \le \mathbf{0}'$ is a high if $\mathbf{h}' = \mathbf{0}''$.

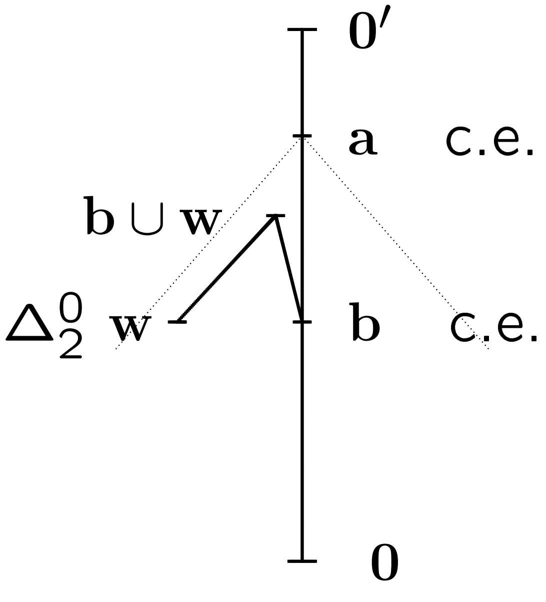
Theorem (Harrington, D. Miller 1981г.). *For every high degree* $\mathbf{h}$ *there exists high c.e. degree* $\mathbf{c} < \mathbf{h}$ *such that* $\mathbf{c}$ *is strongly noncuppable to* $\mathbf{h}$ *in the class* **R**.



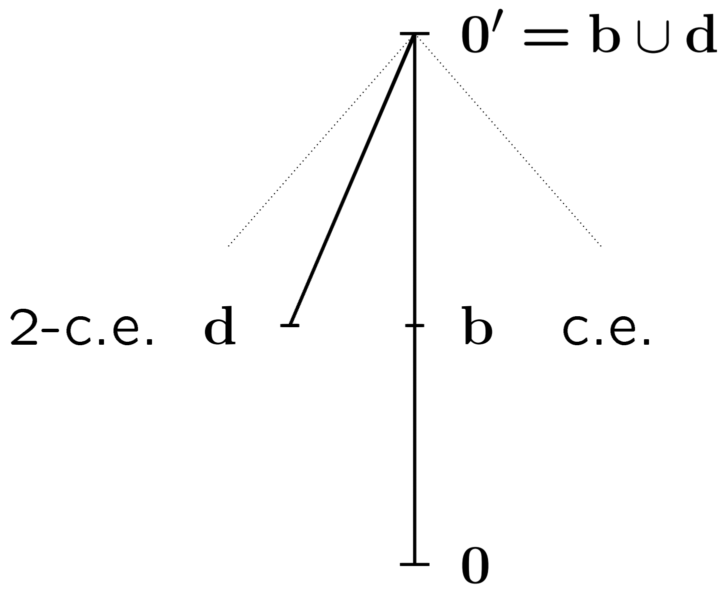
Theorem (Harrington, Fejer and Soar 1981г.). *There exists a noncomputable c.e. degree* $\mathbf{a}$ *such that for every noncomputable c.e. degree* $\mathbf{b} < \mathbf{a}$ *and for every c.e. degree* $\mathbf{d} \geq \mathbf{a}$ *there exists c.e. degree* $\mathbf{c} < \mathbf{d}$ *such that* $\mathbf{b} \cup \mathbf{c} = \mathbf{d}$.



Theorem (Cooper; Slaman and Steel; 1989г.). *There exist noncomputable c.e. degrees* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b}$ *is noncuppable to* $\mathbf{a}$ *in the class of* Δ^0_2 *degrees.*

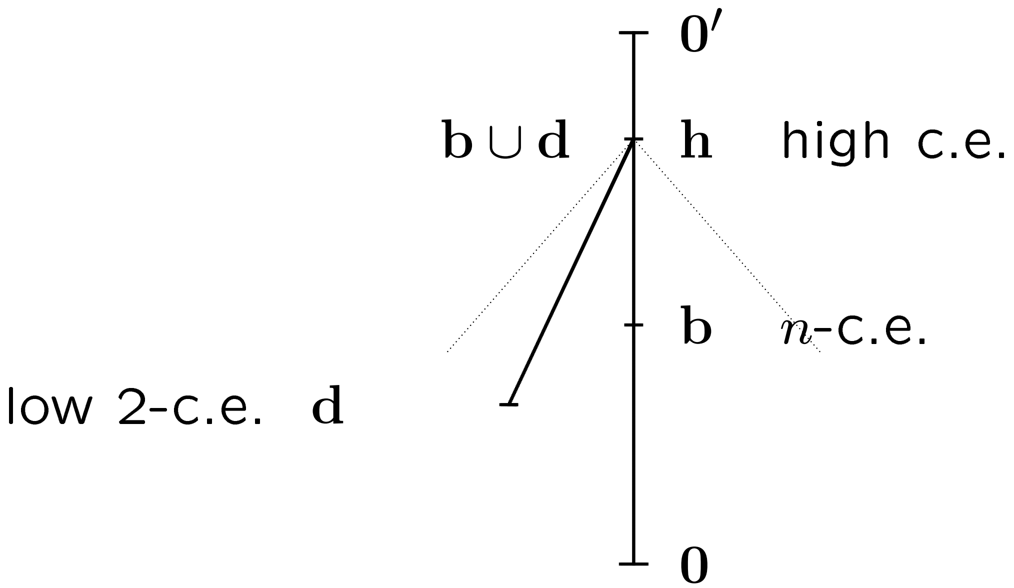


Theorem (Arslanov; 1988г.). *For every noncomputable 2-c.e. degree* $\mathbf{b}$ *there exists 2-c.e. degree* $\mathbf{d}$ *such that* $\mathbf{0}' = \mathbf{b} \cup \mathbf{d}$.

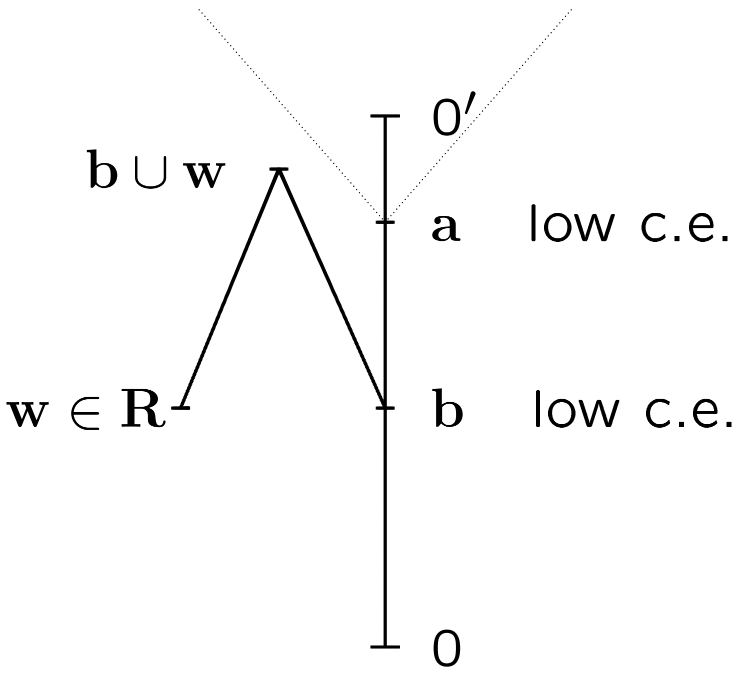


Remind that a degree $\mathbf{d} \leq \mathbf{0}'$ is a low if $\mathbf{d}' = \mathbf{0}'$.

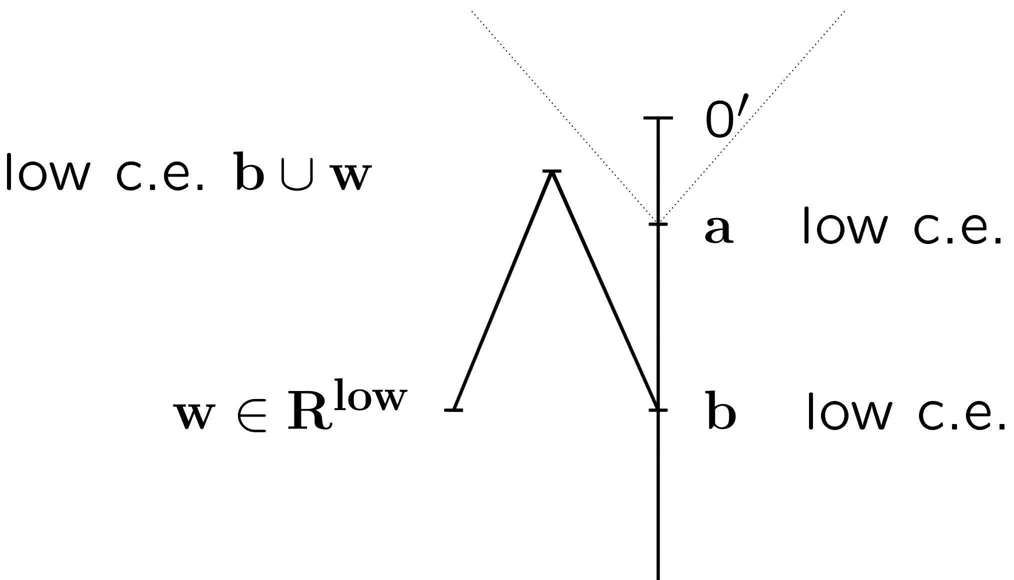
Theorem (Cooper, Lempp and Watson; 1989г.). *For every high c.e. degree* $\mathbf{h}$ *and for every noncomputable* n*-c.e. (*$n \geq 1$*) degree* $\mathbf{b} < \mathbf{h}$ *there exists a low 2-c.e. degree* $\mathbf{d}$ *such that* $\mathbf{h} = \mathbf{b} \cup \mathbf{d}$*.*



Theorem 1. *There exist noncomputable low c.e. degrees $\mathbf{b} < \mathbf{a}$ such that $\mathbf{b}$ is strongly noncuppable to $\mathbf{a}$ in the class $\mathbf{R}$.*



Theorem 2. *There exist noncomputable low c.e. degrees* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b}$ *is strongly noncuppable to* $\mathbf{a}$ *in the class* $\mathbf{R^{low}}$ *and for any low degree* $\mathbf{w}$ *the degree of* $\mathbf{b} \cup \mathbf{w}$ *is low again.*



CONSEQUENCES

Firstly consider the well known consequence of the following two theorems: theorem (Cooper; Yates; 1974г.) and theorem (Arslanov; 1988г.) Remind the that $\mathbf{R}$ is the class of all c.e. degrees and $\mathbf{D}_2$ is the class of all 2-c.e. degrees.

Consider the sentence

$$\varphi = \exists \mathbf{b}\, \forall \mathbf{w}\, [(\mathbf{0} < \mathbf{b}) \wedge [(\mathbf{w} < \mathbf{0}') \to (\mathbf{b} \cup \mathbf{w} < \mathbf{0}')]]$$

By theorem (Cooper; Yates; 1974г.) we have

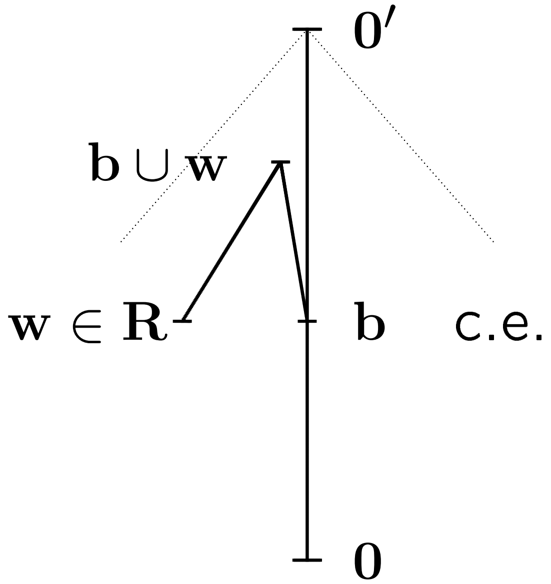
$$\mathbf{R} \models \varphi.$$

On other hand by theorem (Arslanov; 1988г.) we can see that

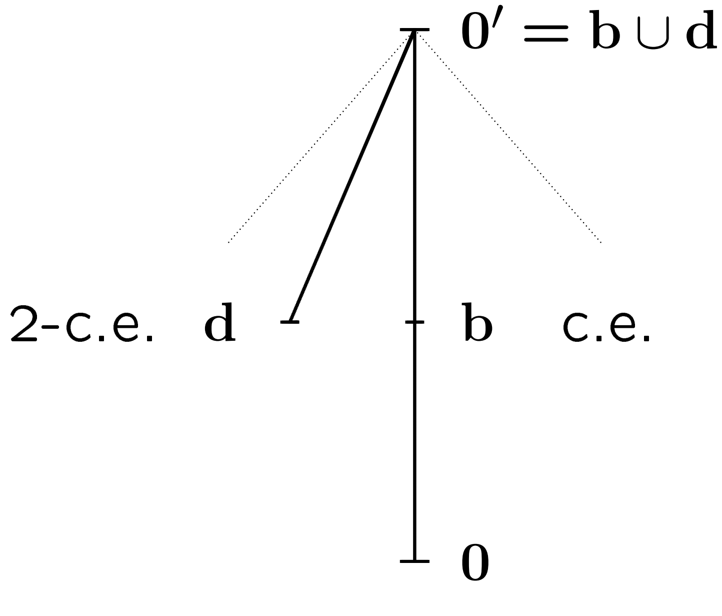
$$\mathbf{D}_2 \not\models \varphi.$$

So, the upper semilattices $\mathbf{R}$ and $\mathbf{D}_2$ are not elementarily equivalent.

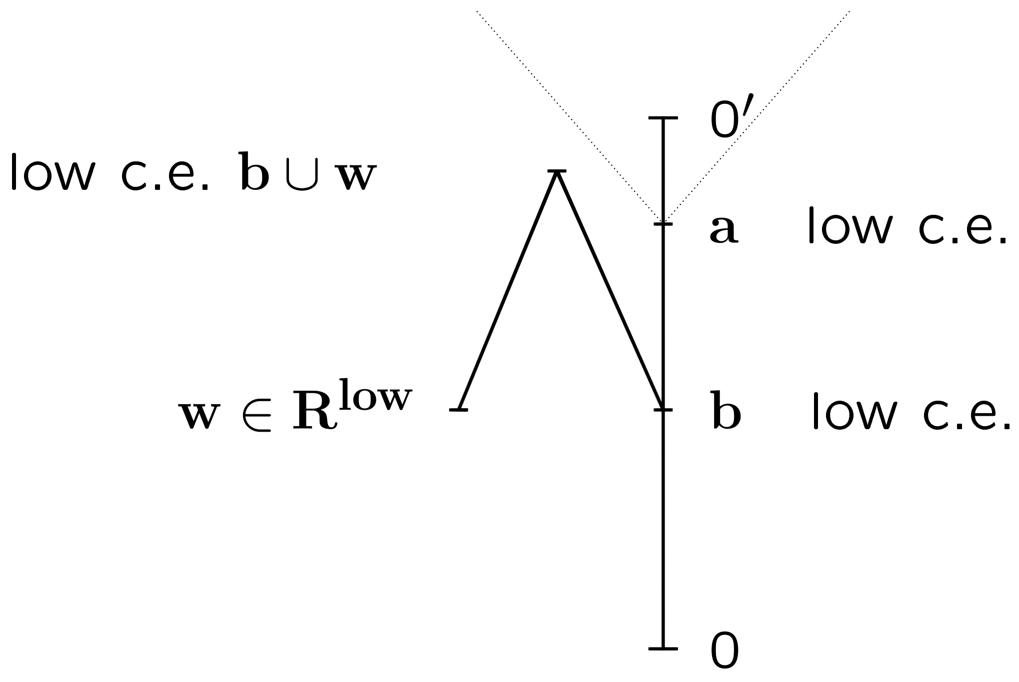
Theorem (Cooper; Yates; 1974г.). *There exists noncomputable c.e. degree* $\mathbf{b}$ *such that it is noncuppable to* $\mathbf{0}'$ *in the class of computably enumerable (c.e.) degrees* $\mathbf{R}$.



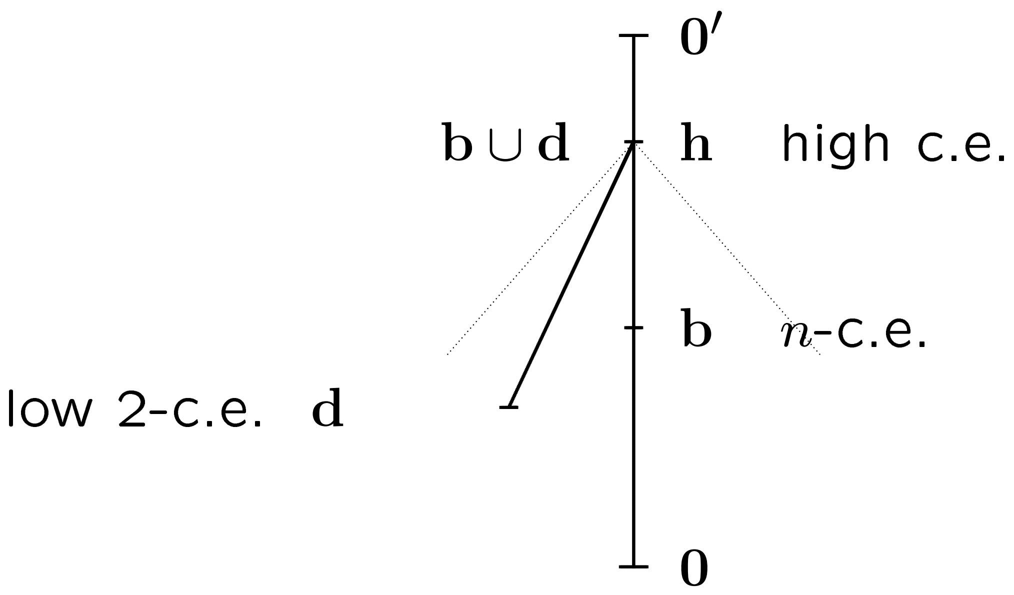
Theorem (Arslanov; 1988г.). *For every noncomputable 2-c.e. degree* $\mathbf{b}$ *there exists 2-c.e. degree* $\mathbf{d}$ *such that* $\mathbf{0}' = \mathbf{b} \cup \mathbf{d}$.



Theorem 2. *There exist noncomputable low c.e. degrees* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b}$ *is strongly noncuppable to* $\mathbf{a}$ *in the class* $\mathbf{R^{low}}$ *and for any low degree* $\mathbf{w}$ *the degree of* $\mathbf{b} \cup \mathbf{w}$ *is low again.*



Theorem (Cooper, Lempp and Watson; 1989г.). *For every high c.e. degree* $\mathbf{h}$ *and for every noncomputable* n*-c.e. (*$n \geq 1$*) degree* $\mathbf{b} < \mathbf{h}$ *there exists a low 2-c.e. degree* $\mathbf{d}$ *such that* $\mathbf{h} = \mathbf{b} \cup \mathbf{d}$.



Let $\mathbf{R}^{\mathbf{low}}$ and $\mathbf{D_2^{low}}$ be the classes of all low c.e. and all low 2-c.e. degrees, respectively. Consider the sentence

$$\psi = \exists\, \mathbf{a},\ \mathbf{b}\, \forall\, \mathbf{w}\ [(\mathbf{0} < \mathbf{b} < \mathbf{a}) \wedge [\mathbf{a} \leq \mathbf{w} \vee \mathbf{a} \not\leq \mathbf{b} \cup \mathbf{w}]].$$

By theorem 2 this sentence is true in the partial order of $\mathbf{R}^{\mathbf{low}}$. But by the theorem (Cooper, Lempp and Watson; 1989г.) for every noncomputable low 2-c.e. degrees $\mathbf{b} < \mathbf{a}$ there exists low 2-c.e. degree $\mathbf{d}$ such that $\mathbf{a} \leq \mathbf{b} \cup \mathbf{d}$. It is enough for

$$\mathbf{D_2^{low}} \not\models \psi.$$

This gives that partial orders of $\mathbf{R^{low}}$ and $\mathbf{D_2^{low}}$ are not elementarily equivalent. At the end show the level of elementarily difference. Transform the sentence ψ to

$$\varphi = \exists\, \mathbf{a},\ \mathbf{b}\ \forall\, \mathbf{w}\ [(\mathbf{0} < \mathbf{b} < \mathbf{a}) \wedge \{(\mathbf{a} \leq\ \mathbf{w}) \vee (\exists\ \mathbf{g}\ [\mathbf{a} \not\leq \mathbf{g} \wedge \mathbf{b} \leq \mathbf{g} \wedge \mathbf{w} \leq \mathbf{g}])\}].$$

So, we see that partial orders of c.e. and 2-c.e. degrees are not elementarily equivalent on the Σ_3 level.