Fuzziness and Probability

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I. Introduction.

In recent developments of theories concerning fuzzy logic and fuzzy set, the relation between fuzziness and probability has often been discussed. In this short article I would like to discuss the problem from a more general point of view, and at the same time would like to touch on related philosophical problems.

In chapter II we resume the similarities and dissimilarities of fuzziness and probability. In chapter III we discuss some philosophical problems involved. The purpose of the present article is to clarify the problems and prepare for future discussion. The implications related to the concept of probability in quantum mechanics will be touched upon.

II. Similarity and Dissimilarity.

From the beginning of the development of fuzzy logic, the relation between probability and fuzzy logic has been discussed.

One of the obvious similarities is that both are dealing with uncertainty. The theory of probability was developed originally from the calculation involved in gambling, where the outcome of the combination of dice or cards was uncertain. It has been pointed out sometimes that uncertainty cannot be the object of mathematics nor the exact sciences in general. A well known example of this kind of objection is the calculation of the probability of the existence of animals on a planet. If you consider the probability of the existence of a dog on that planet, as we have no knowledge of the existence of a dog on that planet[(1)], following the rule of calculation of probability of an object about which we have no knowledge, the probability should be one half. In the same way, if we calculate the probabilities of, say, ten other different animals, following the same rule of calculation, the probability for the existence of each animal should be one half. Again following the rule of calculation, we get the joint probability that the existence of any kind of animal on that planet should be one minus the product of the probabilities of existence of each animal, namely almost one. It means that it is almost sure that some kind of animal does exist on that planet. This is totally absurd.

It has been argued that this absurdity came from the false principle of the rules of calculation of probability. In this case, ignorance of the existence of an animal on that planet is the reason for giving the value of one half to it.

On the other hand, the same calculation can be applied quite reasonably to another case. Take for example, a box with ten coins in it. We calculate the probability of finding at least one coin with the head turned up. As we do not know whether each coin is the position with head or tail up, the probability of finding at least one coin with head up, following exactly the same rule of calculation, is almost one, i.e., we are almost sure that we can find at least one coin with head up. This result is quite satisfactory. In other words, it is not absurd to calculate a probability, even when it is based on ignorance!

As a result of this kind of "paradox", as I understand it, the discussion turned toward the problem whether probability should be defined as an a priori or a posteriori quantity. I will not enter into this problem further. I may come back to it later.

At any rate, it is obvious that the concept of probability is based on uncertainty about an object in question.

As for the concept of fuzziness, it is also obvious that it is based on uncertainty about an object in question. This "fuzzy" concept was originally proposed by L. Zadeh in 1965[(2)]. His idea was to formulate a control system which has some uncertainty in programming, like "about 20°C", or "about 1 meter per second" etc. I discussed these ideas in another article, so I will not go into the details of the problem[(3)]. However, it is important to point out that the basic idea is to deal with uncertainty. On this point, I see a similarity between the concept of fuzziness and that of probability.

Because of this uncertainty, as in the case of probability, fuzzy logic is criticized on the grounds that to quantify a concept with uncertainty, like "tallness" of a man, is unreasonable.

Other scholars, like Prof. S. Watanabe, point out that if the concept with uncertainty is quantified, it is not fuzzy any more[(4)].

The theory of probability is developed from the initial, rather practical, stage to a bona fide mathematical theory with a well defined function, which has a value between 0 and 1.

Likewise the theory of fuzziness is going to be developed. It has a membership function, which has a value between 0 and 1. There have been discussions on the relation between fuzzy logic and the logical system of Łucasiewicz. Recently a vigorous systematization of fuzzy logic is being studied by scholars like Prof. G. Takeuti and Dr. S. Titani, using intuitional complete Heyting algebra[(5)].

In the theory of probability, the discussion is still going on, it seems, about the status of the probability function, whether it is a priori or a posteriori.

Likewise, the status of the membership function is not clear, at least to me, whether it is a priori or a posteriori.

So far I have described some similarities between the two theories of probability and fuzziness.

Now it is easy to find that there are also dissimilarities between them.

Because of the similarities just described, they appear to be very close, but if we compare them more in detail, we notice that they are quite different from each other.

First of all, the uncertainty of probability lies, by definition, in the outcome of future phenomena. You are not certain whether a thrown coin will stop with the head up or the tail. It is not certain whether tomorrow will bring rain or shine. Therefore we predict these phenomena with some probability. If we see no probability of the outcome, we give zero probability, and if we are sure of it, we give the value one to that probability, and in uncertain cases we give values between zero and one, according to the degree of uncertainty.

This is the quantification of uncertainty. However, in this case, consciously or unconsciously, there are clear foundations for quantifying. If they claim a priori probability, it seems, the foundation lies in symmetry consideration, or other mathematical principles.

If the phenomenon is the outcome of the number on a dice, it is one sixth because the dice has a cubic shape, and a cubic shape has a hexagonal symmetry.

Therefore the probability of getting a special number, say three, should be one sixth. This is a priori probability.

If the phenomenon is tomorrow's weather, they calculate relative probability, from past statistical data of similar weather. This is a posteriori probability.

Any uncertain phenomena which lack the foundation for quantifying in order to calculate the probability are excluded. We should have certain knowledge as the foundation for quantifying uncertain phenomena. Therefore ignorance itself, per se, is not a condition for the exclusion of considerations to calculate probability. The existence of an animal on a planet should be out of consideration for the calculation of probability, because it lacks knowledge as its foundation.

In the case of throwing a coin, even if we are ignorant of the outcome, we have knowledge both a priori and a posteriori.

For the foundation to quantify its probability, we have a priori knowledge because we know the shape of a coin with two dimensional symmetry. We have a posteriori knowledge telling us that all the statistical data in the past show that the relative probability of the outcome of the coin's head facing up is very close to one half.

Now the uncertainty of the fuzzy concept lies, not in the outcome of a phenomenon, but in the belongingness of an object to a certain set.

In the example of "tallness", a man one meter tall surely does not belong to the set of "tall" men. Then we give the value zero as the degree of belongingness to this man. To a man two meters tall, we give him the value one for his belongingness.

We give a value between zero and one to a man of medium height, according to his height.

We can see that, even though the so-called membership function, which gives the degree of the belongingness of an individual sample of a set, and varies from zero to one as in the probability function, the meaning of this function is totally different from that of the probability function. This is easy to understand.

The next point is the foundation for quantifying uncertainty in the case of a fuzzy concept.

In fact this is one of the most controversial questions under discussion, as I understand it. What does it mean, for example, that the man who is 1.5 meters tall has a 0.5 degree of belongingness to the concept "tall"? It may be argued in the case of the "tallness" that the conept itself has already a quantitative element, namely, that of height. So in this case, we have knowledge of the height of each individual, which serves as the foundation for quantifying the fuzzy conept "tallness".

Then, as in the statistics, we can collect the data of the population, and from relative frequencies of various heights, we can get the membership function for "tallness", through some mathematical treatment. This corresponds to an a posteriori method of quantification in the case of probability.

Prof. Watanabe's criticism touches on this point[4]. He argues, as mentioned before, that this procedure of quantification of a fuzzy concept makes a fuzzy concept not fuzzy any more, because we know already exactly how tall a man is.

It seems to me that the exactness lies in the background knowledge as the foundation of fuzziness, and not in the fuzzy concept itself.

Is there any counterpart to the a priori method in probability, also in fuzziness? I think there is. Take, for example, "roundness", or "circle". Starting from a very flat ellipse, which cannot be called a circle, you decrease the distance between two foci. The ellipse becomes closer and closer to a circle. Again through some simple mathematical treatment, we can get a nice membership function for "roundness" or "circle". In this case, too, we have exact knowledge as a foundation for fuzziness.

Are these quantified fuzzy concepts interesting or useful? This is another question, to which I will come back later.

The treatment becomes more difficult in a fuzzy concept, if we want to talk about concepts like "beauty", or "kindness", for which, apparently, we have no exact quantitative knowledge as foundations.

What does it mean, for example, that this man has 0.6 degree of "kindness"? There are articles which actually give quantified membership functions to such concepts and give each individual a grade between zero and one[2]. But quantifying such concepts looks too artificial, because of the lack of quantitative foundations. Nevertheless, to consider precisely, these concepts from the view point of fuzzy logic seems to me challenging and important.

In the theory of probability, as mentioned before, the corresponding cases with no exact quantified foundation are excluded from theoretical consideration. Should it be the same in the case of fuzzy concepts ?

This is the question. One possible way to look at the problem is the extension or generalization of the membership function. G. Takeuti proposed considering this possibility using complete Heyting algebra[5].

Another possibility might be to give up quantifying the fuzziness of these concepts, which have no exact quantitative foundations, and treat them qualitatively. I am of the opinion that both possibilities should be explored and we should see how far we can go.

III. Philosophical Considerations.

After reviewing rather technical problems concerning fuzziness and probability, I want to discuss more philosophical problems related to these two concepts.

First of all, let us consider where the uncertainty lies in both concepts.

In the case of probability, as was explained above, uncertainty lies in the outcome of some event or phenomenon. In other words, it deals with the existence of some entities. These entities might be attributes of something, like the head or tail of a coin. They might also be things themselves, like rain. In any case, we look for the existence of these entities. But the concept of these entities should have a definite meaning, namely, they should be identified as one definite entity.

In the case of fuzziness, on the other hand, uncertainty lies in the essence of entities. We do not ask about their existence. So we ask how "tall" this man is, and not whether this man exists or not.

In traditional scholastic philosophy, the distinction between existence and essence played an important role. In recent times, too, scholar like M. Matsumoto has given a detailed analysis of these two modes of entities, in relation to logic[6]. In other articles, I have already discussed the problem of probability and fuzzy concepts in this context, namely the distinction between the essence and existence of entities[3]. But I want to stress this point once more.

As M. Matsumoto explained, the verb "to be" ("aru" in Japanese) has two meanings. One refers to the existence of an entity ; e.g. This dog is (exists) ("ga aru" in Japanese). The other refers to the essence of an entity ; e.g. This dog is black ("de aru" in Japanese).

In scholastic philosophy, the second use of the word "to be" -(in Latin "esse")- is called "esse ut copula" -("to be" as used for copula)-. Whereas the first use of the word "to be" is not called "ut copula". And only the second use, i.e. "ut copula" is related directly to logic. This is the reason why from fuzzy concepts a new kind of logic, namely fuzzy logic, is developed, but the probability is not related directly to logic, therefore no such development is observed.

Because fuzziness is related to fuzzy logic, a new problem has to be discussed. This is the problem of the meta-logic of fuzzy logic. I hold the opinion that the meta-logic of fuzzy logic is formal logic. There are some articles, claiming that the meta-logic of fuzzy logic can again be fuzzy logic[7]. But I do not agree with this opinion on the ground that, as mentioned before, the foundation of fuzziness should be exact knowledge of the entities concerned.

There is another obvious reason for this opinion. In any logical system, whether it is two valued or multi-valued, its meta-logic should always be two-valued formal logic. I do not see any other way, for any logical system to be a bona fide one.

Now I would like to discuss the concept of probability in Quantum Mechanics. One of the most important achievements in the course of the developent of Quantum Mechanics is the interpretation of the state function in the Schrödinger equation as probability amplitude of an entity in question.

When Schrödinger first developed his theory of the Schrödinger equation, he took the model of the simple harmonic oscillation of a one-particle system.

His ψ-function in this case was considered as a wave in three dimensional Euclidian space.

Afterwards, it was Born, who gave a new interpretation to the ψ-function not as wave, but as state-function of a particle in this model[8].

In a one-particle system, Schrödinger's "wave"-equation and "state"-describing equation look the same, but if a many-particle system is treated, the difference between these two kinds of equations becomes clear. Born interpreted the state function as probability amplitude for the existence of a particle or particles at a point of space in a given time. The probability itself of these entities should be the "Hermitian" product of ψ, i.e. $\psi^*\psi$, which satisfies the definition of probability: it is a positive definite, and has a value between zero and one, when properly normalized. $0 \leqq \psi^*\psi \leqq 1$.

The concept of probability amplitude was a new concept in Physics. Probability has a definite meaning, both in Physics and Mathematics, whereas probability amplitude has no direct reference to physical entities. This is the reason why discussions on ψ-function became so controversial.

After quantum field theory was developed, ψ function regained the status of "wave" in three dimensional Euclidian space, even with the status of an operator in Hilbert space as a physical quantity. It means that ψ-function has now a definite physical meaning as wave. On the other hand, in the quantum field theory, state function, — now we write as Ψ to show the difference from "wave" function with small letter: Ψ still holds the status of probability amplitude. The above description of probability amplitude was again rather technical, but what I intend to say is the following.

With the development of Quantum Mechanics, as well as that of the theory of relativity, several new concepts or new interpretations of existing concepts were introduced. An example of the latter is that of space and time, whereas the concept of probability amplitude gives a good example of the former.

The question is: What is the status of Ψ, i.e. probability amplitude in the real world?

To discuss this question, we should know what the real world means. On this point, I have given opinion elsewhere[(3)]. I only stress at this point that concept like probability amplitude is apparently a mathematical tool to calculate the probability of physical objects in a given space-time point, but have foundations in the physical world. By physical world—not the real world—I mean the world where any physicist agrees that there we observe phenomena which are inter-subjective.

In traditional philosophy, these entities are called "rational entities with a foundation in the real" (entia rationis cum fundamento in re). If we follow the dualistic idea and separate the so-called res-cogitans and res-extensa rigorously, then we cannot find any status in physics for those entities like probability amplitude.

If, on the other hand, we stick to the positivistic way of thinking, and consider these concepts as "constructs", and if we follow logical conclusions of this way of thinking to the very end, then again we will lose all informations concerning the physical world as a product of "construction", including electrons, for example, which nowadays play such an important role in our daily life.

Similar discussions on the status of existing, but newly interpreted concepts like space and time should be carried on.

The use of fuzzy concepts for Quantum Mechanics has been discussed before[(9)]. There are also some interesting problems for further study.

There might be a question, whether there will be a similar counterpart of probability amplitude in the context of fuzziness. For this we should wait for a further investigation of the problem.

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Footnotes

(1) J. Suetuna: *Theory of Probability* (in Japanese), Iwanami-zensho, Tokyo. Introduction.

(2) A. Zadeh: cf. Recent interview with Zadeh, Communication of the ACM, April 1984, Vol. 27, No. 4, p. 304-311.

(3) M.M. Yanase: *Annal. of Japan Association for Phil. of Sci.* Vol. 5, No. 5, March 1980, p. 225-244.

(4) S. Watanabe: *J. of Japan Association for Phil. of Sci.* Vol. 13, No. 4, 1978, p. 135-139 (in Japanese).

(5) G. Takeuti & S. Titani: *Journal of Symbolic Logic* (in Print).

(6) M. Matsumoto: *Logic of Existence* (in Japanese), Iwanami, Tokyo, 1968.

(7) cf. references in (2).

(8) cf. any textbook on Quantum Mechanics, e.g. S. Tomonaga: *Quantum Mechanics* (in English, and Japanese, Misuzu-Shobō, Tokyo).

(9) M.M. Yanase: *Annal. of Japan Association for Phil. of Sci.* Vol. 6, No. 2, March 1982.