# Quantization as a guide to ontic structure

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**Abstract**

A framework is presented for the provision of a structural realist ontology as dictated by the implications of simultaneously accepting both inter-formulation and classical-quantum species of 'metaphysical' underdetermination. The example of non-relativistic particle mechanics is considered, and it is argued that, modulo certain mathematical ambiguities, a viable and consistent candidate structural ontology can be constituted in terms of a Lie algebra morphism between algebras of observables and the relationship between the corresponding state spaces.

## 1 Introduction

The viewpoint of ontic structural realism (OSR)(Ladyman 1998; French and Ladyman 2003; Ladyman and Ross 2007) is in part motivated by two arguments towards the underdetermination of the ontology associated with traditional realist descriptions of science. The first is *formulation underdetermination*, and springs from the multiplicity of empirically equivalent formulations of a given theory (Jones 1991; Bain 2009; French 2011). The second is *pessimistic meta-induction*, and springs from the historical superseding of one empirically well-confirmed theory by another (Laudan 1981; Worrall 1989). Under OSR the ontology of a physical theory is constituted by mathematical structures rather than objects and entities. To avoid formulation underdetermination, one is limited to to an ontology constituted by the structures common between two different formulations of a theory; and to avoid pessimistic meta-induction one is limited to the structures common to a theory and its successor. For OSR to be viable: i) these structures must be substantial enough to constitute a viable alternative ontology; and ii) the structures used to avoid the two arguments must be consistent.

Here we will construct a proposal to evaluate OSR with respect to i) and ii) in the context of the mathematical structures which underlie quantization (the procedure for turing a classical theory into a quantum theory). It will be argued that through the close investigation of the formal techniques that lie behind quantization we can gain a detailed understanding of fundamental principles towards the construction of a ontic structural framework for physical theory. The first

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step in the implementation of this proposal is the consideration of the case of non-relativistic particle mechanics. There it will be demonstrated, modulo certain mathematical ambiguities, that a viable and consistent candidate for an OSR framework does exist. Future work will consider gauge theories – in particular classical/quantum field theories – with a view towards exploring the problem of quantizing gravity within our particular perspective on structuralism.

Our purpose here will not be to defend the *necessity* of the various arguments towards underdetermination of ontology or the ontic structural realist response. Rather our analysis will examine, in as concrete and explicit terms as possible, whether a viable and consistent candidate for structural ontology can or cannot be constructed in the context of an explicit case study. If it transpires that the proponent of OSR *is not* able to point to a consistent, generalising and dynamical structural framework, in even the simplest of cases, then the position will be substantially undermined, irrespective of the strength or weakness of the relevant motivating arguments. Conversely, if a suitable framework can be found, then we have good grounds to examine that frameworks extension to more general theories, over and above the question of granting it a privileged metaphysical status. In the end, if the pursuit of the principles underlying the consistent construction of structural ontology gives us *any* genuine insight into the foundations of physical theory, then it will have proved a worthwhile endeavour.

Moreover, within the gravitational arena, which is intended to be our eventual object of study, the twin questions of mediating between competing formalisms and discovering the proper framework for quantization are long standing problems, in want of new insights. Thus, it may prove that the pursuit of ontological structural realism can be motivated by the *pragmatic* task of assisting the physicist in their endeavours, over and above how such hand-maidening should be characterised in an *ontologically thick* sense.

## 2 Formulation underdetermination and structural realism

Following the analysis of (French 2011), we can consider three scenarios which can be grouped together under the heading ontological (or metaphysical) underdetermination: theoretical underdetermination, interpretational underdetermination and formulation underdetermination. The first is the most familiar within the philosophy of science and is when we are presented with distinct theories each consistent with the same set of phenomena but each entailing an ontology incompatible with the other. The classic example of such an underdetermination case is that between special relativity and Lorentzian ether theory. The second interpretational notion of underdetermination is particularly familiar within the philosophy of physics and relates to the existence or two or more competing candidate ontologies for the same physico-mathematical formalism. The classic example of such an interpretational underdetermination is quantum mechanics where multiple ontologies (e.g. non-local hidden variables or many worlds) may be associated with the same Dirac-Von Neumann mathematical structure via starkly different interpretational stances.

The third variant of underdetermination is perhaps the most neglected and shall be the main focus of our analysis. In addition to the underdetermination entailed by the existence of multiple *interpretations* of a physical theory there is a subtly different class of underdetermination which grows out the existence of multiple *formulations*. We can understand the formulation of a theory as different ways in which the theoretical (i.e., non-representative) structure of a theory can be expressed. The crucial hallmark of distinct formulations as opposed to distinct interpretations is that (as well as being confined to the non-representative aspect of the theory) they are necessarily accompanied with a rigorous translation dictionary which allows us to transform from the language of one formalism to the language of the other. The interpretation and the formulation of a theory are closely related. A given interpretation may make use of a particular formulation of a theory and it may even be the case that a particular formulation is conducive to or exclusive of a particular interpretation. The strength of the relationship may not be particularly strong, however – such as in the case of quantum mechanics where the various possible formulations (e.g. Schrödinger vs. Heisenberg pictures) are found to licence most, if not all, of the various interpretations equally. However, there is definite scope for the choice between competing formalisms to be restrictive enough to mandate only certain interpretations and therefore only certain ontologies.

The key to genuine cases of formulation underdetermination is the possibility of cases where two distinct formulations of the same theory place different bounds on the cast of viable interpretations. The strength of these bounds demarcates three distinct notions of formalism underdetermination: First, they may be *strict*, meaning that they are such that there is no single interpretation that can be applied to both formalisms. Second, they may *exclusive*, meaning that there exists at least one interpretation which is applicable to one formulation but not to another. Third they may be *loose*; meaning that they are such that they make a particular interpretation more natural to one formulation than to another. All three variants are philosophically interesting since each (to a varying degree) leads to a situation whereby the nature of our ontology is dictated not by a choice between empirically consistent theories, nor even between interpretations of the same theory, but rather by the seemingly arbitrary choice between different formulations of the same theory.

What notion of realist ontology can be constituted in response to such cases of underdetermination? According the proponents of ontic structural realism (OSR) the answer is a structural one. Whereas, traditional variants of object orientated realism (i.e. that based upon a distinct class of entities and things) must, in order to remain coherent, seek to break any genuine case of formulation underdetermination by appeal to some external criteria; by endorsing OSR one may side-step the undermining cut of underdetermination scenarios altogether.[1] Under OSR the ontology of a physical theory is constituted by mathematical structures rather than objects and entities, and thus, one may avoid formulation underdetermination entirely by limiting oneself to an ontology constituted by the structures common between two different formulations of a theory.

Notwithstanding the question as to whether the traditional realist responses of breaking

[1] The conventional realist might of course alternatively simply dispute that *genuine* cases of underdetermination can every actually occur.

underdetermination by appeal to external criteria may constitute a better alternative strategy, the onus is of course upon the proponent of OSR to be a little more specific about exactly what kind mathematical structures they have in mind. In particular, it remains to be seen whether or not the specification of a structural ontology is on its own be sufficient to resolve a genuine case of formulation driven ontological underdetermination (Pooley 2006). As is well illustrated by the type of 'structural realism' defended by (North 2009) (her position is, without our terminology, a variant of conventional realism within which a geometric simplicity criteria is invoked in order to break formalism underdetermination), it is quite possible for the structural ontological vocabulary itself to be underdetermined if it is characterised in such a way as to be particular to each formalism. Thus, the structures that the defenders of OSR are looking to endorse must be such that they span between the appropriate formulations – it must be common structure.

An obvious candidate for such structure is the mathematical transformations and interrelations that constitute the translation dictionary between two formulations. However, such a characterisation of the structural ontology is also problematic. As further noted by (Pooley 2006), such interrelations between formulations offer only a very thin notion of structure that alone seems insufficient to be the fundamental furniture of the world: what is needed is a 'single, unifying framework [which we can] interpret as corresponding more faithfully to reality than do its various [conventional] realist representations' (p.7). Thus, the challenge to the ontic structural realist is to offer more than a purely set or group theoretic *characterisation* of the common underlying structure invoked to dissolve cases of underdetermination. What is needed, in essence, is a physico-mathematical framework that *generalises* the structures relevant to each formulation in such a way as to illustrate that each formulation is merely a different representation of the same underlying 'reality'. Such a framework must reasonably be taken to include dynamical as well as purely mathematical aspects (Bain 2009) and must therefore be expected to be constituted by structures intimately connected to measurable, dynamical quantities.

There is thus a rather complex challenge to the proponent of ontic structural realism in balancing the need to find structures that are abstract enough to transcend particular formulations but concrete enough to be considered genuinely dynamical. The extent to which such a balance is achievable in practice is best evaluated though the consideration of a series of explicit examples, the simplest available of these will be presented in the final part of this paper. Before then however, we must consider some important additional constraints on our principles for structural ontology construction which result from the interrelation of classical and quantum theories.

## 3 Quantization and structural realism

Quantization is a bridge between classical and quantum theories and thus provides a direct and rigorous way of linking historically successive theories. Formally, the quantization of a classical theory can be understood within a powerful and general geometric framework the essence of which is contained within the relationship between classical symplectic structure and quantum mechanical Hilbert space structure.[2] Explicitly, under the geometric quantization programme (Echeverria-Enriquez, Munoz-Lecanda, Roman-Roy, and Victoria-Monge 1999) we seek a correspondence between the sets of pairs constituted by: symplectic manifolds $(\mathcal{M}, \Omega)$ together with smooth real functions $C^{\infty}(\mathcal{M})$, on the one hand; and complex Hilbert spaces $\mathcal{H}$ together with self-adjoint operators $\mathcal{A}(\mathcal{H})$, on the other. We define the full quantization of a classical system $(\mathcal{M}, \Omega)$ as a pair $(\mathcal{H}_Q, A)$ under certain conditions on $\mathcal{H}_Q$ and the map, $A$, which takes us between classical and quantum observables. The conditions are:

1. $\mathcal{H}_Q$ is a separable complex Hilbert space. The elements $\mid \psi\rangle \in \mathcal{H}_Q$ are the quantum wavefunctions and the elements $\mid \psi\rangle_{\mathbb{C}} \in \mathbf{P}\mathcal{H}_Q$ are the quantum states where $\mathbf{P}\mathcal{H}_Q$ is the projective Hilbert space;

2. $A$ is a one to one map taking the classical observables $f \in \Omega^0(\mathcal{M})$ to the self adjoint operators $A_f$ on $\mathcal{H}_Q$ such that: i) $A_{f+g} = A_f + A_g$ ii) $A_{\lambda f} = \lambda A_f \; \forall \lambda \in \mathbb{C}$ iii) $A_1 = Id_{\mathcal{H}_Q}$;

3. $[A_f, A_g] = i\hbar A_{\{f,g\}}$ (i.e., $A$ is a Lie algebra morphism up to a factor);

4. For a complete set of classical observables $\{f_j\}$, $\mathcal{H}_Q$ is irreducible under the action of the set $\{A_{f_j}\}$.

Of particular significance for our analysis is the third condition which encodes the relationship between the classical Poisson bracket and the quantum mechanical commutator. The former is defined implicitly by the symplectic structure $\Omega$, and thus the pivotal role of that structure in anchoring one end of the quantization bridge becomes apparent.

That quantization itself is found to point to certain structures within the classical predecessor theory as in some way essential to the quantization of that theory is extremely interesting when considered in the context of ontic structural realism. One of the principle motivating arguments for the position (Ladyman 1998; Ladyman and Ross 2007), is that it is observed that throughout the history of science empirically successful theories are often, if not always, replaced by theories which include starkly different types of theoretical entities and objects. From this we may make the pessimistically meta-inductive leap to the conclusion that the terms included within our current best theories that relate to theoretical entities and objects should not be thought of as constituting a genuine, robustly referential ontological vocabulary.[3] Rather, the proponents of OSR contend, we should focus our attention on the structural aspects of physical theory and attempt to reconceive the notion of what constitutes the ontological vocabulary in terms of the structure common between successive theories. If the formal structure of quantization techniques itself points to certain key structural facets of classical theory then it seems natural to ask what these structures correspond to within the quantum theory.

[2]Briefly put, a symplectic manifold is a smooth locally Euclidean space, equipped with a closed non-degenerate two form. A Hilbert space is an abstract vector space equipped with an inner product.

[3]There are, of course, more sophisticated varieties of scietifc realism – such as that defended by (Psillos 1999) – that can be argued to circumvent the cut of the pessimistic meta-induction. Since our primary focus is upon examining the viability of OSR in of itself, rather than the strengths of its rivals we will neglect a full discussion of such nuanced versions of ‘conventional’ realism.

We may then be able to specify precisely the structures that, according to OSR, should be reified when constituting a structural scientific ontology at the classical/quantum boundary such that it is robust to the challenge of pessimistic meta-induction. Clearly, the common structure motivated by such historical succession between theories may or may not be consistent with common structure between formulations discussed in the context of underdetermination above. Quantization provides us with exactly the mechanism to examine exactly such questions since it allows us to consider two different formulations of a classical theory and directly compare them to their quantum correlates.[4] If the OSR framework is a consistent one then one should be able to constitute a structural ontology that simultaneously transcends both inter-formational and inter-theoretic distinctions.

We can formalise this idea more explicitly. Let us assume we are given two formulations of a classical theory which have been quantized (perhaps by different methodologies). We would presume that from the two classical formulations will result the same quantum theory (although this is not guaranteed) and we would thus then have two quantum formulations of this single theory. Let us denote these formulations as $C_1, C_2, Q_1, Q_2$. A genuine implementation of the OSR programme for resolving cases of underdetermination would then provide us with a unifying framework for each of the pair of formulations, $C_{UF}$ and $Q_{UF}$. Furthermore, a genuine implementation of the OSR programme for confronting the challenge of pessimistic meta-induction would give us a structural bridge between each of the classical and quantum formulations: $CQ_1, CQ_2$. It should also give us such a bridge between our two classical and quantum frameworks: $CQ_{UF}$. And furthermore, these two unifications should cohere. We can illustrate the situation graphically (committing a small abuse of mathematical diagrammatic convention) as follows:

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d] & C_{UF} \arrow[d] & C_1 \arrow[l] \arrow[d] \\
CQ_1 \arrow[r] & CQ_{UF} & CQ_2 \arrow[l] \\
Q_1 \arrow[u] \arrow[r] & Q_{UF} \arrow[u] & Q_1 \arrow[u] \arrow[l]
\end{tikzcd}$$

Implementation of such a complex schema might be assumed to be impractical in general terms. However, armed with mathematically well defined quantization procedures and interrelations between formulations we may perhaps be able to make some progress by considering a

[4] An alternative structural approach to conceiving of an ontology at the classical/quantum boundary would be to focus upon the classical limit of the relevant quantum theories. We will here neglect a detailed consideration of this option since it would provide little insight into the inter-formulation issue which we wish to investigate in parallel.

test case. The most basic case available is that of classical/quantum non-relativistic particle mechanics, and it is to this that we now turn.

# 4 The case of non-relativistic particle mechanics

## 4.1 The Formulation and Interpretation of Newtonian Mechanics

Let us consider a classical system consisting of a finite number of degrees of freedom and assume that this system does not display any local symmetry.[5] The physical theory describing such a system is Newtonian mechanics and in modern terms the two principal formulations available are Lagrangian and Hamiltonian (unfortunately, we do not here have space to consider the Hamilton-Jacobi formulation also). The Lagrangian formulation of Newtonian mechanics is framed within the space of solutions to the Euler-Lagrange equations which are dynamical curves, $\gamma_{EL} : T\mathcal{C} \to \mathbb{R}$, in the velocity-configuration space (the tangent bundle to configuration space), $(q_i, \dot{q}_i) \in T\mathcal{C}$. The Hamiltonian formulation of Newtonian mechanics is framed within phase space (a symplectic manifold defined as the cotangent bundle to confirguation space), $(q_i, p_i) \in T^{\star}\mathcal{C}$, with Hamilton's equations, $\omega(X_H, \cdot) = \mathbf{d}H$, picking out a preferred tangent vector field on phase space, $X_H$, which is sufficient to define the set of dynamical curves for any specification of instantaneous initial data.

By the criteria and definitions detailed above what we are dealing with here is two distinct formulations of the Newtonian theory of mechanics: neither Lagrangian nor Hamiltonian formalism furnishes us with an ontology without a further interpretation and the two are connected by a rigorous translation dictionary provided by the Legendre transformation together with the set of maps (parameterised by a one dimensional time parameter) that exists between a given solution in the Lagrangian formulation and the corresponding sequence of instantaneous states in the Hamiltonian formulations. The crucial question, in light of our above analysis, is then whether we should understand these formulations as leading to an underdetermination of the relevant ontology. This depends on the nature of the relevant interpretations available and their relationship to these two formulations.

Focusing in particular on the temporal ontology of Newtonian mechanics, two candidate interpretations are available. The first is constituted by the classic Newtonian picture of instantaneous states of the world together with deterministic laws sufficient to fix all past and future states given an *initial* state. We will call this the *instantaneous* picture of the world and understand it as specifying an ontology which posits instantaneous states as part of the fundamental furniture of the world. Supplementary to this picture we can ascribe additional and *more metaphysical* structure such as a dynamical notion of time and an ontologically privileged present (Markosian 2011). Our concern here is not with the detailed philosophical analysis of these additional interpretational structures and the extent to which they prove acceptable additions to the project

[5]Here and below we neglect the role of global symmetries for the sake of brevity. An analysis of the structural connections relevant to them would follow straightforwardly from what we say about observables and state spaces.

of furnishing the relevant theory with an ontology. They are certainly not generally taken to be precluded by Newtonian mechanics at least.[6] Rather, what we shall assume to be at the very least non-controversial is that given the viability of an interpretation in terms of a instantaneous picture, one may – if it is deemed reasonable – supplement this interpretation with additional temporal ontic structure such as a dynamic time.

A second interpretation of Newtonian mechanics that provides us with a distinct temporal structure is in terms of entire four dimensional histories which are specified by *atemporal* laws (i.e., laws that are not defined at a given time) together with initial and final boundary conditions. We will call this the *teleological* picture of the world since it implies the final boundary data is fundamental in determining the laws. Unlike the instantaneous picture is does not posit instantaneous states as part of the fundamental furniture of the world and, relatedly, it is not as amenable to supplementation with the additional *more metaphysical* structure mentioned above and discussed in more detail below. We do not mean this as necessarily a particularly strong claim and will not therefore seek to make a justification of it in a strong sense. Rather, we believe it is at the very least reasonable to assume that an interpretation of Newtonian mechanics in terms of a teleological picture is, at face value, going to look more like a non-dynamic, 'eternalist' type stance as to the metaphysics of time and less like the dynamic/privileged present type stances.

Just as there is a clear intuitive relationship between the aspects essential to the instantaneous picture and the presentist stance, there is a clear intuitive relationship between the aspects essential to the teleological picture and the anti-presentist stance. It would seem, furthermore, that the teleological picture is such that it is inherently hostile to presentism – the laws, boundary conditions and fundamental objects are things that, by the presentist lights, do not exist. Thus, at a superficial level of analysis at least, there is a natural way of *cashing out* the difference between our two pictures in terms of a substantive ontological difference.[7]

Even if we were to be more minimalist as to the level of metaphysical structure we wish to permit, then we may still end up with genuine differences between the two pictures. Whereas the instantaneous picture is predicated upon an ontology that necessarily includes instantaneous states as fundamental, the teleological picture is not *necessarily* predicated upon such an ontology. Thus any approach to space-time ontology which *precludes* fundamental instantaneous states can only be reconciled with the teleological picture – and is thus more naturally at home within the Lagrangian formalism. Such an argument is of course not sufficient to establish that there is no reasonable conventional realist ontology that transcends the Lagrangian/Hamiltonian divide. Rather, we have that there are at least *some* notions of ontology that are underdetermined by the case in hand, and thus that there is a requirement for the proponent of OSR to provide a viable alternative ontology.

Let us then proceed to examine our case in terms of our framework for formalism under-

[6]See (Wüthrich 2010) for an interesting discussion of the extent to which the presentist view *is* precluded by theories of quantum gravity of exactly the type that have been extensively detailed in this work.

[7]There is also reasonable scope to understand the difference between the instantaneous and teleological pictures as possibly grounding a fundamental metaphysical difference as to the laws of nature. For example, it has been claimed that the disposition essentialist viewpoint on laws of nature is inconsistent with the principle of least action that is fundamental to the teleological picture (Katzav 2004). See (Smart 2012, §8) for further discussion.

determination. We have two formulations of a theory together with two viable and distinct interpretations (or interpretation types). Above we listed three ways in which such a situation may lead to formalism underdetermination. Firstly, the underdetermination may be *strict*, meaning that there is no single interpretation that can be applied to both formalisms. Secondly, it may *exclusive*, meaning that there exists at least one interpretation which is applicable to one formulation but not to another. Thirdly it may be *loose*; meaning that one or more of the interpretations are more natural to one formulation than to another.

Since the teleological interpretation (or interpretation type) is applicable to both Hamiltonian and Lagrangian formulations the first does not apply. One could argue for the second on the grounds that the instantaneous interpretation might seem not to be applicable to the Lagrangian formulation. However, one may reconstruct the Lagrangian formulation such that it is based upon points rather than curves within the tangent bundle and such that the dynamical equations are differential equations giving a unique specification of dynamics at such a point rather than restrictions on possible curves. Such a re-conception means that it is possible to apply a instantaneous interpretation to the Lagrangian formalism. However, the historically prior and arguably most fundamental way of understanding Lagrangian mechanics is in the context of action principles and variational calculus and such formal structure does necessarily lead to a formulation which is in terms of curves with two boundary conditions. This point will be further born out when we come to discuss the quantization of Lagrangian mechanics in terms of path integral methods as well as the intimately related issue of symplectic structure. There is therefore a good case for the Lagrangian formulation being more naturally interpreted in teleological rather than instantaneous terms and thus for us being confronted with a loose case of formalism underdetermination.

Given that the solution space of the Lagrangian formulation is that of curves with two boundary conditions, the natural interpretation is one in terms of a histories based ontology; with the *furniture* of the universe entire four dimensional spacetimes along with the appropriate initial and final conditions (i.e., the teleological picture). Given that the solution space of the Hamiltonian formulation is an initial data space, the natural interpretation is in terms of a instantaneous state based ontology; with the *furniture* of the universe three dimensional spatial states with appropriate instantaneous data (i.e., the instantaneous picture). Since the two formulations are empirically equivalent and yet, to an extent, furnish us with distinct ontologies we have a good case within which to examine the problem of constructing a viable structural ontology.

Clearly, although a positive result would not be sufficient to demonstrate superiority of OSR over its conventional realist rivals – since we have not yet convincingly proved a case for the underdetermination of their ontology – a negative result would be serve to severely weaken the tenability of the OSR position: if one cannot find a consistent structuralist ontology for the simplest of cases then the prospects for the programme to find wider application look grim.

## 4.2 The classical structural ontology

The ontic structural realist response to cases of formalism underdetermination is to seek to reconceive the relevant notion of ontology in structural terms such that it is no longer underdetermined. For such structure to genuinely constitute an ontology it is required to consist of more than a mere interrelation between formulations, we need to find a suitably generalising physico-mathematical framework which includes the requisite level of dynamical structure. Is this possible for the case of Lagrangian and Hamiltonian mechanics?

Considering the analysis of (Belot 2007) and following the arguments of (French 2011), we can make a good argument that the answer is yes. Belot's work illustrates that for all standard theories of classical mechanics the space which represents unique solutions within a Lagrangian formulation of mechanics, has a close formal relationship with the space which uniquely represents instantaneous states within a Hamiltonian formulation. Within Newtonian mechanics these two spaces are simply the space of curves solving the Euler-Lagrange equation, $\gamma_{EL} \in \mathcal{S}$, and phase space, $T^{\star}\mathcal{C}$. Not only are these two spaces connected by a set of maps between time slices of Lagrangian solutions and instantaneous canonical states, but since $\mathcal{S}$ is, like phase space, a symplectic manifold, it is possible to prove that the two relevant dynamical arenas are *symplectically isomorphic*. The existence of this symplectic isomorphism then allows us to fix a precise relationship both between functions representing observable quantities within the two formalisms and between the representation of dynamical change in the observable quantities. Given a preferred slicing of a Lagrangian solution, for every moment of time we can construct a symplectic isomorphism between a phase space function and a corresponding function on $\mathcal{S}$ – and this relationship allows us to understand both functions as representing the same underlying physical quantity as it changes over a dynamical history.

Thus the mutual symplectic structure of Lagrangian and Hamiltonian mechanics provides us with exactly the kind of generalising framework, including dynamical structure, which we are looking for and, although we will certainly not claim that this analysis is complete, there is a convincing case for an ontic structural realist account of the Hamiltonian and Lagrangian formulations of Newtonian mechanics in symplectic terms. This ontology is not constituted by the symplectic isomorphism itself but by the interconnections between dynamical structures that it encodes at the level of both observables and the state spaces. To accept this ontology is not to endorse either the instantaneous or teleological interpretations, rather through OSR we are able conceive of a fundamental reality that stands behind these two contrasting pictures of the world in terms of precise structural framework.

### 4.3 Quantization and structural ontology building

We then come to the question most crucial to our analysis. Is this prospective structural ontology of the suitable type to deal with *both* formulation underdetermination issues and the historical undermining of pessimistic meta-induction? Would it be appropriate to conceive of some aspect of symplectic structure as being preserved between the classical and quantum mechanical arenas?

The essence of the answer to this question has been already given. In our discussion of geometric quantization techniques §3 it was noted that one of the key steps was defining the map $A : f \to A_f$ which takes us between classical algebra of observables, defined by functions on a symplectic manifold, and the quantum algebra of observables, defined by self adjoint operators on a Hilbert space. One of the restrictions on this map was that $[A_f, A_g] = i\hbar A_{\{f,g\}}$ and thus we see that by definition the geometric quantization scheme is such that the classical Poisson bracket structure is carried over into the quantum context in terms of the commutator. We can therefore justifiably argue that there exists a structural bridge between the observables of classical Newtonian mechanics and non-relativistic quantum mechanics at a formal level, precisely in terms of the link between the binary operations constituted by the Poisson bracket and the commutator. This analogy is also reflected at the level of dynamics since when combined with the Hamiltonian observable it is the binary operation that is responsible for generating evolution in both the classical and quantum realms – i.e., we have that $\dot{A}_f = i\hbar[A_f, A_H]$ and $\dot{f} = \{f, H\}$. Of course the Poisson bracket is itself defined implicitly by the classical symplectic structure, $\Omega$, and thus we have that it is the symplectic structure of the algebra of classical observables that provides the foundations of the quantum mechanical algebra.

Further to this structural bridge from the symplectic form to the commutator, there is also a suggestive resemblance between the classical state space (a manifold equipped with a symplectic structure) and the quantum state space (a vector space equipped with an inner product structure). The question of the nature of the fundamental mathematical structures that lie behind relationship is a deep one, and its full consideration would involve introduction of a considerable amount of additional formal machinery. A simple and illustrative demonstration of the connection, that shall suffice for our current purposes, is provided by the decomposition of the inner product into real an imaginary parts (Corichi 2008). If we consider two states $\mid \psi\rangle, \mid \phi\rangle \in \mathcal{H}_Q$, and view $\mathcal{H}_Q$ as a real vector space equipped with an inner product, then we may decompose the inner product as: $\langle\psi \mid \phi\rangle = G(\psi, \phi) - i\Omega(\psi, \phi)$, where G is a Riemannian inner product on $\mathcal{H}_Q$ and $\Omega$ is a symplectic two form. The structures which define the classical and quantum state spaces can thus be straightforwardly seen to be connected through the mutual use of symplectic two form. Independently of anything to do with formalism underdetermination, a proponent of OSR would therefore argue that the fundamental structure of a classical or quantum theory is related to maps between algebras of observables, the relevant binary operations and the relationship between the symplectic/inner product structure of the classical and quantum state spaces. Fundamentally this is what is structurally consistent between the classical and quantum theories. It is therefore what OSR implies we should seek to reify in the face of pessimistic meta-inductive arguments. However, this is also the type of structure which we were driven towards when considering the ontology of the classical theory alone so there would seem to be *prima facie* coherence in our approach.

Let us then examine the case in hand more carefully. Given our two classical formulations we arrived at a structural ontology encoded by a symplectic isomorphism between both the relevant observables and state spaces. Given a generalised, geometric picture of classical and quantum theory we arrived at a structural ontology encoded by: i) a Lie algebra morphism (up to a factor) connecting the algebra of observables and the relevant binary operation; and ii) the connection between the symplectic and inner product structures. Although these classical-classical and classical-quantum bridges are not *the same* structures, they are clearly closely related – a sympletic isomorphism can be understood a type of algebra morphism.

One way to refine our analysis a little is to consider two different formulations of quantum theory, look at the common structure, and compare this to both the classical-classical formulation common structure and the general classical-quantum common structure. If we presume to have quantized the Lagrangian formulation classical mechanics using a path integral methodology and the Hamiltonian formulation using canonical quantization (which just amounts to a concrete implementation of geometric quantization) then we would have two formulations of quantum theory, each based on a formulation of classical theory. We will label these two formulations after their principle originators – Feynman on the one hand and Dirac-von Neumann on the other. Our desired consistent structural ontology could then expressed using the diagram that was introduced above:

$$\begin{tikzcd}
C_{Lag.} \arrow[r] \arrow[d] & C_{UF} \arrow[d] & C_{Ham.} \arrow[l] \arrow[d] \\
CQ_{LagFey} \arrow[r] & CQ_{UF} & CQ_{HamDvN} \arrow[l] \\
Q_{Fey.} \arrow[r] \arrow[u] & Q_{UF} \arrow[u] & Q_{DvN} \arrow[l] \arrow[u]
\end{tikzcd}$$

In this notation, our discussion thus far has already effectively covered $C_{UF}$ and $CQ_{HamDvN}$. We will now briefly consider the rest of the diagram in order to give at least an evaluation of the extent to which the relevant structural notion of ontology is suitably 'commutative'.

The fundamental dynamical equation within Feynman path integral quantum mechanics is, for a single particle:

$$Z = \langle q_f | e^{-\frac{i}{\hbar}H(t_f - t_i)} | q_i \rangle = \int Dx e^{\frac{i}{\hbar}\int_0^T \mathcal{L}(q,\dot{q})dt} \tag{1}$$

Where $D$ is the functional measure. This path integral expression describes quantum mechanical behaviour in a configuration space in that, roughly speaking, it gives us a probabilistic weighting to paths through that space between an initial position $q_i$ and a final position $q_i$. We thus see that, under the Feynman approach, a quantum system is associated with a space of possible histories (i.e., the space over which the integral is taken) and the nature of the path integral is such that it gives (in an informal sense) an inner product structure to that space.

Within the classical theory we also focused upon a space of histories as fundamental to the Lagrangian formulation; and it was the symplectic structure of that space which we took to constitute one side of the structural bridge between Lagrangian and Hamiltonian theory. Furthermore, in the generalised abstract case and the case of Hamiltonian theory, there is an extent to which the symplectic structure within the classical theory is analogous to the inner product structure within the relevant Hilbert space. It is natural therefore to ask whether the symplectic structure of the classical history space in Lagrangian theory can be connected with a

Hilbert space, together with the necessary inner product structure, within Feynman path integral quantum mechanics.

Unfortunately, although its heuristic, intuitive and practical value is undoubtably great, the Feynman path integral as it has been introduced, is insufficiently mathematically well defined for us to be able to answer this question. Consideration of the project of providing a more rigorous mathematical basis to it would take us far beyond the limits of our current discussion, but we may at least note that according to (Albeverio, Hoegh-Krohn, and Mazzucchi 2008) the Feynman path integral for the solution the Schrödinger equation can be interpreted rigorously as a *Fresnel integral*[8] over a Hilbert space of continuous paths. Thus, given a suitable formalisation, it does appear to be correct to think of path integral quantum mechanics in terms some form of *Hilbert space for histories.* There is, therefore, some formal support for a tentative proposal that a structural bridge may be made between Lagrangian classical mechanics and path integral quantum mechanics in terms of a connection between: a classical space of histories with symplectic structure, on the one hand; and a quantum space of histories with an inner product structure, on the other. We do not, however, have the Lie algebra morphism that can be demonstrated to connect the observables and dynamics of the classical Hamiltonian theory with the Dirac-von Neumann quantum theory (as arrived at via canonical quantization). Relating the classical Lagrangian notion of observable to some precisely analogous structure within path integral quantum theory – if it is possible – is a highly non-trivial challenge.

In addition to seeking this structural connection between classical Lagrangian and quantum path integral formalisms, consistency with the OSR philosophical framework drives us to look for a similar connection between path integral and Dirac-von Neumann quantum formalisms. Not least this is because these two quantum formalisms would appear to be naturally associated with interpretation in terms of disparate ontologies – a quantum teleological type and quantum instantaneous type picture respectively. Further to this, in order to establish the relevant commutativity we need to find a quantum unifying framework to parallel our classical unifying framework and then hope that the structural commonalities between these two frameworks (the middle edge of our diagram) mirror those between the individual classical and quantum formulations (the two outside edges).

Unfortunately, our progress is once more hampered by the unsolid mathematical basis of Feynman's approach. Again what is desired would be a well defined Hilbert space of histories which could then be connected to the traditional Hilbert space of instantaneous quantum states. In such circumstances, if the two Hilbert spaces could be shown to be unitary isomorphic *and* the relevant isomorphism can be understood as entwining the representations of two sets of quantum observables, then we would have established, despite the apparently fundamental interpretational difference, that the two quantum formulations are fundamentally manifestations of the same underlying physico-mathematical framework/structure. The situation with regard to the Hilbert space aspect of our problem is again promising. According to (Dowker, Johnston, and Sorkin 2010) we may formalise a histories approach to quantum theory using the framework of quantum measure theory (Sorkin 1994) and proceed to construct a histories Hilbert space which can be

[8] A special type of oscillating integral defined on a real vector space equipped with a norm.

proved (given a unitary quantum theory with a pure initial state) to be isomorphic to the conventional Hilbert space of the Dirac-von Neumann formalism. However, despite this success at the level of state-spaces, the situation with regard to observables is less promising as there is currently not a sufficiently general procedure for constructing an observables algebra within a histories Hilbert space formalism, let alone a proof that such histories observables are suitably related to their conventional Dirac-von Neumann counterparts.

We are, therefore, not in a position to reach a strong conclusion with regard to the observables aspect of a cross-formulation quantum mechanical structural framework – and according to our own criteria this means we have not quite met the necessary conditions for an adequate structural ontology at the quantum level. However, through the relevant state space connections we have suggestive evidence that our application of OSR in terms of the digram above is leading us in a promising direction. In particular, for all of the four outer nodes of the diagram – i.e., the Lagrangian and Hamiltonian formulations of classical mechanics and the path integral and Dirac-von Neumann formulations of quantum mechanics – all the necessary structural connections can be seen to hold with regard to the state spaces involved.

## 5 Conclusion

When considered in the context of pessimistic meta-induction and formalism underdetermination, a viable framework for an ontic structural realist understanding of non-relativistic particle mechanics can be established as follows:

- The symplectic structure and Poisson bracket algebra of observables are what is fundamental at a classical level.
- The inner product structure and commutator algebra of observables are what is fundamental at a quantum level.
- The classical and quantum structures are analogous in the case of the state spaces and, modulo the difficulties mention, connected directly by a Lie algebra morphism in the case of the observables.

We thus have good evidence for the fundamental consistency within an OSR reading of non-relativistic particle mechanics. Furthermore, we have laid the foundations for a set of structures that is substantial enough to constitute a viable ontology. A Lie algebra morphism between algebras of observables is more than a mere set of interconnections or group type object – it is exactly the type of physico-mathematical structure that the proponent of OSR must look for, since it cleavers our formalism at precisely at the basic dynamical joints. Together the interconnections between the relevant observable algebras and state spaces form a framework that generalises all of non-relativistic particle mechanics. As such, what we have found constitutes the fundamental architecture of the physical theories within the scope of our present analysis – and, moreover, provides a basis from which to extend our investigation into the realms classical and quantum field theories.

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## References

Albeverio, S. A., R. J. Hoegh-Krohn, and S. Mazzucchi (2008). *Mathematical Theory of Feynman Path Integrals*. Springer-Verlag.

Bain, J. (2009). Towards structural realism.

Belot, G. (2007, Jan). The representation of time and change in mechanics. In J. Butterfield and J. Earman (Eds.), *Handbook of Philosophy of Physics*, Chapter 2. Elsevier.

Corichi, A. (2008). On the geometry of quantum constrained systems. *arXiv gr-qc*.

Dowker, F., S. Johnston, and R. D. Sorkin (2010). Hilbert spaces from path integrals. *Journal of Physics A: Mathematical and Theoretical 43*(27), 275302.

Echeverria-Enriquez, A., M. Munoz-Lecanda, N. Roman-Roy, and C. Victoria-Monge (1999). Mathematical foundations of geometric quantization (arxiv.org/abs/math-ph/9904008).

French, S. (2011). Metaphysical underdetermination: why worry? *Synthese 180*(2), 205–221.

French, S. and J. Ladyman (2003). Remodelling Structural Realism: Quantum Physics and the Metaphysics of Structure. *Synthese 136*, 31–56.

Jones, R. (1991). Realism about what? *Philosophy of Science 58*(2), 185–202.

Katzav, J. (2004). Dispositions and the principle of least action. *Analysis 64*(283), 206–214.

Ladyman, J. (1998). What is structural realism? *Studies In History and Philosophy of Science Part A 29*(3), 409 – 424.

Ladyman, J. and D. Ross (2007). *Every thing must go*. Oxford University Press.

Laudan, L. (1981). A confutation of convergent realism. *Philosophy of Science 48*, 19–49.

Markosian, N. (2011). Time (plato.stanford.edu/entries/time).

North, J. (2009). The "structure" of physics: A case study? *The Journal of Philosophy 106*, 57–88.

Pooley, O. (2006). Points, particles and structural realism. In S. French, D. Rickles, and J. Saatsi (Eds.), *Structural Foundations of Quantum Gravity*, pp. 83–120. Oxford University Press.

Psillos, S. (1999). *Scientific Realism: How Science Tracks Truth*. London: Routledge.

Smart, B. (2012). *A Critique of Humean and Anti-Humean Metaphysics of Cause and Law*. Ph. D. thesis, Univeristy of Nottingham.

Sorkin, R. D. (1994). Quantum mechanics as quantum measure theory (arxiv: gr-qc/9401003v2).

Worrall, J. (1989). Structural realism: The best of both worlds? *Dialectica 43*(1-2), 99–124.

Wüthrich, C. (2010). No presentism in quantum gravity. In *Space, Time, and Spacetime*, Volume 167 of *Fundamental Theories of Physics*, pp. 257–278.