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The Mathematical Description of a Generic Physical System

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Abstract. When dealing with a certain class of physical systems, the mathematical characterization of a generic system aims to describe the phase portrait of all its possible states. Because they are defined only up to isomorphism, the mathematical objects involved are "schematic structures". If one imposes the condition that these mathematical definitions completely capture the physical information of a given system, one is led to a strong requirement of individuation for physical states. However, we show there are not enough qualitatively distinct properties in an abstract Hilbert space to fulfill such a requirement. It thus appears there is a fundamental tension between the physicist's purpose in providing a mathematical definition of a mechanical system and a feature of the basic formalism used in the theory. We will show how group theory provides tools to overcome this tension and to define physical properties.

Keywords. Group Theory, Individuation, Quantum Mechanics, Structuralism.

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1. Introduction

The progressive mathematization of both Classical and Quantum Mechanics witnessed over more than a century has provided increasingly abstract ways of thinking of physical systems. It has become the norm amongst theoretical and mathematical physicists to characterize a generic physical system by appeal to some specific class of mathematical objects. For example, in Carlo Rovelli's book on Quantum Gravity, one finds that "a [classical, non-relativistic] dynamical system is determined by a triple $(\Gamma_0, \omega_0, H_0)$, where Γ_0 is a manifold, ω_0 is a symplectic two-form and H_0 is a function on Γ_0" (Rovelli 2004, p. 100) whereas "a given quantum [non-relativistic] system is defined by a family (generally an algebra) of operators A_i, including H_0 [the Hamiltonian], defined over an Hilbert space $\mathcal{H}_0$" (Rovelli 2004, p. 165). Statements of the sort can be found in almost any book that introduces the mathematical formalisms of Classical and Quantum Mechanics.

This form of discourse, that "defines", "determines" or "characterizes" physical systems by the specification of some abstract mathematical object, offers a particularly nice setting where to study the relation between the ontology of Physics and that of Mathematics. On a first reading, *defining* a system as e.g. a symplectic manifold seems a markedly stronger claim than simply *characterizing* it by appeal to the notion of symplectic manifold. The former can be read as an *ontological* claim about the actual nature of physical systems, whereas the latter seems to involve only a (sophisticated) way of making *reference* to a given system. Adopting the former reading of "definition" in the context of mathematical physics seems to imply the radical claim that there is no ontological distinction between mathematical and physical objects. And—it is known—this position leads to difficult epistemological problems (see Resnik 1990 for a review of these). However, I consider that any attempt to draw metaphysical conclusions from the formalism of Mechanics should be preceded by a careful study of the mechanisms enabling mathematical objects to *refer* to physical ones. Thus, it seems to me a more prudent attitude, at least at the outset and tentatively, to consider "definition", "determination" and "characterization", as these occur in modern texts, as intended synonyms.

In considering the mathematical descriptions of physical systems, prior to the ontological question "Are physical systems really mathematical objects?", I hence want to pose the epistemic question “Can a physical system be *fully and unambiguously* characterized by some mathematical object? And for the answer to this question to be yes, which properties is this mathematical object required to have?”

It is to the investigation of this last question, particularly in the context of Mechanics, that this work is devoted. Hopefully, from the perspective of a philosopher, this should lead to a better appreciation of what is involved in the mathematical characterization of physical systems, insight that cannot but benefit those trying to adopt an ontological understanding of these characterizations. In particular those defending a realistic interpretation of the wave function in Quantum Mechanics could be led by such an analysis to a better grasp of their commitments.

The remainder of this paper has two main parts. The first, more general one, attempts further to clarify the question I posed above. In Section 2.1., I compare the expectations of different theoretical physicists regarding the descriptive power of the mathematical characterizations of mechanical systems. In Section 2.2., mathematical structuralism enters into the picture by the crucial observation that most often definitions in mathematical physics are stated “up to isomorphism”. In Section 2.3., I propose a precise requirement for the individuation of physical states of a system which any mathematical object must meet in order to be a candidate for the description of that system. Then, the second part of the paper is restricted to Quantum Mechanics and aims to study some techniques used to implement this requirement for the individuation of states. Section 3.1 is a quick introduction to three different approaches to this problem—algebraic, group-theoretic and geometric. Section 3.2 studies how individuating properties may emerge from the introduction of groups and Section 3.3. presents in some detail the example of the group *SO(3)*. I conclude with some remarks in Section 4.

2. Requirement of individuation for physical states

2.1. The descriptive power of mathematical definitions of physical systems

In modern treatments of theoretical physics, a definition of a physical system is typically a statement of the following form:

Definition: A physical system *S* is given by $\{A_1, \ldots, A_n\}$ such that …

Here $\{A_1, \ldots, A_n\}$ is a set of mathematical objects verifying some particular conditions. In Mechanics, definitions of this sort are not intended to apply to all physical systems but just to a restricted class—say, non-relativistic, classical, holonomic systems with a finite number of degrees of freedom. However, inside this class, these definitions do apply to any system and it is in this sense that I say they are definitions of a *generic* physical system.

It does not seem too risky to say that most physicists would agree on this point. Starting from here, there are two major aspects where they may disagree. First, obviously, they may not agree on the *content* of the definition, but it cannot be the aim of this paper to decide which content is "*the* correct one". Instead, we will simply try to establish, by means of general considerations, some criteria of what it means for a mathematical definition of a system to be *acceptable*. Thus, we need to know which is the *purpose* of these definitions; it is here that we encounter a second, subtler source of disagreement.

There are two principal contrasting attitudes one can adopt here. On the one hand, one can follow a "*formalist perspective"* and consider these definitions simply as a device to detect the *minimal framework* necessary to develop most of the (mathematical) techniques used in the study of physical systems. For example, as soon as one has a manifold Γ equipped with a symplectic structure ω and a preferred function H, one can write Hamilton's equations of motion. In other words, the equations of motions do not provide any supplementary information that is not already contained in the triple $\{\Gamma, \omega, H\}$. Here, *mathematical definitions point to the theoretical locus where all the information about a system is stored*; *but there is a fundamental gap between the mathematical description and the physical interpretation*. To actually refer to a given, particular physical system, it is not enough to have the formal, abstract description of it: one needs to add external information that conveys to it its physical interpretation. An explicit example of this viewpoint is found in Strocchi's book on Quantum Mechanics:

> In the mathematical literature, given a C*-algebra A, any normalized positive linear functional on it is *by definition* a state; here we allow the possibility that the set *S* of *states with physical interpretation* (briefly called physical states) is […] smaller than the set of all the normalized positive linear functionals on A. (Strocchi 2005, pp. 22-23, my emphasis)

There is a sharp contrast between what is declared by definition in Mathematics and what is to be interpreted in Physics.

On the other hand, one can pursue a "*descriptive perspective"*, and consider that the abstract mathematical characterization of a physical system encapsulates all the information about the system without the need of any further external interpretation. Whereas the formalist perspective downplays the descriptive role of the mathematical definition of a physical system by demanding that a physical interpretation be also given, this other contrasting perspective confers full descriptive power on the mathematical formalism. On this viewpoint, one considers that if a physicist is given the abstract description of a given physical system and nothing more, *she will nonetheless be able to recognize which physical system is being described*. For example, if she studies the classical mechanical system defined by the triple $\{\Gamma, \omega, H\}$, where (Γ, ω) = $T^*\mathbf{R}$ equipped with its canonical symplectic form and $H(q,p) = p^2/2m$, one expects her to recognize a free massive (non-relativistic) particle moving in a one-dimensional space. To me, this is what Rovelli has in mind when he declares: "a dynamical system is *completely* defined by a presymplectic space (Σ, ω)" (Rovelli 2004, p. 101, my emphasis). It is also the perspective adopted by Landsman when he criticizes the use of Hilbert spaces in Quantum Mechanics:

> "all Hilbert spaces of a given dimension are isomorphic, so that one cannot characterize a physical system by saying that 'its Hilbert space of (pure) states is $L^2(\mathbf{R}^3)$'." (Landsman 1998, p.6),

For it is only when expecting full descriptive power to flow from the mathematical characterization of a physical system that worries about the fact there is only one infinite-dimensional Hilbert space arise. From the formalist perspective, there is no reason why the fact that all Hilbert spaces are isomorphic should be seen as supplying the motivation to build an alternative, possibly more sophisticated, formalism for Quantum Mechanics. Indeed, from that point of view one should never expect to fully characterize a physical system just by "saying that 'its Hilbert space of (pure) states is $L^2(\mathbf{R}^3)$'", since an additional physical interpretation, that transcends the formal language, is needed.

The difference between both perspectives can be rendered precise using the elementary language of type theory as follows[3]. Consider first two different types: the "Physics type" T_{phys} whose tokens are the mechanical systems, and the "Mathematics type" T_{math} whose tokens are the mathematical objects used to describe the systems under consideration. The mathematical definition of a physical system may be seen as a map

[3] For our purposes, type theory is very similar to set theory: the type/token relation is the analogue of the set/element relation. The reason why I choose the language of type theory rather than of set theory will become clear in the next section.

from T_{phys} to T_{math} that, to every physical system S, associates the mathematical object $D(S)$ describing it. Now, the properties ascribed to this map are what fundamentally distinguish the two perspectives. Indeed, the main claim of the descriptive perspective, as I understand it, is the *injectivity* of this map:

> **Faithfulness requirement (descriptive perspective):** consider two physical systems S and S' defined by the mathematical objects $D(S)$ and $D(S')$. We have $D(S) =_{\mathrm{M}} D(S')$ if and only if $S =_P S'$.

This is of course tantamount to saying that any difference between two physical systems should be reflected in their respective mathematical descriptions—a requirement which is not imposed in the formalist perspective.

One might argue that the descriptive perspective is too naïve and that it is *in principle* impossible for an abstract, formal, mathematical description to capture "all there is" about a real, concrete, physical system. However, in common with other moves that attempt to undermine a research program by appeal to such extremely general arguments, I find this position sterile. Rather I believe that adopting the descriptive perspective as a *working hypothesis* and seeking to push it to its limits can yield interesting insights in theoretical physics, even if we are eventually led to reject the hypothesis. As Catren beautifully said of a different though not altogether unrelated topic: "It is necessary to be programmatically ambitious in order to fail in a productive way" (Catren 2009, p. 470).

Thus, we will try to travel as far as possible along the road that the descriptive perspective suggests. As it will become progressively clearer, the faithfulness requirement imposes some strong conditions on the mathematical formalisms to be used for Mechanics. One of the main points of this paper is to show how some of the technical developments in the mathematical foundations of Mechanics arise as attempts to meet these conditions.

2.2. Mathematical objects as structures

It is fundamental to remark that the faithfulness requirement, imposed by the descriptive perspective, *presupposes two underlying notions of identity*: a first one between mathematical objects (denoted by $=_M$), and another *independent* notion of identity between physical systems (denoted by $=_P$). There is not an absolute notion of identity and every given type has its own criteria of identification[4]. To incorporate this requirement into Mechanics, it is of crucial importance to know when we can say, for two mathematical objects A and A', that we have $A =_M A'$. When pursued to its furthest reaches and ramifications, this simple question about identity in Mathematics raises very far-reaching issues connected to the development of n-categories and homotopy type theory, while in the Philosophy of Mathematics it leads to perennial questions about the ontology of Mathematics. Since our concern here is with the mathematics of theoretical Mechanics, we do not dare to address these questions in their full generality but confine ourselves to those aspects relevant to the practice of theoretical physics.

Von Neumann's proof of the mathematical equivalence of Heisenberg's 'matrix mechanics' and Schrödinger's 'wave mechanics' was a celebrated landmark in the development of Quantum Mechanics. Then, the Stone-von Neumann theorem secured that indeed a certain quantum system was *uniquely* defined by the requirement of being an irreducible representation of (Weyl's form of) the canonical commutation relations: the formalisms of Heisenberg (using the abstract Hilbert space of infinite complex matrices), Schrödinger (using the Hilbert space $L^2(\mathbf{R})$) and Fock (using the Hilbert space $\ell^2(\boldsymbol{N})$ of all square-summable sequences) appeared as three equivalent realizations of one and the same mathematical object describing one and the same physical system[5].

The important point is that the founders of the quantum theory considered the physical system to be uniquely described because *they considered equivalent representations as identical*. Otherwise stated, two abstract descriptions of a quantum system had to be considered identical if isomorphic[6]. Thus, the above example shows how it clearly emerged, from the historical development of the theory, that for the purposes of Physics, the relevant notion of identity for mathematical objects was in fact isomorphism:

$$D(S) =_M D(S') \Leftrightarrow D(S) \simeq D(S')$$

As is well explained in Rodin (2011), in dealing with mathematical objects defined up to isomorphism one enters into the realm of mathematical structuralism. In fact, this

[4] This point on identity is precisely one of the deepest differences between type theory (contextual identity) and set theory (absolute identity). It is because of this difference that the language of type theory seems better adapted to our discussion.

[5] For an explicit treatment of these formalisms, see for example (Gazeau 2009, pp. 13-18).

[6] Indeed, when working in the category of all representations (of a given C*-algebra) "equivalence of representations" is just another name for the general concept of "isomorphism".

move—of *extending* the notion of equality to that of isomorphism—has been recently dubbed the "*Principle of Structuralism"* (Awodey 2013). One is thereby forced to endorse the Principle of Structuralism in the mathematical foundations of Quantum Mechanics.

It is striking that consciousness of this development—of always having to read the abstract mathematical definitions of physical systems 'up to isomorphism'—has not so clearly being carried over from Quantum to Classical Mechanics. In the latter context, it is for example rare to consider the phase space of a system to be given by a symplectic manifold 'up to symplectomorphism'[7]. Instead, one generally has the impression of working with a particular symplectic manifold and sticking to it, without considering any 'change of representation'—think of the free massive non-relativistic particle: one will almost universally work in the cotangent bundle $T^*\mathbf{R}^3$. However, a brief reflection shows this is not the case, and that one is in fact working up to isomorphism. Indeed, continuing with the same example, one can instead decide to work in the space $\mathbf{R}^3\times\mathbf{R}^3$ and impose $\boldsymbol{\omega} = dq^i \wedge dp_i$ as an *ad hoc* definition; the physical system ought to be the same, but this symplectic space is identical to the cotangent bundle only insofar as it is symplectomorphic to it[8]. Thus, the Principle of Structuralism is also apparent in the Classical theory. But there, it seems to be less explicitly recognized than in the Quantum case. It seems to me that one reason for this oversight is that in Classical Mechanics it is much easier to adopt a directly realist reading of the mathematical formalism—the configuration space $\mathbf{R}^3$ **'**really is the Euclidean space out there'. Contrary to what happens in Quantum Mechanics, the Classical Realm typically does not force us into higher levels of abstraction and one can still retain the impression of the 'materiality' of the mathematical constructions involved. But it is an impression we must abandon.

Precisely, the whole point of mathematical structuralism is to insist that, beyond certain explicitly stated properties, the specific nature of the elements in a structure—i.e., the 'materiality' I just referred to—is completely irrelevant:

> …there is a certain degree of 'analysis' or specificity required […], and beyond that, it does not matter what the structures are supposed to be or to 'consist of'— the elements […] are simply *undetermined*. (Awodey 2004, p. 59)

Von Neumann himself also stressed this point from the outset:

[7] A 'symplectomorphism' is an isomorphism in the category of symplectic manifolds.

[8] As sets, these two spaces are different. Whence, *sensu stricto*, they are not identical.

> …a unified theory, *independent of the accidents of the formal framework selected at the time*, and exhibiting only the really essential elements of quantum mechanics, will then be achieved if we […] investigate the intrinsic properties (common to $L^2(\mathbf{R})$ and $\ell^2(\boldsymbol{N})$) […], and choose these properties as a starting point (von Neumann 1955, p. 33, my emphasis).

As long as the canonical commutation relations are implemented, it does not matter whether the Hilbert space describing the quantum particle is made of functions over a space, infinite sequences of complex numbers, or sections of a certain fiber bundle. It does not matter either whether the points of the space describing the classical particle are points of a cotangent bundle, of a Cartesian space or of the dual of a Lie algebroid. This is why Awodey calls them "schematic structures" (Awodey 2004, p. 62). The mathematical objects involved in the definitions of mechanical objects are schematic structures.

2.3. The requirement of individuation for physical states

We have already seen how Landsman expresses his dissatisfaction with the use of Hilbert spaces as the mathematical basis for the characterization of physical systems. Of course, many others share this view: it is at the root of von Neumann's motivation in studying rings of operators and in introducing the so-called von Neumann algebras (Rédei 1997). We are now in a position to understand, in very simple terms, the source of this dissatisfaction. For, if you combine the seemingly inescapable—at least for the mathematics of Mechanics—Principle of Structuralism with the faithfulness requirement of the descriptive perspective, the fact that there is only one infinite-dimensional Hilbert space (up to isomorphism) shows that either there is only one unique Quantum system; or that Hilbert spaces do not provide enough descriptive resources for the characterization of mechanical systems.

The classical analogue of this is Darboux's theorem: any two symplectic manifolds of the same dimension are *locally* isomorphic. For a given dimension, we thus get infinitely many non-isomorphic symplectic manifolds, but their differences are only of a topological nature. Whereas Hilbert spaces evidently fall short of accounting for the diversity of the Quantum realm and hence cannot be taken as an *acceptable characterization* of physical systems, it is not so clear whether the same conclusion can be drawn about the use of symplectic manifolds in Classical Mechanics. Undoubtedly, we are missing here further criteria that would allow us to decide when a type of structure is "descriptive enough" to be an acceptable candidate for characterizing physical systems. The existence of numerically many non-isomorphic structures of a given type is surely one criterion, but it cannot be the only one.

In all situations considered so far, there was a two-fold move in order to mathematically capture all the physical information: first, the system was said to be characterized by the set of all its possible states; second, it was this set that was meant to be described by some mathematical space. At the end, one gets the so-called "phase portrait" of the system (Abraham and Marsden 1978, p. xviii). A state was intended to be described by a point of the symplectic manifold Γ in the classical Hamiltonian formalism, and by a ray of the Hilbert space $\mathscr{H}$ in the standard quantum formalism[9]. Now, recall that in the descriptive perspective, one expects the theoretical physicist to be able to extract, simply from the given abstract mathematical structure describing the system, all the relevant physical information. In particular, given an element of this structure—a point of the symplectic manifold or a ray of the Hilbert space—one expects her to be capable of recognizing the specific state of the system. But, for this to be possible, *the mathematical structure describing a physical system must be such that its different elements can be properly distinguished*. In other words, if the mathematical definition is all there is to know in order to completely determine a physical system, one is confronted with the following requirement:

> **Requirement of individuation for physical states:** it must be possible, in practice, to *qualitatively identify* any specific physical state within the mathematical structure used to define the system.

As we now show, this requirement imposes, on the structures that can in principle characterize mechanical systems, much stronger conditions than the previous faithfulness requirement. To see this, let us unpack what the requirement of individuation is actually saying. Firstly, it is important to understand the difference between "being able to *identify* a physical state" and "being able to *distinguish* between two physical states". In a short paper from 1976, Quine introduced three different ways of distinguishing two objects. According to him, two objects are (in decreasing order of discernibility)

[9] A *ray* of a Hilbert space is a one-dimensional subspace.

- *absolutely discernible* if there exists a one-place predicate that is true of one object but not of the other (e.g. two spheres of different color),
- *relatively discernible* if there exists a two-place relation that is true of them in one order but not in the other (e.g. two spheres of the same color but different size),
- *weakly discernible* if there exists a two-place irreflexive relation that is true of them (e.g. two qualitatively identical spheres, as considered in (Black 1952)).[10]

One important motivation for introducing this distinction was reconciling Leibniz's Principle of Indiscernibles with the existence of some highly homogeneous mathematical objects—such as the Euclidean plane, where all points seem indiscernible from each other. The (somewhat irritating) question was then: "Since any predicate true of one given point of the Euclidean plane will also be true of any other point, how can you possibly know there is more than just one point in this plane?" Weak discernibility was meant to provide a rigorous answer to this: it allows you to determine *how many* identical iron spheres there are in Black's otherwise empty universe. However, a physicist dealing with a particular system will not only want to describe the number of different possible states. He will also need to make objective reference to a certain, particular state in such a way that any other physicist will understand which state he is referring to.

I say an element in a structure can be *identified (or individuated)* if there exists a one-place predicate that allows absolute discernibility from any other element. In this way, paraphrasing Weyl, we find "a conceptual fixation [of the element] that would enable one to reconstruct [it] when it has been lost"[11].

Secondly, this identification needs to be "qualitative". As Dieks (2014) rightly points out, Quine's whole distinction implicitly depends on the kind of predicates allowed. For, if among the accepted predicates are included "referential devices as proper names, proper adjectives and verbs", then any two objects possessing a thisness will automatically become absolutely discernible, and the grades of discriminability become

[10] In (Quine 1960), the author introduced the distinction between absolute and relative discernibility. The third term was introduced in (Quine 1976), but there he changed "relative discernibility" into "moderate discernibility". However, I follow the terminology that has been adopted in the philosophy of physics literature (Saunders 2006, Dieks 2014).

[11] In the original, Weyl writes: "A conceptual fixation of points by labels […] that would enable one to reconstruct any point when it has been lost, is here possible only in relation to a *coordinate system*, or frame of reference, that has to be exhibited by an individual demonstrative act." (Weyl 1949, p. 75)

useless[12]. Therefore, a criterion, for what Dieks calls the "scientific respectability of relations [and predicates]", has to be introduced. Fortunately, since we are here dealing with mathematical objects that are schematic structures, we know precisely which predicates are "scientifically respectable" in this context. As Esfeld and Lam stress,

> It goes without saying that there is in [structuralism] no question of identity conditions for an object independently of other objects. But this does not mean that relations cannot provide identity conditions. Which relations make up for identity conditions for which types of objects depends obviously on the case under consideration. (Esfeld and Lam 2009, p. 8)

More precisely, to insure the relations —and predicates built from relations—do not depend on the superfluous nature of the elements in the structure, the allowed, respectable properties have to be *invariant under any isomorphism*. These properties are called "structural" by Awodey (2013, p.5) and "objective" by Weyl (1949, p. 73).

Thus, if a mathematical structure is to fulfill the requirement of individuation, physical states have to be individuated by 'objective' or 'structural' properties. In particular, these properties have to be invariant under any *auto*morphism. This simple remark is actually very fruitful, for it furnishes a practical tool to detect the "amount of individuation" that can be provided within a given mathematical structure. Indeed, by the above definitions, it follows that two elements related by such an automorphism cannot be absolutely distinguished—and hence cannot be individuated either. In fact, the orbits of the group of automorphisms are the smallest subsets of the structure that can be identified or individuated[13]. The (internal) descriptive power of a structure is thus encapsulated in the action of its group of automorphisms, and we arrive at the following consequence: *if a physical system is to be completely characterized by a mathematical structure* S *(that constitutes its phase portrait), then its possible physical states ought to be described by the orbits of the automorphism group* Aut(S).

To conclude this section, notice how this new point of view provides yet another way of understanding the lack of descriptive power of Hilbert spaces. Indeed, given an n-dimensional Hilbert space $\mathscr{H}$, its group of automorphisms is the group *U(n)* of unitary

[12] Adams (1979) introduced a distinction between "thisness" and "suchness". Intuitively, the thisness (or haecceity) is the property of an object that allows one to point at it and say in a meaningful way '*this* object'. On the other hand, "suchness" is a synonym of "qualitative property"—and also, in this paper, of "objective property" and "structural property".

[13] Given the left action of a group G on a set E, the *orbit* O_x of an element x is the subset of elements of E to which x can be transformed by some element of G.

transformations. Now, a pure state of a quantum system is supposed to be described by a ray of this Hilbert space, but the action of *U(n)* on the set of all rays is transitive: the projective Hilbert space is completely homogeneous and no physical state can be individuated. Indeed it turns out this property of homogeneity or maximal symmetry can even be used to define projective Hilbert spaces (Ashtekar and Schilling 1997, section III.B.).

3. Methods for introducing individuality into Quantum Mechanics

Let me briefly summarize what has been said so far. When dealing with a certain class of physical systems, the mathematical characterization of a generic system aims to describe the phase portrait of all its possible states. The mathematical objects involved are defined only up to isomorphism and are thus, in Awodey's terms, "schematic structures". If, rather than describing the minimal framework, one expects these mathematical definitions to completely capture all the physical information of a given system—endorsing hence what I called the descriptive perspective—one is led to the strong requirement of individuation for physical states. On the other hand, the main ingredients of the standard formalisms of both Classical and Quantum Mechanics—namely, symplectic manifolds and Hilbert spaces—do not meet the latter demand. There thus emerges a fundamental tension between the intended purpose of the mathematical definitions of mechanical systems and the basic formalism used in the theory.

The main thesis of the present paper is that this tension has been one of the driving forces in the Foundations of (Quantum) Mechanics, in the sense that many of the developments in this field can be retrospectively understood as attempts to overcome it. The second part of the work is devoted to a brief survey of some of the mathematical notions introduced in the quantum formalism in the light of this understanding.

3.1. Three approaches to introducing individuality

At this point, the essential problem we face in the standard quantum formalism is the lack of enough qualitative properties: given only an abstract Hilbert space, it is impossible to make unambiguous reference to a specific ray without appealing to any primitive thisness—this is the content of Weyl's previous citation[14]. But "science is averse to the use of the notion of *haecceity*, "primitive thisness", in order to individuate objects" (Dieks 2014, p. 43). To avoid this, one option would be to discard Hilbert spaces from the outset and start looking for completely new mathematical structures that would do the job. However, the use of Hilbert spaces in the *practice* of Quantum Mechanics is so widespread and deep-rooted, that an alternative may be sought—namely, to retain Hilbert spaces but enrich them with further structure: i) consider projective Hilbert spaces as a sort of homogeneous underlying receptacle involved in the description of any physical system; and ii) describe a concrete and specific system with a (slightly) more sophisticated mathematical setting, that would somehow break this homogeneity and convey to the different physical states qualitative properties sufficient to distinguish them. In other words, address the problem of how to transform a mere numerical multiplicity into a multiplicity of qualitatively distinct elements[15].

In the standard formalism, the key technical concept needed to implement this idea of "adding extra structure" is that of a *representation*: to describe a system, one should not consider a *bare* Hilbert space but, instead, a Hilbert space only insofar as it is the canvas on which some *external* information is instantiated. There are at least three different strategies that have been followed in mathematical Physics.

i) *Physical systems as representations of algebras.* First, one can claim the crucial information about a physical system lies in the algebraic structure of the observables. Thus, a physical system would be described by the representation of an abstract C*-algebra A—that is, by a triple $(A, \mathcal{H}, \pi)$ where H is a Hilbert space and π is a morphism of C*-algebras from A to $B(\mathcal{H})$, the algebra of all bounded operators. This road leads to the algebraic formulation of Quantum Mechanics (Strocchi 2005) and the theory of algebraic quantum fields (Haag 1996).

ii) *Physical systems as representations of groups.* On the other hand, one can try to build up the theory by focusing on the notion of group, in which case the set of possible states is to be mathematically described by a unitary group representation—that is, by a triple $(G, \mathcal{H}, \rho)$ where G is now a group and ρ is a

[14] See footnote 9.

[15] By "numerical multiplicity" I mean a multiplicity of elements that are only weakly discernible. A mathematical structure is a numerical multiplicity if the action of its group of automorphisms is transitive.

morphism of groups from G to $U(\mathcal{H})$, the group of all unitary operators. This is the road famously followed by Wigner (1959), Eddington (1939) and Souriau (2005), among many others.

iii) *Physical systems as systems of imprimitivity.* In Mackey's approach to Quantum Mechanics, the central notion is that of a G-space (*i.e.* a space X equipped with a group action) and an (elementary) quantum system is then defined as an (irreducible) representation of a G-space. A representation of a G-space is more commonly called a system of imprimitivity[16].

Paraphrasing Castellani, I call these the algebraic, group-theoretical and geometrical approaches to the problem of individuation of physical states[17]. To take an example, consider again the description of the non-relativistic spin-zero particle moving in a three-dimensional space. This quantum system could not be characterized solely by the Hilbert space $L^2(\mathbf{R}^3)$. For an algebraist—who considers *properties* (observables) to be primitive—it is to be determined as an irreducible representation of the Weyl algebra[18] and it is well-defined because of the Stone-von Neumann uniqueness theorem; for a group-theorist—who considers *symmetries* to be primitive—the system is defined as the only irreducible unitary representation of the Heisenberg group $\boldsymbol{H}_7$; finally, for a geometer—who considers *space* to be primitive—the system is defined as an irreducible representation of the action of translations on the Euclidean space, and the uniqueness of the definition is secured through Mackey's theorem of imprimitivity.

Despite their differences, there is a clear technical sense in which the definitions are equivalent for this particular quantum system (for example, the Weyl algebra is isomorphic to the group C*-algebra associated to the Heisenberg group (Strocchi 2005, p.60)). Moreover, the three approaches share the same underlying idea: they add something to the initial abstract Hilbert space, and, by doing so, they define a new structure with a reduced group of automorphisms. In turn, this reduction entails an

[16] Hence, in this third option, a system is described by a tuple $(H, G, X, \rho_X, \rho_H, \pi)$ where ρ_X is the action of G on X, ρ_H is the action of G on H and π is a C*-algebra morphism from $C_0(X, □)$ to $B(H)$. For a modern introduction to Mackey's approach, see Landsman 2006 and Varadarajan 2007.

[17] In (Castellani 1998), the author considers the problem of *constitution* of physical objects: "What kind of properties and prescriptions do we need in order to construct an object?", and then studies "the group-theoretic approach to the problem […] grounded on the idea of invariance." (p. 182).

[18] In Strocchi's words: "The abstract algebra generated by (abstract) elements $U(\alpha), V(\beta), \alpha, \beta$ □ $\mathbf{R}$ […] satisfying $U(\alpha) V(\beta) = V(\beta) U(\alpha) exp(-i\alpha\beta)$, $U(\alpha) U(\beta) = U(\alpha+\beta)$ and $V(\alpha) V(\beta) = V(\alpha+\beta)$ is called the *Weyl algebra*." (Strocchi 2005, pp.58-59) Notice his insistence on the abstract character of this definition.

emergence of some qualitative properties: not all rays will be related to each other by an automorphism, and a certain amount of individuality has thereby been introduced.

Nonetheless, the question still remains: Are these new definitions acceptable? Do these newly introduced structures provide the means for conferring individuality on physical states? To answer this, we need to investigate how structural properties are defined and which substructures they allow us to individuate. In the next section, we shall explain the mechanism of individuation in the group-theoretical approach, leaving the analysis of the other two strategies for future research.

3.2. Properties in the group-theoretical approach

Let us restate the problem from the group-theoretic viewpoint. Essentially, the difficulty encountered with Hilbert spaces is that their group of automorphisms is too big: there is only one orbit (the action of $Aut(\mathscr{H})$ is transitive) and it is thus impossible to find invariant properties that would differentiate different subspaces of $\mathscr{H}$. To break the homogeneity of this given Hilbert space, an *abstract* group G, external and independent to $\mathscr{H}$, is added. This allows to select, among all available transformations of the space state ($Aut(\mathscr{H})$), those that should be considered as meaningful ($\rho(G)$). More precisely, whereas an automorphism of $\mathscr{H}$ will not necessarily be an automorphism of $\mathscr{H}_G = (G, \mathscr{H}, \rho)$, the elements in $\rho(G)$ will[19]. Therefore, in this perspective, *abstract groups enter the picture, not in order to introduce symmetries, but in order to break them*—this is quite the opposite of what is usually thought.

In particular, since structural properties are by definition invariant under isomorphisms, they need to be invariant under the action of the abstract group G. Put differently, it is possible to individuate a subspace of $\mathscr{H}$ only if it is stable under this action. Now, recall $\mathscr{H}_G$ is meant to characterize the phase space of the physical system. This means the *smallest* subspaces of $\mathscr{H}$ that can be individuated are to be regarded as describing physical states. By the above argument, physical states ought then to be described by G-invariant subspaces that contain no smaller invariant subspaces. But this is precisely the technical definition of an *irreducible representation*! Hence, the group-

[19] To see this, consider an element g of the group G. The unitary representation (G, H, ρ') defined by $\rho'(g')=\rho(g)\rho(g')\rho(g^{-1})$ for any g' in G is equivalent to (G, H, ρ), and the intertwining operator that achieves the isomorphism is precisely $\rho(g)$.

theoretical approach to the individuation of physical states leads naturally to the following conclusion:

> **Description of states and properties:** if a specific quantum system is (described by) the schematic structure *(G, $\mathcal{H}$, ρ)*, then a state is necessarily an irreducible representation and physical properties are indices[20] of these representations.

Of course, this should recall Wigner's famous definition of particles as irreducible representations of the Poincaré group (Wigner 1939), as well as Weyl's insight that "[a]ll quantum numbers […] are indices characterizing representations of groups" (Weyl 1950, p.xxi) (more recently, this has also been at the origin of a careful and ambitious group-theoretical analysis of Mechanics (Catren 2014)). The strength of this whole approach, which rests on an analysis of the theoretical means required to objectively single out specific quantum states, is that, not only do we recover Weyl's observation, but we also understand that *it could not have been otherwise*: quantum numbers must be indices of representations simply because they must be structural properties.

3.3. An example: the group S0(3) and angular momentum states

To illustrate the procedure of individuation in the group-theoretical setting, consider a quantum system whose only property is angular momentum. It can be, for example, a free spherical top. It is then customary to take the Hilbert space of states to be $L^2(SO(3))$ (Ashtekar and Lewandowski 2004, section IV.A.). In the light of what has been said, this actually means that one takes the regular representation of the group *SO(3)*, and that $L^2(SO(3))$, equipped with the natural left action of *SO(3)*, is a realization of this representation.

To distinguish the different possible states of the system, one first needs to look for the irreducible representations. This is done by the Peter-Weyl theorem: it decomposes the (infinite-dimensional) regular representation into the direct sum of all (equivalence classes of) finite-dimensional irreducible representations:

$$L^2(SU(2)) = \oplus\, m_l\, V_l$$

[20] An 'index' is a number that takes different values for different representations.

where $m_l = 2l+1 = dim\ V_l$ is the multiplicity of each irreducible representation V_l. From this decomposition emerges the first quantum property—in physics: the total angular momentum; in mathematics: the highest weight. It is the index l that allows the individuation of the different subspaces V_l.

But this is not enough, for if we want to capture the states, we also need to individuate the one-dimensional subspaces lying inside each V_l. In the mathematical theory of Cartan-Weyl, this move corresponds to considering the maximal abelian subgroup—in this case, $U(1)$—and once again breaking the irreducible representations of the whole group into irreducible representations of the subgroup:

$$V_l = \bigoplus V_{l,m}$$

In this second stage what emerges is the second quantum property—in physics: the magnetic quantum numbers; in mathematics: the weights. It is the index m characterizing these one-dimensional irreducible representations of $U(1)$.

Thus, in this description that only uses Hilbert spaces and groups, there is a basis of states that can be discerned using a set of objective properties. Whereas elements of a bare projective Hilbert space did not possess any qualitative properties, the introduction of an abstract group has brought about an emergence of different qualitative properties and successfully transformed the homogeneous canvas of a numerical multiplicity into a multiplicity of qualitatively discernible states.

4. Conclusion

Among the most debated issues in the Metaphysics of Quantum Mechanics, the interpretation of the wave function occupies a central place (Dorato and Laudisa, 2015). In this context, many of the arguments for or against a realist wave function ontology seem to rely heavily on the definition of a wave function as a complex-valued function over configuration space. But the mathematical developments in the foundations of Mechanics clearly show there is no reason to prefer a description of physical states in terms of functions over configuration space rather than, for example, in terms of abstract square-summable sequences. Therefore, before trying to build an ontology for the theory, it seems to me crucial to understand precisely how physical systems are described in the mathematical formalism of Mechanics. The aim of the paper was to start investigating this question, and to trace the consequences of taking such formal descriptions seriously. The analysis paid special attention to the way (in)discernibility was handled. The results can be summarized as follows:

1. The mathematical objects used in the definitions of physical systems are schematic structures, insofar as they are defined only up to isomorphism. The precise nature of the elements in the structure is either a meaningless notion or an otiose one, and in any case should be irrelevant to any philosophical of the formalism of Quantum Mechanics.
2. Moreover, a quantum system cannot be mathematically characterized by a projective Hilbert space, since the elements of such a space are only weakly discernible. This is technically captured by the transitive action of the group of automorphisms. To introduce individuality and qualitative properties, one needs to add extra structure.
3. In the group-theoretical approach, the emergence of qualitative properties conferring individuality to the different states occurs through a *mechanism of restriction*: an abstract group G is introduced, conveying a physical meaning to a restricted set of the group of automorphisms. States are described as irreducible representations of some group and quantum properties, because of their structural nature, are necessarily indices characterizing them.

Strange as it may sound, this analysis shows there is no clear-cut understanding of what a wave function—i.e. a quantum state—actually is. At any rate, any sound definition will involve highly abstract entities, such as C*-algebras or the Heisenberg group. Therefore, wave function realists seem to be necessarily committed to being realist about these abstract schematic structures. They will thus be realist about a big part of pure Mathematics, and this is a step they may not want to take.

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