Observables in the Hamiltonian Formalism

A dissertation based on the classical field theory formulation of Electromagnetism and General Relativity.

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1 Introduction

The canonical formulation of general relativity, specially as a first step towards a quantised theory of gravity, has been a central problem in theoretical physics since Einstein published his ideas at the beginning of the twentieth century. Due to the covariant aspect of this theory, finding a Hamiltonian formulation for it has been one of the major difficulties that physicists face when attempting quantisation. The present work is concerned with this and related issues. The author presents some general notions about field theories so as to later apply them to the case of electromagnetism and general relativity, focussing on the possibility of finding a coherent and consistent canonical formulation for the field equations that govern each of the aforementioned theories.

Although we discuss some of the classical works of Bergman and Komar and the recent paper by Pons *et al.* on these topics, given the limited timespan available to review all the above technical texts, the author opted for a more personal exploration of the subject. This means that, except for the aforementioned works - which were used only as a source for comparison of approaches - and some basic background literature, the author did not use any further bibliographical resources. For instance, the definition of Hamiltonian electromagnetism in Minkowski space or the definition of observer and observable are, so far as the author knows, original in their form, although, as one should expect, they recover well-known results found in the literature. The motivation for this was to find some, hopefully original, ideas on the conceptual issues introduced above within a reasonable amount of time. The result of making this choice is a somewhat less in-depth treatment of the subject, revisiting most of the foundational aspects of it, but also, the author believes, a fresher one. Perhaps one may say that the result is a less extensive but more intensive work.

As the reader may notice by skimming through the text, a concise but exhaustive review of all the necessary definitions is made throughout this work. Each new concept that

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may require more than elementary mathematical notions to be defined is written in boldface. When the author was approaching the questions that motivate this work from the perspective of looking at the foundations, he felt the need to define every concept in precise mathematical terms in order that a formal critique was possible. This limits the diversity and amount of topics that can be discussed in this short paper but it ensures, or at least the author so believes, a careful and thorough consideration of each idea.

Throughout the present work we use the standard formalisms and notations of modern differential geometry; for a concise account of all the tensor notation used, see appendix A which consists of an extract of a recent paper by the author on the topic of intrinsic tensor notations. This may pose some difficulties in phrasing statements that are made in the physics literature which don't usually employ modern notations and terminology. One is tempted to think that all there is to discuss is the convenience of intrinsic definitions, as opposed to coordinate expressions, or the simplicity of the notation; but there are deep philosophical and physical questions associated to these. If one is focused on carrying out certain computations within a well-established theory, then the choice of notations and formalisms should indeed respond to the pragmatic convenience of these for the intended task. The present work, however, is concerned with foundational questions and hence we should not overlook any hidden philosophical (or even physical) assumptions that may be implied by our choice of notation.

In more concrete terms, we will deal with two distinct theories, namely, electromagnetism and general relativity. Despite the obvious similarities - namely, both are classical field theories and both can be formulated as principal bundle gauge theories - the author believes that there is a fundamental difference between the two that may be causing the conceptual difficulties that one seems to find in the latter but not in the former. This difference lies in the fact that classical relativistic electromagnetism - the modern $M \times U(1)$ principal bundle theory - is based in Minkowski space M, a Lorentzian manifold with a fixed metric[1], whereas general relativity is a theory based on a Lorentzian manifold but with *varying* metric. By *varying* we do not refer to the time evolution of the metric, as this is indeed included in a fixed space+time configuration, but rather to the fact that different physical situations - for instance, different distributions of matter or different cosmological boundary conditions - imply different metric configurations. Because of this radical difference we will treat electromagnetism and general relativity on different footings as field theories.

The author believes that computational similarities between the two should not lead to the conclusion that they are conceptually on a par. As an illustration of the point we are trying to convey here, let us consider the example of the harmonic oscillator. It is clear that any classical conservative system can be locally described as some variation of the standard harmonic oscillator by considering Taylor expansions about equilibria. Indeed computations will look very similar, if not identical, in all cases but this is never considered a reason to think that the phenomena being described have some deep conceptual similarities.[2]

[1]The fact that the metric is flat is completely irrelevant for this argument; in fact, electromagnetism is perfectly well-defined for any Lorentzian manifold.

[2]The dynamics of a bird resting on a branch of a tree is identical in mathematical form to that of a

The notion of observable is, of course, at the heart of both theories and the author believes that its definition should be phrased in basic terms drawn from the foundations of the theory in question. As we argued above, observables in relativistic electromagnetism and general relativity will be treated differently in principle, but in both settings the author believes it is necessary to establish a solid definition of observable from the beginning and then to deduce the traditional definitions, using gauge invariance, as some sort of *characterisation theorem.* This is the author's approach to the question of finding a unifying picture of the notion of observables in authors such as Bergmann and Pons et al. However, due to the limited amount of time and scope of the present work and the extensive work done by the author on the foundational mathematical subjects, in these respects only some general considerations are hinted at. Indeed there is much room for further research on these topics and the author hopes to pursue it.

2 The Mathematical Foundations of Field Theories

In this section we present a review of the mathematical concepts that will be used in subsequent sections. We follow Marsden et al. for both the functional analysis (*Tensor Analysis and Manifolds*) and Poisson geometry (*Introduction to Symmetries in Mechanics*). The last two sections, although based in the standard technical developments of Marsden and Kobayashi, contain the result of the author's reflections and analysis of the physical interpretation[3] of the mathematical objects presented therein. In particular the definitions of *Field Theories*, which are largely original in terminology and construction, are made in such a way that later discussions are easily introduced.

The author believes that a good understanding of the mathematical notions presented in this section and a broad picture of the conceptual landscape in which they are embedded may be of great advantage to the task of formally grounding well-established physical theories or successfully formulating new ones.

2.1 Functional Analysis

The calculus of variations is concerned with extrema of functions over some space of maps; these spaces are always Banach[4] and so the notion of derivative is well-defined. In order to formalise the usual *variational* arguments employed in many physical field theories it is necessary to introduce the notion of **pairing** or **generalised duality** between two Banach spaces V and W. This is conveniently realised by means of a bilinear map $\langle , \rangle : V \times W \to \mathbb{C}$ that is non-degenerate in both arguments, that is

- $\langle v, w\rangle = 0, \forall v \in V \Rightarrow w = 0$

ship floating in a harbour; but, who would be persuaded by this fact alone to think that the two are deeply connected?

[3]Nevertheless, the precise physical meaning of field theories will be discussed in the sections below, distinguishing the case of electromagnetism and the case of general relativity.

[4]A Banach space is a vector space equipped with a norm and in which all Cauchy sequences converge to a limit.

- $\langle v, w\rangle = 0, \forall w \in W \Rightarrow v = 0$

In this context the **functional derivative** of a function $f : W \to \mathbb{C}$ with respect to an element $w \in W$ is defined, if it exists, as the only $\delta f/\delta w \in V$ such that:

$$\langle \frac{\delta f}{\delta w}, w'\rangle = df|_w(w') \quad \forall w' \in W \tag{1}$$

and similarly for the second argument.

Let us consider the special case of the space of continuous complex-valued[5] functions over a region $B \subset \mathbb{R}^n$. For specific applications we may restrict to some differentiability or boundary-constraint subspaces. We denote it by $\mathcal{F}(B)$. The pairing or generalised duality that we consider is the L^2 inner product:

$$\langle \varphi, \psi\rangle = \int_B \varphi^*(x)\psi(x)d^n x \tag{2}$$

Then, given a **functional** $F : \mathcal{F}(B) \to \mathbb{C}$ we will be able to determine its **functional derivative** $\delta F/\delta\varphi$ with respect to a function $\varphi \in \mathcal{F}(B)$ by reading off the expression:

$$dF|_\varphi(\psi) = \int_B \frac{\delta F}{\delta \varphi}^*(x)\psi(x)d^n x \quad \forall \psi \in \mathcal{F}(B) \tag{3}$$

and recall that the directional derivative can be computed as usual in calculus on Banach spaces:

$$dF|_\varphi(\psi) = \frac{d}{d\epsilon}F(\varphi + \epsilon\psi)\bigg|_{\epsilon=0}.$$

From this definition it is obvious that a necessary condition for a functional F to have an **extremal value** at $\varphi \in \mathcal{F}(B)$ is that:

$$\frac{\delta F}{\delta \varphi} = 0. \tag{4}$$

A functional on $\mathcal{F}(B)$ is usually given in terms of a **functional density** of order $k \in \mathbb{N}$, that is, a smooth function $f : B \times \mathbb{C} \times ...^k... \times \mathbb{C} \to \mathbb{C}$ such that the value of the functional on an arbitrary argument $\varphi \in \mathcal{F}(B)$ is

$$F(\varphi) = \int_B f(x, \varphi, \partial_i^{(1)}\varphi, ..., \partial^{(k)}_{i_1\cdots i_k}\varphi)d^n x \tag{5}$$

where $\partial^{(k)}_{i_1\cdots i_k}\varphi$ denotes the partial derivative of order k with respect to the variables $x^{i_1}\cdots x^{i_k}$ and the notation $\partial^{(0)}\varphi \equiv \varphi$ may be used for the sake of economy. We can compute the

[5]In our applications below for electromagnetism and general relativity we will restrict to the subspace of real-valued functions and the results are unchanged. It is worth noting that the fact that the theory developed in this section applies to general complex-valued functions is a big advantage for quantisation.

functional derivative of such a functional by using the directional derivative as above, noting that $\partial^{(k)}_{i_1\cdots i_k}(\varphi+\epsilon\psi)=\partial^{(k)}_{i_1\cdots i_k}\varphi+\epsilon\partial^{(k)}_{i_1\cdots i_k}\psi$ and using Gauss' theorem recursively:

$$\frac{d}{d\epsilon}F(\varphi+\epsilon\psi)\bigg|_{\epsilon=0}=\int_B\left(\sum_{a=0}^{k}(-1)^a\partial^{(a)}_{i_1\cdots i_a}\left(\frac{\partial f}{\partial(\partial^{(a)}_{i_1\cdots i_a}\varphi)}\right)\right)^*\psi d^nx+\int_{\partial B}\beta(\psi,\partial\psi,\ldots,\partial^{k-1}\psi)d^{n-1}x$$

where summation over the multi-index $i_1\cdots i_a$ is assumed, function arguments are omitted and β is linear. Restricting to the subspace of functions whose derivatives up to order $k-1$ vanish on ∂B causes the boundary term to vanish and so it is straightforward to read off the **functional derivative of a functional density**:

$$\frac{\delta F}{\delta\varphi}=\sum_{a=0}^{k}(-1)^a\partial^{(a)}_{i_1\cdots i_a}\left(\frac{\partial f}{\partial(\partial^{(a)}_{i_1\cdots i_a}\varphi)}\right) \tag{6}$$

Indeed: if we had a functional defined over multiplets of fields $F(\varphi_1,...,\varphi_\alpha,...,\varphi_L)$, it is easy to show that the notions introduced above generalise trivially. The **partial functional derivative** is found to be

$$\frac{\delta F}{\delta\varphi_\alpha}=\sum_{a=0}^{k}(-1)^a\partial^{(a)}_{i_1\cdots i_a}\left(\frac{\partial f}{\partial(\partial^{(a)}_{i_1\cdots i_a}\varphi_\alpha)}\right) \tag{7}$$

the **total functional derivative** or **functional differential** being the linear map δF that acts on its arguments as:

$$\delta F|_{(\varphi_\alpha)}(\psi_1,...,\psi_L)=\int_B\sum_{\alpha=1}^{L}\frac{\delta F}{\delta\varphi_\alpha}^*\psi_\alpha d^nx. \tag{8}$$

If a set of functions (φ_α) is an **extremal value** of F then a necessary condition is

$$\delta F|_{(\varphi_\alpha)}=0\quad\Leftrightarrow\quad\frac{\delta F}{\delta\varphi_\alpha}=0,\forall\alpha=1,...,L. \tag{9}$$

As concrete examples, for scalar field theories of order 0 and 1, we have the usual expressions:

- $k=0$, $\frac{\delta F}{\delta\varphi}=\frac{\partial f}{\partial\varphi}$
- $k=1$, $\frac{\delta F}{\delta\varphi}=\frac{\partial f}{\partial\varphi}-\partial_i\frac{\partial f}{\partial(\partial_i\varphi)}$

2.2 Poisson and Symplectic Geometry

A smooth manifold P is called a **Poisson manifold** if it is equipped with a **Poisson bracket** or **Poisson structure**. This is a map of the form $\{,\}:\mathcal{F}(P)\times\mathcal{F}(P)\to\mathcal{F}(P)$ satisfying:

- $(\mathcal{F}(P),\{,\})$ is a Lie algebra: so in particular $\{,\}$ is antisymmetric and Jacobi,
- $\{f,\cdot\}$ is a derivation for all $f\in\mathcal{F}(P)$: $\{f,gh\}=\{f,g\}h+g\{f,h\}$.

From basic differential geometry we know that the space of derivations on a manifold is identified with the space of vector fields $D\mathcal{F}(P) \cong \mathfrak{X}(P)$. We use this isomorphism to construct the unique **Hamiltonian vector field** associated with a function $h \in \mathcal{F}(P)$, usually called Hamiltonian, $X_h \in \mathfrak{X}(M)$ such that:

$$X_h[g] = \{g, h\} \quad \forall g \in \mathcal{F}(P) \tag{10}$$

Note that for infinite-dimensional manifolds this statement is technically more subtle since the isomorphism $D\mathcal{F}(P) \cong \mathfrak{X}(P)$ is harder to prove. The set of all Hamiltonian vector fields, i.e. all vector fields associated to some function, is a Lie subalgebra of the vector field Lie algebra $\mathfrak{H}(P) \subset \mathfrak{X}(P)$, moreover we have the map:

$$\begin{aligned} \eta :&\mathcal{F}(P) \to \mathfrak{X}(P) && (11)\\ &h \mapsto X_h && (12)\\ & && (13) \end{aligned}$$

that induces a Lie algebra (anti)homomorphism between $\mathfrak{H}(P)$ and $\mathfrak{X}(P)$:

$$[\eta(f), \eta(g)] = -\eta(\{f, g\}) \quad \Leftrightarrow \quad [X_f, X_g] = -X_{\{f,g\}} \quad \forall f, g \in \mathcal{F}(P).$$

A map $\varphi : (P_1, \{,\}_1) \to (P_2, \{,\}_2)$ is called a **Poisson map** or **canonical map** if

$$\{\varphi^* f, \varphi^* g\}_1 = \varphi^* \{f, g\}_2 \quad \forall f, g \in \mathcal{F}(P_2) \tag{14}$$

where φ^* represents the pull-back. Consider the flow of a Hamiltonian vector field $F_t^{X_h} : P \to P$ whose one-parameter curves are the integral curves of X_h, then the (time) variation or evolution of an arbitrary function $f \in \mathcal{F}(P)$ along the Hamiltonian flow is given by:

$$\frac{d}{dt}(f \circ F_t^{X_h}) = \{f, h\} \circ F_t^{X_h}; \tag{15}$$

this is usually written shortly as $\dot{f} = \{f, h\}$ and the proof follows directly from the definition of Hamiltonian vector field associated with h. In particular, we see that the Hamiltonian flow preserves the Hamiltonian function h: $\dot{h} = 0$ and $h \circ F_t^{X_h} = h$. As a corollary we find for any arbitrary functions $f, g \in \mathcal{F}(P)$:

g is constant along the integral curves of $X_h \Leftrightarrow \{f, g\} = 0 \Leftrightarrow h$ is constant along the integral curves of X_g.

It is readily checked that the Hamiltonian flow $F_t^{X_h}$ is a Poisson or canonical map.

A particularly interesting kind of Poisson manifold are symplectic manifolds. A a smooth manifold S is called a **symplectic manifold** if it is equipped with a **symplectic form** or **symplectic structure** This is a differential form $\omega \in \Omega^2(M)$ satisfying:

- ω is non-degenerate: $\omega(X, Y) = 0 \;\forall X \in \mathfrak{X}(S) \Rightarrow Y = 0$,
- ω is closed: $d\omega = 0$.

Naturally, every tangent space (T_pS, ω_p) is a symplectic vector space, and all the standard discussion for symplectic vector spaces applies. If the non-degeneracy is such that the map

$$\omega^\flat : \mathfrak{X}(S) \to \Omega^1(S) \tag{16}$$

$$X \mapsto \omega^\flat X = i_X\omega = \xi \text{ such that } \xi(Y) = \omega(X,Y) \tag{17}$$

is an isomorphism, then ω is said to be **strong**. Note that for finite-dimensional manifolds all symplectic forms are strong. Every symplectic manifold (S, ω) has a non-trivial Poisson structure given by the isomorphism $\sharp = (\omega^\flat)^{-1}$ that allows us to relate the one-forms $df, dg \in \Omega^1(S)$ to vector fields $\sharp df, \sharp dg \in \mathfrak{X}(S)$ in order to write:

$$\{f, g\}_\omega = \omega(\sharp df, \sharp dg). \tag{18}$$

The antisymmetry and the Leibniz rule for $\{,\}$ are obvious from the definition and the Jacobi identity comes from $d\omega = 0$. The symplectomorphisms automatically become Poisson maps for this symplectic Poisson structure.

In this context we define **Hamiltonian vector fields** associated with functions as we did in the general case for Poisson structures and so we find for a Hamiltonian $h \in \mathcal{F}(S)$:

$$X_h = \sharp dh \Leftrightarrow i_{X_h}\omega = dh. \tag{19}$$

The set of Hamiltonian vector fields is identified with the space of functions up to a local constant[6], and they are symplectic vector fields. An immediate property of Hamiltonian vector fields under symplectomorphisms $\psi \in$Symp(S, ω): $\psi^* X_h = X_{\psi^* h}$.

The flow along the integral lines of every Hamiltonian vector field is fully determined by the Hamiltonian function associated to it. Hence we define the **Hamiltonian flow** as $F_t^h = F_t^{X_h} : S \to S$, the global existence of these flows is ensured when S is closed, otherwise the existence is only ensured locally. From straightforward computation we find that $F_t^h \in$Symp(S, ω) and $h \circ F_t^h = h$ which ensures that the Hamiltonian flow preserves the level sets of h and hence X_h is everywhere tangent to these. A symplectic manifold (S, ω) together with a choice of Hamiltonian function $h \in \mathcal{F}(S)$ is usually called a **Hamiltonian system**.

2.3 Fibre Bundle Geometry

We define a **differentiable fibre bundle** as the multiplet (B, M, F, G, π) that satisfies the following conditions:

1. B is a differentiable manifold and is called the **bundle or total space**.
2. M is a m-dimensional differentiable manifold and is called the **base space**.
3. F is a differentiable manifold and is called a **typical** or the **standard fibre**.

[6]A constant function on every connected piece of the manifold.

4. G is a Lie Group which acts on F on the left and is called the **structure group**.

5. π is a surjective differentiable map, called the **projection**, such that $\pi^{-1}(p) \equiv F_p \cong F$, $\forall p \in M$. We write:

$$\begin{array}{c} B \\ \downarrow \quad \pi \\ M \end{array}$$

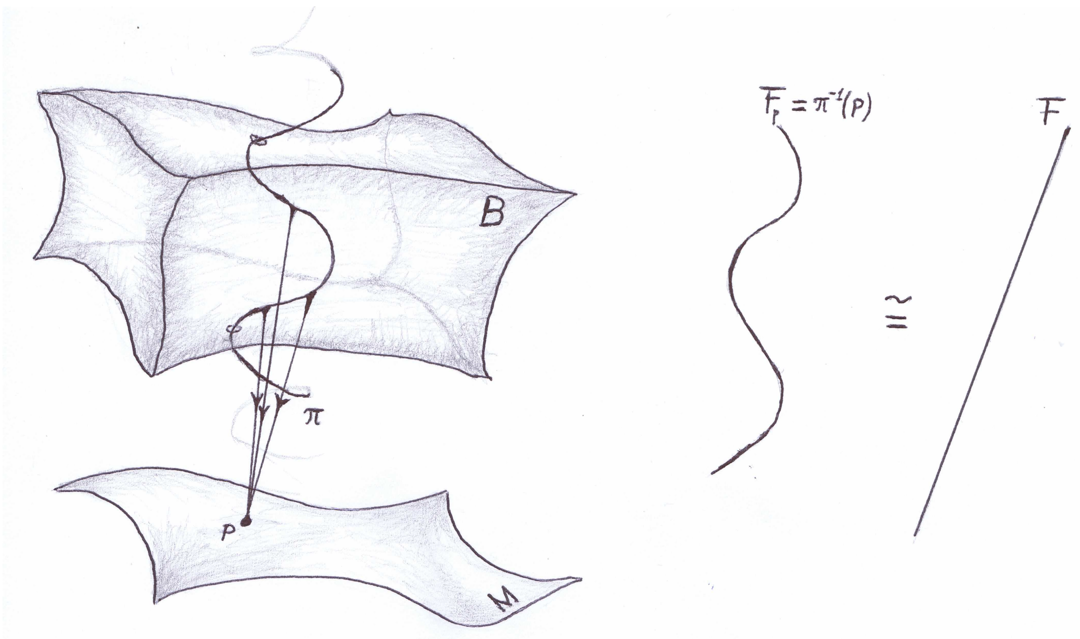

Figure 1: Fibre Bundle: pictorially a 3-dimensional total space manifold over 2-dimensional base space and hence 1-dimensional typical fibre.

6. B must admit a **local trivialisation**, that is, given an open covering $\bigcup_i U_i = M$, there exists a set of diffeomorphisms $\Phi_i^{-1} : U_i \times F \to \pi^{-1}(U_i)$, thereby ensuring that B is locally a cartesian product $U \times F$ with $U \subset M$.

7. The trivialisations must fit together accordingly to the structure group via the **transition functions**. In order to define these consider two open sets of a trivialisation such that $U_i \cap U_j \neq \emptyset$. Any point $u \in \pi^{-1}(U_i \cap U_j)$ will have two distinct images by trivialisation $\Phi_i(u) = (p, f_i)$ and $\Phi_j(u) = (p, f_j)$, where $p = \pi(u)$; for a particular $u \in \pi^{-1}(U_i \cap U_j)$ we impose that the fibre components are related via the (left) action of some element of the structure group G:

$$f_j = g_{ij} f_i, \quad g_{ij} \in G.$$

the discussion is identical for all u in the same fibre, hence, if we set this condition for all $p \in U_i \cap U_j \subset M$ we end up with a map of the form:

$$\begin{aligned} g_{ij} : & U_i \cap U_j \subset M \to G \\ & p \mapsto g_{ij}(p) \end{aligned}$$

and so we have this set of **transition functions** which ensure the following identity $\forall p \in U_i \cap U_j$:

$$\Phi_i^{-1}(p, f) = \Phi_j^{-1}(p, g_{ij}(p) f).$$

For the transition functions $\{g_{ij}\}$ to reliably glue together different trivializations, some natural conditions must hold for any three open sets $U_i, U_j, U_k \subset M$ and any point p of them:

$$g_{ii}(p) = e \tag{20}$$

$$g_{ij}(p) = g_{ji}^{-1}(p) \tag{21}$$

$$g_{ij}(p) g_{jk}(p) = g_{ik}(p). \tag{22}$$

$$\tag{23}$$

We define a **cross-section** or simply a **section** of a fibre bundle as a map of the form:

$$\begin{array}{cc} B & \\ \uparrow & \sigma \\ M & \end{array}$$

that satisfies:

$$\pi(\sigma(p)) = p \quad \forall p \in M. \tag{24}$$

Clearly this means that $\sigma(p) \in F_p$. As a notational remark: we will sometimes also denote the values at single points with the argument as a subindex $\sigma_p, g_{ij,p}, g_{i,p}$... The set of all sections on the base manifold is denoted by $\Gamma(M, B)$. We define a **local section** on an open subset $U \subset M$ as any map of the kind:

$$\begin{array}{cc} B & \\ \uparrow & \sigma \\ U & \end{array}$$

that satisfies:

$$\pi(\sigma(p)) = p \quad \forall p \in U$$

From the above abstract definitions, it is immediately deduced that if (x^μ) are coordinates on the base space M and (y^i) are coordinates on the typical fibre F then it is possible to define an atlas on B with coordinates (x^μ, y^i). In these coordinates, it is obvious to check that a section is simply given by a map of the form $\sigma^i : x^\mu \mapsto \sigma^i(x)$. For the bundle $\pi : B \to M$ we define the **first order jet** at a point $x \in M$ as follows:

$$j_x^1 B = \{[\sigma]_x, \sigma \in \Gamma(M, B)\}, \quad [\sigma]_x = \{\sigma' \in \Gamma(M, B) : \sigma^i(x) = \sigma'^i(x) \text{ and } \partial_\mu \sigma^i(x) = \partial_\mu \sigma'^i(x)\} \tag{25}$$

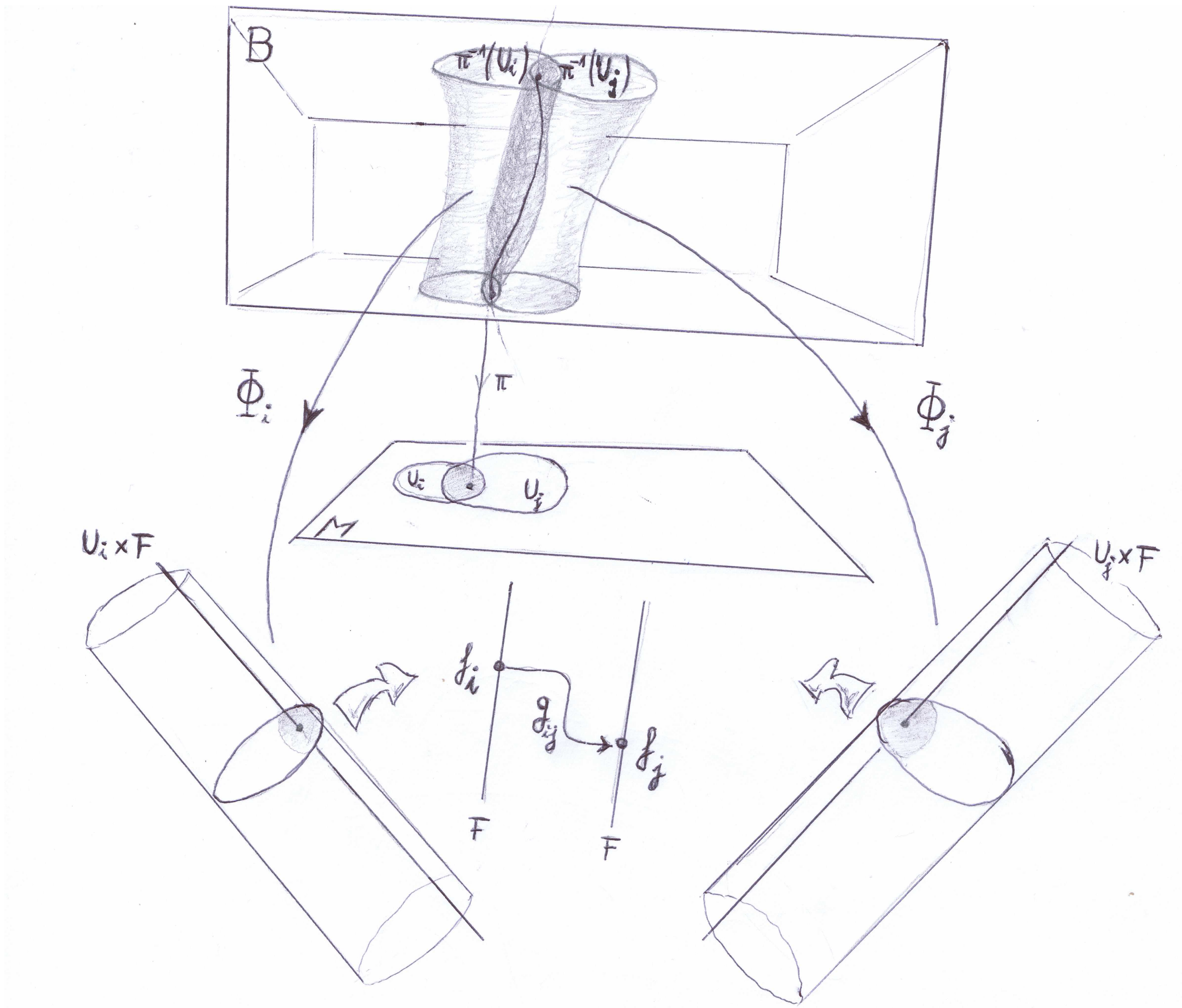

Figure 2: Schematic view of how transition functions operate.

where the bundle coordinates (x^μ, y^i) have been used. The **first-order jet manifold** is then defined as:

$$J^1B = \bigcup_{x \in M} j^1_x B. \tag{26}$$

In order to show that this set is a smooth manifold, we provide the natural set of coordinates (x^μ, y^i, y^i_μ) where y^i_μ simply parametrise all the values of the partial derivatives of sections with respect to the bundle coordinates. Note that this is just a generalisation of the notion of tangent bundle. Indeed we can set $B = \mathbb{R} \times Q$ and $M = \mathbb{R}$ so sections are smooth curves;

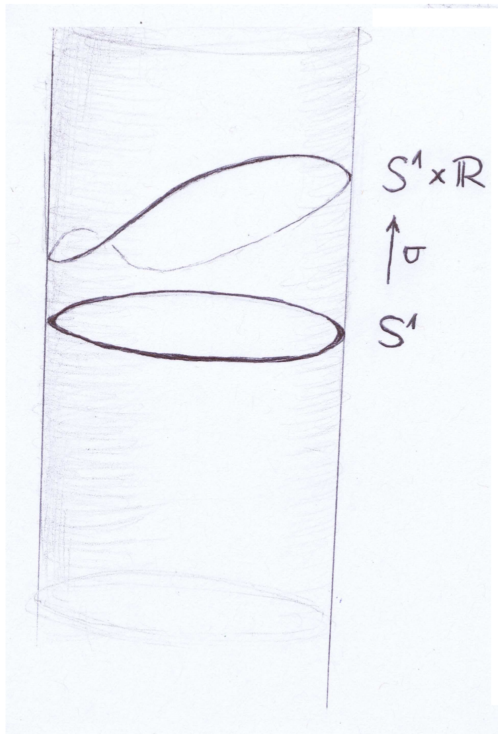

Figure 3: A global section on the cylinder, a trivial bundle $S^1 \times \mathbb{R}$.

then the above construction gives the formal definition of the tangent bundle of Q:

$$J^1(\mathbb{R} \times Q) = TQ.$$

Similarly we can deduce:

$$J^1(Q \times \mathbb{R}) = T^*Q.$$

A special, but still very broad, kind of fibre bundle are the so-called **principal bundles**. As compared to general fibre bundles, the structure groups of principal bundles play a much more prominent role as we see from the definition below. A fibre bundle $\pi : P \to M$ with structure group G, or shortly $P(G, M)$, is a principal bundle if the following conditions hold:

1. There is a right free action of G on P of the form:

$$\begin{aligned} R : & P \times G \to P \\ & (p, g) \mapsto R_g p \equiv pg. \end{aligned}$$

2. The base space is the quotient manifold under the orbit equivalence relation $\sim$

$$M = P/\sim = \{\mathrm{Orb}(p), p \in P\}$$

3. The projection is just the canonical one defined in the quotient:

$$\begin{array}{c} P \\ \downarrow \\ M \end{array} \pi \ , \quad \pi(p) = \mathrm{Orb}(p)$$

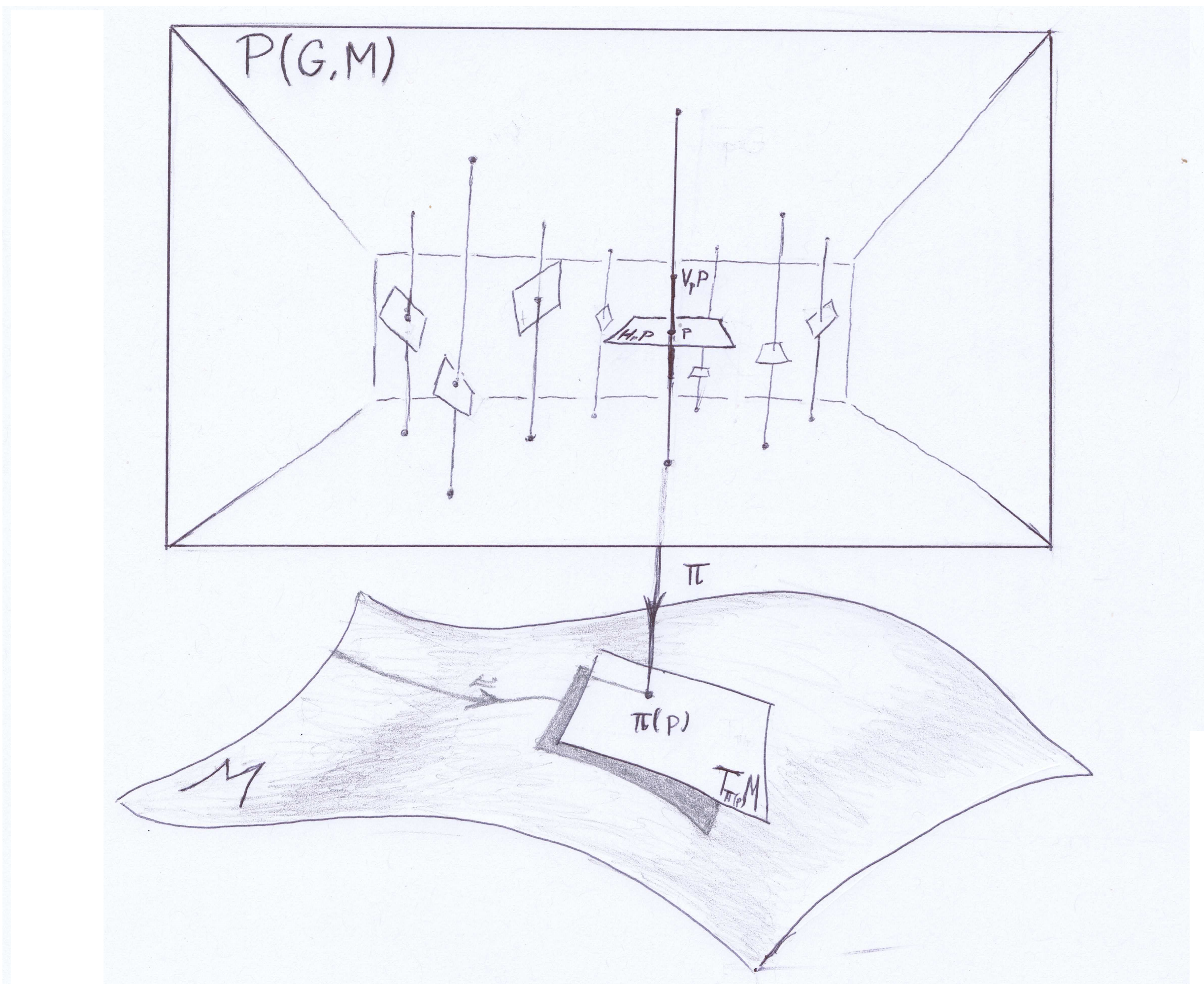

Figure 4: A pictorial representation of the notion of a Connection on a Principal Bundle.

We define a **connection form** on a principal bundle $P(G, M)$ as map $\omega \in \bigwedge^1(P, \mathfrak{g})$ satisfying:

1. $\omega_p(Y_p^V) = \nu_p^{-1}(Y_p^V)$, $\forall p \in P$

2. $T_p\pi(\ker(\omega_p)) = T_{\pi(p)}M$, $\forall p \in P$

3. $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$, $\forall g \in G$.

Where the map ν is defined as

$$\begin{aligned} \nu :& \mathfrak{g} \to \mathfrak{X}(P) \\ & a \mapsto Y^a \end{aligned}$$

here Y^a stands for the infinitesimal generator of transformations induced by a via exponentiation. Since the group action is free, it follows immediately that Y^a is no-where vanishing and that the map ν is injective.

Let us introduce the **horizontal projection map**:

$$\begin{aligned} h :& \mathfrak{X}(P) \to \mathfrak{X}(P) \\ & Y \mapsto Y^H = Y - \nu(\omega(Y)) \end{aligned}$$

which is just the vector field version of the map that takes the horizontal component of a tangent vector defined by the kernel of a connection form ω at a point. We are then in the position to define the **exterior covariant derivative** of pseudotensorial forms as:

$$\begin{aligned} \mathcal{D} :& \bigwedge^k (P, V) \to \bigwedge^{k+1}(P, V) \\ & \alpha \mapsto d\alpha \circ h \end{aligned}$$

Particularly for the connection form we define the **curvature form** simply as:

$$\Omega = \mathcal{D}\omega$$

and so it is a tensorial 2-form of type $(\mathrm{Ad}, \mathfrak{g})$. We say that ω is a **flat connection** if $\Omega = \mathcal{D}\omega = 0$. An important result for these forms is the called **Cartan structural equation**:

$$\Omega = \mathcal{D}\omega = d\omega + \omega \wedge \omega = d\omega + \frac{1}{2}[\omega, \omega] \tag{27}$$

were the usual definitions of products and brackets for Lie algebra-valued forms have been used. We also get the **Bianchi identity**:

$$\mathcal{D}\Omega = \mathcal{D}\mathcal{D}\omega = 0$$

which is a special result for the connection form since, in general, $\mathcal{D} \circ \mathcal{D} \neq 0$.

2.4 Canonical Lagrangian and Hamiltonian Systems

In the canonical formalism of classical mechanics, the set physical states is identified with a manifold, usually called **configuration space** and denoted by $\mathbb{Q}$. Failure to find the minimal space that serves as a faithful representation of the set of physically meaningful or observable states will lead to the discussion of gauge transformations and gauge redundancies. It is always possible to define a Hamiltonian theory on the cotangent bundle $T^*\mathbb{Q}$ and it is always possible to define a Lagrangian theory on the tangent bundle $T\mathbb{Q}$. Here by Hamiltonian or Lagrangian theory we mean a first-order dynamical system for trajectories on $T^*\mathbb{Q}$ or $T\mathbb{Q}$, respectively, that project onto trajectories on $\mathbb{Q}$ that are, therefore, interpreted as the evolution of the system. Although there may seem to be complete symmetry between the two approaches, there is a very important conceptual difference: the tangent bundle of a configuration space is always a physically meaningful entity, after all it is nothing but the union of *all possible rates of change of all the variables*, whereas the cotangent space is not in general. Therefore, all canonical Hamiltonian formalisms of physical significance must come from a Lagrangian formalism under a diffeomorphic fibre derivative, also known as Legendre transform. Let us present these ideas in more technical detail.

Let a smooth manifold $\mathbb{Q}$ and consider its cotangent bundle $T^*\mathbb{Q} = \{\alpha_q\}_{q\in\mathbb{Q}}$, the set of all tangent covectors at all possible points. The projection map π is trivially defined by taking any element $\alpha_q \in T^*\mathbb{Q}$ to its base point $q \in \mathbb{Q}$ and hence we have:

$$\begin{array}{ccc} T^*\mathbb{Q} & & \\ \downarrow & \pi & \\ \mathbb{Q} & & \end{array} \qquad\qquad \begin{array}{ccc} T(T^*\mathbb{Q}) & & \\ \downarrow & & T\pi \\ T\mathbb{Q} & & \end{array}$$

Similarly for the tangent bundle $T\mathbb{Q}$ for which the projection is denoted by τ. Note that elements of $T^*\mathbb{Q}$ act, by construction, on elements of $T\mathbb{Q}$ when they share base point that is if $\pi(\alpha) = q = \tau(v)$ then we obviously have the usual identification $T^*_q\mathbb{Q} = (T\mathbb{Q})^*$ and hence $\alpha(v) \in \mathbb{R}$ is the action. A vector field on the cotangent bundle $X \in \mathfrak{X}(T^*\mathbb{Q})$ is a section of the tangent bundle of the cotangent bundle viewed as a manifold:

$$\begin{array}{ccc} T(T^*\mathbb{Q}) & & \\ \uparrow & & X \\ T^*\mathbb{Q} & & \end{array}$$

There is a natural 1-form on $T^*\mathbb{Q}$ defined using the bundle structure above, this is called the **canonical form** $\theta \in \Omega^1(T^*\mathbb{Q})$ and is defined through its action on vector fields as:

$$\theta(X) \in \mathcal{F}(T^*\mathbb{Q}), \quad \theta(X)(\alpha_q) = \alpha_q((T\pi X)|_q)$$

An equivalent definition is that $\theta \in \Omega^1(T^*\mathbb{Q})$ is the unique 1-form such that:

$$\alpha^*\theta = \alpha \quad \forall \alpha \in \Omega^1(\mathbb{Q})$$

this is why θ is sometimes called the tautological form. As can be easily shown from a coordinate computation, the two form

$$\omega = -d\theta$$

is non-degenerate (and, trivially, closed since it is exact), hence $(T^*\mathbb{Q}, -d\theta)$ is a symplectic manifold and $\omega \in \Omega^2(T^*\mathbb{Q})$ is called the **canonical symplectic form**. It is then obvious that by choosing a function $H \in \mathcal{F}(T^*\mathbb{Q})$ the cotangent bundle with the canonical form is a Hamiltonian system.

Now consider the tangent bundle of $\mathbb{Q}$ and a function on it $L \in \mathcal{F}(T\mathbb{Q})$. Whether we know the dynamics beforehand (as in the case of Newtonian mechanics) or whether we impose a variational principle, the Euler-Lagrange equations are well-defined as a first-order dynamical system on $T\mathbb{Q}$. Let us construct the **fibre derivative** in order to relate to the Hamiltonian formalism: since every fibre of $T\mathbb{Q}$ is a vector space, it is possible to define directional derivatives in the usual way by restricting to each fibre (this is just the usual coordinate construction $p_i = \frac{\partial L}{\partial \dot{q}^i}$). In particular for the Lagrangian we have:

$$\begin{aligned} FL : & T\mathbb{Q} \to T^*\mathbb{Q} \\ & v_q \mapsto (FL)_{v_q}(w_q) = D_{v_q} L(w_q) \end{aligned}$$

If this map is a diffeomorphism, then we can relate the Lagrangian with a Hamiltonian function by means of pull-back:

$$H = (FL^{-1})^* L = L \circ FL^{-1}$$

and we can also pull-back the canonical form:

$$\omega_L = (FL)^* \omega$$

so that the tangent bundle together with this pulled-back symplectic form is a Hamiltonian system with respect to the energy function:

$$\mathcal{F}(T\mathbb{Q}) \ni E : E(v_q) = (FL)_{v_q}(v_q) - L(v_q)$$

Nevertheless, if the map fails to be a diffeomorphism then Hamiltonians will not have a straightforward interpretation in terms of being the pull-back of the energy of some Lagrangian.

Note the similarity of the discussion so far with the usual coordinate treatment of Lagrangians and Hamiltonians (essentially replace any image of the fibre derivative with the momenta p), why would we introduce all this formal approach when we know the coordinate approach works just as well? The answer is, precisely, that we did not use coordinates in any of the derivations and hence we automatically overcame the main limitation of coordinate computations, namely, that they do not apply to the infinite-dimensional case in general. Although the notion of Lagrangian and Hamiltonian will need to be slightly generalised when we deal with field theories - indeed infinite-dimensional theories - in next section, the fundamental result that a degenerate Lagrangian does not allow for a clear interpretation of its dual Hamiltonian will still apply.

2.5 Lagrangian and Hamiltonian Field Theory

Classical Field Theories, that is, physical theories in which the configuration of the system is determined by the extended values of some kind of field over space-time, can be put into what is regarded as the natural generalisation of Lagrangian dynamics. Indeed, in the formulation we present below, setting the *world* manifold to $\mathbb{R}$ results in the formalism introduced in the previous section.

In order to be able to express the full generality of Lagrangian Field Theory and then reduce to the simpler, particular cases that we are concerned about, let us recall the notion of fibre bundle, sections and jet manifold. With these at hand, we have all the necessary tools to precisely define what is meant by a **Field Theory**:

- Let M be an orientable manifold (typically time, space or space-time), and F an other manifold (typically a vector space) that is taken to comprise all the possible values of the fields. Then construct a fibre bundle B with M as base space and F as typical fibre.
- The **space of field configurations**, Φ, is then the set of all sections $\Gamma(M, B)$. A section σ is called a **field**.
- Physical fields, that is, those configurations observed as phenomena for which the field theory is a model, are determined by an **individuation principle**.

Individuation principles may vary from one theory to another but they will involve, either directly or indirectly, a differential operator and some boundary conditions on M. When a variational approach is taken then we define a **Lagrangian Field Theory** as a field theory in which the individuation principle is done by means of an **action**, a function $S : \Phi \to \mathbb{R}$, and whose extrema $\phi \in \Phi$ are the physical fields. This function(al) can be defined from a **Lagrangian** map:

$$L : J^1 B \to \bigwedge^m T^* M$$

Since M was assumed to be orientable, fix $\nu \in \Omega^m(M)$ to be the volume form on M, then the Lagrangian map can be regarded as an ordinary function:

$$L = \mathcal{L}\nu, \quad \mathcal{L} : J^1 B \to \mathbb{R}.$$

Consider a field $\sigma : M \to B$ and pull-back the Lagrangian to M:

$$\sigma^* L = (\mathcal{L} \circ \sigma)\nu \in \Omega^m(M)$$

so the integral over M is well-defined. We define the **Lagrangian action functional** as:

$$S[\sigma] = \int_M \sigma^* L = \int_M (\mathcal{L} \circ \sigma)\nu. \tag{28}$$

Using the natural set of coordinates (x^μ, y^i, y^i_μ) it readily checked that this functional is defined locally as the integral functionals introduced in the Functional Analysis section.

Hence the notion of functional derivative applies to each of the coordinate charts in an intrinsic manner (since the integral doesn't depend on the charts chosen) and so there is a global definition of such operator for the space of fields. We denote it by $\delta/\delta\sigma$ and write:

$$\sigma_0 \text{ is physical iff } \left.\frac{\delta S}{\delta\sigma}\right|_{\sigma=\sigma_0} = 0.$$

Of course, resorting back again to the natural coordinates on the jet bundle, the above equation together with appropriate boundary conditions reduces to the celebrated **Euler-Lagrange equations**:

$$\frac{\partial\mathcal{L}}{\partial\sigma^i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\sigma^i)} = 0$$

where all the local functions derived from $\mathcal{L}$ are evaluated at $(x, \sigma(x), \partial\sigma(x))$.

Although it is possible to present the Hamiltonian approach to field theories in the same generality as above, in doing this we will need to introduce the polysymplectic formalism and that would lead us far beyond the scope of the present work. Nevertheless, the term **Hamiltonian Field Theory** (without an additional *'Covariant'*) usually refers to a Hamiltonian theory (in the sense presented in the previous sections) of fields. This means that in a Hamiltonian Field Theory all the mathematical elements are identical to those of a Canonical Hamiltonian Theory (note that this is, in general, in sharp contrast with the case of Lagrangian Field Theories) with the particularity of having an infinite-dimensional configuration space.

For the sake of simplicity, let us consider a physical system whose state is characterised by a V-valued field configuration over space S at any given time $t \in \mathbb{R}$ where V is a finite-dimensional vector space. Denote the configuration manifold as $\mathbb{Q} = C^\infty(S, V)$, which is indeed a vector space of infinite dimension. As for any vector space we have:

$$T^*\mathbb{Q} = \mathbb{Q} \times \mathbb{Q}^*$$

It is easy to prove that the dual of a V-valued function is a V^*-valued function and so elements of $T^*\mathbb{Q}$ are pairs of fields:

$$T^*\mathbb{Q} = \{(\psi(x), \pi(x)), \psi \in C^\infty(S, V), \pi \in C^\infty(S, V^*)\}$$

In practical examples, of course, once a basis is chosen, we can always identify $V \cong V^*$. A Hamiltonian function in this context is a functional on the pair of fields:

$$H[\psi, \pi] \in \mathbb{R}$$

whether it comes from an integral of a Hamiltonian density or not. Then the canonical symplectic form is defined as for any vector space (finite- or infinite-dimensional):

$$\omega = d\psi \wedge d\pi$$

Now, since the elements of the configuration manifold are fields over S we must use the global version of the functional derivative introduced above as the directional derivative.

Using this and writing the two separate components of the symplectic form we get the celebrated **Hamilton equations** for field theory:

$$\dot{\psi} = \frac{\delta H}{\delta \pi}$$
$$\dot{\pi} = -\frac{\delta H}{\delta \psi}.$$

Note that the tangent vector $(\dot{\psi}, \dot{\pi})$ can be regarded as pair of fields that depend on time. Therefore when a local coordinate chart (x^i) in S is introduced, the above equations are indeed a system of PDEs, involving only first-order time derivatives of the fields $\psi(x^i, t)$ and $\pi(x^i, t)$.

3 Relativistic Classical Electromagnetism

In this section we present the standard modern formalism of classical electromagnetism but in the context of special relativity; this is the some times called covariant formulation of electromagnetism. Although we partially base our discussion about observables on the work by Pitts and Bergmann, the approach we follow in this section is largely original and no other bibliographical sources have been used.

3.1 Special Relativity and Observables

The theory of special relativity addresses kinematics and dynamics of particles observed to be moving at speeds close to the speed of light. The historical evidence supporting Einstein's Principle of Relativity, in its original form stating that the speed of light was measured to have the same value for all observers, was undeniable and the theory was established. The tradition in which this theory was formulated did not use an intrinsic geometric language and, hence, physicists became used to the original formulations in terms of coordinate transformations between observers and the concept of Lorentz invariance before a fully geometric theory was developed. Luckily for the mathematical physics student, today we have a formalism[7] that unifies all the ideas contained in the previous formulations in a way that is more than just restating the old results in a compact and elegant manner. This formalism, that we briefly sketch below, adds nothing new to the theory itself but provides a very natural setting in which old ambiguities and cumbersome coordinate expressions disappear.

One of the main advantages, which may be regarded by some as a disadvantage, is the fact that this formulation requires for a careful definition of observer. In the traditional formalism this is, of course, not necessary, since the coordinates in which statements are formulated are assumed to be parameters that can be measured, either directly or indirectly,

[7] The author knows about this formalism from Miquel Portilla, a member of the Astrophysics Department at the University of Valencia. He claimed that the original ideas were due to separate work by Marzke, Wheeler and Fletcher circa 1960.

by the observer. However, the confusion begins when coordinate transformations are introduced. In the old formalism, there is a need to explicitly distinguish between two types of coordinate transformations, which are mathematically identical entities, one related to the change of observer and others simply regarded as the result of an active transformation. Indeed the introduction of active transformations is an influence of modern geometric approaches and when carelessly mixed with the old formalism it leads to ambiguities, in the best case, and, eventually, also mistakes. Dr Brian Pitts describes some of these misconceptions in his paper of October 2013 and the author believes that, although a thorough investigation following the traditional approach may reveal all the mistakes, the approach suggested by the ideas introduced in the present work may overcome some of the difficulties from the start. Again, due to the limited scope of this work, a detailed exploration of these possibilities was not carried out but it is left as a subject for future research.

Let us then briefly introduce the basics. Consider a 4-dimensional affine space[8] M on which a constant Lorentzian metric $\eta \in \mathcal{T}_2^0(M)$ is defined. This induces a light-cone structure in the sense that, for every $p \in M$, the zero locus of the norm between two points $\eta(\vec{pq}, \vec{pq}) = 0$, defines a 3-dimensional cone with vertex at p. This geometric fact allows for a classification of all possible curves, and lines in particular, in the usual time-like, null and space-like. A principle of individuation is introduced to rule out space-like curves as physical objects and to select only the future-directed time-like and null curves. The result is the usual identification of time-like lines as massive free particles and null lines as light rays. What is the difference then with the traditional formulation? Note that the only objects that have been introduced are algebraic subsets of M, that we shall appropriately call regions of space-time, and the metric that allows for a classification of curves that are identified with the most fundamental constituents of physical reality, therefore any further notion, such as observer or measurement, needs to be specified.

An **observer** is a time-like curve u together with an open neighbourhood of it $u \subset U$. The curve itself is identified with the observer's conciousness world-line - in practice, where the person body lies - and the neighbourhood is identified with the measurement devices that are available to her and so a smooth prolongation of tangent vectors from u is assumed; for this reason u can be equivalently defined as a unit vector field restricted to U that coincides with the unit tangent of the observer's curve. It is very important to stress that, even in the setting of special relativity, the neighbourhoods cannot be taken to be the whole space-time if we want to construct a realistic model of *relative change* between observers. The proper length along u is identified with the proper time t of the observer and the map $\dot{u}^\flat : U \to \mathbb{R}$ defines the simultaneity subspace as the orthogonal complement of the unit tangent to the observer. The intersection of the 3-dimensional simultaneity subspace with the neighbourhood of the observer defines a 3-dimensional open set U_t at every point of the curve with proper time t. This can be pictured as the instantaneous space slice of the observer.

An **observable** is any mathematical entity that can be uniquely characterised by physi-

[8] With the appropriate caveats it is possible to start from a connected 4-dimensional Lorentzian manifold, but this adds nothing in the case of electromagnetism so we keep it as simple as possible.

cal measurements. In the present context, where only particles and light rays are considered, this means that any curve with non-empty intersection with U will be an observable for the observer u. Note that the notion of observability is subordinate to the notion of observer, as intuition suggests. At this level, it seems rather redundant to insist in any transformation properties of the objects that will be observables but as soon as a pragmatic realisation of any of the above geometric entities includes some kind of redundancy, the discussion of gauge invariance will become important. But we stress that, using the present formalism, such issues are a limitation of the concrete realisation not of the theory itself.

As an example[9] of **gauge redundancy** let us consider a time-like curve c that is identified with a massive particle and hence will be observed by several observers. Note that as an algebraic subset of M this object is completely determined. Nevertheless, when we consider an observer u who uses some system of rods and mirrors to detect the position of the particle, the mathematical description of c becomes $c^\mu(\tau) \in \mathbb{R}^4$ and, of course, this is far from unique. Both the coordinates used by u, i.e. the specific setting of rods and mirrors, and the parametrisation of the curve have a degree of arbitrariness that will appear as gauge invariances in the equations describing the dynamics. We hope that this simple example helps to emphasise the idea that a well-defined geometric object may have a non-unique realisation. The general case in which fields are included is very similar in this respect.

3.2 Electromagnetism in Space-Time

Classical electromagnetism is a field theory in $\mathbb{R}^3 \times \mathbb{R}$ for any given observer. The fundamental objects are two time-dependent vector fields, $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$, whose reality can be checked when (massive) matter is associated with a scalar quantity, called electric charge $\rho(\vec{x}, t)$, so that Maxwell's equations hold:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{B} = 0 \tag{29}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \rho \vec{V} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \tag{30}$$

$$\tag{31}$$

and the dynamics of matter are coupled to the fields through the Lorentz force:

$$\frac{\partial \vec{V}}{\partial t} = \frac{\rho}{\rho_{\text{mass}}} (\vec{E} + \vec{V} \times \vec{B}) \tag{32}$$

here $\vec{V}(\vec{x}, t)$ is the mass distribution velocity vector field. It is well known that this field theory was very much anticipating the dawn of relativistic or covariant formulations since the equations above have the structure of a coordinate expression derived from a more general setting.

[9] So far in the development of the theory, indeed the only possible example, since we have not introduced any other physical entities.

Our aim is to find a well-defined formulation of electromagnetism in the context of special relativity that reduces, for any observer, to the classical Maxwell-Lorentz theory. In other words, if **covariant electromagnetism**, as we shall call this generalisation, is a field theory on M, then, given an observer u that constructs her instantaneous inertial frame from $\dot{u}$ so that each space slice is $U_t \cong \mathbb{R}^3$, the coordinate expression for the field equations must be exactly equivalent to Maxwell's equations. Such fields will be called observable in general since, by construction, they reduce to electric and magnetic fields for each observer which, in turn, are assumed to be physical.

It is a well-known result that using the exterior algebra of space-time $\Omega(M)$ together with the Hodge star operator $\star$ induced by the metric, provides an elegant formulation of electromagnetism that satisfies the requirements specified above. The **electromagnetic field strength** is a 2-form tensor field $F \in \Omega^2(M)$ and the **electric current** is a 1-form tensor field $J \in \Omega^1(M)$. These are defined, as usual, from the local Lorentzian frame expressions; indeed the electric and magnetic fields as measured by an observer (u, e_i)[10] are $E_i = F(u, e_i)$ and $B_i = \epsilon_{ijk} F(e_j, e_k)$ respectively and, also, the electric density and electric current as measured by the same observer are $\rho = J(u)$ and $J_i = J(e_i)$. The field equations are imposed to be:

$$\begin{aligned} dF &= 0 \\ d \star F &= \star J \end{aligned}$$

where the second equation unavoidably involves the metric through the Hodge star operator $\star$. We can express these in terms of the covariant derivative as follows:

$$\mathcal{A}(\nabla F) = 0 \tag{33}$$
$$\mathrm{DIV} F^\sharp = J \tag{34}$$
$$\tag{35}$$

or for an observer u using instantaneous innertial coordinates (U, x^μ):

$$\partial_{[\lambda} F_{\mu\nu]} = 0 \tag{36}$$
$$\partial_\mu F^\mu_\nu = J_\nu \tag{37}$$
$$\tag{38}$$

Note that, in the coordinate expression, the unavoidable presence of the metric in the second equation is less obvious but it is still there in the form of a raised index, indeed an operation that can be only carried out using the metric tensor. The dynamics of a charged particle are described by means of the **Lorentz force tensor field** which in this case is simply $F^\sharp$ and so the equations of motion on a general space-time will be:

$$m \nabla_u u = q F^\sharp(u) \tag{39}$$

[10]Here e_i is a space-like orthonormal basis for the instantaneous space slice of the observer u.

where m is the particle's mass and q the particle's electric charge.

Let us consider a particular observer u and let us restrict the field to the observable open subset U. It is obvious that $U \subset M$ is topologically trivial, i.e. homeomorphic to $\mathbb{R}^4$, then in virtue of Poincaré lemma we could always write the strength field $F \in \Omega^2(M)$ as an exact form in terms of the potential form $A \in \Omega^1(M)$ as $F = dA$ and so defining the **electromagnetic potential** as any form satisfying this equation. This trivially solves the first equation since $ddA = 0$ and the second reduces to:

$$d \star dA = \star J. \tag{40}$$

Here is yet another example of **gauge redundancy**: suppose that the observer u somehow forgets that the field A in her description of electromagnetism really plays de role of a potential of F and consider that physics reduces to solving the above equation. Of course, she will find that the equation has no unique solution, even when written as a Cauchy problem, and this is because the operator $d \star d$ is not invertible. In other words, it is not legitimate to forget that A is the potential of the physical field F since any $A' = A + d\chi$ will be a valid potential representation of the same F. This is, just as we saw for the massive particle before, an example of gauge redundancy that appears only when a concrete description of a unique geometric object is chosen by an observer with the intention of solving the equations that individuate physical entities - in our case, the covariant version of Maxwell equations.

This apparent inconvenience can be elevated to a defining principle when we formulate electromagnetism as a gauge theory. To this end, recalling the formalism of connections on a principal bundle introduced above, we consider the trivial principal bundle $P = M \times U(1)$. With this choice[11] of principal bundle, we propose one of the basic ingredients of the theory to be a connection on the bundle $\alpha \in \bigwedge^1(P, \mathfrak{u}(1))$, where $\mathfrak{u}(1)$ denotes the Lie algebra of the Lie group $U(1)$. If this is to be the case, we examine $U(1)$ in some detail and given the standard definition $U(1) = \{\xi \in \mathbb{C} : |\xi| = 1\}$ we can provide a local chart on $U(1)$ in an equivalent way as we would do for S^1, since it is obvious from definition that $S^1 \cong U(1)$. Following this reasoning we will parametrise elements $\xi \in U(1)$ as $\xi = e^{i\theta}$ provided that $\theta \in \mathbb{R}$ belongs to an open subset of $[0, 2\pi]$ as part of a covering for S^1. This may resemble the notation used for the exponential map that takes elements of the Lie algebra onto the Lie group, this is no accident since we can readily check that:

$$\mathfrak{u}(1) \equiv T_1 U(1) = T_p S^1 \cong \mathbb{R}$$

We see that a $\mathfrak{u}(1)$-valued form will be just an ordinary differential form and so our connection form is just $\alpha \in \bigwedge^1(P)$. As it was defined in section 4.6, we identify the set of gauge transformations for our trivial principal bundle $P(U(1), M)$ as the set of functions:

$$\mathcal{C}(P(U(1), M)) = \{R_\xi, \xi : P \to U(1)\}$$

[11] Although the best justification for the choice is that we do recover covariant Maxwell's equations of course, there is a hint towards this seemingly mysterious choice of gauge group. In the view of the author, the group $U(1)$ is the right minimal choice for the gauge group of electromagnetism since the field equations are well-known to present a polarisation redundancy that responds precisely to a plane-rotational, i.e. $SO(2)$, symmetry. Indeed $SO(2) \cong S^1 \cong U(1)$.

The requirement that α is indeed a connection form, which encompasses the adjoint transformation law $R^*_\xi \alpha = \mathrm{Ad}(\xi^{-1})\alpha$, implies for our case that α is invariant under gauge transformations. Let us check this result by computing the adjoint representation of $U(1)$: let $a \in \mathfrak{u}(1)$ and $\xi \in U(1)$, the adjoint representation of ξ acting on a is defined by:

$$\mathrm{Ad}(\xi)a = T_1(L_{\xi^{-1}} \circ R_\xi)$$

but, as for any abelian group, $L_\xi = R_\xi$ and so $L_{\xi^{-1}} \circ R_\xi = R_{\xi^{-1}\xi} = R_1 = id_{U(1)}$. Using the identity property of the tangent map $T_e(id_G) = id_{\mathfrak{g}}$ we find $\mathrm{Ad}(\xi)a = a$, $\forall \xi \in U(1)$. Leaving us with the anticipated gauge invariance of the connection form:

$$R^*_\xi \alpha = \alpha, \quad \forall R^*_\xi \in \mathcal{C}(P) \tag{41}$$

We need to bear in mind that any construction we make will have to reduce to a field theory on space-time, hence we need to relate our objects defined on $P(U(1), M)$ to objects defined on M. In order to do this we consider sections $\Sigma = \{\sigma : M \to P\}$ and notice that they can be all generated by means of the gauge transformations $\xi : P \to U(1)$ simply defining the natural global section $\sigma_1 : x \to (x, 1)$ and so $\Sigma = \{\sigma_1 \xi, \xi \in \mathcal{C}(P)\}$. With these sections at hand, whose physical interpretation correspond to different choices of gauge by different observers, we are able to give a form on M from the connection form and an arbitrary section σ, we denote:

$$\sigma^* \alpha \equiv iA \in \bigwedge^1(M)$$

note that the i factor has been introduced for convenience as we may regard $\mathfrak{u}(1) \cong i\mathbb{R}$. Let us define a different potential from the same connection but using a different global section $iA' = \sigma'^* \alpha$ - this can be regarded as a gauge transformation of the first potential. The sections are related by the gauge transformation ξ which we parametrise as $\xi = e^{i\theta}$ with $\theta : M \to \mathbb{R}$, using the theorem of local connections, we could write in the simplified notation:

$$iA' = i\mathrm{Ad}(\xi^{-1})A + \xi^{-1} d\xi$$

but we know that $\mathrm{Ad}(\xi^{-1}) = 1$ so we finally recover the transformation law for the potentials under gauge transformations:

$$A' = A + d\theta. \tag{42}$$

This result serves as a first indication that we are in the right track to recovering electromagnetism from a connection on a $U(1)$-bundle over M. Since we have identified $\sigma^* \alpha \sim A$, we are now interested in finding the bundle object that will recover the field strength F. Let us first notice that the curvature form of our connection form $\Phi \equiv \mathcal{D}\alpha$ is simply:

$$\mathcal{D}\alpha = d\alpha + \alpha \wedge \alpha$$

but since $\alpha \in \bigwedge^1(P)$ we have $\alpha \wedge \alpha = -\alpha \wedge \alpha = 0$. Take a section so that we have $\sigma^* \alpha = A$, then we define $\sigma^* \mathcal{D}\alpha \equiv F$; now, if we use the property of exterior derivatives $\phi^* d = d\phi^*$ for any differentiable map ϕ, we find:

$$F = \sigma^* \mathcal{D}\alpha = \sigma^* d\alpha = d\sigma^* \alpha = dA$$

finally recovering the definition of the field strength from the potential. This confirms that the bundle object corresponding to the field strength is the curvature form $\Phi = \mathcal{D}\alpha$, again a gauge independent object.

Regardless of any further developments, we find ourselves in the position to formulate a physical fact in a precise and simple bundle terminology:

- $\mathcal{D}\alpha = 0$ i.e. the connection is flat $\Leftrightarrow$ the space-time M will not present any observable electromagnetic effects.
- $\mathcal{D}\alpha \neq 0$ i.e. the connection is not flat $\Leftrightarrow$ the space-time M will present observable electromagnetic effects.

In fact, the general result for connection forms $\mathcal{D}\mathcal{D}\alpha = 0$, i.e. Bianchi identity, constitutes the bundle equivalent of the first source-independent Maxwell equation $dF = 0$.

3.3 Electromagnetism as Lagrangian Field Theory

As it has been shown in the previous section, the principal bundle formalism formulation of electromagnetism as a gauge theory automatically accounts for the *source-independent* Maxwell equation $dF = 0$ but the other, namely $d \star F = 0$ if we focus in the free theory for the time being, requires further constructions to be fixed. In order to do this we formulate the remaining field equations as the natural individuation principle of a Lagrangian Field Theory. The fields are, of course, sections of the exterior algebra tensor bundle $\bigwedge(M)$ and if we set for finding an action functional from which the field equations could be derived, the first step should be to find a way to construct Lagrangian maps from the space of field configurations $F \in \Omega^2(M)$.

Let us recall that Lagrangian maps are constructed from a volume form on M and a $\mathbb{R}$-valued function on the space of fields, $L = \mathcal{L}\omega$. The volume form is easily found since M is Minkowski space-time and the metric provides the natural volume form, which in the inertial coordinates (x^μ) of any given observer can be globally written as:

$$\omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

Finding what the function $\mathcal{L}$ should be is slightly trickier. Since we are working in an intrinsic formalism, our only concern is to find a geometrically well-defined real function on the space of field configurations $\Omega^2(M)$ but we know that this is possible due to the existence of the form inner product derived from the metric:

$$\begin{aligned} \langle \cdot, \cdot \rangle : &\Omega^k(M) \times \Omega^k(M) \to \mathbb{R} \\ &(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle = C^{1\ldots k}_{1\ldots k}(\alpha \otimes \beta^\sharp) \end{aligned}$$

where $\beta^\sharp$ denotes the tensor with all its indices risen. Note that using an arbitrary coordinate chart this inner product is easily written as:

$$\langle \alpha, \beta \rangle = C^{1\ldots k}_{1\ldots k}(\alpha \otimes \beta^\sharp) = \alpha_{i_1\ldots i_k}\beta^{i_1\ldots i_k}.$$

In particular we have the norm of any form defined as $||\alpha||^2 = \langle \alpha, \alpha \rangle$, which is indeed a well-defined real-valued function.

Field configurations are 2-forms $F \in \Omega^2(M)$ so, without considering any extra structures, the Lagrangian functions would be of the form:

$$\mathcal{L} = f(||F||^2)$$

It may appear as if we have derived the most general Lagrangian Field Theory for 2-form fields so we may be tempted to write the action functional:

$$S[F] = \int_M f(||F||^2)\omega$$

But it is clear that the Euler-Lagrange equations for this action are, in some coordinate chart:

$$f'(||F||^2)F^{\mu\nu} = 0$$

which are far from resembling the desired Maxwell equation $d \star F = 0$ or $\partial_\mu F^{\mu\nu} = 0$. The main problem, as it turns out, is that there are no derivatives of the field involved in the action functional and hence Euler-Lagrange equations will never include the first order derivatives that we require. Also any geometric derivatives will make no contribution to the equations since they vanish automatically by construction $dF = 0$. This is when Poincaré lemma and the discussion about potentials for an observer become useful. In order to properly formulate electromagnetism as a Lagrangian Field Theory a particular observer must be considered.

Let u be an observer and let U be its observable open neighbourhood, then the electromagnetic field F is expressed as the exterior derivative of some potential $F = dA$ in virtue of Poincaré lemma. We are now dealing with a field theory where the basic fields are 1-forms $A \in \Omega^1(M)$ and so the most general action may include first order exterior derivatives dA and not higher order ones (since they all vanish due to antisymmetry). At this point we may devise the most general form of an action as:

$$S[A] = \int_M f(||A||^2, ||dA||^2)\omega$$

We have included a general dependence on the field itself in addition to its exterior derivative for purely mathematical reasons, we have just encountered yet another example of the consequences of **gauge redundancy**. First let us note a simple fact from the individuation principle used in Lagrangian Field Theories: the physical field configurations extremise the action functional hence they must be gauge invariant. Or put in more precise terms, when the functional is restricted to the subspace of field configurations that correspond to a unique physical state[12] it should be constant. Therefore we require $S[A] = S[A + d\chi]$, and since

[12] In the case of $F \in \Omega^2(M)$ this is a zero-dimensional subspace, just a point, since F is a physically distinct field from all its neighbours and in the case of $A \in \Omega^1(M)$ it is an infinite-dimensional subspace of all fields related to A by $A + d\chi$.

$\chi \in \Omega^0(M)$ is an arbitrary function the dependence on the term $||A+d\chi||^2 < ||A||^2 + ||d\chi||^2$ must be trivial otherwise we could always find a suitable function χ_0 such that $S[A] \neq S[A + d\chi_0]$. We have then reduced the general form of the action to:

$$S[A] = \int_M f(||dA||^2)\omega$$

And so we are only left with the task to find the appropriate form for f so that the Euler-Lagrange equations reduce to the Maxwell equation. The simplest choice, $f(||dA||^2) = ||dA||^2$, does indeed recover the desired expressions, using an arbitrary coordinate chart (x^μ) for the observer's open neighbourhood:

$$\frac{\delta S}{\delta A} = 0 \quad \Leftrightarrow \quad \partial_\mu \frac{\partial ||dA||^2}{\partial(\partial_\mu A_\nu)} = 0 \quad \Leftrightarrow \quad \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \tag{43}$$

And these are indeed the coordinate versions of $d \star F = 0$ provided $F = dA$, which is certainly the case by construction.

Although our discussion was based on a choice of observer u, it is clear that for any Lorentzian manifold M with trivial 2-form cohomology group $H^2(M) \cong \mathbb{R}^0$, in particular for Minkowski space-time, the above Lagrangian Field Theory describes electromagnetism intrinsically and in a global way. One can think of fields F and A living on the entire space-time and then observers restricting them to their observable neighbourhoods and using an instantaneous inertial frame to interpret the components of F as the electric and magnetic fields. Therefore, in a sense, one may argue that **the Lagrangian Formulation of electromagnetism is independent of the observer**.

3.4 Electromagnetism as a Canonical Formalism

Moving to a canonical - either Lagrangian or Hamiltonian - formalism means fixing a very precise geometric setting when we are dealing with a field theory. As presented in sections 2.4 and 2.5, Hamiltonian and Lagrangian formalisms have a strongly pre-relativistic flavour; indeed, they attribute a very special role to the time parameter which acts as a universal label for the physical state of a system. This is why Hamiltonian techniques are so naturally satisfactory with Newtonian mechanics and non-relativistic classical field theories. However, electromagnetism has been introduced in a fully relativistic setting and so the canonical formulation is, in principle, far from obvious to define. It is worth noting here again the terminological ambiguity between Lagrangian Field Theories and the canonical Lagrangian formulation of a Field Theory. The first refers to very general formalisms dealing with spaces of maps between general manifolds that involve some kind of variational principle, the second, however, is the result of a broad geometric reformulation of Newtonian mechanics, very much the heritage from the Analytical Mechanics tradition. Since relativistic mechanics mathematically redefine the notions of time and space an identification of canonical, either Lagrangian or Hamiltonian, formalisms in such context will require careful consideration.

The author believes that this problem is at the heart of many conceptual and notational issues that arise when carelessly resorting to the canonical Hamiltonian formalism from the Field-Theoretic Lagrangian formalism. In this final section of our discussion of electromagnetism, we will tackle this problem, namely, how one constructs a canonical Hamiltonian formulation of electromagnetism once we are given a precise Field-Theoretic Lagrangian formulation as the one developed in the previous section. In order to do this the notions of **observer** and **observable** will become of vital importance. In fact, despite the considerable notational and terminological gap between the present work and the classical literature, we will be in the position to judge Bergmann's definition of observable as introduced in the *Il Nuovo Cimento* paper of 1956. In addition to this central topics, in this section we discuss a tangential issue that is also a source of many misconceptions: the space-time point individuation problem and the meaning of active diffeomorphisms.

Let us begin by considering an observer u and its observable neighbourhood $U \in M$. As we discussed above, the instantaneous inertial frame provides a map $\varphi_t^{-1} : U_t \to \mathbb{R}^3$ that allows us to identify the usual euclidean space in the vicinity of the observer. Note that this map is defined for each value of the proper time t of the observer and, from the hypothesis that the observer follows a time-like curve, it will have always varying domain. This means that the observable neighbourhood of the observer is diffeomorphic to a 4-dimensional cylinder $\mathbb{R}^3 \times \mathbb{R}$ in which the first components are interpreted as some system of orthonormal coordinates on a static euclidean space. This is, however, certainly not the case at all. If the observer interprets the coordinates as points in a static euclidean space, rather than numerical labels, as they really are, of space-time events, then she will live in the fiction that physics is actually happening on her $\mathbb{R}^3$ rather than on a spatial slice of her instantaneous inertial frame. This is a vital point to clarify when working in the intrinsic formalism and the author recognises, as it was already pointed out, several conceptual mistakes stemming from this issue.

For instance the discussion on **active diffeomorphisms** of space-time becomes rather trivial if this point of view is taken seriously. Before we try to elucidate where the confusion may lie, let us comment on the nature of active diffeomorphisms: what is an active diffeomorphism? In this present work we have constantly used the notion of a map between sets $\phi : M \to N$ and the notion of a map of a set to itself $\phi : M \to M$, these are of course well grounded in the mathematical language and they are second-nature for anyone who works in mathematical sciences. But let us briefly stop and reflect about the conceptual meaning of $\phi : M \to M$. It is customary to interpret these kind of maps as *moving* the points of M and, although it is a very useful perspective for developing intuitions about the abstract concepts they represent, specially in geometry, when it comes to formally dealing with maps as objects by themselves, this **active** view becomes less appealing. Indeed a set M cannot move[13] and so a map from the set to itself is properly interpreted as a correspondence between points. The author believes that the best picture for the formal treatment of maps is that of a set of arrows linking pairs of points in M. It is obvious then that space-time

[13] We could even argue that motion or evolution has to be mathematically characterised by a fibration of the real line, that is interpreted as time. This is obviously not the case when discussing general maps of the form $\phi : M \to M$.

points cannot be formally linked with a diffeomorphism without leading to inconsistencies in the physical interpretation of relativistic theories.

Take a diffeomorphism of space-time regarded as a manifold consisting of all space-time events $\phi : M \to M$ and consider a suitable observer $u \subset U$ for which $\phi|_U$ is also a non-trivial diffeomorphism. The set of arrows connecting space-time points will translate, through the map φ_t, into a set of arrows connecting points of $\mathbb{R}^3$. Naturally, there is information that is lost when the observer considers $\mathbb{R}^3$ to be a static euclidean space, since arrows no longer connect two distinct physical points. If the observer uses her proper time to label the points connected by the set of arrows ϕ, this loss of information disappears, since the observer's open neighbourhood is diffeomorphic to the coordinate chart she is using, but there is another aspect of the original information contained in ϕ that is lost, namely, coordinate independence. The observer has a very concrete set of arrows, constructed by restricting ϕ to U and using the map φ_t, and she may think it has physical meaning by itself and that it should be coordinate independent, i.e. that using any other set of coordinates will reproduce the set of arrows just as well. This is a fallacy, though, since the set of arrows is tied down to the particular chart used by the observer in the first place.

After this last conclusion we see the importance to continue to work in the intrinsic approach once we have started to do so, since neglecting the geometric origin of a given construction will lead to ambiguities as the one explained above. Although we based our discussion on Minkowski space-time these considerations also apply to general Lorentzian manifolds.

Let us go back to the matter in hand and show how to express electromagnetism as a Hamiltonian theory. First consider an observer $u \subset U$ and the Lagrangian Field Theory that is defined in terms of the action $\int_U \mathcal{L}\omega$ with $\mathcal{L} = ||dA||^2$. Consider the construction detailed above in which the observer uses the map φ_t to identify each spatial slice with $\mathbb{R}^3$. It is obvious that using this coordinate chart the observer sees electromagnetism as a Lagrangian Field Theory on $\mathbb{R}^4$ with action $\int_{\mathbb{R}^4} d^4x\varphi^*\mathcal{L}$. Since the observer's interpretation of the first 3 real numbers of this description is in terms of a spatial position in a static euclidean space, the only conceivable configuration space that the observer may define for a canonical Lagrangian or Hamiltonian theory is the space of fields over $\mathbb{R}^3$. More precisely, the space of differential forms on $\mathbb{R}^3$, indeed those pulled back from the forms defined over space-time φ_t^*F and φ_t^*A. The exterior algebra of $\mathbb{R}^3$ is quite special in that all information is encoded in terms of scalar functions and vector fields: $\Omega^0(\mathbb{R}^3)$ are functions, $\Omega^1(\mathbb{R}^3)$ are 1-forms isomorphic to vector fields through the standard euclidean metric, $\Omega^2(\mathbb{R}^3)$ are 2-forms isomorphic to 1-forms through Hodge duality and hence isomorphic to vector fields and finally $\Omega^3(\mathbb{R}^3)$ are top forms that are obviously isomorphic to functions. For this reason the configuration space is taken to be that of vector fields over $\mathbb{R}^3$, or possibly many copies of these, from the start since any other choice will turn out to be equivalent given the particularity of the exterior algebra of 3-dimensional spaces.

Recall from previous sections that the electromagnetic field F is *projected* into a particular observer's description in two different objects that recover Maxwell's equations. If the observer's vector field is $u \in \mathfrak{X}(U)$ with $U \subset M$ the observer's open neighbourhood, then the form F is *projected* as:

$$E_u = F(u) \quad B_u = \star F$$

where $\star$ stands for the Hodge star operator of the 3-dimensional euclidean space U_t. Note that the above are, by construction and noting the peculiarities of the exterior algebra of 3-dimensional manifolds explained above, one-forms on U_t so the euclidean metric rising index operator can be used to identify them with vector fields. This is the formal definition of the **electric and magnetic fields**. If we consider the potential A, a similar approach gives us:

$$\phi_u = A(u) \quad A_u = \star A$$

which is taken to be the formal definition of the **electric and vector potentials**. Although one may think that this indeed recovers the traditional observer's description of electromagnetism, there is one further consideration regarding time evolution that is, to the knowledge of the author, always neglected in the literature. The observer's description takes a *static* euclidean space $\mathbb{R}^3$ on which fields evolve in time, however, note that this not quite the case in the above construction since U_t is a different space slice for every time value t. Recall that the observer is provided with a map φ_t that allows her to diffeomorphically map U_t to the fixed euclidean space $\mathbb{R}^3$ and so we can finally define the mathematical objects that the observer interprets as electromagnetic fields

$$\vec{E}_u = \varphi_t^* F(u) \quad \vec{B}_u = \varphi_t^* \star F \quad \phi_u = \varphi_t^* A(u) \quad \vec{A}_u = \varphi_t^* \star A.$$

These are indeed time-dependent fields over a fixed,static euclidean space.

The key point in the setting presented above that contrasts with many of the classical texts on the subject, including Bergmann's work, is the definition of electromagnetic fields as time-dependent pull-backs of intrinsic tensor fields over Minkowski space-time. As we shall see in what follows, this will help us clarify the classical ambiguities of transformations generating time translations. But first let us go back to our original aim of formulating electromagnetism as a canonical theory.

Let us first introduce the Lagrangian formulation that could be regarded as the one found in the classical literature but phrased in terms of the notions presented in the current setting. Consider the coordinate chart of the observer $(U, \varphi(p) = x^\mu)$ and remember that the Lagrangian Field Theory action is written as:

$$S[A] = \int_U \mathcal{L}\omega = \int_{\mathbb{R}\times\mathbb{R}^3} d^4x \varphi^* \mathcal{L}$$

with $\mathcal{L} = ||dA||^2$. Moreover we can expand $A = A_\mu dx^\mu$ and so this becomes a Lagrangian Multi-Field Theory with simply 4 real-valued fields being the components $A_\mu(x,t)$ in the observer's chart. It is obvious that we can rewrite the action as:

$$S[A] = \int_U \mathcal{L}\omega = \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3x \varphi^* ||dA||^2 = \int_{\mathbb{R}} dt L[A_\mu, \dot{A_\mu}]$$

where L is a functional over 3-dimensional field configurations and where, after a quick analysis check, the time evolution of the fields correspond with the point-wise partial derivatives with respect to time. Note that this is now in the form of a usual canonical Lagrangian formalism with configuration space $\mathbb{Q} = \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3)$, i.e. the four components of A in the observer's coordinates. The Euler-Lagrange equations are:

$$\frac{\delta L}{\delta A_\mu} - \frac{d}{dt}\frac{\delta L}{\delta \dot{A}_\mu} = 0 \tag{44}$$

which, after a straightforward computation are shown to be equivalent to the Field-Theoretic version:

$$\frac{\delta S}{\delta A_\mu} = 0. \tag{45}$$

These computations have traditionally given enough reasons for physicists to think that a proper Hamiltonian system (the canonical one) must exist associated with this Lagrangian formalism. In what follows we show the traditional approach first in order to discuss the limitations and difficulties that arise and how, in the author's view, they may be overcome when the right perspective, indeed following the ideas of the present work, is taken.

The canonical Hamiltonian formulation follows naturally as it was defined in section 2.5. The cotangent bundle elements for this configuration space are given by pairs (A_μ, π^μ) where π^μ are $\mathbb{R}^* \cong \mathbb{R}$-valued fields that correspond to the algebraic duals of A_μ as fields over $\mathbb{R}^3$. The fibre derivative or Legendre transform is then trivially defined since we are dealing with a vector space:

$$(FL)_* L = \frac{\delta L}{\delta \dot{A}_\mu}(\dot{A}_\mu) - L$$

the first term stands for the action of the functional derivatives on tangent vectors to the field configuration space and, hence, by definition they correspond to dual vectors. This is usually referred to as **generalised momenta** that act under the integral giving the Hamiltonian:

$$H = (FL)_* L = \int_{\mathbb{R}^3} \pi^\mu A_\mu d^3x - L.$$

This is indeed a well-defined function on $T^*\mathbb{Q}$ but does this mean that its physical interpretation is clear? The answer is no. Recall that in section 2.4 the connection between the always-physical Lagrangian formulation and the powerful Hamiltonian formulation was dependent on the non-degeneracy of the fibre derivative of the Lagrangian. A straightforward computation gives the Jacobian of the fibre derivative in our case:

$$\frac{\delta L}{\delta \dot{A}_\nu \delta \dot{A}_\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \partial_i^2 A & \partial_i^2 A & \partial_i^2 A \\ 0 & \partial_i^2 A & \partial_i^2 A & \partial_i^2 A \\ 0 & \partial_i^2 A & \partial_i^2 A & \partial_i^2 A \end{pmatrix}$$

where $\partial_i^2 A$ stands for some non-vanishing derivatives of the fields A_μ. This Jacobian is obviously degenerate and hence the physical interpretation of both the Hamiltonian H and the generalised momenta $p^\mu = \int_{\mathbb{R}^3} \pi^\mu$ cannot be automatically linked to that of the Lagrangian. This, of course, is equivalent to the observation that, in fact, one of the generalised momenta is identically zero $p^0 = 0$ since $\pi^0 = 0$.

There is something that did not work well in this approach, as has been noted by many authors in the classical literature, and one could take, roughly, one of the following standpoints:

- *The Hamiltonian formulation is useful only when is useful*, that is, if it doesn't work or its interpretation is unclear we should stick to the original Lagrangian formulation that surely works well.
- Insist in the convenience of having a Hamiltonian formulation (maybe because we want to canonically quantise the system) and analyse the traditional formulation until the source of the *problem* is found.
- Question the traditional formalism and revise its foundations so this apparent *problem* can be put in context and given a precise meaning.

Since the first will require no further discussion, we will tackle the remaining two. Of course, the second standpoint has been discussed extensively in the classical literature and we refer to Pitts (2013) for a detailed account of the history of this problem both in electromagnetism as exposed here and in the context of general relativity. In the traditional approach, the vanishing momentum is always attributed to an underlying constrained mechanical system and then, the theory of constrained Hamiltonian dynamics is applied to fully interpret the degeneracy of the Legendre transform. The author, taking the third standpoint, aims to provide a different approach to this problem based on the developments presented in this work.

Before getting into the details of the author's take on this problem and related issues concerning observables, we stress that, in particular for electromagnetism, the classical approach using constraints on the Hamiltonian taken carefully, suffices to explain all the ambiguities. The point in presenting the author's approach is to give a precise idea of something that is necessary for a Hamiltonian formulation to be derived from a completely covariant theory and which works for the case of electromagnetism[14] but fails in the case of general relativity, as we shall see in the sections below.

Our strategy here will be to present an argument based on the ideas exposed above that shows how the constraint viewpoint of the classical approach is equivalent to simply developing the premises introduced earlier in the present work. Recall that, for a canonical formalism, potentials are required from purely mathematical reasons, after all one could argue that the Hamiltonian formalism is the very generalisation of the concept of potential. Also recall that the observer interprets electromagnetic fields as time-dependent fields over a fixed euclidean space only if we define:

$$\vec{E} = \varphi_t^* F(u) \quad \vec{B} = \varphi_t^* \star F \quad \phi = \varphi_t^* A(u) \quad \vec{A} = \varphi_t^* \star A.$$

[14] And, indeed, for any compact Lie group gauge theory over space-time in general.

Where we have simplified the subindices indicating the observer since no other observers will be considered. Therefore, the observer may consider a configuration space of fields $(\phi, \vec{A})$ over $\mathbb{R}^3$, note that this is roughly what was done before if we identify $A_0 \sim \phi$ and $A_i \sim \vec{A}$. However, we must impose that the theory is well-defined for observables and hence the condition $F = dA$ must be projected down to the observer chart in some way. Indeed by direct computation we can check that $\vec{E} = \varphi_t^* dA(u)$ and $\vec{B} = \varphi_t^* \star dA$ correspond precisely to the pre-relativistic definitions of electric and magnetic fields in terms of electric potential and vector potential. This is a reality check that confirms that we are indeed in the right track and we are reassured in our definitions. Also, once the theory is reformulated in Hamiltonian terms, the second Maxwell equation, which in space-time reads $d \star dA = 0$, needs to be recovered to finally confirm that we have found a valid description of electromagnetism as a Hamiltonian theory.

First let us note that we get the following time evolution formulas for the fields simply using the geometry of the definitions provided:

$$\frac{d}{dt}\phi = \varphi_t^* d(A(u))(u) \tag{46}$$

$$\frac{d}{dt}\vec{A} = \varphi_t^* d(\star A(u)) + \varphi_t^* d \star A(u) \tag{47}$$

$$\frac{d}{dt}\vec{E} = \varphi_t^* d(F(u,u)) + \varphi_t^* (d(F(u)))(u) \tag{48}$$

$$\frac{d}{dt}\vec{B} = \varphi_t^* d \star F(u) + \varphi_t^* d(\star F(u)) \tag{49}$$

$$\tag{50}$$

In order to derive these expressions we have simply used the time-dependent pull-back formula, noting that the flow with the parameter t corresponds, as it should by construction, to the flow of the observer's vector field u, and the Lie derivatives that appear therein are subsequently rewritten in terms of contraction with u and the exterior derivative by means of Cartan magic formula $\mathcal{L}_u = di_u + i_u d$. The above expressions are a good starting point for formulating a Hamiltonian theory since they are first-order equations of time-dependent fields over $\mathbb{R}^3$ and this is precisely what Hamilton's equations will be if we find an appropriate Hamiltonian function on an appropriate configuration space. The above equations can be further manipulated using the antisymmetry of F, the fact that $d\varphi^* = \varphi^* d$ and reinserting the expressions that arise from the potential definition $F = dA$. After doing so and imposing the second Maxwell equation $d \star dA = 0$ they become exactly Maxwell equations on $\mathbb{R}^3$.

Writing the Hamiltonian theory requires for an appropriate choice of set of dynamical variables and the list equations written above provides an obvious way to do so, as opposed to the conventional way that made no distinction, in principle, between A_0 and A_i as fields. Now the situation is radically different, and here is precisely when the approach proposed by the author finds its utility above the traditional one. The potentials ϕ and $\vec{A}$ are forms of order 0 and 1, respectively, and hence, this poses an asymmetry between them when it comes to the exterior calculus. Indeed, we have $d \star \phi = 0$ identically since $\star\phi$ is a top form, whereas both dA and $d\star A$ need not be zero in general, this is reflected in the particularly simple form of the evolution equation for ϕ:

$$\frac{d}{dt}\phi = \varphi_t^* d(A(u))(u).$$

Let us rewrite this equation using the standard results from tensor calculus $\varphi^*\{\alpha(X)\} = \varphi^*\alpha(\varphi^* X)$ and $d\varphi^* = \varphi^* d$:

$$\frac{d}{dt}\phi = d(\varphi_t^* A(\varphi_t^* u))(\varphi_t^* u)$$

also recall that, by construction, $df(X) = X[f]$ as directional derivatives and so renaming the time-dependent vector field of $\mathbb{R}^3$, $w = \varphi_t^* u$, and identifying the definition of the vector potential $\vec{A} = \varphi_t^* A$, we finally have:

$$\frac{d}{dt}\phi = w[\vec{A}(w)]. \tag{51}$$

Therefore, noting that the time-dependence of $w = \varphi_t^* u$ is fixed simply by definition of observer, we find the dynamics of ϕ to be completely determined once the dynamics of $\vec{A}$ are known; more concretely, note that the equation above is an identity between real functions over $\mathbb{R}^3$ that depend on time and since these are elements of an (infinite-dimensional) vector space, we could write the formal solution as usual $\phi = \int w[\vec{A}(w)]dt + \phi_0$. This has some resemblance to the classical discussion of Hamiltonian electromagnetism when A_0 is said to be *non-dynamical*, we have just shown what is meant by this in precise terms. The difference between the classical approach and ours is that now this is a result of the construction of the theory instead of a happy coincidence that cannot be tracked down to a precise physical principle.

Keeping this result in mind, we take the configuration space to be the set of vector fields over $\mathbb{R}^3$, indeed a space isomorphic to the space of *all possible* $\vec{A}$, which are 1-forms, through the musical map induced by the euclidean metric $\varphi_t^*\eta$. The rest of the procedure follows the standard developments that can be found in classical texts for a field theory on the space $\mathbb{R}^3 = \varphi_t(U_t)$. Following Appedix E of Wald's book, for instance, we see that the cotangent bundle that is required for the canonical formulation is that of pairs of fields $(\vec{A}, \vec{\pi})$ such that $\mathrm{div}\vec{\pi} = 0$, this is indeed what connects with the aforementioned idea of regarding electromagnetism as a constrained theory. Then Maxwell equations are easily recovered from Hamilton equations once the Hamiltonian function:

$$H = \int_{\mathbb{R}^3} ||\vec{\pi}||^2 + ||d\vec{A}||^2$$

is used, note that this is precisely the standard Hamiltonian function for the cotangent bundle of any normed vector space.

The above recovers electromagnetism only if the identification $\vec{\pi} = -\vec{E}$ is made but, as one often finds in the classical literature, no justification other than *because it works* is given for doing so. The argument exploits, again, the particularly symmetric structure of the exterior algebra of $\mathbb{R}^3$. Recall that $\vec{A} \in \Omega^1(\mathbb{R}^3)$ by construction from $A \in \Omega^1(M)$, also recall that $\vec{E} \in \Omega^1(\mathbb{R}^3)$ as $F(u) \in \Omega^1(M)$. Since the euclidean metric $\varphi_t^* \eta = \delta$ provides an inner product for each $\Omega^k(\mathbb{R}^3)$, the dual spaces of the spaces of forms are identified with themselves. This implies that the canonical momenta conjugate to $\vec{A}$, in general an element of the dual of $\Omega^1(\mathbb{R}^3)$, will be an element of of $\Omega^1(\mathbb{R}^3)$ itself. Now, the possible field variables are those written in time derivatives above[15], namely $\phi, \vec{A}, \vec{E}$ and $\vec{B}$, but the only variable, other than $\vec{A}$, that corresponds to $\Omega^1(\mathbb{R}^3)$ is $\vec{E}$ indeed. The minus sign $\vec{E} = -\vec{\pi}$ is caused by conventions in the definition of the Hodge star and the raising and lowering of indices. This provides a precise justification for the above identification and hence we conclude that **there is a formal way to cast relativistic electromagnetism as a purely Hamiltonian theory but always subject to particular observers in space-time**.

Finally, to close this section, and with it, our discussion on electromagnetism, we give a brief account of the notion of observable presented by Bergmann in his paper *Introduction of True Observables into the Quantum Field Equations* (1956). As the title indicates, Bergmann's work is concerned with finding admissible canonical variables in the Hamiltonian version of relativistic electromagnetism that will serve as observables, in the quantum-canonical way, for a quantised theory. We must note that the concept of observable presented by the author, although very closely related to that of Bergmann, differs slightly in that we do not have a quantisation scheme in mind when defining it. As one should expect, if the definition by the author is sensible enough, it should fit nicely in any quantisation scheme. With this in mind, let us quote Bergmann on what he calls *true observables*:

The true observables are the physically meaningful variables of a theory. Their values (at a given time) are independent of the choice of the frame of reference (including the gauge frame). Their values can be predicted from one time to another by integration of the canonical equations [...].

This a very broad definition that would indeed apply to any physical theory but he goes on and specifies for the case of electromagnetism:

[...] because all the constraints are first-class constraints, the true observables are those dynamical variables that are left over after we have eliminated not only the constraints themselves but also their canonical conjugates.

This is stated after giving a formulation equivalent to what we have described earlier in this section as being the classical approach to Hamiltonian electromagnetism in which the conjugate momentum π^0 vanishes. This vanishing momentum is the constraint that Bergmann mentions[16] in the previous quote. Note that, except for the fact that Bergmann does not use an intrinsic approach, his definition of observables as *dynamical variables that are left over after we have eliminated* the constraints coincides with the result we derived

[15] A gauge theory with monopoles will include two additional fields corresponding to a 3-form in space-time describing the monopole current.

[16] He refers to constraints in plural since in his paper sources are considered and so there are further conditions imposed by the presence of electric charges that are also interpreted as first-class constraints.

above that the meaningful configuration space should only include the vector potential as pulled-back with a time-dependent diffeomorphism. Therefore, without further considerations, we see that Bergmann adheres to what we have called the classical approach, and so, since a precise, intrinsic reformulation of such approach has been presented in this work, we should expect any canonical computation carried out in his work to be correct. Nevertheless, the issue of not being an intrinsic theory still remains and ambiguities stemming from considering time-evolution as active transformations may appear. Indeed, as discussed before, a careful formulation is required if the Hamiltonian flow is interpreted as an active transformation of space-time since one should never forget that a Hamiltonian formulation is tied down to a particular observer and not to space-time itself.

4 General Relativity

In this section we aim to discuss the notion of observability in the context of general relativity. We first introduce the formal construction of the theory, as it was done for special relativity, and we point out the key similarities and differences between these two. We critically review the viewpoints on the topic of observables as found in the various Bergmann papers and in the recent Pons *et al.* paper by providing a comparison with the intrinsic definition that the author suggests.

4.1 General Relativity and Observables

General Relativity can be understood as a direct generalisation of Special Relativity by considering **Einstein's equivalence principle**. The early formulations of this principle, which used statements relating free fall and accelerated inertial frames, led to many predictions of the final theory of gravitation, namely: red-shift, gravitational clock delay and light bending. Since Special Relativity has been introduced as precisely formulated in 4-dimensional affine space, the corresponding formulation of the equivalence principle will be simply the usual mathematical setting for General Relativity.

Space-time M, or the universe set where particles and light-rays are taken to be subsets, will be an oriented 4-dimensional smooth manifold that admits a Lorentzian metric g. The unique Levi-Civita connection ∇ will be employed. The metric or invariant volume form ν, as defined from the metric, will be used.

The Lorentzian metric g induces a causal structure on each tangent space T_pM and tangent vectors are correspondingly called **time-like**, **null** or **space-like** if their norm is negative, zero or positive, respectively. This notion extends naturally to vector and covector fields defining everywhere properties with the metric and inverse metric tensor fields, respectively. It is possible to further extend these notions to curves and hypersurfaces by considering the tangent vector and the normal covector. Let us characterise a curve by means of a map $c : \lambda \mapsto c(\lambda)$ and a hyper surface as the sub manifold defined by the regular function f as $\Sigma = \{p \in M : f(p) = 0\}$, in this case we have a natural vector field restricted to c, indeed the tangent $\frac{d}{d\lambda}$, and a natural covector field restricted to Σ, indeed the differential

$df|_\Sigma$ which gives zero when acting on any $S \in T_p\Sigma$; by virtue of the musical isomorphism induced by the metric, we can express the differential as $df|_\Sigma = N^\flat$ which will be a vector field restricted to Σ called the normal. Therefore we have the following definitions:

- A curve $c(\lambda)$ is said to be **time-like**, **null** or **space-like** when its tangent $\frac{d}{d\lambda}$ is everywhere time-like, null or space-like, respectively.
- A hypersurface Σ is said to be **time-like**, **null** or **space-like** when its normal N is everywhere space-like, null or time-like, respectively.

The **free particles** and **light rays** are taken to be affinely-parametrised time-like geodesics and null geodesics, respectively. Equivalently, free particles are taken to be integral lines of vector fields $u \in \mathfrak{X}(M)$ satisfying $g(u,u) = -1$ and light rays are taken to be integral lies of vector fields $k \in \mathfrak{X}(M)$ satisfying $g(k,k) = 0$.

The mathematical formalism proposed in the previous section proves to be the most adequate if we want to sensibly implement the **principle of equivalence** to our geometric theory. The fundamental result that underlies this fact is the **local flatness theorem** that can be easily formulated by using the normal coordinates of the Lorentzian metric g. The key idea is to construct an isomorphism between some sufficiently small neighbourhood of an arbitrary point $p \in U_p \subset M$ and a convex open set of the tangent space at that point T_pM by means of the exponential map. In this construction, it is easy to check that the neighbourhood U_p has the precise structure of an affine space, where straight lines are the geodesics, together with a constant Lorentzian metric, simply the metric evaluated at p; then becoming Minkowski space and hence being able to exactly recover special relativity on U_p.

The discussion above, together with the fact that free particles are represented by geodesics with unitary, time-like tangent, motivates the following definition of **inertial observer**: physicists, or entities that are capable of measuring physical magnitudes, are confined to operate in Minkowski space and so an inertial observer will be a geodesic characterised by the flow of a unitary, time-like vector field. The fundamental object in this definition is the geodesic C itself and the vector field will be restricted to a neighbourhood of this curve $C \subset U_C \subset M$, called the **observer's domain**, such that for any point of the geodesic the corresponding neighbourhood that realises the Minkowski space isomorphism lies inside the observer's domain, that is, $\forall p \in C \Rightarrow U_p \subset U_C$. More precisely, we can construct **local Lorentz frames at a point** $p \in M$ as coordinate charts restricted to a certain inertial observer's domain by considering the affine parameter along the time-like geodesic given by the observer with tangent u_p and the three affine parameters given by three space-like geodesics generated by an orthonormal system of space-like vectors at p.

Restricting to an inertial observer's domain, it is possible to define a coordinate time smooth function, essentially the time-like parameter of the local Lorentz frame above, in very much the same way as we did for Minkowski space-time. This construction allows for a local and, in general, only local, notion of **simultaneity** embodied in the simultaneity submanifold $\Sigma_{u,p}$ defined as the zero locus of the coordinate time function and with normal u.

Note that the notion of simultaneity thus implemented is manifestly local, which is another an important supporting argument for the use of the present formalism in the description of cosmological theories.

The first statements of the **principle of general covariance** were concerned with the equations used in the expression of physical laws and how they depended upon the choice of coordinates in which they were formulated. Since, originally, there was no alternative but to use some particular set of coordinates the question of whether the expression of a physical law should depend on the coordinates used was justified. In a modern formulation this is no longer a problem since the fundamental setting is already geometric and any fields describing physical systems are taken to be defined over space-time without any mention of coordinates at all.

Nevertheless, the fact that the physical theory, or rather the *background* formalism for general physical theories, that we want general relativity to become uses the principle of equivalence as a cornerstone for all its experimental validity leads to a set of coordinates that is somehow singled out, indeed these are local Lorentzian coordinates as introduced above. In the context of general relativity, a **physical law** should be naturally global and geometric if it is intended to describe matter in large regions of space-time, as any astrophysical or cosmological theory will certainly aim to do. This amounts for considering physical models constructed with submanifolds in space-time (classical particle dynamics and string theories serve as one- and two-dimensional examples) or sections of some fibre bundle over space-time (any field theory both classical and quantum is an example) and, indeed, any relations between them that are described solely by means of the fundamental objects available: the metric g and the corresponding Levi-Civita connection ∇. Of course, no reference to a special coordinate chart is made whatsoever.

The crucial conceptual leap comes when we realise that real observations will be made by inertial observers and so the physical laws will be formulated only in the corresponding local Lorentzian frame. This poses a complicated situation for the **generalisation of *flat* physical theories** since the only constraint for a valid generalisation will be that it takes the expected form when expressed in local Lorentzian coordinates. For many examples, such as those concerned with field theories which involve derivatives, a kind of *Ockham razor's principle* may be invoked and the simplest possible generalisation comes by replacing the partial derivatives with covariant derivatives. Similarly, one could imagine that only the simplest laws should be considered and so, in a more general case, the generalisation will be such that we end up with the simplest intrinsic equation that reduces to the observed flat law when expressed in local Lorentzian coordinates. Recall that curvature-related objects such as the Ricci scalar or the connection components vanish in local Lorentzian coordinates so, in principle, arbitrary terms involving these may be added without loss of generality. This procedure is usually summarised under the name of **minimal-coupling principle**. In other circumstances a gauge theory is already defined over an arbitrary Lorentzian manifold and the generalisation is trivial, recall the discussion about classical electromagnetism in the previous sections.

The final prescription for these ambiguities is indeed **experiment verification**. If a generalisation is proposed it should come along with a way to measure the possible effects of it in an unexplored experimental regime (or accounting for previously known data in a novel way), there is no other way to check the validity of such generalisations.

These ideas relate to that of observable and we can conclude, taking inspiration from the considerations made in previous sections, that an **observable** is any tensor field, restricted to any space-time submanifold, that has non-empty intersection with an observer's local inertial frame.

A serious formal issue does arise, though, when dealing with general relativity in the modern mathematical formulation, this is: the diffeomorphism independence of physical theories. This could be regarded as a **global, active viewpoint of the principle of general covariance** and it is succinctly stated in terms of an equivalence relation for Lorentzian manifolds together with all their structure (connection, metric, matter fields...) related by diffeomorphism and push-forward of the objects. Recall that it is possible to define a push-forward of a connection ∇ through a diffeomorphism ϕ as $\tilde{\nabla}_X Y = \phi_*(\nabla_{\phi^* X}\phi^* Y)$ that is consistent with all the Riemannian structure. Nevertheless, as for the case of special relativity discussed in previous sections, the active viewpoint of diffeomorphisms is better avoided when establishing the foundations of space-time theories. The argument we would use here will simply reproduce the ideas of the relevant paragraph of section 3.4 replacing Minkowski space-time with a general Lorentzian manifold.

A **particle** is defined as an affinely parametrised time-like or null curve or, equivalently, an integral line of a vector field $X \in \mathfrak{X}(M)$ satisfying $g(X, X) = -1$ or $g(X, X) = 0$. Let us first focus on the **kinematics** of time-like particles: we denote the **trajectory** or **world-line** followed by such particles as $x(\tau)$ and the **velocity** as $u(\tau)$ or $\dot{x}(\tau)$, the velocity of a particle is, of course, a vector field restricted to the particle's trajectory. A general **observer** is defined likewise an inertial observer but allowing for a general time-like curve as the observer's world-line. Although general observers are of course not always inertial, it is possible to define the **instantaneous inertial frame** for any given point of the observer's world-line $x(\tau_0)$, at this point the velocity will be $u(\tau_0)$ and so we can consider the local Lorentzian frame in the tnagent space $T_{x(\tau_0)}M$ and define the simultaneity subspace as $u(\tau_0)^\perp \subset T_{x(\tau_0)}M$. A useful tool for latter applications will be the **instantaneous projection operator** of a time-like vector u defined as a map $T_pM \to T_pM$ at every point and hence characterised by a tensor field $\perp \in \mathcal{T}^1_1(M)$ given by:

$$\perp = \delta + u \otimes u^\flat$$

An observer carrying a clock that sits with an arbitrary time-like particle will measure a **time span** between to events of its world-line given by proper time $\Delta\tau$. The **acceleration** of a particle is defined as the vector field restricted to its trajectory given by the covariant derivative:

$$a = \nabla_u u = \frac{Du}{d\tau}$$

Considering a many-particle-system scenario as described in an inertial frame we see that a rank 2 tensor field is suitable for encapsulating all the information of all types of matter in space-time. We, therefore, define **stress-energy tensor** of matter on space-time, which is indeed a sufficient characterisation for the matter-space-time dynamics to be fully determined. Then matter on space-time is described by a tensor field $T \in \mathcal{T}_2^0(M)$ that is symmetric $\mathcal{S}T = T$ and conserved $\mathrm{DIV}T = 0$. The symmetry is required in order to fully recover the interpretation of this tensor field as matter observed in a local Lorentzian frame, as we shall see below. The divergence-free condition is more profound, it is tied to the fact that the matter system described will be taken to be only self-interacting, that is only internal forces will be assumed and in case interactions were included they will be represented by some other fields. The stress-energy tensor is a density of the dynamical variables that characterise matter and from the equivalence principle we have the following interpretation for measurements made by an inertial observer (u, e_i):

- **Energy density**: $\rho = T(u, u)$
- **3-Momentum density**: $s_i = T(u, e_i)$
- **Stress Tensor**: $t_{ij} = T(e_i, e_j)$
- **Energy Current**: $j = T^\sharp(u)$.

The energy current is indeed conserved in the Minkowski space-time of the inertial observer using the local Lorentzian frame and so we have the corresponding energy-momentum conservation for the system described with T as seen by the inertial observer. Note that this notion of energy conservation is purely local (only fully enforced in Minkowski space-time) and, in general, global conservation laws are difficult to formulate or, if given, to interpret physically.

General relativity was conceived to account for gravitational phenomena in the first place with the foundational idea by Einstein that space-time geometry is affected by the presence of matter. As it was discussed above, matter is fully described by the stress-energy tensor T which is symmetric and conserved. If we aim to recover Newtonian gravity when the metric differs slightly from Minkowski we will need to relate T to a tensor field that is also symmetric and conserved and which depends only on the metric, its first and second derivatives and only linearly on the latter. A theorem by Lovelocke states that the most general such tensor field is of the form $H = \alpha G + \beta g$ with G the Einstein tensor, g the metric and α, β real constants. Therefore, the general form of **Einstein's equation** is:

$$G + \Lambda g = kT$$

here Λ is called the cosmological constant and for most astrophysical applications must be taken to be zero, giving the usual form of Einstein's equation:

$$G = kT$$

and k is a constant that is left to be determined. Einstein's equation, when expressed in a coordinate chart, comprises 10 non-linear partial differential equations $G_{\mu\nu}(g, \partial g, \partial^2 g) = T_{\mu\nu}(g)$, one for each independent component of the symmetric metric $g_{\mu\nu}$, with 4 constraints imposed by the conservation condition $\nabla^\mu G_{\mu\nu} = 0$. These are very complicated to solve in general and so we often look for situations in which there are enough symmetries that simplify the problem down to a tractable one. There is a very remarkable fact about Einstein's equation: non-linearity. This is, of course a clear departure from Newtonian gravity and it is precisely this feature of Einstein's theory that accounts for phenomena unexplained by previous models of gravity, for instance the well-known perihelion transit of Mercury. Gravity, as described by general relativity, self-interacts, note how the coordinate equation above makes explicit that the stress-energy tensor always involves the metric. This is, again, radically different from the Newtonian case in which Poisson equation $\partial_i \partial_i \phi = 4\pi G\rho$ does not couple the dynamics of the gravitational field ϕ with the values of the matter field ρ. If we regard general relativity as a gauge theory, the non-linearity of the field equations can be understood as a direct consequence of the gauge group, in this case, $GL(\mathbb{R}^4)$ or some subgroup of it, not being abelian.

Since the left-hand-side of Einstein's equation only involves $G(g, \partial g, \partial^2 g)$, an entirely geometric object determined solely from the metric, solutions for the equations may be found by simply writing an appropriate stress-energy tensor $T(g)$ in terms of the metric. Of course, this perspective makes the general relativity description of gravity vacuous and totally non-physical. Consequently, the issue is to find **realistic stress-energy tensors** that will faithfully describe matter and account for the observed gravitational phenomena. This will be done by imposing further constraints in a symmetric, conserved tensor T but let us first consider the **vacuum** stress-energy tensor $T = 0$. The inertial observer interpretation of a general stress-energy tensor makes it obvious why we should take such definition for the absence of matter. Therefore any metric g satisfying $G(g) = 0$ is called a **vacuum solution** and if the solutions are taken to be global the Lorentzian manifold (M, g) is called a **vacuum** or **empty space-time**. Note that the vacuum metrics satisfy $G = Ric - \frac{1}{2}Rg = 0$, if we act with the contraction $C_1^1 \sharp_1$ we find that it implies $R = 0$ and so an equivalent condition for a vacuum solution is $R(g) = 0$.

For the remainder of this work we will assume a scenario of an - at least locally - empty space-time with zero cosmological constant. Therefore **Einstein's equation** will be taken to involve the metric tensor field equation:

$$G(g) = Ric(g) - \frac{1}{2}R(g)g = 0.$$

4.2 General Relativity as a Lagrangian Field Theory

Simply following the above statement we can clearly see that general relativity is a Field Theory on the tensor bundle $T_2^0 M$ and with individuation principle given by the Einstein equation $G(g) = 0$. In this sense there is no formal difficulty to cast general relativity in the form of a Field Theory of the general kind defined in section 2.5. From a purely mathematical perspective this is nothing exceptional and given a Lorentzian manifold M one could look for a Lagrangian action for a metric g such that its variation would yield the field equations $G(g) = 0$. Before reviewing some of the standard procedures to effectively do this locally, let us explain what the author believes to be a major difficulty when attempting to keep physical consistency and globally describe general relativity as a Lagrangian Field Theory - and, therefore, also as a Hamiltonian Field Theory.

Note that the Field-Theoretic Lagrangian formalism requires integration over the entire space-time manifold M and so a global volume form must be fixed. The metric is the unknown of the problem, hence, we cannot fix the metric or invariant volume form ν_g as in doing so we would have fixed the metric itself leading to a trivial situation. Physically realistic theories will aim to describe gravitational phenomena when arbitrary matter content is considered and, as may examples such as cyclic cosmologies or black holes confirm, the global topology of the manifold will indeed depend on this content. Recall that volume forms are top differential forms which are, by de Rham duality, the dual objects of top-dimensional submanifolds of M and so, if the manifold admits a certain non-zero top form, the manifold's top homology is constrained. Therefore, even if we fix an arbitrary volume form other than the invariant one, we still lose a crucial degree of topological freedom that will limit the outcome of any field theory formulation of general relativity with respect to such volume form. This argument does not affect any local considerations, of course, but it is worth keeping in mind that a conventional - both Lagrangian and Hamiltonian - field theory formulation of general relativity, as discussed in the present work as well as in the classical literature, is inherently incomplete in that it lacks the ability to determine the global topology of space-time; rather, it assumes a precise global topology in which a fixed non-metric volume form is chosen for integrals to be defined.

As we already mentioned, the above issue does not affect the local formulation of the theory since given an arbitrary coordinate chart (U, x^μ) the local volume form $\nu_U = dx^1 \wedge \cdots \wedge dx^\mu$ is always defined. Under this perspective, formulating general relativity as a local Lagrangian Field Theory is a systematic procedure very similar indeed to the one described in section 3.3 for electromagnetism. We will not reproduce the details here, which can be found, for instance, in Appendix E of Wald's book, but simply state the main results using the terminology introduced in this work.

Let a Lorentzian manifold M and a coordinate chart (U, x^μ) on which the local standard volume form $\nu_U \in \Omega^m(U)$ is defined. A local metric tensor field $g \in \mathcal{T}^0_2(U)$ will satisfy Einstein's equation iff it is an extremal value of the functional:

$$S[g] = \int_U \sqrt{-\det(g)} R(g) \quad \nu_U.$$

Note that here $\det(g)$ stands for the determinant of the metric as computed in the coordinate chart, therefore it is readily checked that this corresponds to an integral with respect to the metric volume form:

$$S[g] = \int_U R(g) \quad \nu_g.$$

Also note that U is assumed to admit boundary conditions for g and its derivatives that cancel all contributions from boundary terms arising from the variational computation.

4.3 Hamiltonian Formulation of General Relativity

If we aim to move from the Lagrangian formalism to the Hamiltonian formalism, as we did in the case of electromagnetism, we must consider an observer's time slicing and a similar pull-back with a map $\varphi_t^{-1} : U_t \to \mathbb{R}^3$ where $U = \bigcup_{t \in \mathbb{R}} U_t$ is the observable open set of the observer. Before imposing a hypothesis, that is not often mentioned in the classical literature, regarding the coordinate chart induced by such construction, we find a difficulty at the very start of an attempt to find a Hamiltonian formulation.

An observer, inertial or otherwise, is defined as a local unitary vector field. Of course, the notion of unitary refers to the metric and hence its is absurd to assume a fixed metric in which the observer is defined and then try to find this metric. Indeed, the concept of observer is tied down to the metric tensor field and so, in a first approach, it seems absurd to formulate a Hamiltonian theory, based on *an* observer, in order to find the dynamics of what allows us to even define the observer in the first place. This circularity is not easily overcome and, indeed, ignoring it may be the cause of many conceptual difficulties. The author believes this to be the case in some classical texts, where a precise intrinsic formalism is not enforced, and the notion of observer is, as a result, blurred, to say the least. The ability to become aware of this issue is, of course, not an exclusive feature of the intrinsic approach but it is one that stands out very clearly.

As we shall show later in this section, there are mathematical resources, available as part of the geometry of general Lorentzian manifolds, that allow us to cast the above Lagrangian Field Theory as something very much resembling a canonical Hamiltonian theory, but this should not distract us from the fact that there is a foundational conceptual aspect of the notion of observer that conflicts the Hamiltonian approach. This is the point where general relativity proves to be radically different from electromagnetism and this is precisely the reason why the author chose to treat them separately. As we saw in section 3.4, the ambiguities in the case of electromagnetism were resolved by clearly identifying the configuration space and noting that observables in that context matched the original ones when taking into account the time slicing map of the observer. However, it is obvious that both the metric and the volume form were fixed globally, something that is, clearly, no longer possible in the case of general relativity.

Let us give an example of how this can be done following the standard procedure that can be found in the literature, here we follow Appendix E of Wald's book. Again, since this is standard material, we omit the details and limit ourselves to restating the results with the terminology introduced in the present work. The main idea of this procedure is to take a nowhere-vanishing vector field $T \in \mathfrak{X}(M)$ and a nowhere-singular function $t \in \mathcal{F}(M)$, what we may call a *pre-observer*, such that $T[t] = 1$ and that will coincide with a true observer, by construction, once the metric is found. Note that the existence of such a vector field and function is ensured by the fact that M is assumed to be Lorentzian. The level sets of the function t are identified with the time slicing of the observer. Then, as for a real observer, the tensor $\perp \in \mathcal{T}_1^1(M)$ together with the associated unit normal $n \in \mathfrak{X}(M)$ is defined with respect to an arbitrary metric and similarly for the usual lapse function $l = -g(T, n)$ and shift vector $S = \perp(T)$, it can be shown that the information determining g is equivalent to $(\perp, l, N)$. From the defining assumptions for t and T we can see that there always exists a coordinate chart (U, x^μ) such that $x^0 = t|_U$ and that x^i are coordinates for each time slice U_t. Of course this is just the usual 3+1 decomposition. Choosing such a chart means fixing a volume form both for all of U and for each U_t individually, and so the field theory discussion can be introduced. From this point the procedure is reduced to a long, but straightforward, computation in which the Lagrangian action from the previous section is rewritten in terms of $(\perp, l, S)$ as:

$$I[\perp, l, S] = \int_U \sqrt{\det(\perp)}\, l\{Ric(\perp) + C^{12}_{12} K \otimes K^{\sharp\sharp} - (C^1_1 K^\sharp)^2\}$$

where the intrinsic curvature tensor can be written as:

$$K = \frac{1}{2l}(\mathcal{L}_T \perp' - \mathcal{A}(\nabla^\perp S^\flat)).$$

In the above expression, the term $\mathcal{L}_T \perp'$ stands for a linear combination of derivatives with respect to time and with factors only involving $\perp$ and $\nabla^\perp$ is the Levi-Civita connection induced in U_t by $\perp$ for each t. It is obvious from this result that, following an argument parallel to that exposed in section 3.4, when the configuration space is taken to be the fields $(\perp, l, S)$ over each slice U_t then the cotangent space is given by pairs $(\perp, \pi, l, \lambda, S, \sigma)$ with:

$$\pi = \frac{\delta I}{\delta \dot{\perp}} \quad \lambda = \frac{\delta I}{\delta \dot{l}} \quad \sigma = \frac{\delta I}{\delta \dot{S}}$$

where the dot notation stands for partial derivatives with respect to the t coordinate. From the expression given for I it is clear that no time derivatives are present so we have, like in the case of electromagnetism, identically vanishing momenta:

$$\lambda = 0 \quad \sigma = 0.$$

This is, again, interpreted as a result of a constrained Hamiltonian, which in fact turns out to be the case when it is explicitly computed. When this is done, the constraints which must be considered in addition to the Hamilton's equations in order to recover Einstein's equation, are:

$$\begin{aligned} &Ric(\perp) - C^{12}_{12}\pi \otimes \pi_{\flat\flat} - \tfrac{1}{2}(C^1_1 \pi_\flat)^2 \\ &\mathrm{div}^\perp \pi = 0 \end{aligned}$$

where, again, $\mathrm{div}^\perp$ stands for the divergence operator defined from the Levi-Civita connection $\nabla^\perp$ as above. The second constraint can be very well understood from the perspective of the principal bundle formulation of general relativity. The gauge group for a metric-compatible connection $\nabla^\perp$ in each slice is a subgroup of $GL(\mathbb{R}^3)$ and therefore $\mathrm{div}^\perp \pi = 0$ can be interpreted as the pulled-back Bianchi identity. In more pedestrian terms, this equation arises as a result of the freedom of choice of coordinates on each U_t, as it can be shown after a relatively short computation, and therefore the redundancy disappears after we quotient by diffeomorphism of U_t. However, the first constraint has no gauge theory equivalent and is, in fact, very much what distinguishes general relativity from other Hamiltonian Field theories. Since the constraint equation is quadratic in the conjugate momentum π, there is no obvious way to restrict the phase space so we can reformulate the theory as an unconstrained Hamiltonian theory. This difficulty is recognised by many, including Wald himself, to be the computational barrier between Einstein's general relativity and an equivalent canonical Hamiltonian formulation.

As we have seen above, even after exploiting all the resources that a Lorentzian manifold has to offer, there are issues that limit our capacity to meaningfully formulate general relativity as a Hamiltonian theory. The author believes that this fact, and therefore all questions related to a Hamiltonian formalism, are, somehow, a result of the aforementioned fundamental distinction between electromagnetism and general relativity regarding how observers are defined.

To close this section we give an account of the notion of observable in the context of general relativity as introduced by several authors. One of the definitions by Bergmann has been already discussed in section 3.4 and, although it was applied to electromagnetism , the statement was general enough to be considered as one of the multiple definitions that Bergmann seems to provide in his work. There are three other sources in which Bergmann and collaborators provide definitions for the notion of observables, for the sake of economy of space let us label each paper as follows:

- **0** : *Introduction of True Observables into the Quantum Field Equations* (1956)
- **A** : *Observables and Commutation Relations* (1960)
- **B** : *The General Theory of Relativity* (1960)
- **C** : *Observables in General Relativity* (1961)

In A Bergmann is concerned with finding observables for a quantised theory of gravity and he states that observables are *dynamical variables(field variables or functionals of the field variables) whose unmodified Poisson brackets with all the constraints of the theory vanish.* This is, as it was in 0, in accordance with the results found in the present work. Nevertheless, he goes on and relates observables to *invariant quantities* as is seen clearly in the statement *If our theory deals with physically meaningful quantities, invariants should be observable.* In the lines preceding that statement Bergmann distinguishes *invariants* to *scalars* and characterise the former as *a quantity so defined that its value in every coordinate system is the same.* Here we notice a link between Bergmann's definition and the present work's intrinsic formulation: Bergmann does not work in an intrinsic formalism but does refer to it indirectly by imposing coordinate invariance as a requisite for observability. Note that we did the same but this condition was simply ensured by construction, another instance where the intrinsic approach proves useful. The discussion in B follows a very similar scheme to that of A, observables are again directly related to invariant quantities to the extent of saying *the term observable (which is physically motivated) and invariant (which is formally defined)* *are interchangeable*. Bergmann then focuses on the problem of finding *intrinsic coordinates* as a way to parametrise space-time with coordinates that are observables themselves. The author would like to comment on this effort, which is also found in C, by saying that if the intrinsic approach presented in this work is taken seriously, providing *intrinsic coordinates* is a highly unnecessary thing to do since, by construction, all the physical entities are already intrinsic. Also, related to this issue, in all 0, A, B and C, coordinates are said to bear no true physical content by themselves, this statement while probably referring to the freedom of choice of local charts, may be misinterpreted as implying that coordinates are not observable or physical in nature, except, perhaps, for those identified as *intrinsic coordinates*. This is clearly not the case we one follows the intrinsic approach: coordinates always reduce to functions that are completely characterised by any observer. The author believes that referring to the numerical value of functions on specific points of space-time may be what is causing some of the apparent contradictions: a function over space-time, as any tensor field, is an assignment of values to points of space-time not the collection of values alone. Again, when phrased in the intrinsic language this sounds rather tautological but when one does not use this approach additional considerations, which are cumbersome and tend to complicate the terminology, are required. A perfect example is found in B page 251, here Bergmann needs to make formal and notational distinctions between objects of the same mathematical nature (multivariable real functions).

The approach followed in C highlights a feature of observables that relate them more closely to the Hamiltonian formalism, namely, the dynamical point of view of the definition *We shall call a quantity an observable if it can be predicted uniquely from initial data*. Bergmann then connects with previous notions of observable, as those of A or B, to finally state *An observable is then a dynamical variable that has vanishing Poisson brackets with all the generators of infinitesimal coordinate transformations*. In what follows, the constrained Hamiltonian perspective is taken again to discuss the subject in the context of general relativity. This new emphasis in dynamics leads Bergmann to state *[...] within any equivalence class all observables are constant* where he has identified the gauge classes as being connected by all the possible gauge transformations on the system. This connects with the statement made in section 3.3 that any Lagrangian action should have the same value on all the elements of every gauge class. When it comes to finding a unifying view of observables as deduced from the contents of 0, A, B and C, we may say that the first three 0, A and B present a reasonably common grounding for the concept. Nevertheless, C introduces the dynamical component, which is necessarily liked to an observer's projection and, as we showed above, no obvious way to intrinsically identify a Hamiltonian for the system is found. This, the author believes, may cause a contradiction when blindly computing canonical quantities in one special observer's frame and then trying to generalise the results to the entirety of space-time.

Another recent paper that addresses the problem of observables in general relativity is *Observables in classical canonical gravity: folklore demystified* (2010) by Pons, Salisbury and Sundermeyer. In this paper, observables are discussed mainly in the context of what is known as the frozen time paradox, which refers to the apparent paradox that, in the canonical formulation of general relativity, the standard requirement of observables having vanishing Poisson brackets with the gauge generators further implies that these are independent of the space-time coordinates, in particular, independent of the time coordinate and hence the idea that *nothing happens*. Pons *et al.* show how the above is indeed only an apparent paradox and not a real problem of the canonical formulation of general relativity in terms of the Ronsenfeld-Dirac-Bergmann formalism, as we see from the quote:

[...] observables must have vanishing Poisson brackets with the the gauge generator and hence with the Dirac Hamiltonian. This is entirely correct and indisputable because, by its very definition, observables are gauge invariant. But the [...] claim [...] asserting that an observable can not depend explicitly on the coordinates is wrong. It originates from a confusion between the active and passive view of diffeomorphism invariance. In the passive view the fields are considered always the same, their mathematical description changing according to the use of different coordinatizations. Instead, in the active view, which is the view taken in phase space, a gauge transformation moves from one configuration to a different one without changing the coordinates. Although both views are equivalent, the coordinates themselves, not being variables in phase space, are gauge invariant in the active view. This explains why an observable may depend explicitly on the coordinates. Requiring the observables to be independent of the coordinates and at the same time to have vanishing Poisson brackets with the gauge generators is to mix the two pictures of passive and active diffeomorphisms. It is too much of a requirement and therefore it is little wonder that paradoxes occur.

The definition of observable, as we see, follows that of Bergmann in 0, A and B quite closely, the novelty here is the inclusion of phase space transformation as part of the gauge action. Note that the above text refers to the same conceptual difficulties faced when active and passive views are carelessly mixed. This is the precise example that the author had in mind when making the decision to work in the intrinsic formalism since confusions as the one described by Pons *et al.* are automatically avoided.

5 Conclusions and Further Thoughts

Our discussion of electromagnetism has, in a sense, simply recovered known facts and results. This is, nevertheless, what should be expected since, after all, it is a theory whose Hamiltonian formulation has been consistently implemented in several successful physical theories such as quantum electrodynamics or the standard model of particle physics. The main point, however, was to give a mathematical and conceptual framework that consistently links the covariant formulation of the theory on Minkowski space-time and its pre-relativistic formulation as a field theory over a 3-dimensional euclidean space.

Another remarkable achievement of the present work, the author believes, was the fact that we developed our formalism, both recovering well-known facts as well as providing new insights for former misconceptions and misunderstandings, in a purely geometric and intrinsic manner. This required the author to make a considerable effort to first review all the mathematical ideas and then develop the physical theories following, to a certain extent, the classical formulations and being guided by intuition and conceptual clarity for any developments that were original.

A direct consequence of taking the intrinsic approach was the rapid identification of a fundamental distinction between the field theory formulations of electromagnetism and general relativity. Indeed, we found that consistency of field formulations - both Lagrangian and Hamiltonian - for electromagnetism very much depends on the fact that the metric on the space-time manifold over which the theory is defined is fixed. This is, of course, no longer the case in general relativity, since this is a field theory for the metric itself. The author identified many difficulties found in the classical literature on the subject to be natural consequences of this fact.

The author could not devise, especially given the limited scope of the present work, a way to formally reformulate general relativity as an intrinsic Hamiltonian theory. However, due to the success of several non-intrisinc approaches, as shown by Pitts (2013) or Pons *et al.* (2010), he believes that further research following the ideas outlined in the present work may yield fruitful results in the future. In the view of the author, the first issue to be considered to this end would be the problem of defining observers, and hence also observables, in a setting where the metric is not known.

The second major issue, which will still remain problematic even if the above difficulty with the definition of observer is overcome, will be to allow for global topological freedom of the space-time manifold, as it was pointed out at the beginning of section 4.2. The author believes that traditional field theory approaches are insufficient for this purpose and thinks that more sophisticated, maybe new, mathematics are required to consistently assess this problem.

6 Acknowledgements

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A An Intrinsic Formalism for Tensor Calculus on a Riemannian Manifold

In this appendix we provide a brief description of an intrinsic, index-free formalism for tensor field operations on a Riemannian manifold M, namely: contractions, partial filling, symmetrisations, permutations, rising of indices, lowering of indices, Lie derivatives, covariant derivatives and exterior derivatives. Then a simple example from GR is used to show how the formalism is applied in practice.

As for any smooth manifold M the space of all tensor fields $\mathcal{T}(M) = \bigotimes_{r,q=1}^{\infty} \mathcal{T}_q^r(M)$ is a $\mathcal{F}(M)$-module and together with the tensor product it constitutes the **tensor algebra** of a smooth manifold $(\mathcal{T}(M), \otimes)$. The particularity that a Riemannian manifold has a metric tensor and the associated Levi-Civita connection makes it worth devoting an entire section to the formalisation all tensor operations and the derivation of some very useful identities.

The **contraction** operation is a linear map:

$$\begin{aligned}C^i_j :&\mathcal{T}^r_q(M) \to \mathcal{T}^{r-1}_{q-1}(M)\\ &t \mapsto C^i_j t\end{aligned}$$

whose image is defined by:

$$(C^i_j t)(\dots,\omega_{i-1},\omega_{i+1},\dots,v_{j-1},v_{j+1},\dots) = \sum_{n=1}^{\dim M} t(\dots,\omega_{i-1},\theta^n,\omega_{i+1},\dots,v_{j-1},e_n,v_{j+1},\dots)$$

where $\{e_n\}$ is a basis for vector fields $\mathfrak{X}(M)$ and $\{\theta^n\}$ is its dual basis satisfying $\theta^n(e_m) = \delta^n_m$, this is, of course, checked to be a basis independent definition. Equivalently one could define the contraction of the tensor product of elementary tensors, i.e.vectors and covectors, such that $C^1_1(\omega \otimes v) = \omega(v)$ and then extend it to the entire tensor product space $\mathcal{T}^1_1(M) \otimes \mathcal{T}^1_0(M)$ by linearity. Then, providing a basis and its dual and writing an arbitrary tensor in the product basis the general definition of the contraction above is recovered. Note that we always employ the canonical tensor product isomorphism $V \otimes V^* \otimes V^* \otimes V \otimes V \otimes V^* \otimes V \otimes ... = V^* \otimes V^* \otimes V^* \otimes ... \otimes V \otimes V \otimes V \otimes V \otimes ...$. The contraction operator can be regarded as a generalised way of writing the action of a tensor on some of its arguments, indeed let a tensor field $t \in \mathcal{T}^r_q(M)$ and two possible arguments $\omega \in \mathcal{T}^0_1(M)$ and $X \in \mathcal{T}^1_0(M)$ then we can write the **partial filling of arguments** as:

$$\mathcal{T}^{r-1}_q(M) \ni t(\omega)^i \equiv t(\dots,\cdot,\omega,\cdot,\dots) = C^i_{q+1}(t \otimes \omega)$$

$$\mathcal{T}^r_{q-1}(M) \ni t(X)_i \equiv t(\dots,\cdot,X,\cdot,\dots) = C^{r+1}_i(t \otimes X)$$

Let us define the **Kronecker delta tensor** as

$$\delta \in \mathcal{T}^1_1(M) \quad : \quad \delta(\omega, v) = \omega(v)$$

From this definition we can expect some special properties regarding the contraction with other tensors, indeed let $t \in \mathcal{T}^r_q(M)$ then the following hold:

$$C^1_j(\delta \otimes t) = t \quad , \quad C^i_1(\delta \otimes t) = t$$

Consider two tensors $t \in \mathcal{T}^r_q(M)$ and $s \in \mathcal{T}^{r'}_{q'}(M)$ and let two contractions $C^{i \leq r}_{j \leq q}$ and $C^{i' > r}_{j' > q}$ then:

$$C^i_j(t \otimes s) = (C^i_j t) \otimes s \quad , \quad C^{i'}_{j'}(t \otimes s) = t \otimes (C^{i'}_{j'} s)$$

For a given tensor space $\mathcal{T}^r_q(M)$ a natural **permutation group action** can be defined on both covectorial and vectorial arguments, let us see how this is done:

$$\begin{aligned}&S_r \times \mathcal{T}^r_q(M) \to \mathcal{T}^r_q(M)\\ &(\sigma, t) \mapsto S^\sigma t\end{aligned}$$

$$
\begin{aligned}
&S_q \times \mathcal{T}^r_q(M) \to \mathcal{T}^r_q(M)\\
&(\sigma, t) \mapsto S_\sigma t
\end{aligned}
$$

being the resulting tensors defined by:

$$
\begin{aligned}
(S^\sigma t)(\omega_1, ..., \omega_r, v_1, ..., v_q) = t(\omega_{\sigma(1)}, ..., \omega_{\sigma(r)}, v_1, ..., v_q)\\
(S_\sigma t)(\omega_1, ..., \omega_r, v_1, ..., v_q) = t(\omega_1, ..., \omega_r, v_{\sigma(1)}, ..., v_{\sigma(q)})
\end{aligned}
$$

Obviously one can restrict the number of arguments to be permuted and find up to $r!$ different actions of r covectorial or vectorial arguments. In the special case of two-element permutations or transpositions we define the **swap** operators as before simply by noting the arguments to be swapped S^{ij}, S_{ij}. We can further define the **symmetrisation** and **antisymmetrisation** operators for vectorial arguments as

$$
\begin{aligned}
\mathcal{S}_{()} :& \mathcal{T}^r_q(M) \to \mathcal{T}^r_q(M)\\
& t \mapsto \mathcal{S}_{()} t = \sum_{\sigma \in S_r} S_\sigma t
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_{[]} :& \mathcal{T}^r_q(M) \to \mathcal{T}^r_q(M)\\
& t \mapsto \mathcal{A}_{[]} t = \sum_{\sigma \in S_q} \mathrm{sgn}(\sigma) S_\sigma t
\end{aligned}
$$

Similarly, $\mathcal{S}^{()}$ and $\mathcal{A}^{[]}$ are defined for covectorial arguments. Again one can restrict to any of the ordered subsets of arguments and define the same operations, therefore always bearing in mind that the set of arguments should be made explicit at some point we may simply write $\mathcal{S}$ and $\mathcal{A}$. Consider the space of covariant tensors $\mathcal{T}^0_q(M)$ then it is easy to show the symmetrisation and antisymmetrisation operators define a set of projectors:

$$
\begin{aligned}
\mathcal{S}\mathcal{S} = \mathcal{S} \quad , \quad \mathcal{A}\mathcal{A} = \mathcal{A}\\
\mathcal{S}\mathcal{A} = 0 = \mathcal{A}\mathcal{S}
\end{aligned}
$$

Therefore, we have a split tensor space in its **totally symmetric** $\mathcal{S}^q(M)$ and **totally antisymmetric** $\mathcal{A}^q(M) = \Omega^q(M)$ parts:

$$
\mathcal{T}^0_q(M) = \mathcal{S}_q(M) \oplus \mathcal{A}_q(M)
$$

And so every tensor $t \in \mathcal{T}_q^0(M)$ can be written as $t = s + \alpha$ for unique $s \in \mathcal{S}^q(M)$ and $\alpha \in \mathcal{A}^q(M)$. These components further satisfy the **symmetry** and **antisymmetry property**:

$$
\begin{aligned}
S_\sigma s &= s \\
S_\sigma \alpha &= \mathrm{sgn}(\sigma)\alpha
\end{aligned}
$$

Since we are considering tensor fields over a Riemannian manifold the symmetric, non-degenerate metric tensor field $g \in \mathcal{T}_2^0(M)$ is available in order to introduce an isomorphism between $\mathcal{T}_0^1(M)$ and $\mathcal{T}_1^0(M)$ and so we introduce the **flat** map:

$$
\begin{aligned}
\flat :&\mathcal{T}_0^1(M) \to \mathcal{T}_1^0(M) \\
&X \mapsto \flat X \equiv X^\flat \quad : \quad (\flat X)(Y) = g(X, Y)
\end{aligned}
$$

and its inverse the **sharp** map:

$$
\begin{aligned}
\sharp :&\mathcal{T}_1^0(M) \to \mathcal{T}_0^1(M) \\
&\omega \mapsto \sharp\omega \equiv \omega^\sharp \quad : \quad g(\sharp\omega, Y) = \omega(Y)
\end{aligned}
$$

And for arbitrary tensor fields we naturally define:

$$
\begin{aligned}
\flat^i :&\mathcal{T}_q^r(M) \to \mathcal{T}_{q+1}^{r-1}(M) \\
&t \mapsto \flat^i t : \flat^i t(\ldots, \omega_{i-1}, \omega_{i+1}, \ldots, v_1, \ldots, v_q, v_{q+1}) = t(\ldots, \omega_{i-1}, \flat v_{q+1}, \omega_{i+1}, \ldots)
\end{aligned}
$$

and

$$
\begin{aligned}
\sharp_i :&\mathcal{T}_q^r(M) \to \mathcal{T}_{q-1}^{r+1}(M) \\
&t \mapsto \sharp_i t : \sharp_i t(\omega_1, \ldots, \omega_r, \omega_{r+1}, \ldots, v_{i-1}, v_{i+1}, \ldots) = t(\ldots, v_{i-1}, \sharp\omega_{r+1}, v_{i+1}, \ldots)
\end{aligned}
$$

This are, of course, the index-free definitions for the usual **lowering** and **rising of indices** and are sometimes called the **musical operators**, note that we have used the convention that new indices appear at the rightmost place. It is obvious that the maps $\flat$ and $\sharp$ can we regarded as $\mathcal{T}_2^0(M)$ and $\mathcal{T}_0^2(M)$ tensor fields respectively; indeed by definition we see that $\flat \cong g \in \mathcal{T}_2^0(M)$ and so we define the **inverse metric** as $\sharp \cong \gamma \in \mathcal{T}_0^2(M)$. Note that both tensors are symmetric and hence any operations performed on one of the indices will be equivalent for the other. From the fact that $\flat \circ \sharp = id = \sharp \circ \flat$ we find the following identities:

$$
\sharp_1 g = \delta \quad , \quad \flat_1 \gamma = \delta \quad , \quad C_1^1(g \otimes \gamma) = \delta
$$

From the above we see that if we write in a basis $g = g_{\mu\nu}\theta^\mu \otimes \theta^\nu$ and $\gamma = g^{\mu\nu} e_\mu \otimes e_n u$ then the components will be related by the inverse matrix relation $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$. With the aid of these explicit tensors it is also possible to rewrite the flat and sharp maps for an arbitrary tensor $t \in \mathcal{T}_q^r(M)$ as:

$$\flat^i t = C^i_{q+1}(t \otimes g) \quad , \quad \sharp_i t = C^{r+1}_i (t \otimes \gamma)$$

It is worth noticing at this point that the several operations defined thus far interact in a rather complicated manner when composed. A convention regarding multiple compositions of contractions, swaps, flats and sharps may be adopted as follows: *Let $A \circ B \circ \cdots \circ C$ be a sequence of operations performed on a tensor field $t \in \mathcal{T}_q^r(M)$ the result will be written $AB \ldots Ct$ as opposed to $A'(B'(\ldots C(t) \ldots))$. We need to carefully distinguish between the two notations since the in the first version all operations are defined for the same type of tensor (r, q) whilst in the second, nested version each operation is defined for a potentially different type of tensor.* Therefore, **commutativity** of these operations can be discussed but care should be taken with the potential relabelling of the indices identifying the operations. In this convention, and for values of the indices such that the expressions are defined, we have:

- $C^i_j C^m_n = C^m_n C^i_j$
- $\flat^i \sharp_j = \sharp_j \flat^i$
- $\flat^i C^m_n = C^m_n \flat^i$
- $\sharp_i C^m_n = C^m_n \sharp_i$

We emphasise again that these pairs of operations should be taken as acting *at once* on the argument and not one after the other, following the convention above. Although it is possible to incorporate the swap operators to this formalism the practical value of doing such a thing is highly diminished by the complications arising from the convention used. Therefore, for long-winded computations of tensor expressions the **intrinsic notation** introduced in this text is suggested for all steps of computations that do not involve swap operations and then the **abstract index notation** or **Penrose's diagrammatic notation** is encouraged for the remaining steps involving swaps.

Let us now consider the natural derivatives of the tensor algebra of a Riemannian manifold:

$$\nabla_X : \mathcal{T}_q^r(M) \to \mathcal{T}_q^r(M)$$
$$\mathcal{L}_X : \mathcal{T}_q^r(M) \to \mathcal{T}_q^r(M)$$

where ∇ is the torsion-free, metric-compatible Levi-Civita connection. The important common feature of these operations is that they satisfy the product rule for $\otimes$ and they commute with contractions. Also, when these are defined, it is noted that from the primary objects - the connection for ∇ and the Lie bracket of vector fields for $\mathcal{L}$ - and their defining properties, it is possible to find the the action of the derivatives on arbitrary tensors. Employing this construction, and the fact that the connection is torsion-free, it is possible to express the Lie derivative of an arbitrary tensor field in terms of its covariant derivative. For a vector, a covector and a covariant tensor we have:

- $\mathcal{L}_X Y = [X,Y] = \nabla_X Y - \nabla_Y X$
- $\mathcal{L}_X \omega(Y) = \nabla_X \omega(Y) + \omega(\nabla_Y X)$
- $\mathcal{L}_X t(Y_1, T_2, \dots) = \nabla_X t(Y_1, Y_2, \dots) + t(\nabla_{Y_1} X, Y_2, \dots) + t(Y_1, \nabla_{Y_2} X, \dots) + \cdots$

Recall that, contrary to the case for $\mathcal{L}_X$, the covariant derivative ∇_X can be used to define a derivation that upgrades covariant index of a tensor. When restricted to differential forms, a similar operator is found and so we have two derivations:

$$\nabla : \mathcal{T}^r_q(M) \to \mathcal{T}^r_{q+1}(M)$$
$$d : \mathcal{A}_q(M) \to \mathcal{A}_{q+1}(M)$$

And, again making use of the Levi-Civita connection properties, we find for any differentiable form $\eta \in \mathcal{A}_q(M)$:

$$d\eta = \mathcal{A}(\nabla\eta)$$

Finally, let us consider the interactions of the covariant derivative with all the tensor operators defined above. It is important to bear in mind the **covariant derivative index convention** which reads: *The additional covector argument arising from ∇ is always defined to be the rightmost one. If more than one ∇ appear in an expression then the arguments will be assigned reading from the rightmost in a left-right order. For example ,the tensor $\nabla t \otimes \nabla s$ will act on all its arguments as* $(\nabla t \otimes \nabla s)(\dots, X, Y) = \nabla_X t(\dots) \cdot \nabla_Y s(\dots)$. Under this convention we find the following identities:

- $\nabla C^i_j = \nabla C^i_j$ if j is not the rightmost index,
- $\nabla S^\sigma = S^\sigma \nabla$
- $\nabla S_\sigma = S_\sigma \nabla$ if σ does not permute the rightmost index,
- $\nabla \delta = 0$, $\mathcal{L}_X \delta = 0$
- $\nabla g = 0 \Rightarrow \nabla \gamma = 0$
- $\nabla \flat^i = \flat^i \nabla$
- $\nabla \sharp_i = \sharp_i \nabla$.

An Example of Computation: The Relativistic Perfect Fluid

If we are presented the stress-energy tensor of a perfect fluid as:

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$$

with $u^a u_a = -1$ and we are asked to show the implication:

$$\nabla_c T^c_a = 0 \Rightarrow \left\{ \begin{array}{c} u^c \nabla_c \rho + (\rho + p) \nabla_c u^c = 0 \\ (\rho + p) u^c \nabla_c u^a + \nabla^a p + (u^c \nabla_c p) u^a = 0 \end{array} \right.$$

we can restate this problem in intrinsic notation as follows:

$$\mathcal{T}_1^1(M) \ni T = q(u \otimes u^\flat) + p\delta$$

with $q = \rho + p$ and $g(u,u) = -1$ which implies $g(u, \nabla_u u) = 0$. The divergence of a (r,q)-tensor is defined as $\mathrm{div}t = C^r_{q+1}\nabla t$ and so the implication now reads:

$$\mathrm{div}T = 0 \Rightarrow \begin{cases} \nabla_u \rho + q\mathrm{div}u = 0 \\ q\nabla_u u + \nabla p^\sharp + (\nabla_u p)u = 0 \end{cases}$$

We simply compute the divergence of T carefully following the rules presented in this paper, first simply the product rule:

$$\nabla T = \nabla q \otimes u \otimes u^\flat + q \otimes \nabla u \otimes u^\flat + q \otimes u \otimes \nabla u^\flat + \nabla p \otimes \delta + p\nabla\delta$$

recall that $\nabla\delta = 0$. We further contract the required indices and using the properties of contraction we write:

$$C_2^1\nabla T = u^\flat \otimes C_1^1(u \otimes \nabla q) + q(C_1^1\nabla u) \otimes u^\flat + qC_2^1(u \otimes \nabla u^\flat) + C_2^1(\nabla p \otimes \delta)$$

recovering the definition of divergence and the action of the delta tensor we write:

$$\mathrm{div}T = C_2^1\nabla T = (\nabla_u q)u^\flat + (q\mathrm{div}u)u^\flat + q\nabla_u u^\flat + \nabla p$$

This is indeed a covector field and so it being vanishing is equivalent to yielding zero when acted on any argument. In particular take u and write $\mathrm{div}T(u) = 0$:

$$\mathrm{div}T(u) = (\nabla_u q)u^\flat(u) + (q\mathrm{div}u)u^\flat(u) + q(\nabla_u u^\flat)(u) + \nabla p(u) = 0$$

Now recall that $u^\flat(u) = g(u,u) = -1$, also $\nabla p(u) = \nabla_u p$, and by definition of covariant derivative $(\nabla_u u^\flat)(u) = u(u^\flat(u)) - u^\flat(\nabla_u u) = g(u,u) - g(u, \nabla_u u) = -1$, then rewriting $q = \rho + p$ we find the fist equation:

$$\mathrm{div}T(u) = 0 \Rightarrow \nabla_u\rho + (\rho + p)\mathrm{div}u = 0$$

If we plug this identity in the initial equation $\mathrm{div}T = 0$ we are left with:

$$\mathrm{div}T = (\nabla_u p)u^\flat + q\nabla_u u^\flat + \nabla p = 0$$

Now simply applying sharp to the equality we get:

$$(\nabla_u p)\sharp\flat u + q\sharp\nabla_u\flat u + \sharp\nabla p = 0$$

and finally, since the covariant derivative commutes with the musical operators, we find the desired equation:

$$(\nabla_u p)u + (\rho + p)\nabla_u u + \sharp\nabla p = 0.$$