SMALL FORCINGS AND COHEN REALS

Jindřich Zapletal

The Pennsylvania State University

Abstract. We show that all posets of size $\aleph_1$ may have to add a Cohen real and develop some forcing machinery for obtaining this sort of results.

0. Results.

Theorem A. *Cons (ZFC) implies Cons (ZFC+every forcing of size $\aleph_1$ adds a Cohen real).*

The proof of the Theorem A is a routine iteration argument based on the following two theorems which the author considers interesting in their own right.

Theorem B. *If P is an $\aleph_0$-distributive forcing of size $\aleph_1$ then there is a proper forcing Q such that $Q \Vdash$ "$\check{P}$ is nowhere $\aleph_0$-distributive".*

Remark 1. *Actually we get $Q \Vdash$"$RO(\check{P}) = RO(Coll(\omega, \omega_1))$". Also if CH holds then our Q will be $\aleph_1$-centered.*

Corollary 1. *PFA proves that there are no $\aleph_0$-distributive forcings of size $\aleph_1$.*

Proof. Assume for contradiction that PFA holds and $|P| = \aleph_1$, P is $\aleph_0$-distributive. The Theorem B yields a Q proper, $Q \Vdash$"$\langle \dot{O}_n : n \in \omega \rangle$ is a family of dense open subsets of P with empty intersection". For $n \in \omega$, $p \in P$ we define $D_{n,p} = \{q \in Q : \exists p' \leq p \ q \Vdash$"$p' \in \dot{O}_n$"$\}$ and $D_{\omega,p} = \{q \in Q : \exists n \in \omega \ q \Vdash$ "$\check{p} \notin \dot{O}_n$"$\}$. Now $\langle D_{n,p}, D_{\omega,p} : n \in \omega, p \in P \rangle$ is a family of $\aleph_1$-many open dense subsets of Q and by PFA there is a filter $F \subset Q$ meeting all of them. The reader can verify that if $\dot{O}_n/F = \{p \in P : \exists q \in F \ q \Vdash$"$\check{p} \in \dot{O}_n$"$\}$ then $\langle \dot{O}_n/F : n \in \omega \rangle$ is a family of open dense subsets of P with empty intersection, contradiction.

Theorem C. *If P is a forcing adding a real then there is a c.c.c. forcing Q such that $Q \Vdash$ "$\check{P}$ adds a Cohen real".*

Remark 2. *If $|P| = \kappa$ then we obtain $|Q| = \kappa$ and in any case our Q satisfies the Knaster condition.*

Corollary 2. *If P is nowhere $\aleph_0$-distributive then there is a c.c.c. Q such that $Q \Vdash$"$\check{P}$ adds a Cohen real".*

Proof. Let $P \Vdash$"$\langle \dot{\alpha}_n : n \in \omega \rangle \subset \kappa$ is a new ω-sequence of ordinals". Then if Q_0 is the forcing adding κ Cohen reals we have $Q_0 \Vdash$ "$\check{P}$ adds a real", namely the real coding the sequence of $\dot{\alpha}_n$-th Cohen reals. By the Theorem C, there is a c.c.c. $\dot{Q}_1 \in V^{Q_0}$ such that $Q_0 \Vdash \dot{Q}_1 \Vdash$"$\check{P}$ adds a Cohen real". $Q_0 * \dot{Q}_1$ is c.c.c. and we are done.

Corollary 3. *$MA_{<\kappa}$ proves that all nowhere $\aleph_0$-distributive forcings of size $< \kappa$ add a Cohen real.*

This uses something specific from the proof of the Theorem C and so we postpone the demonstration to the Section 2.

Question 1. *What if we want to embed more complicated forcings? For instance, if $C_{\aleph_1}$ denotes the BA adding $\aleph_1$ Cohen reals, is it consistent that every forcing of size $\aleph_1$ embeds $C_{\aleph_1}$?*

To show that the mechanism of giving a positive answer to this question would be different from the proof of the Theorem A we give

Example 1. *$MA_{\aleph_1}$ is consistent with existence of a forcing of size $\aleph_1$ adding a real but not embedding $C_{\aleph_1}$.*

Question 2. *What if we want to embed Cohen real into bigger forcings? Our proof of the Theorem B is specific for $\aleph_1$. Is it consistent that all forcings of size $\aleph_2$ add a Cohen real?*

1. Proof of Theorem B.

We use two simple lemmas.

Lemma 1. *If P is an $\aleph_0$-distributive forcing of size $\aleph_1$ then there is a normal tree T of height ω_1 such that $T \subset P$ is dense. (The order of T is inherited from P.)*

Lemma 2. *Forcings of the form $R * \dot{S}$, where R is the Cohen real and $R \Vdash$ "$\dot{S}$ is σ-closed", do not add new branches to trees of height ω_1 in the ground model.*

Remark 3. *Lemma 1 is specific for $\aleph_1$. Lemma 2 holds true for many other standard generic reals in place of R. Generalizations are left to the reader.*

Granted the lemmas, we prove the Theorem B: fix P, an $\aleph_0$-distributive forcing of size $\aleph_1$. Due to Lemma 1, we can chose $T_0 \subset P$, a dense normal tree of height ω_1. We construct $Q = Q_0 * \dot{Q}_1 * \dot{Q}_2$:

(0) Q_0 is the Cohen real.
(1) $\dot{Q}_1 \in V_{Q_0}$ is the following set: $\{\langle s, A\rangle : s \in [T_0]^{\aleph_0}$, A is a countable set of cofinal branches of $T_0\}$.

Ordering is defined by $\langle s_0, A_0\rangle \geq \langle s_1, A_1\rangle$ if $s_0 \subset s_1$, $A_0 \subset A_1$ and $\forall b \in A_0$ $b \cap s_1 \subset s_0$. Obviously, $\dot{Q}_1$ is σ-closed in V^{Q_0}. Now if $\dot{G}_0 * \dot{G}_1$ is the canonical term for a $Q_0 * \dot{Q}_1$-generic filter (with the obvious meaning) we set $T_1 \in V^{Q_0 * \dot{Q}_1}$ to be $\bigcup\{s : \exists q \in \dot{G}_1\ q = \langle s, A\rangle$ for some $A\}$.

Claim 1. *$Q_0 * \dot{Q}_1 \Vdash$ "$\dot{T}_1 \subset \check{T}_0$ is a dense subtree without cofinal branches".*

Given the claim,

(2) $\dot{Q}_2 \in V^{Q_0 * \dot{Q}_1}$ is the standard c.c.c. forcing specializing T_1 ([She]).

Obviously, $Q \Vdash$ "P adds a cofinal branch through a special tree of height ω_1" and so $Q \Vdash$ "$\check{P}$ collapses $\aleph_1$". For future reference we record

Lemma 3. *If Q is as above then*

(1) *Q is proper*
(2) *(CH) Q is $\aleph_1$-centered*
(3) *(GCH) $|Q| = \aleph_2$*
(4) *Q has $\aleph_2$-p.i.c.* ([She Ch. VIII])

As for the Remark 1, note that all complete BA's of density $\aleph_1$ collapsing $\aleph_1$ are isomorphic to $\mathrm{RO}(Coll(\omega,\omega_1))$.

Proof of the Lemma 1. Let $P = \langle p_\alpha : \alpha < \omega_1\rangle$ be an $\aleph_0$-distributive forcing. We construct $T \subset P$ by induction on its levels. The induction hypothesis at $\alpha < \omega_1$ is that we have constructed $T^\beta \subset P, \beta < \alpha$ so that:

(1) T^β is a tree of height β
(2) $\beta' < \beta < \alpha$ implies that $T^{\beta'}$ constitutes precisely the first β' levels of T^β
(3) the levels of T^β's are maximal antichains in $P, \leq$
(4) $\beta + 1 < \alpha$ implies $\exists t \in T^{\beta+1}\ t \leq p_\beta$.

How do we proceed with the induction?

(1) α limit. Set $T^\alpha = \bigcup_{\beta<\alpha} T^\beta$ and the induction hypotheses continue to hold.
(2) $\alpha = \beta + 1$. As P is $\aleph_0$-distributive, $D = \{p \in P : \forall \beta' < \beta\ \exists t$ in the β'-th level of $T^\beta\ p \leq t\}$ is open dense in P. (Remember (3)!) Choose $A \subset D$, a maximal antichain such that there is $t \in A$, $t \leq p_\beta$. Set $T^\alpha = T^\beta \cup A$ and the induction hypotheses again continue to hold.

Finally, set $T = \bigcup_{\alpha<\omega_1} T^\alpha$. Checking the needed properties of T is trivial and we leave it to the reader.

Proof of the Lemma 2. Fix $\dot S$, $R \Vdash$"$\dot S$ is σ-closed", and T, a tree of height ω_1. Assume $\langle r_0, \dot s_0\rangle \in R * \dot S$, $\dot b$ are such that $\langle r_0, \dot s_0\rangle \Vdash$"$\dot b$ is a new cofinal branch through $\check T$".

Claim 2. *For all $r_1, \dot s_1, t$ such that $r_1 \leq r_0$, $r_1 \Vdash \dot s_1 \leq \dot s_0$ and $\langle r_1, \dot s_1\rangle \Vdash$ "$\check t \in \dot b$" there is $\alpha < \omega_1$, $r_2 \leq r_1$ and $\langle \dot s^i, t^i : i \in \omega\rangle$ such that t^i's are distinct elements of the α-th level of T, $r_2 \Vdash_R$ "$\dot s^i \leq \dot s_1$" and $\langle r_2, \dot s^i\rangle \Vdash$ "$\check t^i \in \dot b$".*

Proof. Let $r_1, \dot s_1, t$ witness the failure of the claim. Then immediately $T_0 = \{x \in T : \exists \langle r_2, \dot s_2\rangle \leq \langle r_1, \dot s_1\rangle\ \langle r_2, \dot s_2\rangle \Vdash \check x \in \dot b\}$ is a tree of height ω_1 and all levels countable. Also $\langle r_1, \dot s_1\rangle \Vdash$"$\dot b \subset \check T_0$ is a new cofinal branch". Now R does not add new cofinal branches to T_0 and in V^R, S does not add new branches to T_0 either, since S is σ-closed and T_0 is still an ω_1-tree there. So $\langle r_1, \dot s_1\rangle \Vdash$ "$\dot b \in V$", a contradiction.

Now fix θ large regular and $\langle M_i : i < \omega\rangle$, a sequence of countable submodels of H_θ such that $M_i \subset M_{i+1}, \dot b, \langle r_0, \dot s_0\rangle$ in M_0 and let $\alpha_i = M_i \cap \omega_1$. Enumerate intersections of α_i-th levels of T with M_{i+1} by $\langle x_i^j : j \in \omega\rangle$. Fix $G \subset R$ generic and work in $V[G]$.

We let $f : \omega \to \omega$ be unbounded with respect to functions in V. By the Claim 2, there are $\langle r_1, \dot s_1\rangle \leq \langle r_0, \dot s_0\rangle$, $r_1 \in G$, and $j_0 \in \omega$, $j_0 > f(0)$ such that $\langle r_1, \dot s_1\rangle \Vdash$ "$\check x_0^{j_0} \in \dot b$". By elementarity we can find this $\langle r_1, \dot s_1\rangle$ in M_1. Now using this argument repeatedly together with the Claim 2, by induction on $i \in \omega$ we can build a sequence $\langle\langle r_i, \dot s_i\rangle, j_i : i \in \omega\rangle$ such that $r_i \in G$, $j_i > f(i)$, $\langle r_i, \dot s_i\rangle$ in M_i, $\langle r_{i+1}, \dot s_{i+1}\rangle \leq \langle r_i, \dot s_i\rangle$, $\langle r_{i+1}, \dot s_{i+1}\rangle \Vdash$"$\check x_i^{j_i} \in \dot b$". Since $\dot S/G$ is σ-closed, the decreasing sequence of conditions $\langle \dot s_i/G : i \in \omega\rangle$ has a lower bound. Let $\dot s_\omega$ be an R-name for it.

Back in V, let $\alpha = sup_{i\in\omega}\alpha_i$ and find any x in the α-th level of T and $\langle r_{\omega+1}, \dot s_{\omega+1}\rangle \leq \langle r_0, \dot s_\omega\rangle$ with $\langle r_{\omega+1}, \dot s_{\omega+1}\rangle \Vdash$ "$\check x \in \dot b$". Then we define $g : \omega \to \omega$ by $g(i) = j$ if $x_i^j > x$. This function should not be bounded by any function in V, since it is forced to be greater than our $f \in V[G]$. However, g is clearly in V, and we have obtained a contradiction.

Proof of the Claim 1. Work in V^{Q_0}. The density of $\dot{T}_1 \subset \check{T}_0$ is clear since countably many branches cannot cover all of $T_0 \restriction t$ for any $t \in T_0$. Let $p \in Q_1$, $\dot{b}$ be such that $p \Vdash_{Q_1}$ "$\dot{b} \subset \dot{T}_1$ is a cofinal branch". We distinguish two cases.

(1) $\exists p_0 \leq p \ \forall p_1, p_2 \leq p_0$, $t_1, t_2 \in T$, if $p_1 \Vdash$"$\check{t}_1 \in \dot{b}$" and $p_2 \Vdash$"$\check{t}_2 \in \dot{b}$" then t_1, t_2 are compatible. Then $c = \{t \in T : \exists t_1 \leq t, p_1 \leq p_0 \ p_1 \Vdash$"$\check{t}_1 \in \dot{b}$"$\}$ is a cofinal branch through T and $p_0 \Vdash$"$\dot{b} \subset \check{c}$". So if $p_0 = \langle s, A \rangle$, we may set $p_1 = \langle s, A \cup \{c\} \rangle$ to obtain $p_1 \leq p_0$, $p_1 \Vdash$"$\dot{T}_1 \cap \check{c} \subset \check{s}$" and therefore $p_1 \Vdash$"$\dot{b} \subset \check{s}$ and thus $\dot{b}$ is not cofinal", contradiction.

(2) Otherwise. Then setting $\dot{c} = \{t \in T_0 : \exists t_1 \leq t, t_1 \in \dot{b}\}$ we have $p \Vdash$"$\dot{c} \notin V$ is a cofinal branch through $\check{T}_0$" contradicting the Lemma 2 for iteration $Q_0 * Q_1$ and the tree T_0.

We have a contradiction in both cases and the Claim is proven.

Proof of the Lemma 3.

(1) Q is an iteration of three proper forcings.

(2) If CH holds, the centeredness of Q follows from some cardinal arithmetic. Let $\langle \tau_\alpha : \alpha < \omega_1 \rangle$ enumerate the Q_0-names for elements of $[T_0]^{\aleph_0} \in V^{Q_0}$. Let $F_{\sigma,\alpha,q} = \{\langle \sigma, \langle \tau_\alpha, \eta \rangle, q \rangle : \sigma \Vdash_{Q_0}$ "$\langle \tau_\alpha, \eta \rangle \in \dot{Q}_1$", $\langle \sigma, \langle \tau_\alpha, \eta \rangle \rangle \Vdash_{Q_0 * \dot{Q}_1}$ "$\check{q} \in \dot{Q}_2$"$\}$ for $\sigma \in {}^{<\omega}2$, $\alpha < \omega_1$, $dom(q) \in [T_0]^{<\omega}$, $q : dom(q) \to \omega$. Then there are $\aleph_1$-many $F_{\sigma,\alpha,q}$'s, each $F_{\sigma,\alpha,q} \subset Q$ is a centered system and $\bigcup_{\sigma,\alpha,q} F_{\sigma,\alpha,q} \subset Q$ is dense and the $\aleph_1$-centeredness of Q follows.

(3) $Q_0 \Vdash$"all cofinal branches of $\check{T}_0$ are in V" and by GCH there are only $\aleph_2$-many of them. Again by GCH, one can enumerate all Q_0-names for pairs $\langle s, A \rangle$ as in the definition of Q_1 by $\langle \tau_\alpha : \alpha < \omega_2 \rangle$. Then $D = \{\langle \sigma, \tau_\alpha, q \rangle : \sigma \in {}^{<\omega}2$, $\alpha < \omega_2$, $dom(q) \in [T_0]^{<\omega}$, $q : dom(q) \to \omega$, $\sigma \Vdash_{Q_0}$"$\tau_\alpha \in \dot{Q}_1$", $\langle \sigma, \tau_\alpha \rangle \Vdash_{Q_0 * \dot{Q}_1}$"$\check{q} \in \dot{Q}_2$"$\} \subset Q$ is dense. By the cardinal arithmetic and $\aleph_2$-c.c. of Q we have $|RO(Q)| = \aleph_1$ and since $Q \subset RO(Q)$ we are done.

(4) Let us repeat what this means.

Definition. [She,Ch.VIII,§2] *Q has ω_2-p.i.c. if for some θ large regular, $\Delta \in H_\theta$, for every $i < j < \omega_2$, q, h, M_i, M_j countable submodels of $\langle H_\theta, \in, \Delta \rangle$ with $i \in M_i$, $j \in M_j$, $Q \in M_i \cap M_j$, $M_i \cap i = M_j \cap j$, $M_i \cap \omega_2 \subset j$, $q \in Q \cap M_i$, $h : M_i \to M_j$ an isomorphism which is identity on $M_i \cap M_j$ there is $q' \leq q$, a master condition for M_i such that $q' \Vdash$ "$h''(\check{M}_i \cap \dot{G}) = \check{M}_j \cap \dot{G})$".*

To prove this we show that Q_0, $\dot{Q}_1$, $\dot{Q}_2$ have ω_2-p.i.c. in the respective models and by [She,Ch.VIII,Lemma 2.3] we will be finished. Certainly Q_0 has ω_2-p.i.c. since for any isomorphism h as in the definition of p.i.c. $h \restriction Q_0 = id$. For $\dot{Q}_1$ work in V^{Q_0} and fix q, h, M_i, M_j as in the definition of p.i.c. with $T_0 \in M_i \cap M_j$. Choose $q = q_0 = \langle s_0, A_0 \rangle \geq q_1 = \langle s_1, A_1 \rangle \geq \cdots \geq q_i = \langle s_i, A_i \rangle \geq \ldots, i \in \omega$, a strongly generic sequence for M_i. Set $q_\omega = \langle s_\omega, A_\omega \rangle$, where $s_\omega = \bigcup_{i \in \omega} s_i$ and $A_\omega = \bigcup_{i \in \omega} A_i$. Also set $q_{\omega+1} = \langle s_\omega, A_\omega \cup h'' A_\omega \rangle$. Then $q \geq q_\omega \geq q_{\omega+1}$ and we claim that q_{ω_1} is what we are looking for. Certainly it is a master condition for M_i. It is enough to show $q_{\omega+1} \Vdash$"$\check{h}''(\check{M}_i \cap \dot{G}) \subset (\check{M}_j \cap \dot{G})$". Now if $r = \langle s, A \rangle \in M_i \cap Q_1$ then $\exists q_{\omega+2} \leq q_{\omega+1} \ q_{\omega+2} \Vdash$"$\check{r} \in \dot{G}$" iff $q_{\omega+1} \Vdash$ "$\check{r} \in \dot{G}$" iff $r \geq q_{\omega_1}$ iff $\forall b \in A \ b \cap s_\omega \subset s$ iff $\forall b \in A \ h(b) \cap s_\omega \subset s$ iff $h(r) = \langle h(s) = s, h(A) = h'' A \rangle \geq q_{\omega+1}$. Here the first and second equivalences hold by the strong genericity of $q_{\omega+1}$ and the fourth is due to the fact that $s \subset s_\omega \subset M_i \cap M_j$ and for $b \in M_i$, $b \subset T_0$ a cofinal branch we have

$b \cap M_i \cap M_j = h(b) \cap M_i \cap M_j$ as h is an isomorphism identical on $M_i \cap M_j$. We are finished for $\dot{Q}_1$ and the case of $\dot{Q}_2$ is easy again: ω_2-p.i.c. follows from the c.c.c. of Q_2 and from $|Q_2| = \aleph_1$. (Any h as in the Definition has to be identical on $M_i \cap Q_2$.)

2. Proof of the Theorem C. Fix $P, \dot{r}$ such that $P \Vdash$ "$\dot{r} \in {}^{\omega}2 \setminus V$". We define Q as the set of ordered pairs $\langle f, g \rangle$ satisfying the following conditions:

1. $\exists n \in \omega \; dom(f) = n$. n is called the *height* of the condition.
2. $\forall i < n \; f(i) = \langle I_i, W_i \rangle$, where $I_i, i < n$ are subsequent finite intervals of ω and $W_i \subset {}^{I_i}2$.
3. $dom(g) \in [P]^{<\omega}$.
4. $\forall p \in dom(g) \; g(p) \in {}^{\leq n}2$, p decides $\dot{r} \restriction \bigcup_{i<lth(g(p))} I_i$ and $\forall \sigma \in {}^{n}2 \; g(p) \subset \sigma$ implies $\exists p' \leq p \; \forall i < n \; p' \Vdash$ "$\dot{r} \restriction \check{I}_i \in \check{W}_i$ iff $\sigma(i) = 1$".

The order is by coordinatewise extension.

Motivation. The first coordinates will generically compose a sequence $\langle I_i, W_i : i < \omega \rangle$ and the future Cohen real will then be read off $\dot{r}$ as $\dot{c}(i) = 1$ iff $\dot{r} \restriction I_i \in W_i$. The second coordinate is (approximately) a finite fragment of a future projection of P into the Cohen real algebra.

Lemma 4. *Q is c.c.c.*

Proof. We aim for the Knaster condition of Q. Let $\langle q_\alpha = \langle f_\alpha, g_\alpha \rangle : \alpha < \omega_1 \rangle$ be a sequence of conditions in Q. We can thin this sequence out to $\langle q_\alpha : \alpha \in S \rangle$ for some $S \subset \omega_1$ of full cardinality such that $|\{f_\alpha : \alpha \in S\}| = 1$, $\{dom(g_\alpha) : \alpha \in S\}$ is a Δ-system with root r and $|\{g_\alpha \restriction r : \alpha \in S\}| = 1$. (First we use countability of the set of candidates for f_α's, then a Δ-system argument on $dom(g_\alpha)$'s and finally countability of the set of candidates for $q_\alpha \restriction r$'s.) By the definition of Q, if α_0, $\alpha_1 \in S$ then $q_{\alpha_0}, q_{\alpha_1}$ are compatible: their common lower bound is $\langle f_{\alpha_0}, g_{\alpha_0} \cup g_{\alpha_1} \rangle = \langle f_{\alpha_1}, g_{\alpha_0} \cup g_{\alpha_1} \rangle$. We are done.

For future reference notice that if $|P| = \aleph_1$ then $|Q| = \aleph_1$ and Q has ω_2-p.i.c.

Let $H \subset Q$ be generic. In $V[H]$, set $F = \bigcup\{f : \exists \langle f, g \rangle \in H\}$, $G = \bigcup\{g : \exists \langle f, g \rangle \in H\}$.

Lemma 5.

1. *$dom(G) \subset P$ is dense.*
2. *$\forall p_0, p_1 \in dom(G)$ if p_0, p_1 are compatible in P then $G(p_0)$, $G(p_1)$ are compatible elements of ${}^{<\omega}2$.*
3. *$\forall p_0 \in dom(G) \; \forall \sigma \in {}^{<\omega}2$ if $G(p_0) \subset \sigma$ then $\exists p_1 \leq p_0 \; p_1 \in dom(G)$ and $\sigma \subset G(p_1)$.*

Granted the lemma, we show how in $V[H]$, P adds a Cohen real: if $K \subset P$ is generic over $V[H]$, set $c = \bigcup_{p \in H \cap dom(G)} G(p)$. (2) makes sure that this is a function. c is Cohen over $V[H]$: let $p \in P$, $D \subset {}^{<\omega}2$ open dense. Using (1), find $p_0 \leq p$, $p_0 \in dom(G)$. There is $\sigma \in D$ extending $G(p_0)$. By (3) we can find $p_1 \leq p_0$ such that $\sigma \subset G(p_1)$; thus $p_1 \Vdash_P$ "$\dot{c}$ meets $\check{D}$" and by genericity we are done.

To verify the claim of the Corollary 3, note first that if $MA_{<\kappa}$ holds and $|P| = \lambda < \kappa$ and P is nowhere $\aleph_0$-distributive then P adds a real. (This is because P adds a countable sequence to λ and as $2^{\aleph_0} > \lambda$ this new sequence can be coded over V by a real, which then has to be new as well.) Now we know that then there is a c.c.c. poset Q adding a function G with properties described in the Lemma 4. It is a simple exercise in counting necessary open dense subsets of Q to show that then G with these properties exists in V. The same proof as above then shows that in V, P adds a Cohen real.

Proof of the Lemma 5.

(1) Let $q = \langle f, g\rangle \in Q$, $dom(f) = n \in \omega$ and $p \in P$. Find $p' \leq p$, p' deciding $\dot{r} \restriction \bigcup_{i<n} I_i$, where for $i < n$ $f(i) = \langle I_i, W_i\rangle$. If $\sigma \in {}^n2$ is such that $\sigma(i) = 1$ iff $p' \Vdash$ "$\dot{r} \restriction \check{I}_i \in \check{W}_i$" one can easily check that $q' = \langle f, g \cup \{\langle p', \sigma\rangle\}\rangle \in Q$, $q' \leq q$ and $q' \Vdash$"$p' \in dom(G)$," and density of $dom(G)$ follows.

(2) Easy.

(3) Choose $\langle f, g\rangle \in Q$, $p \in dom(g)$ and $g(p) \subset \sigma \in {}^{<\omega}2$. First we show

Claim 3. *For every $n \in \omega$ there is $\langle f', g\rangle \leq \langle f, g\rangle$ such that $n \subset dom(f')$.*

Proof. This can be proven by induction on $n \in \omega$. Obviously for $n = 0$ there is nothing to prove. Assume now that we have $\langle f', g\rangle \leq \langle f, g\rangle$ with $n \subset dom(f')$. If $n + 1 \subset dom(f')$ then $\langle f', g\rangle$ witnesses the claim for $n + 1$ and the induction step follows. So assume $n = dom(f')$. For each $p' \in dom(g)$, $\eta \in {}^n2$, $g(p') \subset \eta$ we choose $r_{\eta,p'} \leq p'$ such that $r_{\eta,p'} \Vdash_P$"$\forall i < n$ $\sigma(i) = 1$ iff $\dot{r} \restriction \check{I}_i \in \check{W}_i$" where for $i < n$ $f'(i) = \langle I_i, W_i\rangle$. This is possible by the definition of Q and the induction hypothesis. Now choose

$$\bigotimes_{\tau\in 2} K_{\eta,p',\tau} \subset \bigotimes_{\tau\in 2} P_{\eta,p',\tau}$$

generic, where $P_{\eta,p',\tau}$ are just distinct copies of P and $r_{\eta,p'} \in K_{\eta,p',\tau}$. Our initial assumption about $\dot{r}$ now gives

$$\dot{r}/K_{\hat{\eta},\hat{p}',\hat{\tau}} \notin V[\bigotimes_{\langle\eta,p',\tau\rangle\neq\langle\hat{\eta},\hat{p}',\hat{\tau}\rangle} K_{\eta,p',\tau}]$$

for any $\hat{\eta}, \hat{p}', \hat{\tau}$ and thus one can find I_n, a finite interval of ω starting at $\bigcup_{i<n} I_i$, such that $\dot{r}/K_{\eta,p',\tau} \restriction I_n$ are all different elements of ${}^{I_n}2$. This is possible since there are only finitely many reals to take care of. Here is the only place where we use the forced novelty of $\dot{r}$. Now we set $W_n = \{\dot{r}/K_{\eta,p',1} : \eta \in^n 2,\ p' \in dom(g)$ and $g(p') \subset \sigma\}$ and $f'' = f' \cup \langle n, \langle I_n, W_n\rangle\rangle$. The attentive reader can check that $\langle f'', g\rangle \in Q$ and thus finish the induction step on his own.

Given the claim we can easily complete the proof of (3): let n be the length of σ. Choose $\langle f', g\rangle \leq \langle f, g\rangle$ such that $n \subset dom(f') = m$. Choose $\eta \in {}^m2$, $\sigma \subset \eta$. By the definition of Q there is $p' \leq p$, p' deciding $\dot{r} \restriction \bigcup_{i<n} I_i$ and such that $\eta(i) = 1$ iff $p' \Vdash$ "$\dot{r} \restriction \check{I}_i \in \check{W}_i$" where for $i < n$ $f'(i) = \langle I_i, W_i\rangle$. Then as in (1) $\langle f, g\rangle \geq \langle f', g \cup \{\langle p', \eta\rangle\}\rangle \in Q$, and since $p' \leq p$, $\sigma \subset \eta$, (3) follows.

3. Proof of the Theorem A.

Let us start with a model of GCH. Fix $\langle x_\alpha : \alpha < \omega_2\rangle$, an enumeration with repetitions of objects of the form $x_\alpha = \langle A^\alpha_z : z \in \omega_1 \times \omega_1\rangle$, where $A_z \in [\{f : dom(f) \in [\omega_2]^{\aleph_0}, rng(f) \subset \omega_2\}]^{\aleph_1}$. By induction on $\alpha < \omega_2$ we build a countable support iteration

$$P = \langle P_\alpha : \alpha \leq \omega_2, \dot{Q}_\alpha : \alpha < \omega_2\rangle$$

together with sequences $\langle \tau^i_\alpha : i < \omega_2\rangle$ with the following induction hypothesis: for $\beta < \alpha$

(1) $P_\beta \Vdash$"$\dot{Q}_\beta$ is a proper ω_2-p.i.c. poset of size $\aleph_2$ and we assume that the universe of $\dot{Q}_\beta$ is $\check{\omega}_2$."

(2) $|P_\beta| = \aleph_2$, P_β is $\aleph_2$-c.c., $P_\beta \Vdash GCH$
(3) $\langle \tau^i_\beta : i < \omega_2 \rangle$ is an enumeration of P_β-names for elements of $\dot{Q}_\beta$ (resp. elements of ω_2). Moreover, defining $\prec_\beta \subset \omega_1 \times \omega_1$ in V^{P_β} by $\gamma_0 \prec \gamma_1$ iff there is $f \in A^\beta_{\langle \gamma_0, \gamma_1 \rangle}$ such that g given by $g(\delta) = \tau^\delta_{f(\delta)}$ if $\delta \in dom(f)$ and $g(\delta) = 1$ otherwise, we have
(4) if $P_\beta \Vdash$"$\prec_\beta$ is an $\aleph_0$-distributive poset" then $\dot{Q}_\alpha \in V^{P_\alpha}$ is any proper ω_2-p.i.c. forcing of size $\aleph_2$ such that $P_\beta * \dot{Q}_\beta \Vdash$"RO($\langle \omega_1, \prec_\beta \rangle$)=RO($Coll(\omega, \omega_1)$)" (see the Theorem B and Lemma 3).
(5) if $P_\beta \Vdash$"$\prec_\beta$ is a poset adding a real" then $\dot{Q}_\beta \in V^{P_\beta}$ is any c.c.c. ω_2-p.i.c. forcing of size $\aleph_1$ such that $P_\beta * \dot{Q}_\beta \Vdash$"$\langle \omega_1, \prec_\beta \rangle$ adds a Cohen real" (see the Theorem C).
(6) otherwise. Then $P_\beta \Vdash \dot{Q}_\beta = 1$.

For α limit [She,Ch.VIII,§2] takes care about preservation of (2). By GCH and (2), there are only $\aleph_2$-many P_α-names for elements of ω_2 (or elements of $\dot{Q}_\alpha$) and (3) continues to hold. For (1),(4),(5) there is nothing to check. The successor step is handled similarly.

Now by [She] $P = P_{\omega_2}$ is a proper $\aleph_2$-c.c. notion of forcing. We show $P \Vdash$"all forcings of size $\aleph_1$ add a Cohen real". Let $p \in P$, $p \Vdash$"x is a poset with universe ω_1". W. l. o. g. either

(1) $p \Vdash$"x is $\aleph_0$-distributive", or
(2) $p \Vdash$"x adds a real"

since nowhere $\aleph_0$-distributive forcings af size $\aleph_1$ add reals. For the first case, by the ω_2-c.c. of P and preservation of $\aleph_1$ there is $dom(p) < \alpha < \kappa$ such that $p \Vdash_{P_\alpha}$"$\prec_\alpha$ is $\aleph_0$-distributive poset of size $\aleph_1$" and $p \Vdash_P$"$x = \prec_\alpha / G \cap P_\alpha$". But then $p \Vdash_P$"$V^{P_{\alpha+1}} \models$ RO(x)= RO($Coll(\omega, \omega_1)$)" and since the equality is absolute upwards as long as ω_1 stays in place the same holds in V^P. The second case is taken care of in the same way, observing that the formula "x adds a Cohen real" is absolute upwards.

This leaves us with only one thing to demonstrate, the Example 1. We define the following forcing $P : P = \{f : dom(f) \in \omega_1, rng(f) \subset {}^{<\omega}2\}$. The ordering is defined by $f \geq g$ if $dom(f) \subset dom(g)$, $\forall \beta \in dom(f)\ f(\beta) \subset g(\beta)$ and $\{\beta \in dom(f) : f(\beta) \neq g(\beta)\}$ is finite. As far as we know, P has not been explicitly defined before, so we list some of its simplest properties:

(1) The P-generic G is unambiguously given by $F : \omega_1 \to {}^\omega 2$, where $F(\beta) = \bigcup\{f(\beta) : f \in G, \beta \in dom(f)\}$. Each $F(\beta)$ is Cohen generic over the ground model.
(2) $|P| = 2^{\aleph_0}$.
(3) P is proper; actually, P embeds into (Cohen subset of ω_1 by countable conditions)$\times C_{\aleph_1}$.
(4) P embeds (Cohen subset of ω_1 by countable conditions).
(5) If Q is c.c.c. then $Q \Vdash$"$C_{\aleph_1}$ does not embed into $\check{P}$".

Only (2) and (5) are relevant for our purposes, and we leave the proof of the other items to the reader. Notice that the consistency statement in Example 1 follows immediately: just start with $V \models$CH and force $MA_{\aleph_1}$ by a c.c.c. poset. (2) and (5) together show that in the resulting model $|P^V| = \aleph_1$ and P^V does not embed $C_{\aleph_1}$. Obviously, P^V adds many new reals.

Now (2) is trivial; we concentrate on proving (5). For contradiction, assume we have a c.c.c. forcing Q, $q \in Q$, $p \in P$ and $\dot{h}$, a Q-name for a P-name such that $q \Vdash_Q p \Vdash_P$ "$\dot{h} : \omega_1 \to 2$ is $C_{\aleph_1}$-generic over V^Q". By induction on $\alpha < \omega_1$ we construct a sequence $\langle f_\alpha, s_\alpha, t_\alpha, i_\alpha, q_\alpha : \alpha < \omega_1 \rangle$ so that

(1) $f_\alpha \in P$, $s_\alpha \in [dom(f_\alpha)]^{<\omega}$, $t_\alpha : s_\alpha \to {}^{<\omega}2$, $i_\alpha \in 2$ and $q_\alpha \in Q$, $q_\alpha \leq q$.
(2) $f_0 = p$ and the f_α's are continuously increasing with respect to ordinary inclusion. Also $\forall \beta \in s_\alpha\ f_\alpha(\beta) \subset t_\alpha(\beta)$.
(3) For two functions k, l define $k \nearrow l$ to be $\{\langle x, y \rangle : x \in dom(k) \setminus dom(l), k(x) = y$ or $x \in dom(l),\ l(x) = y\}$. Then for each $\alpha < \omega_1$ we want $q_\alpha \Vdash_Q f_{\alpha+1} \nearrow t_\alpha \Vdash_P$ "$\dot{h}(\check{\alpha}) = i_\alpha$".

There is no problem in the induction. Once we are done, by a Fodor-style argument we find stationary $S \subset \omega_1$ such that $|\{s_\alpha : \alpha \in S\}| = 1$, $|\{t_\alpha : \alpha \in S\}| = 1$. Now Q is c.c.c. and so there is $q' \leq q$, $q' \Vdash_Q$ "$|\{\alpha \in S : q_\alpha \in \dot{K}\}| = \aleph_1$", where $\dot{K}$ is the term for a Q-generic. Once more by c.c.c.-ness of Q there is $\beta < \omega_1$ such that $q' \Vdash_Q$ "$\dot{Z} = \{\alpha \in S \cap \beta : q_\alpha \in \dot{K}\}$ is infinite". Now set $p' \in P$, $p' \leq p$ to be $f_\beta \nearrow t$, where t is the only element of $\{t_\alpha : \alpha \in S\}$. Then $q' \Vdash_Q p' \Vdash_P$ "$\forall \alpha \in \dot{Z}\ \dot{h}(\alpha) = i_\alpha$" and so $q' \Vdash_Q p' \Vdash_P$ "$\dot{h} \restriction \dot{Z} \in V^Q$". Since $\dot{Z} \in V^Q$ is an infinite set this contradicts our assumption about $\dot{h}$ being $C_{\aleph_1}$-generic over V^Q.

References

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Department of Mathematics, The Pennsylvania State University, University Park, PA 16802

E-mail address: zapletal@math.psu.edu