# Vagueness and Roughness

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**Abstract.** The paper proposes a new formal approach to vagueness and vague sets taking inspirations from Pawlak's rough set theory. Following a brief introduction to the problem of vagueness, an approach to conceptualization and representation of vague knowledge is presented from a number of different perspectives: those of logic, set theory, algebra, and computer science. The central notion of the vague set, in relation to the rough set, is defined as a family of sets approximated by the so called lower and upper limits. The family is simultaneously considered as a family of all denotations of sharp terms representing a suitable vague term, from the agent's point of view. Some algebraic operations on vague sets and their properties are defined. Some important conditions concerning the membership relation for vague sets, in connection to Blizard's multisets and Zadeh's fuzzy sets, are established as well. A classical outlook on a logic of vague sentences (vague logic) based on vague sets is also discussed.

## 1 Introduction

Logicians and philosophers have been interested in the problem area of vague knowledge for a long time, looking for some logical foundations of a theory of vague notions (terms) constituting such knowledge. Recently vagueness and, more generally - imperfection, has become the subject of investigations of computer scientists interested in the problems of AI, in particular, in the problems of reasoning on the basis of imperfect information and in the application of computers to support and represent such reasoning in the computer memory (see e.g. Parsons [15]).

Imperfection is considered in a general information-based framework, where objects are described by an agent in terms of attributes and their values. Bonissone and Tong [5] indicated three types of imperfections relating to information: *incompleteness*, *uncertainty* and *imprecision*. Incompleteness arises from the absence of a value of an attribute for some objects. Uncertainty arises from a lack of information; as a result, an object's attribute may have a finite set of values rather than a single value. Imprecision occurs when an attribute's value cannot be measured with adequate precision. There are also other classifications of imperfect information (e.g. Słowiński, Stefanowski [26]).

Marcus [12] thought of imprecision more generally. He distinguished e.g. such types of imprecision as *vagueness*, *fuzziness* and *roughness*. Both fuzziness and roughness are mathematical models of vagueness.

Fuzziness is closely related to Zadeh's fuzzy sets [28]. In fuzzy set theory, vagueness is described by means of a specific membership relation. Fuzziness is often identified with vagueness, however, Zadeh [29] noted that vagueness comprises fuzziness. Roughness is connected with Pawlak's rough sets [19].

Classical, set-theoretical sets (orthodox sets) are not sufficient to deal with vagueness. Non-orthodox sets - rough sets and fuzzy sets - are used in two different approaches to vagueness (Pawlak [22]): while Zadeh's fuzzy set theory represents a quantitative approach, Pawlak's rough set theory represents a qualitative approach to vagueness.

Significant results obtained by computer scientists in the scope of imprecision and vagueness: the Zadeh's fuzzy set theory [28], the Shafer's theory of evidence [24] and the Pawlak's rough sets theory [19], [21] greatly contributed to actualization and intensification of research into vagueness.

This paper is an extended version of a previous article by the same authors [4]. It proposes a new approach to vagueness taking into account the main ideas of roughness. Roughness considered as a mathematical model of vagueness is here replaced by an approach to vagueness in which vague sets, defined in this paper, play the role of rough sets. Vague sets are connected with vague knowledge and, at the same time, are understood as denotations of vague notions. The paper also attempts to lay logical foundations to the theory of vague notions (terms) and thus brings an essential contribution to research in this area.

The structure of the paper is as follows. In Section 2, we introduce the notion of unit information (unit knowledge) and vague information (vague knowledge). The central notion of the vague set, inspired by the Pawlak's notion of a rough set, is defined in Section 3. Section 4 is devoted to the problem of multiplicity of an object's membership to a vague set. In Section 5 some operations on vague sets and their algebraic properties are given. A view on the logic of vague concepts (terms) is discussed in Section 6. The paper ends with Section 7 which delivers some final remarks.

## 2 Unit Knowledge and Vague Knowledge

In the process of cognition of a definite fragment of reality, the cognitive agent (a man, an expert, a group of men or experts, a robot) attempts to discover information contained in it or, more adequately, about its objects. Each fragment of reality recognized by the agent can be interpreted as the following relational structure:

$$\Re = \langle \mathcal{U}, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n \rangle,$$

where $\mathcal{U}$, the *universe of objects of reality* $\Re$, is a nonempty set, and $\mathcal{R}_i$, for $i = 1, 2, \ldots, n$, is the set of $i$-ary relations on $\mathcal{U}$. One-ary relations are regarded as subsets of $\mathcal{U}$ and understood as properties of objects of $\mathcal{U}$, and multi-argument relations as relationships among its objects. Formally, every $k$-ary relation of $\mathcal{R}_k$ is a subset of $\mathcal{U}^k$.

We assume that reality $\Re$ is objective with respect to cognition. Objective knowledge about it consists of pieces of unit information (knowledge) about objects of $\mathcal{U}$ in relation to all particular relations of $\mathcal{R}_k$ ($k = 1, 2, \ldots, n$).

We introduce the notion of knowledge and vague knowledge in accordance with some conceptions of the second co-author of this paper ([27]).

**Definition 1.** ***Unit information (knowledge)*** *about the object $o \in \mathcal{U}$ with respect to the relation $R \in \mathcal{R}_k$ ($k = 1, 2, \ldots, n$) is the image $\overrightarrow{R}(o)$ of the object $o$ with respect to the relation $R$*[1]*.*

Discovering unit knowledge about objects of reality $\Re$ is realized through asking questions including certain aspects, called ***attributes***, of the objects of the universe $\mathcal{U}$. Then, we usually choose a finite set $U \subseteq \mathcal{U}$ as the universe and we put it forward as a generalized *attribute-value system* $\Sigma$, also called an *information system* (cf. Codd [6]; Pawlak [16], [18], [19]; Marek and Pawlak [13]). Its definition is as follows:

**Definition 2.** *$\Sigma$ is an* ***information system*** *iff it is an ordered system*

$$\Sigma = \langle U, A_1, A_2, \ldots, A_n \rangle,$$

*where $U \subseteq \mathcal{U}$, card$(U) < \omega$ and $A_k$ ($k = 1, 2, \ldots, n$) is the set of $k$-ary attributes understood as $k$-ary functions, i.e.*

$$\forall_{a \in A_k} a \colon U^k \to V_a,$$

*where $V_a$ is the set of all values of the attribute $a$.*

*Example 1.* Let us consider the following information system: $\mathbf{S} = \langle S, A_1, A_2 \rangle$, where $S = \{p_1, p_2, \ldots, p_5\}$ is a set of 5 papers and $A_1 = \{$IMPACT FACTOR ($IF$), QUOTATIONS ($Q$)$\}$, $A_2 = \{$TOPIC CONNECTION ($TC$)$\}$. The attribute $IF$ is a function which assigns to every paper $p \in S$ an impact factor of the journal, in which $p$ was published. We assume that $V_{IF} = [0, 100]$. The value of the attribute $Q$ for any paper $p \in S$ is the number of quotations of $p$. We assume that $V_Q = \{0, 1, 2, \ldots, 2000\}$. We also assume that $TC$ assigns to every pair of papers a quotient of the number of common references by the number of all references, and that $V_{TC} = [0, 1]$.

The information system $\mathbf{S}$ can be clearly presented in the following tables:

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[1] $\overrightarrow{R}(o) = \begin{cases} R, \text{ if } o \in R, \\ \emptyset, \text{ otherwise.} \end{cases}$ for $R \in \mathcal{R}_1$.

$\overrightarrow{R}(o) = \{\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \rangle : \langle x_1, \ldots, x_{i-1}, o, x_{i+1}, \ldots, x_k \rangle \in R\}$ for $R \in \mathcal{R}_k$ ($k = 2, \ldots, n$).

| | $IF$ | $Q$ |
|---|---|---|
| $p_1$ | 0.203 | 125 |
| $p_2$ | 0.745 | 245 |
| $p_3$ | 0.498 | 200 |
| $p_4$ | 0.105 | 150 |
| $p_5$ | 1.203 | 245 |

| $TC$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $p_1$ | 1 | 3/10 | 0 | 6/7 | 0 |
| $p_2$ | 3/10 | 1 | 0 | 0 | 4/17 |
| $p_3$ | 0 | 0 | 1 | 0 | 1/12 |
| $p_4$ | 6/7 | 0 | 0 | 1 | 0 |
| $p_5$ | 0 | 4/17 | 1/12 | 0 | 1 |

Every attribute of the information system $\Sigma$ and every value of this attribute explicitly indicates a relation belonging to the so-called ***relational system determined by*** $\Sigma$. The unit information (knowledge) about an object $o \in U$ should be considered with respect to relations of the system.

**Definition 3.** *$\Re(\Sigma)$ is a **system determined by the information system** $\Sigma$ iff*

$$\Re(\Sigma) = \langle U, \{R_{a,W} : a \in A_1, \emptyset \neq W \subseteq V_a\}, \ldots, \{R_{a,W} : a \in A_n, \emptyset \neq W \subseteq V_a\}\rangle,$$

*where* $R_{a,W} = \{(o_1, o_2, \ldots, o_k) \in U^k : a((o_1, o_2, \ldots, o_k)) \in W\}$
*for any* $k \in \{1, 2, \ldots, n\}$, $a \in A_k$.

Let us see that $\bigcup\{R_{a,\{v\}} : a \in A_1, v \in V_a\} = U$, i.e. the family $\{R_{a,\{v\}} : a \in A_1, v \in V_a\}$ is a covering of $U$.

It is easy to see that

**Fact 1** *The system $\Sigma$ uniquely determines the system $\Re(\Sigma)$.*

*Example 2.* Let $\mathbf{S}$ be the above given information system. Then the system determined by the system $\mathbf{S}$ is $\Re(\mathbf{S}) = \langle U, R_{A_1}, R_{A_2}\rangle$, where $R_{A_1} = \{R_{IF,S}\}_{\emptyset \neq S \subseteq V_{IF}} \cup \{R_{Q,S}\}_{\emptyset \neq S \subseteq V_Q}$ and $R_{A_2} = \{R_{TC,S}\}_{\emptyset \neq S \subseteq V_{TC}}$.

For any attribute $a$ of the system $\mathbf{S}$ and any $i, j \in R$ we adopt the following notation:

$S_i^j = \{v \in V_a : i \leq v \leq j\}$, $S^j = \{v \in V_a : v \leq j\}$, $S_i = \{v \in V_a : i \leq v\}$.

Then, in particular, we can easily state that: $R_{IF,S_{0.1}^{0.5}} = \{p_1, p_3, p_4\}$, $R_{IF,S_{0.7}} = \{p_2, p_5\}$, $R_{IF,S^{0.3}} = \{p_1, p_4\}$, $R_{Q,S_{150}^{150}} = R_{Q,\{150\}} = \{p_4\}$, $R_{Q,S_{200}} = \{p_2, p_3, p_5\}$ and $R_{TC,\{1/12\}} = \{(p_3, p_5), (p_5, p_3)\}$, $R_{TC,\{1\}} = \{(p_i, p_i)\}_{i=1,\ldots,5}$.

The notion of knowledge about the attributes of the system $\Sigma$ depends on the cognitive agent discovering the fragment of reality $\Sigma$. According to Skowron's understanding of the notion of knowledge determined by any unary attribute (cf. Pawlak [17], Skowron et al. [25], Demri, Orlowska [8] pp.16–17), we can adopt the following generalized definition of the notion of ***knowledge*** $K_a$ ***about any k-ary attribute*** $a$:

**Definition 4.** *Let $\Sigma$ be the information system and $a \in A_k$ ($k = 1, 2, \ldots, n$). Then*

(a) $K_a = \{((o_1, o_2, \ldots, o_k), V_{a,u}) : u = (o_1, o_2, \ldots, o_k) \in U^k\}$,

*where $V_{a,u} \subseteq P(V_a)$, $V_{a,u}$ is the family of all sets of possible values of the attribute $a$ for the object $u$ from the viewpoint of the agent and $P(V_a)$ is the family of all subsets of $V_a$.*

(b) *The knowledge $K_a$ of the agent about the attribute $a$ and its value for the object $u$ is*

(0) ***empty*** *if card*$(\bigcup V_{a,u}) = 0$,

(1) ***definite*** *if card*$(\bigcup V_{a,u}) = 1$,

($>1$) ***imprecise****, in particular* ***vague****, if card*$(\bigcup V_{a,u}) > 1$.

Let us observe that vague knowledge about some attribute of the information system $\Sigma$ is connected with the assignation of a ***vague value*** to the object $u$.

*Example 3.* Let us consider again the information system $\mathbf{S} = \langle S, A_1, A_2 \rangle$. The agent's knowledge $K_{IF}, K_Q, K_{TC}$ about the attributes of the information system $\mathbf{S}$ can be characterized by means of the following tables:

| | $V_{IF,p}$ | $V_{Q,p}$ |
|---|---|---|
| $p_1$ | $\{S_{0.2}, S_{0.3}, S_{0.25}\}$ | $\{S_{100}, S_{150}, S_{90}, S_{80}\}$ |
| $p_2$ | $\{S_{0.5}, S_{0.7}, S_{0.8}\}$ | $\{S_{180}, S_{200}, S_{250}, S_{240}\}$ |
| $p_3$ | $\{S_{0.5}, S_{0.6}, S_{0.4}\}$ | $\{S_{170}, S_{230}, S_{180}, S_{150}\}$ |
| $p_4$ | $\{S_{0.1}, S_{0.2}, S_{0.15}\}$ | $\{S_{100}, S_{90}, S_{10}, S_{140}\}$ |
| $p_5$ | $\{S_{0.7}, S_{1.5}, S_{1.0}\}$ | $\{S_{270}, S_{150}, S_{240}, S_{200}\}$ |

| $V_{TC,(p,p')}$ | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| $p_1$ | $\{S_1^1\}$ | $\{S^{0.3}, S^{0.5}\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S_{0.5}, S_{0.8}\}$ | $\{S^{0.1}, S^{0.2}\}$ |
| $p_2$ | $\{S^{0.3}, S^{0.5}\}$ | $\{S_1^1\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S^{0.3}, S^{0.4}\}$ |
| $p_3$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S_1^1\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S^{0.3}, S^{0.1}\}$ |
| $p_4$ | $\{S_{0.5}, S_{0.8}\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S_1^1\}$ | $\{S^{0.1}, S^{0.2}\}$ |
| $p_5$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S^{0.3}, S^{0.4}\}$ | $\{S^{0.3}, S^{0.1}\}$ | $\{S^{0.1}, S^{0.2}\}$ | $\{S_1^1\}$ |

From Definitions 1 and 3 we arrive at:

**Fact 2** *Unit information (knowledge) about the object $o \in U$ with respect to a relation $R$ of the system $\Re(\Sigma)$ is the image $\overrightarrow{R}(o)$ of the object $o$ with respect to the relation $R$, from the viewpoint of the agent.*

Contrary to the objective unit knowledge $\overrightarrow{R}(o)$ about the object $o$ of $U$ in the reality $\Re$ with regard to its relation $R$, the subjective unit knowledge about the object $o$ of $U$ in the reality $\Re(\Sigma)$ depends on an attribute of $\Sigma$ determining the relation $R$ and its possible values from the viewpoint of the knowledge of an agent discovering $\Re(\Sigma)$. The subjective unit knowledge $\overrightarrow{R}(o)$ from the agent's viewpoint depends on the agent's ability to solve the following equation:

$$\overrightarrow{R}(o) = x, \qquad (e)$$

where $x$ is an unknown quantity.

Solutions of $(e)$ for a $k$-ary relation $R$ should be images of the object $o$ with respect to $k$-ary relations $R_{a,W}$ from $\Re(\Sigma)$, where $\emptyset \neq W \subseteq V_a$. Let us note that for each unary relation $R$ solutions of $(e)$ are unary relations $R_{a,W}$, where $\emptyset \neq W \in V_{a,o}$.

A solution of the equation $(e)$ can be correct - then the agent's knowledge about object $o$ is ***exact***. If the knowledge is ***inexact***, then at least one solution of $(e)$ is not an image of the object $o$ with respect to the relation $R$.

**Definition 5.** *Unit knowledge about the object $o \in U$ in $\Re(\Sigma)$ with respect to its relation $R$ is*

(0) ***empty*** *iff the equation $(e)$ does not have a solution for the agent (the agent knows nothing about the value of the function $\overrightarrow{R}$ for the object $o$),*

(1) ***definite*** *iff the equation $(e)$ has exactly one solution for the agent (either the agent's knowledge is exact – the agent knows the value of the function $\overrightarrow{R}$ for the object $o$ – or he accepts only one, but not necessarily accurate, value of the function),*

($> 1$) ***imprecise*** *iff the equation $(e)$ has at least two solutions for the agent (the agent allows at least two possible values of the function $\overrightarrow{R}$ for the object $o$).*

From Definitions 4 and 5 we arrive at:

**Fact 3** *Unit knowledge about the object $o \in U$ in $\Re(\Sigma)$ with respect to its relation $R$ is*

(0) ***empty*** *if the agent's knowledge $K_a$ about the attribute $a$ and its value for the object $o$ is empty,*

(1) ***definite*** *if the agent's knowledge $K_a$ about the attribute $a$ and its value for the object $o$ is definite,*

($> 1$) ***imprecise*** *if the agent's knowledge $K_a$ about the attribute $a$ and its value for the object $o$ is imprecise.*

When the unit knowledge of the agent about the object $o$ is imprecise, then most often we replace the unknown quantity $x$ in $(e)$ with a vague value.

*Example 4.* Consider the relation $R = R_{Q,S_{200}}$ within the previous system $\Re(\mathbf{S})$, i.e. the set of all papers of $S$ that have been quoted in at least 200 papers. The unit knowledge about the paper $p_5$ with respect to $R$ can be the following vague information:

$$\overrightarrow{R}(p_5) = VALUABLE, \qquad (e_1)$$

where $VALUABLE$ is an unknown, indefinite, vague quantity.

Then the agent refers to the paper $p_5$ non-uniquely, assigning to him different images $\overrightarrow{R}(p_5)$ of the paper $p_5$ with respect to the relation R that are possible from his point of view. Then the equation $(e_1)$ usually has, for him, at least two solutions. From *Example 3*, it follows that each of these relations: $R_{Q,S_{270}}, R_{Q,S_{150}}, R_{Q,S_{240}}, R_{Q,S_{200}}$ can be a solution of $(e_1)$.

## 3 Vague Sets and Rough Sets

Let $\Re(\Sigma)$ be the system determined by the information system $\Sigma$. In order to simplify our considerations in the subsequent sections of the paper, we will limit ourselves to the unary relation $R$ (property) – a subset of $U$ of the system $\Re(\Sigma)$.

**Definition 6.** ***Unit knowledge*** *about the object o in* $\Re(\Sigma)$ *with respect to* $R$ *is* ***inexact*** *iff the equation* $(e)$ *has the form:*

$$\overrightarrow{R}(o) = X, \qquad (ine)$$

*where* $X$ *is an unknown quantity from the viewpoint of the agent, and* $(ine)$ *has for him at least one solution and at least one of the solutions is not an image* $\overrightarrow{R}(o)$.

The equation $(ine)$ can be called as the *equation of inexact knowledge of the agent*. All solutions of $(ine)$ are unary relations in the system $\Re(\Sigma)$.

**Definition 7.** ***Unit knowledge*** *about the object o in* $\Re(\Sigma)$ *with respect to* $R$ *is* ***vague*** *iff the equation* $(ine)$ *has the form:*

$$\overrightarrow{R}(o) = VAGUE, \qquad (ve)$$

*where* $VAGUE$ *is an unknown quantity and* $(ve)$ *has at least two different solutions for the agent.*

The equation $(ve)$ can be called the *equation of vague knowledge of the agent*.

**Fact 4** *Vague unit knowledge is a particular case of inexact unit knowledge.*

**Definition 8.** *The family of all solutions (sets) of* $(ine)$, *respectively* $(ve)$, *is called the* ***vague set for the object o determined by*** $R$, *respectively the* ***proper vague set for the object o determined by*** $R$.

*Example 5.* The family of all solutions of $(e_1)$ from *Example 4* is a vague set $\mathbf{V}_{p_5}$ for the paper $p_5$ determined by $R_{Q,S_{200}}$ and $\mathbf{V}_{p_5} = \{R_{Q,S_{270}}, R_{Q,S_{150}}, R_{Q,S_{240}}, R_{Q,S_{200}}\}$.

Vague sets, thus also proper vague sets, determined by a set $R$ are here some generalizations of sets approximated by representations (see Bonikowski [3]). They are non-empty families of unary relations from $\Re(\Sigma)$ (such that at least one of them includes $R$) and sub-families of the family $P(U)$ of all subsets of the set $U$, determined by the set $R$. They have the greatest lower bound (the *lower limit*) and the least upper bound (the *upper limit*) in $P(U)$ with respect to inclusion. We will denote the greatest lower bound of any family $\mathbf{X}$ by $\underline{\mathbf{X}}$. The least upper bound of $\mathbf{X}$ will be denoted by $\overline{\mathbf{X}}$. So, we can note

**Fact 5** *For each vague set* $\mathbf{V}$ *determined by the set (property)* $R$

$$\mathbf{V} \subseteq \{Y \in P(U) : \underline{\mathbf{V}} \subseteq Y \subseteq \overline{\mathbf{V}}\}.$$

The idea of vague sets was conceived upon Pawlak's idea of rough sets [19], who defined them by means of the operations of *lower approximation*: $\underline{\phantom{X}}$ and upper approximation: $\overline{\phantom{X}}$, defined on subsets of $U$. The lower approximation of a set is defined as a union of indiscernibility classes of a given relation in $U^2$ which are included in this set, whereas the upper approximation of a set is defined as a union of the indiscernibility classes of the relation which have a non-empty intersection with this set.

**Definition 9.** *A* ***rough set*** *determined by a set* $R \subseteq U$ *is a family* $\mathbf{P}$ *of all sets satisfying the following condition:*

$$\mathbf{P} = \{Y \in P(U) : \underline{Y} = \underline{R} \wedge \overline{Y} = \overline{R}\}^2$$

Let us observe that because $R \subseteq R \in \mathbf{P}$, the family $\mathbf{P}$ is a non-empty family of sets such that at least one of them includes $R$ (cf. Definition 8). By analogy to Fact 5, we have

**Fact 6** *For each rough set* $\mathbf{P}$ *determined by the set (property)* $R$

$$\mathbf{P} \subseteq \{Y \in P(U) : \underline{R} \subseteq Y \subseteq \overline{R}\}.$$

It is obvious that

**Fact 7** *If* $\mathbf{V}$ *is a vague set and* $\underline{X} = \underline{\mathbf{V}}$ *and* $\overline{X} = \overline{\mathbf{V}}$ *for any* $X \in \mathbf{V}$*, then* $\mathbf{V}$ *is a subset of a rough set determined by any set of* $\mathbf{V}$*.*

For every rough set $\mathbf{P}$ determined by $R$ we have: $\underline{\mathbf{P}} = \underline{R}$ and $\overline{\mathbf{P}} = \overline{R}$. We can therefore consider the following generalization of the notion of the rough set:

**Definition 10.** *A non-empty family* $\mathbf{G}$ *of subsets of* $U$ *is called a* ***generalized rough set*** *determined by a set* $R$ *iff it satisfies the following condition:*

$$\underline{\mathbf{G}} = \underline{R} \text{ and } \overline{\mathbf{G}} = \overline{R}.$$

It is easily seen that

**Fact 8** *Every rough set determined by a set* $R$ *is a generalized rough set determined by* $R$*.*

**Fact 9** *If* $\mathbf{V}$ *is a vague set and there exists a set* $X \subseteq U$ *such that* $\underline{X} = \underline{\mathbf{V}}$ *and* $\overline{X} = \overline{\mathbf{V}}$*, then* $\mathbf{V}$ *is a generalized rough set determined by the set* $X$*.*

[2] Some authors define a rough set as a pair of sets (lower approximation, upper approximation)(cf. e.g. Iwiński [10], Pagliani [14]).

## 4 Multiplicity of membership to a vague set

For every object $o \in U$ and every vague set $\mathbf{V}_o$, we can count the ***multiplicity of membership of o to this set***.

**Definition 11.** *The number i is the multiplicity of membership of the object o to the vague set $\mathbf{V}_o$ iff o belongs to i sets of $\mathbf{V}_o$ ($i \in N$).*

The notion of multiplicity of an object's membership to a vague set is closely related to the so-called ***degree of an object's membership to the set***.

**Definition 12.** *Let $\mathbf{V}_o$ be a vague set for the object o and $card(\mathbf{V}_o) = n$. The function $\mu$ is called a **degree of membership of o to $\mathbf{V}_o$** iff*

$$\mu(o) = \begin{cases} 0, \text{ if the multiplicity of membership of } o \text{ to } \mathbf{V}_o \text{ equals } 0, \\ \frac{k}{n}, \text{ if the multiplicity of membership of } o \text{ to } \mathbf{V}_o \text{ equals } k \ (0 < k < n), \\ 1, \text{ if the multiplicity of membership of } o \text{ to } \mathbf{V}_o \text{ equals } n. \end{cases}$$

*Example 6.* The degree of the membership of the paper $p_5$ to the vague set $\mathbf{V}_{p_5}$ (see *Example 5*) is equal to 3/4.

It is clear that

**Fact 10** *1. Any vague set is a multiset in Blizard's sense* [1]*.*
*2. Any vague set is a fuzzy set in Zadeh's sense* [28] *with $\mu$ as its membership function.*

## 5 Operations on Vague Sets

Let us denote by $\mathcal{V}$ the family of all vague sets determined by relations in the system $\Re(\Sigma)$. In the family $\mathcal{V}$ we can define a unary operation of the negation $\neg$ on vague sets, a union operation $\oplus$ and an intersection operation $\odot$ on any two vague sets.

**Definition 13.** *Let $\mathbf{V_1} = \{R_i\}_{i \in I}$ and $\mathbf{V_2} = \{S_i\}_{i \in I}$ be vague sets determined by the sets $R \subseteq U$ and $S \subseteq U$, respectively. Then*

(a) $\mathbf{V_1} \oplus \mathbf{V_2} = \{R_i\}_{i \in I} \oplus \{S_i\}_{i \in I} = \{R_i \cup S_i\}_{i \in I}$,
(b) $\mathbf{V_1} \odot \mathbf{V_2} = \{R_i\}_{i \in I} \odot \{S_i\}_{i \in I} = \{R_i \cap S_i\}_{i \in I}$,
(c) $\neg\mathbf{V_1} = \neg\{R_i\}_{i \in I} = \{U \setminus R_i\}_{i \in I}$.

*The family $\mathbf{V_1} \oplus \mathbf{V_2}$ is called the union of the vague sets $\mathbf{V_1}$ and $\mathbf{V_2}$ determined by the relations R and S. The family $\mathbf{V_1} \odot \mathbf{V_2}$ is called the intersection of the vague sets $\mathbf{V_1}$ and $\mathbf{V_2}$ determined by the relations R and S. The family $\neg\mathbf{V_1}$ is called the negation of the vague set $\mathbf{V_1}$ determined by the relation R.*

**Theorem 1.** *Let $\mathbf{V_1} = \{R_i\}_{i \in I}$ and $\mathbf{V_2} = \{S_i\}_{i \in I}$ be vague sets determined by the sets R and S, respectively. Then*

(a) $\underline{\mathbf{V_1} \oplus \mathbf{V_2}} = \underline{\mathbf{V_1}} \cup \underline{\mathbf{V_2}}$ *and* $\overline{\mathbf{V_1} \oplus \mathbf{V_2}} = \overline{\mathbf{V_1}} \cup \overline{\mathbf{V_2}}$,

(b) $\underline{\mathbf{V_1} \odot \mathbf{V_2}} = \underline{\mathbf{V_1}} \cap \underline{\mathbf{V_2}}$ *and* $\overline{\mathbf{V_1} \odot \mathbf{V_2}} = \overline{\mathbf{V_1}} \cap \overline{\mathbf{V_2}}$,

(c) $\underline{\neg\mathbf{V_1}} = U \setminus \overline{\mathbf{V_1}}$ *and* $\overline{\neg\mathbf{V_1}} = U \setminus \underline{\mathbf{V_1}}$.

**Theorem 2.** *The structure* $\mathfrak{B} = (\mathcal{V}, \oplus, \odot, \neg, \mathbf{0}, \mathbf{1})$ *is a Boolean algebra, where* $\mathbf{0} = \{\emptyset\}$ *and* $\mathbf{1} = \{U\}$*.*

We can easily observe that the above-defined operations on vague sets differ from Zadeh's operations on fuzzy sets, from standard operations in any field of sets and, in particular, from the operations on rough sets defined by Pomykała [23] and Bonikowski [2]. The family of all rough sets with operations defined in the latter two works is a Stone algebra.

## 6 On Logic of Vague Terms

How to solve the problem of logic of vague terms, logic of vague sentences (*vague logic*) based on the vague sets characterized in the previous sections? Answering this question requires a brief description of the problem of language representation of unit knowledge.

On the basis of our examples, let us consider two pieces of unit information about the paper $p_5$, with respect to the set $R$ of all papers that have been quoted in at least 200 papers:

first, exact unit knowledge

$$\overrightarrow{R}(p_5) = \{p_2, p_3, p_5\}, \qquad (ee)$$

next, vague unit knowledge:

$$\overrightarrow{R}(p_5) = \mathit{VALUABLE}. \qquad (e_1)$$

Let $p_5$ be the designator of the proper name $a$, $R$ – the denotation (extension) of the name-predicate $P$ ('*a paper that has been quoted in at least 200 papers*'), and the vague name-predicate $V$ ('*a paper which is valuable*') — a language representation of the vague quantity *VALUABLE*. Then a representation of the first equation $(ee)$ is the logical atomic sentence

$$a \text{ is } P \qquad (re)$$

and a representation of the second equation $(e_1)$ is the vague sentence

$$a \text{ is } V. \qquad (re_1)$$

In a similar way, we can represent, respectively, $(ee)$ and $(e_1)$ by means of a logical atomic sentence:

$$aP \text{ or } P(a), \qquad (re')$$

where $P$ is the predicate (*'has been quoted in at least 200 papers'*), and by means of a vague sentence

$$aV \text{ or } V(a), \qquad (re'_1)$$

where $V$ is the vague predicate (*'is valuable'*).

The sentence $(re_1)$ (res. the sentence $(re'_1)$) is not a logical sentence, but it can be treated as a *sentential form*, which represents all logical sentences, in particular the sentence $(re)$ (respectively sentence $(re')$) that arises by replacing the vague name-predicate (res. vague predicate) $V$ by allowable sharp name-predicates (res. sharp predicates), whose denotations (extensions) constitute the vague set $\mathbf{V}_{p_5}$ being the denotation of $V$ and, at the same time, the set of solutions of the equation $(e_1)$ from the agent's point of view.

By analogy, we can consider every atomic vague sentence in the form $V(a)$, where $a$ is an individual term and $V$ — its vague predicate, as a *sentential form with $V$ as a vague variable* run over all denotations of sharp predicates that can be substituted for $V$ in order to get precise, true or false, logical sentences from the form $V(a)$. Then, the scope of the variable $V$ is the vague set $\mathbf{V}_o$ determined by the designator $o$ of the term $a$.

All the above remarks lead to a 'conservative', classical approach in searching for logic of vague terms or vague sentences, here referred to as *vague logic* (cf. Fine [9], Cresswell [7]). It is easy to see that all counterparts of laws of classical logic are laws of *vague logic*, even if for the fact that vague sentences have an interpretation in Boolean algebra $\mathfrak{B}$ of vague sets (see Theorem 2).

We can distinguish two directions in seeking such a logic:

1a) all counterparts of tautologies of classical sentential calculus that are obtained by replacing sentence variables with atomic expressions of this logic (in the form $\mathcal{V}(x)$), representing vague atomic sentences (sentential functions in the form $V(a)$), are tautologies of *vague logic*,

1b) all counterparts of tautologies of classical predicate calculus that can be obtained by replacing predicate variables with vague predicate variables, representing vague predicates, are tautologies of *vague logic*;

2) *vague logic* should be a finite-valued logic, in which a value of any vague sentence $V(a)$ represented by its vague atomic expression (in the form $\mathcal{V}(x)$) is the multiplicity of membership of the designator $o$ of $a$ to the vague set $\mathbf{V}_o$ being the denotation of $V$, and the multiplicities of membership of the designators of the subjects of any composed vague sentence, represented by its composed vague formula, to the denotation (a vague set) corresponding to this sentence is a function of the multiplicities of membership of every designator of the subject of its atomic component to the denotation of its vague predicate.

It should be noticed that sentential connectives for *vague logic* should not satisfy *standard conditions* (see Malinowski [11]). For example, an alternative of two vague sentences $V(a)$ and $V(b)$ can be a 'true' vague sentence (sentential form) despite the fact that its arguments $V(a)$ and $V(b)$ are neither 'true' or 'false' sentential forms, i.e. in certain cases they represent true sentences, and in some other cases they represent false sentences. It is not contrary to the statement that all vague sentential forms which we obtain by a suitable substitution of sentential variables (resp. predicate variables) by vague sentences (resp. vague predicates) in laws of classical logic always represent true sentences. Thus they are laws of vague logic.

## 7 Final Remarks

1. The concept of vagueness was defined in the paper as an indefinite, vague quantity or property corresponding to the knowledge of an agent discovering a fragment of reality, and delivered in the form of the *equation of inexact knowledge of the agent*. A vague set was defined as a set (family) of all possible solutions (sets) of this equation and although our considerations were limited to the case of unary relations, they can easily be generalized to encompass any $k$-ary relations.
2. The idea of *vague sets* was derived from the idea of rough sets originating in the work of Zdzisław Pawlak, whose theory of rough sets takes a non-numerical, qualitative approach to the issue of vagueness, as opposed to the quantitative interpretation of vagueness provided by Lotfi Zadeh.
3. Vague sets, like rough sets, are based on the idea of a set approximation by two sets called the lower and the upper limits of this set. These two kinds of sets are families of sets approximated by suitable limits.
4. Pawlak's approach and the approach discussed in this paper both make a reference to the concept of a cognitive agent's knowledge about the objects of the reality being investigated (see Pawlak [20]). This knowledge is determined by the system of concepts that is determined by a system of their extensions (denotations). When the concept is vague, its denotation, in Pawlak's sense, is a rough set, while in the authors' sense – a vague set which, under some conditions, is a subset of the rough set.
5. In language representation, the *equation of inexact, vague knowledge of the agent* can be expressed by means of vague sentences containing a vague predicate. Its denotation (extension) is a family of all scopes of sharp predicates which, from the agent's viewpoint, can be substituted for the vague predicate. The denotation is, at the same time, the vague set of all solutions to the equation of the agent's vague knowledge.
6. Because vague sentences can be treated as sentential forms whose variables are vague predicates, all counterparts of tautologies of classical logic are laws of *vague logic* (logic of vague sentences).
7. *Vague logic* is based on classical logic but it is many-valued logic, because its sentential connectives are not extensional.

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