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Research Article

A Method Adjusting Consistency and Consensus for Group Decision-Making Problems with Hesitant Fuzzy Linguistic Preference Relations Based on Discrete Fuzzy Numbers

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In each hesitant fuzzy linguistic preference relation, experts may express their opinions through comparison linguistic information combined with a discrete fuzzy number. In this paper, a hesitant fuzzy linguistic computational model based on discrete fuzzy numbers whose support is a subset of consecutive natural numbers is proposed, which enriches the flexibility of group decision-making. First, some main concepts related to discrete fuzzy numbers and an aggregation function of individual subjective linguistic preference relations are introduced. Then, a consistency measure is presented to check and improve the consistency in a given matrix. Further, in order to achieve the predefined degree of consensus and to arrive at the final result, a consensus-reaching process based on the interactive feedback mechanism is defined. Meanwhile, a revised formula is introduced to calculate the consistency and the degree of consensus in a preference relation matrix. Besides, an illustrative example and comparative analysis are conducted through the proposed calculation process and the optimization algorithm. Finally, the analysis on the threshold values is made to help the decision-maker determine critical consensus level. The proposed method can address both consistency and consensus, and the results confirmed the effectiveness of the proposed method and its potential use in the qualitative decision-making problems.

1. Introduction

With the development of science and information technology, many decision problems in social and economic life become more and more complicated. It is more difficult for individual decision-makers to consider all relevant aspects of the problem. To reduce the decision-making mistakes, it usually requires more than one expert to make decisions for one problem, which is the so-called group decision-making (GDM) [1]. In GDM, the decision-makers (DMs) often express their opinion in pairwise comparison form with linguistic term sets (LTSs). In previous research, various linguistic models have been proposed, such as the 2-tuple linguistic model [2, 3], the type-2-fuzzy-set-based model [4, 5], the granular method [6, 7], and symbolic linguistic models [8, 9]. These models have been popular; however, they have proved to be inadequate when facing more complex subjective information. The decision-maker may be hesitant about the linguistic variables like "better than good," "between fair and very good," or even more complex expressions. Rodriguez et al. [10] propose the concept of the hesitant fuzzy linguistic term set (HFLTS) which increases the flexibility and richness of linguistic elicitation in hesitant situations under qualitative settings.

After that, HFLTS has garnered considerable attention from researchers [11–14]. However, all possible linguistic evaluations provided by experts have equal importance in most of the current approaches about HFLTSs. Obviously, it may be not appropriate in real-life GDM problems, since the DMs may prefer some LTSs to other ones so that the linguistic assessments should have different values. For example, if the forms of importance are taken as discrete fuzzy numbers, then the evaluation sets include not only several possible linguistic terms but also the discrete fuzzy number information; the ignorance of which may lead to erroneous results. Moreover, in the discrete fuzzy number model, the semantics of the linguistic terms are included into the evaluation of the expert and there is no need for defining any underlying membership [15]. In group decision-making, when the individual preferences are aggregated into group preference, most of the aggregate operators are defined in $[0, 1]$ interval and the result of the aggregation is a definite number on the unite level that cannot generally correspond to the linguistic terms in the original linguistic set. So the hesitant fuzzy linguistic preference relations (HFLPRs) based on discrete fuzzy numbers provide a greater flexibility to the experts [16–18]. However, until now, few studies were carried out in the GDM with the HFLPRs based on discrete fuzzy numbers: Massanet et al. [15] proposed a linguistic computational model based on discrete fuzzy numbers whose support is a subset of consecutive natural numbers which is presented to ensure the accuracy and consistency of the model; then, Massanet et al. [19] presented a GDM model with the HFLPR based on discrete fuzzy numbers.

These studies above make a great contribution to GDM with the HFLPR based on discrete fuzzy numbers. However, there are still some problems that need to be further studied, and one of them is that the consistency and the consensus must be considered. It is because on the one hand consistency must be considered in preference relations to show that the supplied preferences satisfy some transitive properties. More pairwise comparisons are generated than is necessary, and the provided preferences run the risk of being inconsistent [20–23]. This is especially true when the pairwise comparisons involve intangibles, as it is unrealistic to expect that a given preference relation is perfectly consistent [24]. Because a lack of consistency in preference relations can lead to inconsistent conclusions, consistency tests are a critical step for any kind of preference relations. On the other hand, consensus is another fundamental issue widely employed in GDM [25–29]. Decision-makers with different attitudes, perceptions, motivations, and personalities attempt to reach a collective decision in which the individual preference comparability is as high as possible.

For an overview of the GDM field, a lot of papers focus on the consistency [30–32] and the consensus [33, 34]. However, there have been a few papers which have considered hesitation in a linguistic environment. In the aspects of consistency-considered hesitation in a linguistic environment, Zhu and Xu [35] adopt an automatic improvement process to revise the unacceptable HFLPRs, and the modified linguistic terms are virtual terms which can save time as there is no need for further expert interaction. Zhang and Wu [36] defined the multiplicative consistency of an HFLPR. Wang and Xu [37] presented some consistency measures for an extended HFLPR (EHFLPR). Zhang et al. [38] further consider the use of the consistency improvement process in these HFLPRs. In the aspects of consensus-considered hesitation in a linguistic environment, Dong et al. [13] proposed a two-stage consensus model based on the similarity that was developed to improve the consensus level. Wu and Xu [39] propose a new approach to deal with the consensus-reaching process for multiple attribute group decision-making (MAGDM), and the consensus degree for each expert is defined based on the distance between the individual decision matrix and the collective decision matrix. From these related works above, we may conclude the following:

(1) In a GDM problem with HFLPR, the consistency and the consensus must be considered [15–22]. The researchers begin to do some contributions about this field. But few refer to the HFLPR based on discrete fuzzy numbers, so consistency and consensus for group decision-making problems with HFLPR based on discrete fuzzy numbers need to be further studied [15, 19].

(2) There are few papers that consider considered hesitation with HFLPR [35–37]. In [36, 37], the improvement process was not addressed for HFLPR, and in [35], automatic methods which may substantially change the experts' preferences were proposed. So the interactive improvement process needs to be considered when developing a new adjusting consistency method with HFLPR based on discrete fuzzy numbers.

(3) The related consensus methods [13, 39–41] show that to stimulate interest and increase expert engagement, a feedback mechanism which provides guidelines was used, which is helpful in assisting. So the feedback mechanism needs to be considered when developing a new adjusting consensus method with HFLPR based on discrete fuzzy numbers.

Thus, in this paper, we proposed a new method to adjust the consistency and consensus issues for a GDM with HFLPRs based on the discrete fuzzy numbers. The novelty of this paper is as follows.

(1) We developed a consistency measure and proposed an optimization algorithm to increase the consistency degree for a given linguistic model. To deal with HFLPR based on the discrete fuzzy numbers, we used a local revision strategy with distance conversion formula, and compared to existing automatic approaches, it has distinct characteristics.

(2) We defined a direct consensus-reaching process to assist decision-makers who need to reconsider their preferences for the purpose of achieving the predefined consensus degree. The proposed feedback mechanism is based on the degree of similarity between individual preference relations; therefore, there is no need to calculate the proximity matrices. So the proposed method is computationally simpler when compared with others.

(3) We supply some examples to illustrate the performance of the proposed consistency measure and consensus-reaching processes. And then, a comparative study is conducted to emphasize the potential advantages and characters of the proposed measures.

(4) We conduct the analysis on the threshold values in a way of the simulation method, which can help the decision-maker determine the threshold values according to the number of experts, and this method provides a minimum consensus level.

This paper is structured as follows. In Section 2, we make a brief review of discrete fuzzy numbers and the linguistic model based on hesitant fuzzy linguistic relations and discrete fuzzy numbers; at the same time, an aggregation function constructed from discrete aggregation function (defined on a finite chain) is recalled and it is applied to the aggregation of the individual HFLPR. In Section 3, a HFLPR consistency measure is defined, and for matrices that are of unacceptable consistency, a consistency-improving process is introduced. In Section 4, a HFLPR consensus-reaching process is presented to measure the degree of agreement in the group and find out the expert who should consider his preferences again. In Section 5, we present an example to illustrate the applicability of the proposed method and its advantages. Then, a comparative study is conducted. In Section 6, the threshold values of consensus were analyzed by the simulation method. Section 7 introduces how the proposed method solves large-scale group decision-making (LGDM) problems. Finally, in Section 8, some concluding remarks and future work are proposed.

2. Preliminaries

2.1. Discrete Fuzzy Numbers

Definition 1 [42]. A fuzzy subset A of R with membership mapping $A:\mathbb{R}\rightarrow[0,1]$ means a discrete fuzzy number if its support is finite, and there exists $x_1,\ldots,x_n\in\mathbb{R}$ with $x_1<x_2<\cdots<x_n$ such that $\operatorname{supp}(A)=\{x_1,\ldots,x_n\}$; there are natural numbers s and t with $1\le s\le t\le n$. According to the above analysis, one can know that

(1) $A(x_i)=1$ for any natural number i with $s\le i\le t$ (core),

(2) $A(x_i)\le A(x_j)$ for each natural number i and j with $1\le i\le j\le s$,

(3) $A(x_i)\ge A(x_j)$ for each natural number i and j with $t\le i\le j\le n$.

In Figure 1, we present a graphical representation of a discrete fuzzy number. This is an asymmetric scatter plot, in which the images increase till the core (equal to 1) and then they decrease.

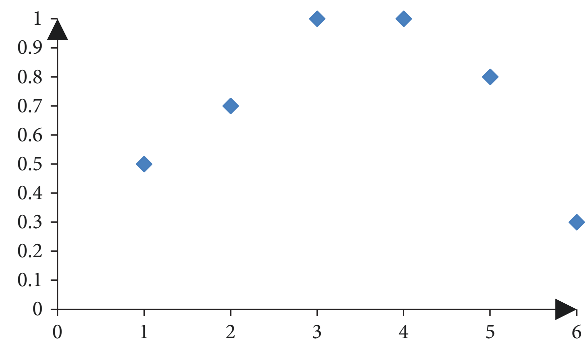

Figure 1: Graphical representation of a general discrete fuzzy number with support $\{1,\ldots,6\}$ and core $\{3,4\}$.

2.2. Hesitant Fuzzy Linguistic Model Based on Discrete Fuzzy Numbers

Remark 1 [21]. Note that we can think of a bijective mapping between the ordinal scale $\mathscr{L}=\{s_0,\ldots,s_n\}$ and the finite chain L_n which keeps the initial order. Furthermore, each normal continuous convex fuzzy subset defined on the ordinal scale $\mathscr{L}$ can be regarded as a discrete fuzzy number belonging to $A_1^{L_n}$ and vice versa. Thus, from now on, a HFLPR A can be also interpreted equivalently as a normal continuous fuzzy set on the ordinal scale $\mathscr{L}$.

Then, we consider the following linguistic hedge:

$$\mathscr{L}=\{\mathrm{AB},\mathrm{VB},\mathrm{MB},\mathrm{E},\mathrm{MG},\mathrm{VG},\mathrm{AG}\},\tag{1}$$

where the letters refer to the linguistic terms absolutely bad, very bad, moderately bad, equality, moderately good, very good, and absolutely good. They are listed in an increasing order as follows:

$$\mathrm{AB}\prec\mathrm{VB}\prec\mathrm{MB}\prec\mathrm{E}\prec\mathrm{MG}\prec\mathrm{VG}\prec\mathrm{AG},\tag{2}$$

and the finite chain is L_6. Thus, the HFLPR $A=\{0.6/\mathrm{VB},1/\mathrm{MB},0.5/\mathrm{MG},0.8/\mathrm{VG}\}$ can be also expressed as $A=\{0.6/1,1/2,0.5/4,0.8/5\}\in A_1^{L_6}$.

Then, let us consider a set of alternatives $X=\{x_1,x_2,x_3,x_4\}$ and an expert e_1 who provides preferences on this set. The following HFLPR p_{e_1} on X is expressed.

$$p_{e_1}=\begin{pmatrix} - & p_1^{12} & p_1^{13} & p_1^{14}\\ p_1^{21} & - & p_1^{23} & p_1^{24}\\ p_1^{31} & p_1^{32} & - & p_1^{34}\\ p_1^{41} & p_1^{42} & p_1^{43} & - \end{pmatrix},\tag{3}$$

Suppose that

$$p_1^{12}=\left\{\frac{0.5}{1},\frac{1}{2}\right\}=\left\{\frac{0.5}{\mathrm{VB}},\frac{1}{\mathrm{MB}}\right\},\tag{4}$$

in that way $p_1^{21}=N(p_1^{12})=\{0.5/5,1/4\}=\{0.5/\mathrm{VG},1/\mathrm{MG}\}$, which satisfies the reciprocal property (see [21, 43]).

2.3. Aggregation of HFLPR Based on the Discrete Fuzzy Numbers

Theorem 1 [44]. *Let one consider an aggregation function F on the finite chain L_n. The binary operation on $A_1^{L_n}$ was developed as follows:*

$$\mathscr{F} : A_1^{L_n} \times A_1^{L_n} \to A_1^{L_n}, \quad (A, B) \longmapsto \mathscr{F}(A, B). \tag{5}$$

As for $\mathscr{F}(A, B)$, the discrete fuzzy numbers whose $\alpha -$ cuts are the sets.

$$\mathscr{F}(x_1, \ldots, x_m) = \max\{\min\{x_1, \ldots, x_m\}, \max\{x_1, \ldots, x_m\} - k\}, \tag{6}$$

where $k \in [0, n-1]$ and $\alpha \in [0, 1]$. One can regard it as an aggregation function on $A_1^{L_n}$.

Example 1. Considering the finite chain $L_8 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{0.3/0, 0.5/1, 1/2, 0.3/3\}$ and $B = \{0.3/2, 0.5/3, 1/4, 0.8/5\} \in A_1^{L_8}$, we obtain $\mathscr{F}(A, B) = \{0.3/0, 0.5/1, 1/2, 0.5/3\}$.

2.4. Measure Distance between Two Discrete Fuzzy Numbers

Theorem 2 [45]. *For each discrete fuzzy number N, the upper and lower limits of the* kth $\alpha -$ cut *for N are defined as*

$$l_{i,k} = \min\{x \mid N(x) \geq \alpha_k\}, \quad r_{i,k} = \max\{x \mid N(x) \geq \alpha_k\}, \tag{7}$$

where $l_{i,k}$ and $r_{i,k}$ are the left and right spreads, accordingly.

The left dominance D_{ij}^{L} or right dominance D_{ij}^{R} of N_i over N_j is developed as the average difference of the left or right spreads at some $\alpha -$ cut levels. They are obtained in the following:

$$D_{i,j}^{L} = \frac{1}{n+1}\sum_{k=0}^{n}\left(l_{i,k} - l_{j,k}\right), \quad D_{i,j}^{R} = \frac{1}{n+1}\sum_{k=0}^{n}\left(r_{i,k} - r_{j,k}\right), \tag{8}$$

where $n + 1$ $\alpha -$ cuts is applied to compute the dominance, the total dominance of N_i over N_j.

Example 2. Considering the two HFLPRs in Example 1, we compute $D_{A,B}^{R}$ in the following.

$$D_{A,B}^{R} = \frac{1}{n+1}\sum_{k=0}^{n}(r_{A,k} - r_{B,k}) = \frac{1}{4} * (3 - 5 + 2 - 5 + 2 - 4 + 2 - 5) = -\frac{5}{2}. \tag{9}$$

With the level of optimism, $\beta \in [0, 1]$ can be introduced as the convex combination of D_{ij}^{L} and D_{ij}^{R} by

Table 1: The right dominance of A and B.

HFLPR	$\alpha -$ cut levels	Left and right spreads
A	0.3	[0,3]
	0.5	[1,2]
	0.8	[2]
	1	[2]
B	0.3	[2,5]
	0.5	[3,5]
	0.8	[4]
	1	[4,5]

$$D_{i,j}(\beta) = \beta D_{ij}^{R} + (1-\beta) D_{i,j}^{L}, \tag{10}$$

From (10), we can determine the comparison between two discrete fuzzy numbers.

If $D_{i,j}(\beta) = 0$, then $A_i = A_j$; if $D_{i,j}(\beta) > 0$, then $A_i > A_j$; and if $D_{i,j}(\beta) < 0$, then $A_i < A_j$.

So we can determine the distance between two preference relations based on the discrete fuzzy numbers by

$$d\left(p_{ij}^{f}, p_{ij}^{k}\right) = \frac{1}{n}\left|D_{i,j}(\beta)\right|, \tag{11}$$

where $n + 1$ is the number of elements in the linguistic term set. The right dominance of A and B is shown in Table 1.

3. Consistency Method

In this section, we firstly introduced a consistency measure for HFLPR based on discrete fuzzy numbers. Then, a consistency improvement process is developed to obtain an acceptable consistency for a matrix.

3.1. Consistency Measure. From the hesitant fuzzy linguistic model proposed above, we know that experts only need to provide pairwise preferences for the upper triangular elements, and because of the reciprocal property, the other elements are obtained. Meanwhile, in the above example,

$$p_1^{11} = p_1^{22} = p_1^{33} = p_1^{44} = \left\{\frac{1}{4}\right\}, \tag{12}$$

which is useful in the following.

Definition 2 [46]. An HFLPR P is called an additive consistent HFLPR if and only if

$$p_f^{ij} = p_f^{ik} \oplus p_f^{kj}, \quad \text{for all } i, j, k = 1, 2, \ldots, n. \tag{13}$$

For convenience, we denote p_k as the individual preference relation and p_c as the collective preference relation. From (6), we can determine p_c^{ij} through the way of aggregating all the decision-makers' subjective evaluations in the position (i, j).

Example 3. Let $S = \{S_1, \ldots, S_t\}$ be a predefined linguistic term set and $X = \{X_1, \ldots, X_n\}$ be a set of alternatives. There are m experts, $\Xi = \{e_1, \ldots, e_m\}$, $(m \geq 2)$. Let $P_k = (p_{ij}^{k})_{n\times n}$ be a

HFLPR based on the discrete fuzzy numbers given by an expert $e_k \in \Xi$, and p_{ij}^k represents a judgement for alternative X_i over alternative X_j. Then, let $P_c = (p_{ij}^c)_{n\times n}$ be the collective preference relation; we can obtain

$$p_{ij}^c = \mathscr{F}\left(p_{ij}^1, \ldots, p_{ij}^m\right), \quad i, j = 1, \ldots, n. \tag{14}$$

Theorem 3 [46]. *Let P_k $(k = 1, 2, \ldots, m)$ and P_c be as before, and if P_k $(k = 1, 2, \ldots, m)$ is additive consistent, then P_c is additive consistent.*

Proof 1. Suppose that P_k $(k = 1, 2, \ldots, m)$ is additive consistent; then, from Definition 2, we can find that

$$p_{ij}^k = p_{iy}^k \oplus p_{yj}^k, \quad \text{for all } i, j, y = 1, 2, \ldots, n. \tag{15}$$

We need to prove that

$$p_{ij}^c = p_{iy}^c \oplus p_{yj}^c, \quad \text{for all } i, j, y = 1, 2, \ldots, n. \tag{16}$$

It follows that

$$\begin{aligned} p_{iy}^c \oplus p_{yj}^c &= \mathscr{F}\left(p_{iy}^1, \ldots, p_{iy}^m\right) \oplus \mathscr{F}\left(p_{yj}^1, \ldots, p_{yj}^m\right) \\ &= \mathscr{F}\left\{\mathscr{F}\left(p_{iy}^1, \ldots, p_{iy}^m\right), \mathscr{F}\left(p_{yj}^1, \ldots, p_{yj}^m\right)\right\} \\ &= \mathscr{F}\left\{p_{iy}^1, \ldots, p_{iy}^m, p_{yj}^1, \ldots, p_{yj}^m\right\} \\ &= \mathscr{F}\left\{p_{iy}^1 \oplus p_{yj}^1, \ldots, p_{iy}^m \oplus p_{yj}^m\right\} = p_{ij}^c. \end{aligned} \tag{17}$$

This completes the proof.

3.2. Consistency Improvement Process. In this section, an algorithm is proposed to compute the consistency degree of a given HFLPR and find out the position which needs to be modified as well as guide the direction to improve the consistency.

Definition 2 implies that for each matrix, there will be a completely consistent matrix, $\mathbf{C}_{P_k} = (c_{ij}^{P_k})_{n\times n}$, which can be obtained from P_k.

$$\begin{aligned} c_{ij}^{P_k} &= \frac{1}{n}\sum_{y=1}^{n}\left(p_{iy}^k \oplus p_{yj}^k\right) \\ &= \mathscr{F}\left\{p_{i1}^k, p_{1j}^k, \ldots, p_{in}^k, p_{nj}^k\right\}, \quad i, j = 1, \ldots, n. \end{aligned} \tag{18}$$

Definition 3 [46]. Let P_k, P_c, C_{P_k} be as before; the consistency degree for P_k is computed by the distance measure between P_k and C_{P_k} in the following:

$$\mathrm{CI}(P_k) = d\left(P_k, C_{P_k}\right) = \sqrt{\frac{1}{n(n-1)/2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} d^2\left(p_{ij}^k, c_{ij}^{P_k}\right)}, \tag{19}$$

where

$$d\left(P_k, C_{P_k}\right) = \frac{1}{n}\left|D_{ij}(\beta)\right|. \tag{20}$$

Table 2: Threshold values for different n ($t = 4$, $\alpha = 0.1$, and $\sigma = 2$).

	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
$t = 4$	0.0980	0.1347	0.1550	0.1667	0.1765	0.1828	0.1876

So it is easy to see that $0 \le \mathrm{CI}(P_k) \le 1$, and the bigger $\mathrm{CI}(P_k)$, the more inconsistent it is. Reference [30] established the consistency threshold $\overline{\mathrm{CI}}$ as shown in Table 2. Note that $t = 4$ means that there are 9 terms in the linguistic term set S. And when $\mathrm{CI}(P_k) \le \overline{\mathrm{CI}}$, one can say that P_k is of acceptable consistency, and when $\mathrm{CI}(P_k) \ge \overline{\mathrm{CI}}$, one can say that P_k is of unacceptable consistency.

Then, we develop an improved algorithm to conduct the consistency improvement procedure, and the original algorithm was proposed in [46].

There are many ways to improve the convergence rate of the algorithm; for example, the experts could revise more than one of their preferences in Step 4. Then, we should note that in order to speed up the improvement process, the revised preference should be as close as possible to the preference in the completely consistent matrix in each modification process, while it is also important to keep the original information, so there should be a tradeoff between them.

In the following theorem, we prove that the proposed algorithm is convergent.

Theorem 4 [46]. *In the consistency improvement process,* CI $(P_{h+1}^k) < \mathrm{CI}(P_h^k)$.

Proof 2. One can know that after each modification cycle, there is at least one position (i, j), where $d(p_{ij,h+1}^k, c_{ij,h+1}^{P_k}) < d(p_{ij,h}^k, c_{ij,h}^{P_k})$, and others remain the same. From (27), it is obvious that $\mathrm{CI}(P_{h+1}^k) < \mathrm{CI}(P_h^k)$ can be determined.

Example 4. Let $S = \{\mathrm{N}, \mathrm{EL}, \mathrm{VL}, \mathrm{L}, \mathrm{M}, \mathrm{H}, \mathrm{VH}, \mathrm{EH}, \mathrm{T}\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be the linguistic term set. An HFLPR based on the discrete fuzzy number is given as follows:

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \left\{\frac{1}{6}, \frac{1}{7}\right\} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \left\{\frac{1}{0}, \frac{1}{1}\right\} & \frac{1}{6} \\ \left\{\frac{1}{1}, \frac{1}{2}\right\} & \left\{\frac{1}{7}, \frac{1}{8}\right\} & \frac{1}{4} & \left\{\frac{1}{1}, \frac{1}{2}\right\} \\ \frac{1}{3} & \frac{1}{2} & \left\{\frac{1}{6}, \frac{1}{7}\right\} & \frac{1}{4} \end{pmatrix}. \tag{21}$$

In the following, Algorithm 1 is developed to examine the consistency index of P. First, we obtain the completely consistent matrix corresponding to P.

$$C_{P_0} = \begin{pmatrix} \frac{1}{4} & \left\{\frac{1}{4}, \frac{1}{5}\right\} & \left\{\frac{1}{3}, \frac{1}{4}\right\} & \left\{\frac{1}{3}, \frac{1}{4}\right\} \\ \frac{1}{4} & \frac{1}{4} & \left\{\frac{1}{3}, \frac{1}{4}\right\} & \frac{1}{3} \\ \left\{\frac{1}{5}, \frac{1}{6}\right\} & \left\{\frac{1}{5}, \frac{1}{6}\right\} & \frac{1}{4} & \left\{\frac{1}{4}, \frac{1}{5}\right\} \\ \left\{\frac{1}{4}, \frac{1}{5}\right\} & \left\{\frac{1}{5}, \frac{1}{6}\right\} & \left\{\frac{1}{4}, \frac{1}{5}\right\} & \frac{1}{4} \end{pmatrix}. \quad (22)$$

Setting $\overline{\mathrm{CI}} = 0.1347$, we obtain $\mathrm{CI}(P_0) = \mathrm{CI}(P) = 0.3166$. Since $\mathrm{CI}(P_0) > \overline{\mathrm{CI}} = 0.1347$, so we need to continue to the next step. In Step 4, we find that $(1, 3)$, $(2, 3)$, $(2, 4)$, and (3,4) should be modified, because when $P^0_{13} > C^{p_0}_{13}$, $P^0_{23} < C^{p_0}_{23}$ and $P^0_{24} > C^{p_0}_{24}$, $P^0_{34} < C^{p_0}_{34}$, the expert should decrease his preferences; the specific results are shown in Table 3.

After 6 iterations, an acceptable consistency is obtained and the revised $\bar{P}$ is

$$\bar{P} = \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \left\{\frac{1}{4}, \frac{1}{5}\right\} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \left\{\frac{1}{2}, \frac{1}{3}\right\} & \frac{1}{4} \\ \left\{\frac{1}{3}, \frac{1}{4}\right\} & \left\{\frac{1}{5}, \frac{1}{6}\right\} & \frac{1}{4} & \left\{\frac{1}{3}, \frac{1}{4}\right\} \\ \frac{1}{3} & \frac{1}{4} & \left\{\frac{1}{4}, \frac{1}{5}\right\} & \frac{1}{4} \end{pmatrix}. \quad (23)$$

From the above results, on the one hand, we can find that the revised preference plays a vital role in the consistency improvement process, while Step 4 is applied to suggest the direction which the expert should take to increase the consistency, not specific preference. Therefore, if the modified preference is inappropriate, the algorithm will continue to execute in the position (i, j), and the terminology span of this experiment is two terms. On the another hand, as shown in the example, if several elements take the same maximum value in the upper triangular, in order to speed up the convergence of the algorithm, the decision-makers can change more than one of their preferences in each round. In the end, note that we set $k = 2$ and $\beta = 0.5$; however, changes in parameter values also affect the performance of the algorithm, which we leave for future research.

Then, we conduct a comparative analysis. In [46], the modified $\bar{P}$ is as follows:

$$\overline{P_0} = \begin{pmatrix} \{S_0\} & \{S_1\} & \{S_2, S_3\} & \{S_1\} \\ \{S_{-1}\} & \{S_0\} & \{S_{-2}, S_{-1}\} & \{S_0, S_1\} \\ \{S_{-3}, S_{-2}\} & \{S_1, S_2\} & \{S_0\} & \{S_{-2}, S_{-1}\} \\ \{S_{-1}\} & \{S_{-1}, S_0\} & \{S_1, S_2\} & \{S_0\} \end{pmatrix}. \quad (24)$$

Zhu and Xu [35] proposed an automatic optimization method, which is used to improve the consistency level of P. In this way, the modified HFLPR denoted by P_{AOM} is

$$P_{\mathrm{AOM}} = \begin{pmatrix} \{S_0\} & \{S_{1.737}, S_{1.737}\} & \{S_{1.263}, S_{1.772}\} & \{S_{1.000}, S_{1.491}\} \\ \{S_{-1.737}, S_{-1.737}\} & \{S_0\} & \{S_{-0.561}, S_{-0.052}\} & \{S_{-0.702}, S_{-0.211}\} \\ \{S_{-1.772}, S_{-1.263}\} & \{S_{0.052}, S_{0.561}\} & \{S_0\} & \{S_{-0.298}, S_{-0.280}\} \\ \{S_{-1.491}, S_{-1.000}\} & \{S_{0.211}, S_{0.702}\} & \{S_{0.280}, S_{0.298}\} & \{S_0\} \end{pmatrix}. \quad (25)$$

We can find that in $\overline{P_0}$, experts only express their preferences through the hesitant fuzzy linguistic terms and there is no specific preference value to measure the degree of the evaluations, while the proposed method can solve this problem. Using the HFLPR based on discrete fuzzy numbers, the decision-makers can be more flexible and delicate to give their own preference relations. In P_{AOM}, the modified preference relations are virtual terms, and they do not belong to the original preference relations. So it can be difficult for experts to accept them as the revised preference relations, and this is not concise to calculate the final program ranking. According to the proposed method, the experts know their current status in each cycle, so it is easy to find the preference relations that need to be reconsidered, and they also can control the speed of convergence. As stated above, compared with existing approaches, the proposed method is acceptable, and the modified HFLPR is easier for experts to describe.

4. Direct Consensus Process

In the previous survey, we find that a rational consensus-reaching process is not just an aggregation or pooling but a procedure whereby rationally motivated changes in individual preferences occur [47]. In this way, some optimization mechanism should be added to a consensus process, which assists experts to consider their thoughts again and change their preferences. In this section, we first define a consensus measure to calculate the degree of agreement in the group; then, we develop a consensus-reaching procedure to solve CDM problems based on the model proposed previously.

Input: An original $P_k = (p_{ij}^k)_{n\times n}$ and the acceptable consistency threshold $\overline{\mathrm{CI}}$.
Output: Modified $\overline{P_k} = (\overline{p_{ij}^k})_{n\times n}$.
Step 1. Construct an additive consistent HFLPR $C_{P_k} = (c_{ij}^{P_k})_{n\times n}$. Set $h = 0$, which represents the number of iterations, and $P_0^k = P_k$.
Step 2. Calculate $\mathrm{CI}(P_h^k)$, and if $\mathrm{CI}(P_h^k) \le \overline{\mathrm{CI}}$, go to Step 5; otherwise, go to the next step.
Step 3. Calculate the distance matrix, $\mathbf{D}_h = (d_{ij,h})_{n\times n}$, and from (20), we can obtain $d_{ij,h} = (1/n)|D_{ij}(\beta)|$.
Step 4. Then, we find the maximum element in the upper triangular part of D_h to modify $d_{xy} = \max_{i<j}\{d_{ij,h}\}$; if $p_{xy}^k < c_{xy}^{P_k}$, which represents $D_{xy}(\beta) < 0$, experts should increase their evaluation associated with the pair (X_x, X_y), and if $p_{xy}^k > c_{xy}^{P_k}$, which represents $D_{xy}(\beta) > 0$, experts should decrease their evaluation. Due to the reciprocal property, the elements in the lower triangular part of P_k are changed. Denote modified HFLPR P_{h+1}^k, and set $h = h + 1$; then, go back to Step 2.
Step 5. Output h and P_h^k.
Step 6. End.

Algorithm 1

Table 3: Consistency-improving process for P.

Iteration	CI(P)	(i, j)	Modified preference	$\mathrm{CI}(P) < \overline{\mathrm{CI}}$
$h = 0$	0.3166	(1, 3), (2, 3) (2, 4), (3, 4)	$P_{13} = \{\frac{1}{4}, \frac{1}{5}\}$, $P_{23} = \{\frac{1}{2}, \frac{1}{3}\}$ $P_{24} = \{\frac{1}{4}\}$, $P_{34} = \{\frac{1}{3}, \frac{1}{4}\}$	No
$h = 1$	0.1301	—	—	Yes

4.1. Consensus Method. At present, there are two kinds of measures to compute consensus level in GDM problems: the first one is based on the distance to the group preference and the other is based on the distance between individual preference relations [48]. We follow the latter definition.

First, from (11), we can use the following formula to define the similarity degree between two HFLPRs:

$$sm_{ij}^{fk} = 1 - \frac{1}{n}\left|D_{i,j}(\beta)\right|, \tag{26}$$

where $n + 1$ is the number of elements in the linguistic term set.

And it is obvious that $0 \le sm_{ij}^{fk} \le 1$. And the sm_{ij}^{fk} is the similarity degree between experts e_f and e_k in their assessments of the position (i, j), $i, j = 1, 2, \ldots, n$. The closer sm_{ij}^{fk} is to 1, the more similar the preference on A_i over A_j between e_f and e_k, while the closer sm_{ij}^{fk} is to 0, the more distant e_f is from e_k.

With the help of the similarity degree, for each pair of experts e_f and e_k, a similarity matrix $\mathbf{SM}_{fk} = (sm_{ij}^{fk})_{n*n}$ can be developed by

$$\mathbf{SM}_{fk} = \begin{pmatrix} - & \cdots & sm_{1n}^{fk} \\ \vdots & \ddots & \vdots \\ sm_{n1}^{fk} & \cdots & - \end{pmatrix}. \tag{27}$$

Suppose that there are m decision-makers, corresponding to $m(m-1)/2$ similarity matrices. Then, an aggregation operator is used to obtain a consensus matrix $\mathbf{CM} = (cm_{ij})_{n*n}$, which is computed by aggregating all the similarity matrices related to each pair of experts; the specific formula is as follows:

$$cm_{ij} = \mathrm{AGG}\left(sm_{ij}^{fk}\right) = \frac{\sum_{f=1}^{m-1}\sum_{k=f+1}^{m} sm_{ij}^{fk}}{m(m-1)/2}, \quad i, j = 1, 2, \ldots, n. \tag{28}$$

Next, we use the following three aspects to implement the arithmetic operator to compute the consensus degree.

(a) Consensus level for the pair of alternatives: in order to determine the consensus level between all the decision-makers for pair of alternatives (x_i, x_j), $(CP)_{n*n} = cp_{ij}$ is defined as follows:

$$cp_{ij} = cm_{ij}, \quad i, j = 1, 2, \ldots, n, i \ne j, \tag{29}$$

which has the same meaning as cm_{ij}; the closer cp_{ij} is to 0, the worse the agreement between all experts for pair of alternatives (x_i, x_j).

(b) Consensus level for the alternatives: the consensus level for an alternative x_i , denoted as ca_i, is computed by an average operator.

$$\mathrm{ca}_i = \frac{\sum_{j=1, j\ne i}^{n} cp_{ij}}{n-1}. \tag{30}$$

This measure is used to access the poorest consensus level between all the alternatives.

(c) Consensus level for the preference relations: the consensus level for HFLPRs, denoted as cr, is calculated as follows:

$$\text{cr} = \min \{\text{ca}_i\}, \quad (31)$$

where the result of this final calculation represents the global consensus level between the experts' preferences; then, we compare it with the predefined consensus threshold value θ; if $\text{cr} \gg \theta$, we can say that the model satisfies the requirement for consensus, so the consensus-reaching process is terminated; otherwise, we should conduct a new round.

4.2. Feedback Mechanism. In general, a feedback mechanism consists of two advice rules: one is the identification rules (IR) and the other is the direction rules (DR). Reference [40] proposed a calculation of proximity measures, while [46] builds a feedback mechanism which involves consensus degrees. In this paper, we adopt the second method of calculation.

Firstly, an IR is conducted, which intends to identify the alternatives, the pairs of alternatives, and the experts. The essence of it is to find out a pair of alternatives that should be modified.

(a) Identification rule for the alternatives: the set of alternatives with an associated consensus level which is lower than the predefined consensus threshold θ is denoted as AIR, and it is utilized to identify which row should be changed; the formula is as follows:

$$\text{AIR} = \{X_i \mid \min \{\text{ca}_i \mid \text{ca}_i < \theta, \quad i = 1, \ldots, n\}\}. \quad (32)$$

(b) Identification rule for the pairs of alternatives: according to the above analysis, we will get the $X_i \in$ AIR; then, this rule is conducted to identify the compared alternatives X_j and determine the position (i, j) that should be modified. In this way, these positions are denoted using the set Pos_i; the specific formula is as follows:

$$\text{Pos}_i = \left\{(i, j) \mid X_i \in \text{AIR} \cap cp_{ij} < \theta\right\}. \quad (33)$$

In the following, we will find which expert should change his ideas that involve the position (i, j).

(c) Identification rule for the experts: this identifies the set of experts that need to receive suggestion on how to change their preferences for each preference relation of Pos_i. To do this, the distance between e_f and the others should be determined firstly; it is given by

$$d_{ij}^{f} = m - 1 - \sum_{k=1,k\neq f}^{m} sm_{ij}^{fk}. \quad (34)$$

Thus, we know the distance between e_f and all the other e_k, $k \neq f$. Secondly, the set of experts, called EXPS_{ij}, is collected through the way of judging the distances; they are those whose preference relations are the most distant from all other experts' preferences in position (i, j); it is given as follows:

$$\text{EXPS}_{ij} = \left\{e_f(i, j) \in \text{Pos}_i \cap d_{ij}^{f} = \max \left\{d_{ij}^{k}\right\}\right\}. \quad (35)$$

Based on the above identification rules, both places at which preferences should be modified and the experts who need to change their thoughts are obtained; they can be summarized as follows:

$$\text{IRS} = \left\{(f, (i, j)) \mid e_f \in \text{EXPS}_{ij} \cap (i, j) \in \text{Pos}_i\right\}. \quad (36)$$

If nothing ever happened, we make the end of the algorithm.

Direction rules (DR), in order to improve the consensus level, are utilized to specify the direction in which decision-makers modify their preferences; they are designed as follows:

(a) DR.1: if $D_{kc}(\beta) < 0$, then e_k should increase his preference relations which involve the pair of alternatives (X_i, X_j).

(b) DR.2: if $D_{kc}(\beta) > 0$, then e_k should decrease his preference relations which involve the pair of alternatives (X_i, X_j).

In the above steps, $(k, (i, j)) \in \text{IRS}$ and p_{ij}^{c} is the group preference for position (i, j), which is computed by the aggregation function that is proposed previously. In the procedure of DR, a collective preference relation for position (i, j) exists as an objective of the reference.

Through the above analysis, we can draw the conclusion that

$$p_{ij,r+1}^{k} \in \left[\min \left\{p_{ij,r}^{k}, p_{ij,r}^{c}\right\}, \max \left\{p_{ij,r}^{k}, p_{ij,r}^{k}\right\}\right], \quad (37)$$

where the experts e_k should modify their ideas for the pair of alternatives (X_i, X_j) in the rth interaction, $p_{ij,r}^{c}$ is the collective preference relations for the rth round, and $p_{ij,r+1}^{k}$ and $p_{ij,r}^{k}$ are the $r + 1$th and the rth interaction, respectively, of the preferences for e_k.

Theorem 5 [46]. *If the identification rules and the direction rules are applied in the consensus-reaching process based on the consensus degrees, for any alternative X_i for which the related preferences need to be revised, the following result is as follows:*

$$\text{ca}_{i,r+1} > \text{ca}_{i,h}, \quad (38)$$

that is, the consensus level in the $(r + 1)$ round is higher than that in the round; in other words, the proposed procedure is convergent.

In order to prove the above conclusions, we use the analytical method; to prove $\text{ca}_{i,r+1} > \text{ca}_{i,h}$, just prove $cp_{ij,r+1} >$

$cp_{ij,h}$ and then $cm_{ij,r+1} > cm_{ij,h}$. From the consensus matrix calculation formula, we shall prove

$$\frac{\sum_{f=1}^{m-1}\sum_{k=f+1}^{m} sm_{ij,r+1}^{fk}}{m(m-1)/2} > \frac{\sum_{f=1}^{m-1}\sum_{k=f+1}^{m} sm_{ij,r}^{fk}}{m(m-1)/2}. \quad (39)$$

Further, according to (26), it is equivalent to

$$\frac{\sum_{f=1}^{m-1}\sum_{k=f+1}^{m}\left\{1/n^*\left(1-\left|D_{ij,r+1}^{fk}(\beta)\right|\right)\right\}}{m(m-1)/2} > \frac{\sum_{f=1}^{m-1}\sum_{k=f+1}^{m}\left\{1/n^*\left(1-\left|D_{ij,r}^{fk}(\beta)\right|\right)\right\}}{m(m-1)/2}. \quad (40)$$

Then, for the purpose of simplifying the proof process, suppose that e_1 is just the expert who needs to modify his preference for the pair (i,j). Thus, only the similarity degree that involves e_1 needs to be recalculated, so we only need to prove that

$$\frac{1}{n}\left(\left|D_{ij,r+1}^{12}(\beta)\right| + \left|D_{ij,r+1}^{13}(\beta)\right| + \cdots + \left|D_{ij,r+1}^{1m}(\beta)\right|\right) < \frac{1}{n}\left(\left|D_{ij,r}^{12}(\beta)\right| + \left|D_{ij,r}^{13}(\beta)\right| + \cdots + \left|D_{ij,r}^{1m}(\beta)\right|\right). \quad (41)$$

The decision-maker who needs to modify his preferences in the position (i,j) will provide new preference relation based on the discrete number that is closer to the collective preference relation involving the position (i,j) in the rth round, for the reason that the collective preference relations are determined through the way of aggregating all the individual preference relations, so we can say that in the $r+1$ round, new preference in the position (i,j) is closer to others, accordingly. The above analysis process implies what (38) holds. The proof is as follows:

We suppose that $d(P_{ij,r}^k) = (1/n)|D_{ij,r}^{k0}(\beta)|$ and $P^0 = \{1/0\}$ firstly.

From (37), we can find $\lambda \in (0,1)$ such that

$$d\left(P_{ij,r+1}^k\right) = \lambda d\left(P_{ij,r}^k\right) + (1-\lambda) d\left(P_{ij,r}^c\right), \quad (42)$$

$$\frac{1}{n}\left|D_{ij,r+1}^{12}(\beta)\right| = \left|d\left(P_{ij,r}^1\right) - d\left(P_{ij,r}^2\right)\right|. \quad (43)$$

In this case, to determine (41), we need to prove

$$\begin{aligned}&\left|d\left(P_{ij,r+1}^1\right) - d\left(P_{ij,r+1}^2\right)\right| + \left|d\left(P_{ij,r+1}^1\right) - d\left(P_{ij,r+1}^3\right)\right| + \cdots + \left|d\left(P_{ij,r+1}^1\right) - d\left(P_{ij,r+1}^m\right)\right| \\ &< \left|d\left(P_{ij,r}^1\right) - d\left(P_{ij,r}^2\right)\right| + \left|d\left(P_{ij,r}^1\right) - d\left(P_{ij,r}^3\right)\right| + \cdots + \left|d\left(P_{ij,r}^1\right) - d\left(P_{ij,r}^m\right)\right|.\end{aligned} \quad (44)$$

According to (42), it follows that

$$\begin{aligned}&\left|d\left(P_{ij,r+1}^1\right) - d\left(P_{ij,r+1}^2\right)\right| \\ &= \left|\lambda d\left(P_{ij,r}^1\right) + (1-\lambda) d\left(P_{ij,r}^c\right) - \lambda d\left(P_{ij,r}^2\right) - (1-\lambda)\mathrm{d}\left(P_{ij,r}^2\right)\right| \\ &= \left|\lambda d\left(P_{ij,r}^1\right) - \lambda d\left(P_{ij,r}^2\right) + \frac{1-\lambda}{m} \cdot \left[\left(d\left(P_{ij,r}^1\right) - d\left(P_{ij,r}^2\right)\right) + \left(d\left(P_{ij,r}^3\right) - d\left(P_{ij,r}^2\right)\right) + \cdots + \left(d\left(P_{ij,r}^m\right) - d\left(P_{ij,r}^2\right)\right)\right]\right|.\end{aligned} \quad (45)$$

Therefore, we have

$$\begin{aligned}&\frac{1}{n}\left(\left|D_{ij,r+1}^{12}(\beta)\right| + \left|D_{ij,r+1}^{13}(\beta)\right| + \cdots + \left|D_{ij,r+1}^{1m}(\beta)\right|\right) \\ &= \left|\lambda d\left(P_{ij,r}^1\right) - \lambda d\left(P_{ij,r}^2\right) + \frac{1-\lambda}{m}\left[\left(d\left(P_{ij,r}^1\right) - d\left(P_{ij,r}^2\right)\right) + \left(d\left(P_{ij,r}^3\right) - d\left(P_{ij,r}^2\right)\right) + \cdots + \left(d\left(P_{ij,r}^m\right) - d\left(P_{ij,r}^2\right)\right)\right]\right| \\ &+ \cdots + \left|\lambda d\left(P_{ij,r}^1\right) - \lambda d\left(P_{ij,r}^m\right) + \frac{1-\lambda}{m}\left[\left(d\left(P_{ij,r}^1\right) - d\left(P_{ij,r}^m\right)\right) + \left(d\left(P_{ij,r}^3\right) - d\left(P_{ij,r}^m\right)\right) + \cdots + \left(d\left(P_{ij,r}^{m-1}\right) - d\left(P_{ij,r}^m\right)\right)\right]\right| \\ &= \left(\lambda + (1-\lambda)\frac{m-1}{m}\right)\frac{1}{n} \cdot \left(\left|D_{ij,r}^{12}(\beta)\right| + \left|D_{ij,r}^{13}(\beta)\right| + \cdots + \left|D_{ij,r}^{1m}(\beta)\right|\right) \\ &< \left(\left|D_{ij,r}^{12}(\beta)\right| + \left|D_{ij,r}^{13}(\beta)\right| + \cdots + \left|D_{ij,r}^{1m}(\beta)\right|\right).\end{aligned} \quad (46)$$

In summary, we can show that Theorem 5 is reasonable and correct.

5. Illustrative Example

In dermatology, it is a tough choice to look for the best treatment for a patient with a severe skin lesion for the reason that it does not only depend on the disease [49]. In order to get the most appropriate treatment for the individual patient, there are a lot of factors to consider, such as the patient's physical condition, character, and beliefs. Although there are some more mature treatment options, it is possible that apparently successful and well-motivated treatments can fail to cure the disease. Many dermatologists are committed to it, and they need to express their opinions in a flexible way to solve this complex issue. Therefore, with the help of the hesitant fuzzy linguistic model based on the discrete fuzzy number, we can find a well solution to the problem in the case of ensuring the consensus of the experts.

Suppose that there are three dermatologists $E = \{e_1, e_2, e_3\}$ and three alternative treatment regimens $X = \{X_1, X_2,$

$X_3\}$, where X_1 is the photodynamic therapy, X_2 is the isotretinoin, and X_3 is the large acne cyst removal. Each dermatologist provides his hesitant fuzzy linguistic preference relations using the linguistic scale L_8.

$$\mathscr{L}=\left\{\begin{array}{c}\text{extremely poor, very poor, poor, slightly poor,}\\ \text{fair, slightly good, good, very good, extremely good}\end{array}\right\},$$
$$\mathscr{L}=\{0,1,2,3,4,5,6,7,8\}. \tag{47}$$

The experts provide their pairwise comparison of X_i over X_j as follows:

$$\begin{aligned}
P_1&=\begin{pmatrix} - & P_1^{12} & P_1^{13}\\ P_1^{21} & - & P_1^{23}\\ P_1^{31} & P_1^{32} & -\end{pmatrix},\\
P_2&=\begin{pmatrix} - & P_2^{12} & P_2^{13}\\ P_2^{21} & - & P_2^{23}\\ P_2^{31} & P_2^{32} & -\end{pmatrix},\\
P_3&=\begin{pmatrix} - & P_3^{12} & P_3^{13}\\ P_3^{21} & - & P_3^{23}\\ P_3^{31} & P_3^{32} & -\end{pmatrix},\\
P_1^{12}&=\left\{\frac{0.5}{0},\frac{0.5}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\},\\
P_1^{13}&=\left\{\frac{0.6}{6},\frac{0.6}{7},\frac{1}{8}\right\},\\
P_1^{23}&=\left\{\frac{0.3}{2},\frac{0.3}{3},\frac{0.6}{4},\frac{0.6}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8}\right\},\\
P_2^{12}&=\left\{\frac{1}{0},\frac{1}{1},\frac{0.7}{2}\right\},\\
P_2^{13}&=\left\{\frac{0.5}{0},\frac{0.8}{1},\frac{1}{2},\frac{0.9}{3}\right\},\\
P_2^{23}&=\left\{\frac{0.6}{5},\frac{0.8}{6},\frac{1}{7},\frac{0.7}{8}\right\},\\
P_3^{12}&=\left\{\frac{0.4}{0},\frac{0.5}{1},\frac{1}{2},\frac{0.6}{3}\right\},\\
P_3^{13}&=\left\{\frac{0.6}{3},\frac{0.7}{4},\frac{1}{5},\frac{0.9}{6}\right\},\\
P_3^{23}&=\left\{\frac{0.7}{5},\frac{0.8}{6},\frac{1}{7},\frac{1}{8}\right\},\\
P_e^{ij}&=N\left(P_e^{ij}\right),\quad e=1,2,3,\ i,j=1,2,3,\ i>j,\\
P_e^{11}&=P_e^{22}=P_e^{33}=\left\{\frac{1}{4}\right\}.
\end{aligned} \tag{48}$$

5.1. Application of the Consensus Process. First, the proposed consensus improvement is applied to achieve the predefined consensus degree; we set $\theta = 0.8$. Then, the consistency level for each matrix is computed in the following:

$$\begin{aligned}
\mathrm{CI}(P_1)&=0.0919,\\
\mathrm{CI}(P_2)&=0.0247,\\
\mathrm{CI}(P_3)&=0.0338.
\end{aligned} \tag{49}$$

From Table 2, we can know that they all satisfy the requirement of consistency, so the consensus process is directly applied.

5.1.1. First Cycle. From (26) and (27), we obtain the similarity matrices between dermatologists as follows:

$$\begin{aligned}
\mathbf{SM}_{12}&=\begin{pmatrix} - & 0.7653 & 0.3100\\ 0.7653 & - & 0.9800\\ 0.3100 & 0.9800 & -\end{pmatrix},\\
\mathbf{SM}_{13}&=\begin{pmatrix} - & 0.9188 & 0.6688\\ 0.9188 & - & 0.9600\\ 0.6688 & 0.9600 & -\end{pmatrix},\\
\mathbf{SM}_{23}&=\begin{pmatrix} - & 0.8700 & 0.6250\\ 0.8700 & - & 0.9750\\ 0.6250 & 0.9750 & -\end{pmatrix}.
\end{aligned} \tag{50}$$

According to (28), the consensus matrix is obtained as

$$\mathbf{CM}_1=\begin{pmatrix} - & 0.8514 & 0.5346\\ 0.8514 & - & 0.9717\\ 0.5346 & 0.9717 & -\end{pmatrix}. \tag{51}$$

The consensus levels of the above matrices are computed in the following three steps.

(a) Consensus level for the pair of alternatives: we obtain

$$cp_{ij}=cm_{ij},\quad i,j=1,2,3,\ i\neq j. \tag{52}$$

(b) Consensus level for the alternatives: from (30), we can compute that

$$\begin{aligned}
\mathrm{ca}_1&=0.6930,\\
\mathrm{ca}_2&=0.9116,\\
\mathrm{ca}_3&=0.7532.
\end{aligned} \tag{53}$$

(c) Consensus level for the preference relations: from (31), it follows that

$$\mathrm{cr}=\min\{\mathrm{ca}_i\}=0.6930. \tag{54}$$

Since $\mathrm{cr} < \theta$, in order to improve the consensus degree, a feedback mechanism should be conducted to assist the decision-makers in modifying their preferences.

According to the proposed measures in Section 4.2, identification rules and direction rules are developed as follows:

(a) Identification rule for the alternatives: through (32), we can find that

$$\mathrm{AIR} = \{X_i \in X \mid \min \mathrm{ca}_i\} = \{X_1\}. \tag{55}$$

(b) Identification rule for the pairs of alternatives: for X_1,

$$\mathrm{Pos}_1 = \left\{(1, j) \mid X_1 \in \mathrm{ALT} \cap cp_{1j} < 0.8\right\} = (1, 3). \tag{56}$$

Due to the reciprocal property, the positions that should be changed are

$$\mathrm{Pos} = \{(1, 3), (3, 1)\}. \tag{57}$$

(c) Identification rule for the decision-makers: according to (34), the elements in the upper triangle are checked in the following:

$$\begin{aligned} d_{13}^1 &= 1.0212, \\ d_{13}^2 &= 1.0650, \\ d_{13}^3 &= 0.7062. \end{aligned} \tag{58}$$

Therefore, we have

$$\mathrm{EXPS}_{13} = \{e_2\}. \tag{59}$$

From (6), we set the $k = 4$; the collective preference relation in position $(1, 3)$ is as follows:

$$\begin{aligned} P_c^{13} &= \left\{\frac{0.6}{2}, \frac{0.6}{3}, \frac{1}{4}\right\}, \\ P_2^{13} &= \left\{\frac{0.5}{0}, \frac{0.8}{1}, \frac{1}{2}, \frac{0.9}{3}\right\}. \end{aligned} \tag{60}$$

Let $\beta = 0.6$; from (10), we can compute that $D_{13}^{2c}(\beta) < 0$, so if $P_2^{13} < P_c^{13}$, the dermatologist e_2 should increase his assessments. Suppose that he provides the following new preferences:

$$P_2^{13} = \left\{\frac{0.5}{3}, \frac{0.8}{4}, \frac{1}{5}, \frac{0.9}{6}\right\}. \tag{61}$$

Accordingly,

$$P_2^{31} = \left\{\frac{0.9}{2}, \frac{1}{3}, \frac{0.8}{4}, \frac{0.5}{5}\right\}. \tag{62}$$

5.1.2. Second Cycle. The same as the first cycle, we get the following similarity matrices:

$$\begin{aligned} \mathbf{SM}_{12} &= \begin{pmatrix} - & 0.7563 & 0.6850 \\ 0.7563 & - & 0.9800 \\ 0.6850 & 0.9800 & - \end{pmatrix}, \\ \mathbf{SM}_{13} &= \begin{pmatrix} - & 0.9188 & 0.6688 \\ 0.9188 & - & 0.9600 \\ 0.6688 & 0.9600 & - \end{pmatrix}, \\ \mathbf{SM}_{23} &= \begin{pmatrix} - & 0.8700 & 1.0000 \\ 0.8700 & - & 0.9750 \\ 1.0000 & 0.9750 & - \end{pmatrix}. \end{aligned} \tag{63}$$

Then, the consensus matrix is computed:

$$\mathbf{CM}_2 = \begin{pmatrix} - & 0.8514 & 0.7846 \\ 0.8514 & - & 0.9717 \\ 0.7846 & 0.9717 & - \end{pmatrix}. \tag{64}$$

The consensus degrees for the matrices on three aspects are in the following:

(a) Consensus degree for the pair of alternatives: we obtain

$$cp_{ij} = cm_{ij}, \quad i, j = 1, 2, 3, i \neq j. \tag{65}$$

(b) Consensus degree for the alternatives: from (30), we can compute that

$$\begin{aligned} \mathrm{ca}_1 &= 0.8180, \\ \mathrm{ca}_2 &= 0.9116, \\ \mathrm{ca}_3 &= 0.8782. \end{aligned} \tag{66}$$

(c) Consensus degree for the preference relations: from (31), it follows that

$$\mathrm{cr} = \min\{\mathrm{ca}_i\} = 0.8180. \tag{67}$$

Because $\mathrm{cr} > 0.8$, the consensus process is in the end. The consistency levels for the modified matrices are

$$\begin{aligned} \mathrm{CI}(P_1) &= 0.0919, \\ \mathrm{CI}(P_2) &= 0.0201, \\ \mathrm{CI}(P_3) &= 0.0338. \end{aligned} \tag{68}$$

Obviously, they are still of acceptable consistency.

Then, we can obtain a ranking of the alternatives from (10). Firstly, from (6), we determine the collective preference relation as follows:

$$P_c = \begin{pmatrix} - & p_c^{12} & p_c^{13} \\ p_c^{21} & - & p_c^{23} \\ p_c^{31} & p_c^{32} & - \end{pmatrix},$$
$$p_c^{12} = \left\{\frac{1}{0}, \frac{1}{1}, \frac{0.7}{2}\right\},$$
$$p_c^{13} = \left\{\frac{0.6}{3}, \frac{1}{4}, \frac{0.9}{5}, \frac{0.8}{6}\right\},$$
$$p_c^{23} = \left\{\frac{0.3}{2}, \frac{0.3}{3}, \frac{0.6}{4}, \frac{0.7}{5}, \frac{1}{6}, \frac{1}{7}, \frac{0.7}{8}\right\}. \quad (69)$$

Then, we compute for each preference the values of the choice functions:

$$P_{X_1} = \mathcal{F}\left(p_c^{12}, p_c^{13}\right) = \left\{\frac{1}{0}, \frac{1}{1}, \frac{0.8}{2}\right\},$$
$$P_{X_2} = \mathcal{F}\left(p_c^{21}, p_c^{23}\right) = \left\{\frac{0.3}{2}, \frac{0.3}{3}, \frac{0.6}{4}, \frac{0.7}{5}, \frac{1}{6}, \frac{1}{7}, \frac{0.7}{8}\right\},$$
$$P_{X_3} = \mathcal{F}\left(p_c^{31}, p_c^{32}\right) = \left\{\frac{0.3}{2}, \frac{0.6}{3}, \frac{1}{4}, \frac{0.9}{5}, \frac{0.3}{6}\right\}. \quad (70)$$

Remark 2. To date, there have been few papers that have achieved both predefined consistency and reasonable consensus. While during the consensus-reaching process, the consistency index may become unacceptable. There are possible two ways to solve this situation. One is to conduct new direction rules in the feedback mechanism, and the other is to develop a new cycle of the consistency measure and consensus improvement process; however, how to fully deal with both of them is still a question, which still needs to be studied.

From Table 4, we can see that the ranking order for the treatments after the modification is changed, the reason for it greatly depends on the dermatologist e_2 and his revised preference, and the group ranking after modification is more reasonable and convincing because of the consensus level. The result also proves the correctness of the consensus process; in order to improve the consensus, the dermatologist e_2 should provide the preference that is closer to that of other experts in the position $(1, 3)$. In the consensus process, the experts and the position that contribute the least to the consensus level are found. At the same time, the proposed mechanism guides the direction for experts to revise their preferences through the above two rules.

5.2. Comparative Study

5.2.1. Comparison with the Consensus Method with Calculation of the Proximity Measures. We examine the consensus-improving process in [40] to implement a comparative study. Considering the same situation, since the consensus degrees computed did not reach the predefined level in the first round, proximity measures are conducted to determine the distance between individual preferences and group preference to find experts and preferences that need to be modified. The group preference and proximity matrix $\mathbf{PM}_k$ for e_k, $k = 1, 2, 3$, are as follows:

$$P_c = \begin{pmatrix} - & p_c^{12} & p_c^{13} \\ p_c^{21} & - & p_c^{23} \\ p_c^{31} & p_c^{32} & - \end{pmatrix},$$
$$p_c^{12} = \left\{\frac{1}{0}, \frac{1}{1}, \frac{0.7}{2}\right\},$$
$$p_c^{13} = \left\{\frac{0.6}{2}, \frac{0.6}{3}, \frac{1}{4}\right\},$$
$$p_c^{23} = \left\{\frac{0.3}{2}, \frac{0.3}{3}, \frac{0.6}{4}, \frac{0.7}{5}, \frac{1}{6}, \frac{1}{7}, \frac{0.7}{8}\right\},$$
$$\mathbf{PM}_1 = \begin{pmatrix} - & 0.7583 & 0.5000 \\ 0.7583 & - & 0.9688 \\ 0.5000 & 0.9688 & - \end{pmatrix},$$
$$\mathbf{PM}_2 = \begin{pmatrix} - & 1.0000 & 0.8100 \\ 1.0000 & - & 0.9400 \\ 0.8100 & 0.9400 & - \end{pmatrix},$$
$$\mathbf{PM}_3 = \begin{pmatrix} - & 0.8700 & 0.8313 \\ 0.8700 & - & 0.9200 \\ 0.8313 & 0.9200 & - \end{pmatrix}. \quad (71)$$

Then, the proximity measures are calculated on three aspects.

(a) Proximity for the pairs of alternatives: this result for e_k is given in $\mathbf{PM}_k$.

(b) Proximity for the alternatives: for e_k, let pa_i^k be the proximity distance for the alternative X_i between their preferences on that alternative and the collective preferences.

$$\begin{aligned} \mathrm{pa}_1^1 &= 0.6291, \\ \mathrm{pa}_2^1 &= 0.8636, \\ \mathrm{pa}_3^1 &= 0.7344, \\ \mathrm{pa}_1^2 &= 0.9050, \\ \mathrm{pa}_2^2 &= 0.9700, \\ \mathrm{pa}_3^2 &= 0.8750, \\ \mathrm{pa}_1^3 &= 0.8506, \\ \mathrm{pa}_2^3 &= 0.8950, \\ \mathrm{pa}_3^3 &= 0.8756. \end{aligned} \quad (72)$$

(c) Proximity for the preference relations: This step, denoted as pr_k for e_k, is defined to compute the global

Table 4: The rankings before and after the modification.

Preference relation	The original ranks	The modified ranks	The ranks of [41]
P_1	$X_2 \succ X_1 \succ X_3$	$X_2 \succ X_1 \succ X_3$	$X_2 \succ X_3 \succ X_1$
P_2	$X_2 \succ X_3 \succ X_1$	$X_2 \succ X_3 \succ X_1$	$X_2 \succ X_3 \succ X_1$
P_3	$X_2 \succ X_1 \succ X_3$	$X_2 \succ X_1 \succ X_3$	$X_2 \succ X_1 \succ X_3$
P_c	$X_2 \succ X_1 \succ X_3$	$X_2 \succ X_3 \succ X_1$	$X_2 \succ X_3 \succ X_1$

proximity between their preferences and the collective preferences for all alternatives.

$$pr_1 = 0.7423,\quad pr_2 = 0.9167,\quad pr_3 = 0.8737. \tag{73}$$

According to the identification rules in [40], it is found that the expert e_1 should decrease his preference relation for the position $(1, 3)$, and suppose that e_1 provides the following preferences:

$$P_1^{13} = \frac{0.6}{3}, \frac{0.6}{4}, \frac{1}{5}. \tag{74}$$

The consensus levels for the alternatives in the second round are

$$\mathrm{ca}_1 = 0.8000,\quad \mathrm{ca}_2 = 0.9116,\quad \mathrm{ca}_3 = 0.8594. \tag{75}$$

Obviously, a minimum consensus level of 0.8 is achieved. The ranks after this consensus-reaching process are given in Table 4. Only the expert e_1 changed his preferences, and accordingly, the individual ranks for P_1 have changed. Note that the individual ranks using the proposed method and [41] differ only for P_1, but the above three methods obtained the same final ranks, $X_2 \succ X_3 \succ X_1$.

Although the identified expert and pair of alternatives that should be modified are different, both consensus-reaching processes can achieve a threshold consensus level; however, there are some differences between the proposed measure and the method in [40]. On the one hand, the proposed feedback mechanism depends directly on the similarity calculation formula, while [40] requires the proximity matrix measures, increasing the complexity of the calculation process and the difficulty. On the another hand, the proposed method develops very different identification rules in which we first find the pair of alternatives that work greatly against the consensus, followed by the decision-makers, but the method in [40] works exactly the opposite. The reason we do this is to avoid misunderstanding that one (decision-maker) is regarded as an authority. In all, the proposed method provides an efficient and new way to deal with GDM problems when consensus level is considered.

5.2.2. Comparison with the Consensus Method with Probabilistic Linguistic Preference Relations. When we normalize the memberships of linguistic variables in one HFLTS based on discrete fuzzy numbers, it will be only described as the form of PLPRs. However, substantially, they are different in both expression and computation. For expression, firstly, for discrete fuzzy numbers, the membership mapping increases till the core, where they are equal to 1, and then, they decrease. It is reasonable, because decision-makers always think that the suitable grades lie in several consecutive linguistic terms, but they cannot discard other grades around them in some level. For PLPRs, the degree of every membership is arbitrary without the restriction of tendency. But the sum of all memberships is 1 or less than 1. Discrete fuzzy numbers have no such restriction. So the decision-makers have more flexibility to express their opinions using discrete fuzzy numbers. Secondly, discrete fuzzy numbers can be switched to PLPRs with no less of information, but not vice versa. Thirdly, for PLPRs, it is hard to obtain complete information of probabilistic distribution of all possible linguistic terms because of not enough knowledge. Thus, probabilistic information is partially known, and the sum of all memberships are always less than one. Obviously, the normalization of PLPRs reduces the accuracy of evaluation.

And for computation, the research on consistency and consensus of probabilistic linguistic preference relations can be seen in literature [50–52]. For PLPRs and linguistic preference relations based on discrete fuzzy numbers, both core ideas of obtaining and improving consistency and consensus are coincident. Their primary differences behave in two aspects. One is the distance computation approach of linguistic preference relations. PLPRs adopt the generalized form of the Hamming distance and the Euclidean distance. Linguistic preference relations based on discrete fuzzy numbers which the paper discusses take the left (right) dominance of one evaluation over another to measure similarity. Compared with Hamming distance and Euclidean distance, the computation is easier. The other difference is the feedback mechanism. Our proposed approach first searches the pair of alternatives which need to be adjusted and then identifies decision-makers who need to change their preferences, contrary to the approach for improving consensus of PLPRs.

6. Analysis on the Threshold Values

In order to assist the decision-maker in setting the threshold of group consensus level reasonably, let us consider it with the corresponding simulation method by MATLAB, which is inspired by [53].

We use the following two formulas to calculate the value of the threshold:

$$\mathrm{GCI}(P_l, P_k) = 1 - \frac{1}{n(n-1)} \sum_{i,j=1,i<j}^{n} \frac{1}{8} \left| D_{ij}^{lk}(\beta) \right|,$$
$$\theta \geq \min \mathrm{GCI}(P_l, P_c) \geq \min \min\mathrm{GCI}(P_l, P_k). \tag{76}$$

The main principle is to obtain the linguistic preference relations randomly; the simulation method is as follows:

Input: m and n.

Output: θ.

Step 1. Generate m linguistic preference relations with n orders, which are randomly selected from $[0, 8]$.

Step 2. Use (76) to obtain the value of θ.

Step 3. Output θ.

Simulating 10,000 times, we obtained the average values of θ for linguistic preference relations with different numbers of experts as shown in Table 5

Example 5. Suppose that there are four experts $E = \{e_1, e_2, e_3, e_4\}$ and four alternatives $X = \{X_1, X_2, X_3, X_4\}$; they make efforts to solve a decision problem based on their own knowledge. When they adjust consensus, they should set the threshold that is greater than 0.6875 according to simulation results.

Remark 3. According to Table 5, we can see that with the increase in the number of experts, the group consensus of the threshold value is gradually declining. Meanwhile, the θ of a linguistic preference relation should not be smaller than the average value in Table 5, and the decision-maker should make a balance between the actual requirement and the result of the simulation method. The simulation results provide a lower limit for setting the threshold. One thing to note is that setting a threshold too high will make experts' preferences change largely, thus losing their originality seriously. There is another point to note; the number of alternatives affects the variation of the threshold value. In the above simulation analysis, we assume that the number of alternatives is a constant and there are four alternatives. We will study the influences of other parameters on the threshold value in the future.

7. Extended Discussion

With the increasing complexity of decision-making problems, LGDM has attracted widespread concern [54, 55]. The number of decision-makers in LGDM problems is very large. We make a discussion about adjusting consistency and consensus in a LGDM problem. There is an extended k-means clustering method, which is described in Algorithm 2 [56].

Assume that there are three subgroups after clustering, so the subsequent calculation process is the same as the method proposed in this paper. First, we aggregate the HFLPRs in each subgroup, which can be obtained from Algorithm 2. Then, we adjust consistency and consensus of the aggregated matrices. Assume that the obtained clusters will not change. We apply the proposed method directly on the aggregation matrix, which is the same as the previous analysis process, so it is omitted here.

8. Concluding Remarks

Since there are few literatures to study both the consistency and consensus of preference relations based on the discrete fuzzy numbers, the proposed method in this paper is necessary and meaningful. With the help of the HFLPR model based on the discrete fuzzy number, decision-makers have more flexibility when expressing their preference in pairwise comparisons. This paper focused on the consistency and consensus in a given linguistic model, which play increasingly important roles in the group decision-making. The calculation of these two indices is highly dependent on the theory of aggregation function and distance formula. The contributions of this paper are as follows:

(1) We developed a consistency measure and proposed an optimization algorithm based on the distance between the given matrix and the complete consistency matrix to increase the consistency index for a linguistic model, and Example 4 proves the feasibility of the algorithm. This approach makes the linguistic model more standardized and improves the accuracy of the final ranking from the perspective of interactive design and improving consistency level. And the revised preferences are still HFLTs that contain original linguistic terms, so it is easy to interpret. Since there is little literature studying the consistency of hesitation fuzzy linguistic sets based on the discrete fuzzy number, the proposed method is meaningful.

(2) We defined a direct consensus-reaching process to assist the decision-maker who needs to reconsider his preferences for the purpose of achieving the predefined consensus degree. This method builds a feedback mechanism to find out the inappropriate preferences and guide the direction of modification. The direction of the modification is based on the similarity between experts; therefore, it is simple to calculate. Further, we conducted the analysis on the threshold values, which provide a reference for the development of the critical value of consensus level.

(3) The illustrative example confirmed that the approaches are convincing; it addressed both consistency and consensus. And comparative analyses were given to discuss the advantages and performance of the proposed method.

(4) We proposed a simulation method to help the decision-maker judge the group consensus in group decision-making with HFLPR. With the increase in the number of experts, the threshold consensus level should be gradually reduced, and the experimental result also proves this assumption, as it provides a clear minimum constraint value. There are also some extensions to LGDM problems.

In the future, we will study the GDM involving multigranular linguistic terms for different decision-makers and the

Table 5: Average values of θ.

M	3	4	5	6	7	8	9	10
θ	0.7147	0.6875	0.6702	0.6586	0.6501	0.6430	0.6373	0.6326

Input: HFLPRs $P_k = (p_{ij}^k)_{n\times n}$, $k = 1, 2, \ldots, m$, and the number of clusters t $(t \geq 2)$.
Output: Clusters $G_1, G_2, \ldots, G_t$.
Step 1. $P_k = (p_{12}^k, p_{13}^k, \ldots, p_{(n-1)n}^k)$, which keeps upper triangular elements of each HFLPR. Then, let $\lambda = n(n-1)/2$, so $P_k = (p_1^k, p_2^k, \ldots, p_\lambda^k)$.
Step 2. Chose t cluster centers as the initial mean vectors randomly, that is, $\{\psi_1, \psi_2, \ldots, \psi_t\}$. Let $\psi_t = (\psi_1^t, \psi_2^t, \ldots, \psi_\lambda^t)$.
Step 3. Assign each P_k to the closest cluster center. Firstly, calculate $d_{kt} = \|P_k - \psi_t\|_2$; then, we get $j = \arg\ \min_{i=1}^t d_{kt}$. Finally, we say $P_k \in G_j$.

$$d_{kt} = \|P_k - \psi_t\|_2 = \sqrt{\sum_{i=1}^{\lambda} (p_i^k - \psi_i^t)^2}.$$

Step 4. Using the membership information from the current clusters to recalculate the cluster center, assume that the new cluster centers are $\{\psi_1', \psi_2', \ldots, \psi_t'\}$ and $\psi_t' = (\psi_1^{t\prime}, \psi_2^{t\prime}, \ldots, \psi_\lambda^{t\prime})$.
Step 5. We set a convergence criterion, and let $\epsilon > 0$; that is, $E \leq \epsilon$.

$$E = \sum_{t=1}^{t} \|\psi_t - \psi_t'\|_2 = \sum_{t=1}^{t} \sqrt{\sum_{\lambda=1}^{\lambda} (\psi_\lambda^t - \psi_\lambda^{t\prime})^2}.$$

If the criterion is met, go to the next step. Otherwise, go to Step 3.
Step 6. Output $G_1, G_2, \ldots, G_t$ and $\{\psi_1, \psi_2, \ldots, \psi_t\}$.
Step 7. End.

Algorithm 2

effect of different aggregation functions on both the consistency and consensus calculation process.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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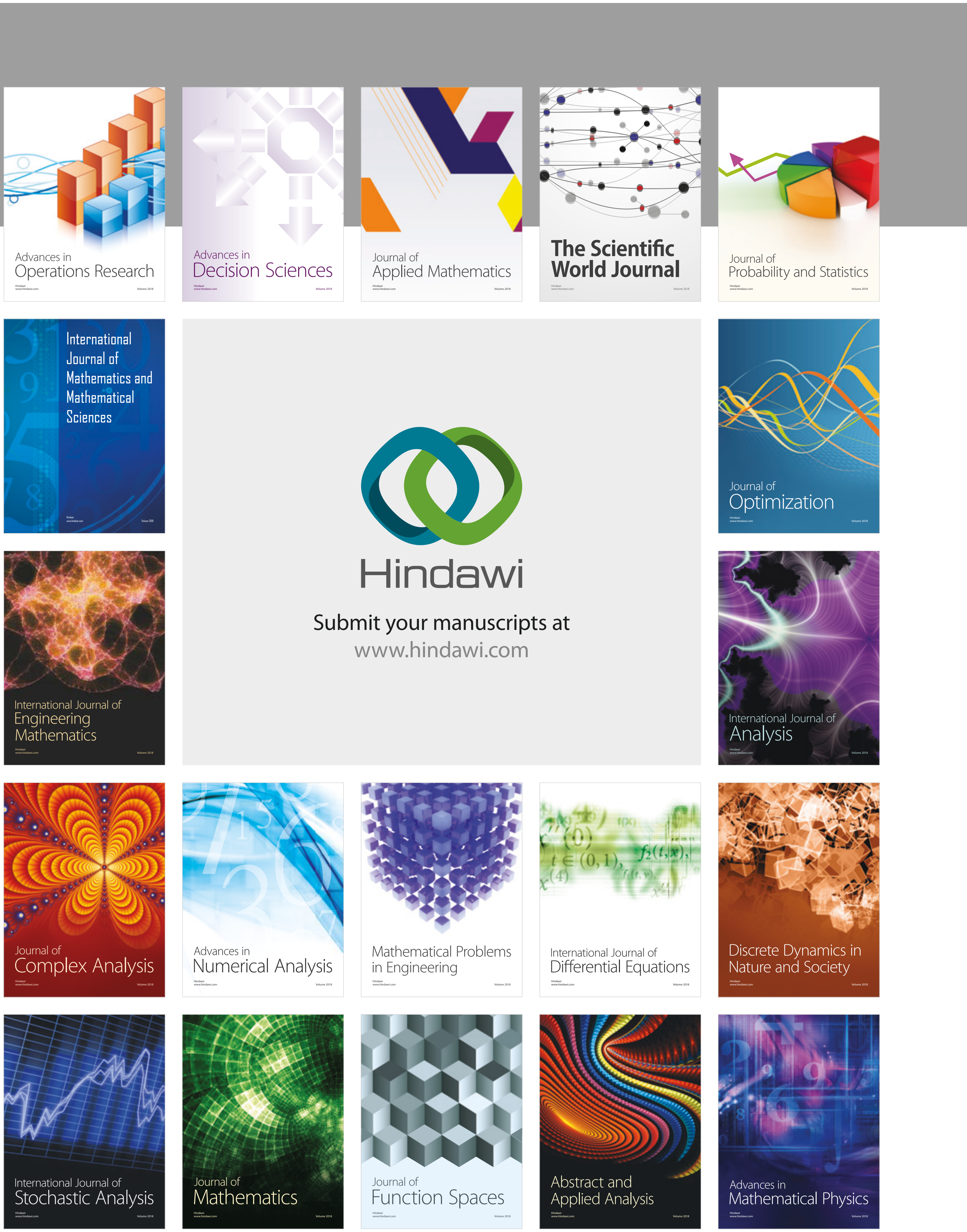
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