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Research Article

Analysis of a Generalized Lorenz–Stenflo Equation

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Although the globally attractive sets of a hyperchaotic system have important applications in the fields of engineering, science, and technology, it is often a difficult task for the researchers to obtain the globally attractive set of the hyperchaotic systems due to the complexity of the hyperchaotic systems. Therefore, we will study the globally attractive set of a generalized hyperchaotic Lorenz–Stenflo system describing the evolution of finite amplitude acoustic gravity waves in a rotating atmosphere in this paper. Based on Lyapunov-like functional approach combining some simple inequalities, we derive the globally attractive set of this system with its parameters. The effectiveness of the proposed methods is illustrated via numerical examples.

1. Introduction

In 1963, Lorenz found the well-known three-dimensional Lorenz model when he studied the dynamics of the atmosphere [1]. Since then, various complex dynamical behaviors of the Lorenz system have been studied by mathematicians, physicists, and engineers from various fields due to various applications in the fields of engineering, science, and technology [2–14]. In order to improve the stability or predictability of the Lorenz system, Stenflo and Leonov derived the following four-dimensional Lorenz–Stenflo system with four parameters to describe the dynamics of the atmosphere [15, 16]:

$$\begin{aligned}\frac{dx}{dt} &= a(y-x) + dw,\\ \frac{dy}{dt} &= cx - y - xz,\\ \frac{dz}{dt} &= xy - bz,\\ \frac{dw}{dt} &= -x - aw.\end{aligned} \tag{1}$$

In order to give a better description of the atmosphere, Chen and Liang propose a generalized Lorenz–Stenflo system with six parameters according to the Lorenz–Stenflo system [17]:

$$\begin{aligned}\frac{dx}{dt} &= a(y-x) + sw,\\ \frac{dy}{dt} &= cx - dy - xz,\\ \frac{dz}{dt} &= xy - bz,\\ \frac{dw}{dt} &= -x - rw,\end{aligned} \tag{2}$$

where x, y, z, and w are state variables and a, b, c, d, r, and s are positive parameters of system (2). System (2) can describe the dynamic behavior of finite amplitude acoustic gravity waves in a rotating atmosphere.

The Lyapunov exponents of the dynamical system (2) are calculated numerically for the parameter values $a = 19.42$, $b = 1.91$, $c = 29.45$, $d = 2.86$, $r = 0.23$, and $s = 9.64$ with the initial state $(x_0, y_0, z_0, w_0) = (2.2, 2.0, 10.5, 20)$. System (2) has Lyapunov exponents as $\lambda_{\text{LE}_1} = 0.0696$, $\lambda_{\text{LE}_2} = 0.0359$, $\lambda_{\text{LE}_3} = 0.0002$, and $\lambda_{\text{LE}_4} = -24.5176$ for the parameters listed above (see [18, 19] for a detailed discussion of Lyapunov exponents of strange attractors in dynamical systems). Thus,

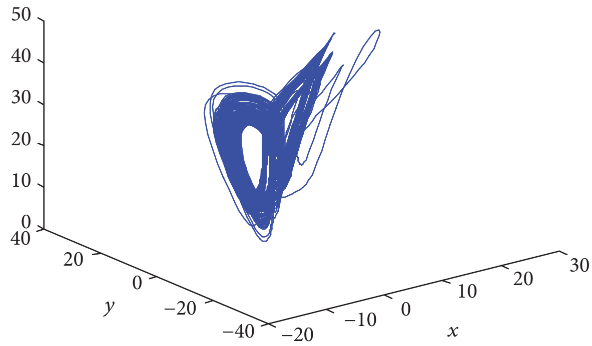

FIGURE 1: Projection of hyperchaotic attractor of system (2) onto the $xOyz$ space with $a = 19.42$, $b = 1.91$, $c = 29.45$, $d = 2.86$, $r = 0.23$, and $s = 9.64$.

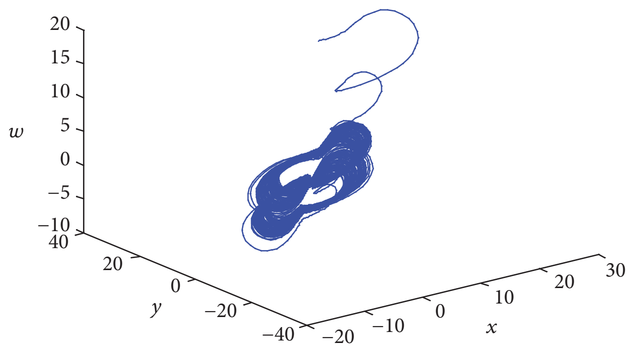

FIGURE 2: Projection of hyperchaotic attractor of system (2) onto the $xOyw$ space with $a = 19.42$, $b = 1.91$, $c = 29.45$, $d = 2.86$, $r = 0.23$, and $s = 9.64$.

system (2) has two positive Lyapunov exponents and the strange attractor, which means system (2) can exhibit a variety of interesting and complex chaotic behaviors. System (2) has a hyperchaotic attractor with $a = 19.42$, $b = 1.91$, $c = 29.45$, $d = 2.86$, $r = 0.23$, and $s = 9.64$, as shown in Figures 1–4.

In this paper, all the simulations are carried out by using fourth-order Runge-Kutta Method with time-step $h = 0.005$.

The rest of this paper is organized as follows. In Section 2, the globally attractive set for the chaotic attractors in (2) is studied using Lyapunov stability theory. To validate the ultimate bound estimation, numerical simulations are also provided. Finally, the conclusions are drawn in Section 3.

2. Bounds for the Chaotic Attractors in System (2)

Theorem 1. *For any $a > 0$, $b > 0$, $c > 0$, $d > 0$, $r > 0$, $s > 0$, there exists a positive number $M > 0$, such that*

$$\Omega_{\lambda,m} = \left\{ X \mid \lambda (x - m_1)^2 + my^2 + m\left(z - \frac{a\lambda + cm}{m}\right)^2 + \lambda s (w - m_3)^2 \le M,\ \forall\lambda > 0,\ \forall m > 0 \right\} \quad (3)$$

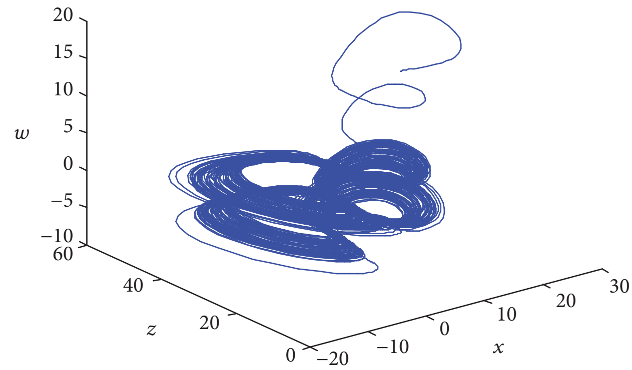

FIGURE 3: Projection of hyperchaotic attractor of system (2) onto the $xOzw$ space with $a = 19.42$, $b = 1.91$, $c = 29.45$, $d = 2.86$, $r = 0.23$, and $s = 9.64$.

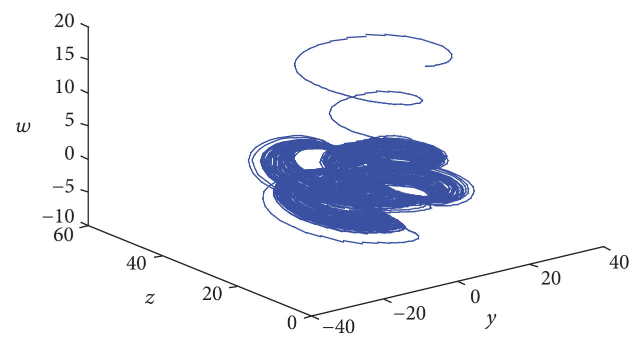

FIGURE 4: Projection of hyperchaotic attractor of system (2) onto the $yOzw$ space with $a = 19.42$, $b = 1.91$, $c = 29.45$, $d = 2.86$, $r = 0.23$, and $s = 9.64$.

is the ultimate bound set of system (2), where $X(t) = (x(t), y(t), z(t), w(t))$.

Proof. Define the following Lyapunov-like function:

$$\begin{aligned} V_{\lambda,m}(X) &= V_{\lambda,m}(x, y, z, w) \\ &= \lambda (x - m_1)^2 + my^2 + m\left(z - \frac{a\lambda + cm}{m}\right)^2 \\ &\quad + \lambda s (w - m_3)^2, \end{aligned} \quad (4)$$

where $\forall\lambda > 0, \forall m > 0, X(t) = (x(t), y(t), z(t), v(t), u(t), \omega(t))$, and $m_1 \in R$, $m_3 \in R$ are arbitrary constants.

And we can get

$$\begin{aligned} \left.\frac{dV_{\lambda,m}(X(t))}{dt}\right|_{(2)} &= 2\lambda (x - m_1)\frac{dx}{dt} + 2my\frac{dy}{dt} \\ &\quad + 2m\left(z - \frac{a\lambda + cm}{m}\right)\frac{dz}{dt} \end{aligned}$$

$$+ 2\lambda s (w - m_3) \frac{dw}{dt}$$
$$= 2\lambda (x - m_1)(ay - ax + sw)$$
$$+ 2my (cx - dy - xz)$$
$$+ 2m \left(z - \frac{a\lambda + cm}{m} \right)(xy - bz)$$
$$+ 2\lambda s (w - m_3)(-x - rw)$$
$$= -2a\lambda x^2 + 2(a\lambda m_1 + \lambda s m_3) x$$
$$- 2dmy^2 - 2a\lambda m_1 y - 2bmz^2$$
$$+ 2bmm_2 z - 2\lambda s r w^2$$
$$- 2(\lambda s m_1 - \lambda s r m_3) w. \quad (5)$$

Let $dV(X(t))/dt = 0$. Then, we can get the surface Γ:

$$X \mid -a\lambda x^2 + (a\lambda m_1 + \lambda s m_3) x - dmy^2 - a\lambda m_1 y - bmz^2 + bmm_2 z - \lambda s r w^2 - (\lambda s m_1 - \lambda s r m_3) w = 0 \quad (6)$$

is an ellipsoid in R^4 for $\forall \lambda > 0, \forall m > 0, a > 0, b > 0, c > 0, d > 0, r > 0, s > 0$. Outside Γ, $dV_{\lambda,m}(X(t))/dt < 0$, while inside Γ, $dV_{\lambda,m}(X(t))/dt > 0$. Thus, the ultimate boundedness for system (2) can only be reached on Γ. Since the Lyapunov-like function $V_{\lambda,m}(X)$ is a continuous function and Γ is a bounded closed set, then the function (4) can reach its maximum value $\max_{X\in\Gamma} V_{\lambda,m}(X) = M$ on the surface Γ that is defined in (6). Obviously, $\{X \mid V_{\lambda,m}(X) \le \max_{X\in\Gamma} V_{\lambda,m}(X) = M, X \in \Gamma\}$ contains solutions of system (2). It is obvious that the set $\Omega_{\lambda,m}$ is the ultimate bound set for system (2).

This completes the proof. □

Theorem 2. *Suppose that $\forall a > 0, b > 0, d > 0, r > 0, c > 0, s > 0, \lambda > 0, m > 0$.*

Let $(x(t), y(t), z(t), w(t))$ be an arbitrary solution of system (2) and

$$L^2_{\lambda,m} = \frac{1}{\theta} \left[\left(\frac{a^2\lambda^2}{md} + \frac{\lambda s}{r} + a\lambda \right) m_1^2 + bmm_2^2 + \left(\frac{\lambda s^2}{a} + \lambda s r \right) m_3^2 \right], \quad \theta = \min(a, b, d, r) > 0,$$
$$V_{\lambda,m}(X) = V_{\lambda,m}(x, y, z, w) = \lambda (x - m_1)^2 + my^2 + m\left(z - \frac{a\lambda + cm}{m} \right)^2 + \lambda s (w - m_3)^2,$$
$$\forall \lambda > 0,\ \forall m > 0,\ \forall m_1 \in R,\ \forall m_3 \in R. \quad (7)$$

Then the estimation

$$\left[V_{\lambda,m}(X(t)) - L^2_{\lambda,m} \right] \le \left[V_{\lambda,m}(X(t_0)) - L^2_{\lambda,m} \right] e^{-\theta(t-t_0)} \quad (8)$$

holds for system (2), and thus $\Omega_{\lambda,m} = \{X \mid V_{\lambda,m}(X) \le L^2_{\lambda,m}\}$ is the globally exponential attractive set of system (2); that is, $\overline{\lim}_{t\to+\infty} V_{\lambda,m}(X(t)) \le L^2_{\lambda,m}$.

Proof. Define the following functions:

$$f(x) = -a\lambda x^2 + 2\lambda s m_3 x,$$
$$h(y) = -dmy^2 - 2a\lambda m_1 y, \quad (9)$$
$$g(w) = -\lambda s r w^2 - 2\lambda s m_1 w.$$

then we can get

$$\max_{x\in R} f(x) = \frac{\lambda s^2 m_3^2}{a},$$
$$\max_{y\in R} h(y) = \frac{a^2\lambda^2 m_1^2}{dm}, \quad (10)$$
$$\max_{w\in R} g(w) = \frac{\lambda s m_1^2}{r}.$$

Construct the Lyapunov-like function

$$V_{\lambda,m}(X) = V_{\lambda,m}(x, y, z, w) = \lambda (x - m_1)^2 + my^2 + m\left(z - \frac{a\lambda + cm}{m} \right)^2 + \lambda s (w - m_3)^2,$$
$$\forall \lambda > 0,\ \forall m > 0,\ \forall m_1 \in R,\ \forall m_3 \in R. \quad (11)$$

Differentiating the above Lyapunov-like function $V_{\lambda,m}(X)$ in (11) with respect to time t along the trajectory of system (2) yields

$$\left.\frac{dV_{\lambda,m}(X(t))}{dt}\right|_{(2)} = 2\lambda (x - m_1) \frac{dx}{dt} + 2my \frac{dy}{dt}$$
$$+ 2m \left(z - \frac{a\lambda + cm}{m} \right) \frac{dz}{dt}$$
$$+ 2\lambda s (w - m_3) \frac{dw}{dt}$$
$$= 2\lambda (x - m_1)(ay - ax + sw)$$
$$+ 2my (cx - dy - xz)$$
$$+ 2m \left(z - \frac{a\lambda + cm}{m} \right)(xy - bz)$$
$$+ 2\lambda s (w - m_3)(-x - rw)$$
$$= -2a\lambda x^2 + 2(a\lambda m_1 + \lambda s m_3) x$$
$$- 2dmy^2 - 2a\lambda m_1 y - 2bmz^2$$
$$+ 2bmm_2 z - 2\lambda s r w^2$$

$$
\begin{aligned}
&- 2(\lambda s m_1 - \lambda s r m_3) w \\
&= -a\lambda x^2 + 2a\lambda m_1 x - a\lambda x^2 \\
&\quad + 2\lambda s m_3 x - dmy^2 - dmy^2 \\
&\quad - 2a\lambda m_1 y - bmz^2 - bmz^2 \\
&\quad + 2bmm_2 z - \lambda srw^2 \\
&\quad - 2\lambda s m_1 w - \lambda srw^2 \\
&\quad + 2\lambda s r m_3 w \\
&= -a\lambda \left(x^2 - 2m_1 x\right) - a\lambda x^2 \\
&\quad + 2\lambda s m_3 x - dmy^2 - dmy^2 \\
&\quad - 2a\lambda m_1 y - bm\left(z^2 - 2m_2 z\right) \\
&\quad - bmz^2 - \lambda sr\left(w^2 - 2m_3 w\right) \\
&\quad - \lambda srw^2 - 2\lambda s m_1 w \\
&= -a\lambda (x - m_1)^2 + a\lambda (m_1)^2 \\
&\quad + f(x) - dmy^2 + h(y) \\
&\quad - bm(z - m_2)^2 + bm(m_2)^2 \\
&\quad - bmz^2 - \lambda sr(w - m_3)^2 \\
&\quad + \lambda sr(m_3)^2 + g(w) \\
&= -a\lambda (x - m_1)^2 - dmy^2 \\
&\quad - bm(z - m_2)^2 \\
&\quad - \lambda sr(w - m_3)^2 + f(x) \\
&\quad + h(y) + g(w) - bmz^2 \\
&\quad + a\lambda (m_1)^2 + bm(m_2)^2 \\
&\quad + \lambda sr(m_3)^2 \\
&\le -\theta V_{\lambda,m}(X) + \max_{x\in R} f(x) \\
&\quad + \max_{y\in R} h(y) + \max_{w\in R} g(w) \\
&\quad + a\lambda (m_1)^2 + bm(m_2)^2 \\
&\quad + \lambda sr(m_3)^2 \\
&= -\theta V_{\lambda,m}(X) + \frac{\lambda s^2 (m_3)^2}{a} \\
&\quad + \frac{a^2\lambda^2 (m_1)^2}{md} + \frac{\lambda s (m_1)^2}{r} \\
&\quad + a\lambda (m_1)^2 + bm(m_2)^2 \\
&\quad + \lambda sr(m_3)^2 \\
&= -\theta\left[V_{\lambda,m}(X) - L_{\lambda,m}^2\right].
\end{aligned}
\tag{12}
$$

Thus, we have

$$
\begin{aligned}
&\left[V_{\lambda,m}(X(t)) - L_{\lambda,m}^2\right] \\
&\quad \le \left[V_{\lambda,m}(X(t_0)) - L_{\lambda,m}^2\right] e^{-\theta(t-t_0)}, \\
&\overline{\lim_{t\to+\infty}} V_{\lambda,m}(X(t)) \le L_{\lambda,m}^2,
\end{aligned}
\tag{13}
$$

which clearly shows that $\Omega_{\lambda,m} = \{X \mid V_{\lambda,m}(X) \le L_{\lambda,m}^2\}$ is the globally exponential attractive set of system (2).

The proof is complete. □

Remark 3. (i) In particular, let us take $m_1 = 0$, $m_3 = 0$ in Theorem 2, we can get the conclusions below.

Suppose that $\forall a > 0, b > 0, d > 0, r > 0, c > 0, s > 0, \lambda > 0, m > 0$.

Let $(x(t), y(t), z(t), w(t))$ be an arbitrary solution of system (2) and

$$
\begin{aligned}
M_{\lambda,m}^2 &= \frac{b(a\lambda + cm)^2}{\theta m}, \quad \theta = \min(a, b, d, r) > 0, \\
V_{\lambda,m}(X) &= V_{\lambda,m}(x, y, z, w) \\
&= \lambda x^2 + my^2 + m\left(z - \frac{a\lambda + cm}{m}\right)^2 + \lambda s w^2, \\
&\qquad \forall \lambda > 0,\ \forall m > 0.
\end{aligned}
\tag{14}
$$

Then the estimation

$$
\begin{aligned}
&\left[V_{\lambda,m}(X(t)) - M_{\lambda,m}^2\right] \\
&\quad \le \left[V_{\lambda,m}(X(t_0)) - M_{\lambda,m}^2\right] e^{-\theta(t-t_0)}
\end{aligned}
\tag{15}
$$

holds for system (2), and thus

$$
\begin{aligned}
\Sigma_{\lambda,m} = \Bigg\{&(x, y, z, w) \mid \lambda x^2 + my^2 \\
&+ m\left(z - \frac{a\lambda + cm}{m}\right)^2 + \lambda s w^2 \le M_{\lambda,m}^2,\ \forall \lambda \\
&> 0,\ \forall m > 0\Bigg\}
\end{aligned}
\tag{16}
$$

is the globally exponential attractive set and positive invariant set of system (2); that is,

$$
\overline{\lim_{t\to+\infty}} V_{\lambda,m}(X(t)) \le M_{\lambda,m}^2. \tag{17}
$$

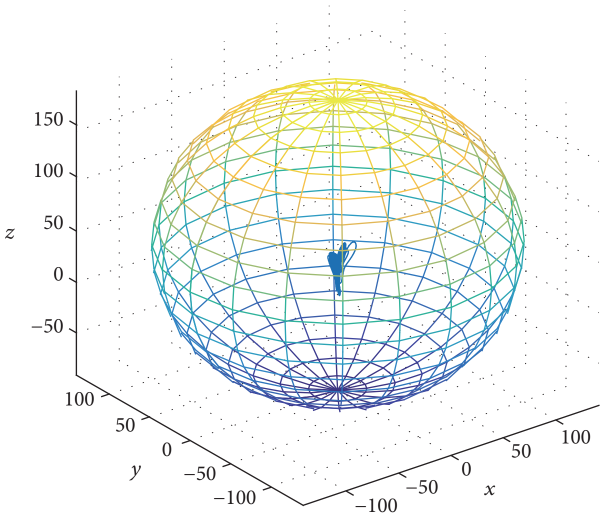

Figure 5: Localization of hyperchaotic attractor of system (2) in the $xOyz$ space defined by $\Omega_{1,1}$.

(ii) Let us take $m_1 = 0, m_3 = 0, \lambda = 1, m = 1$; then we can get

$$\Sigma_{1,1} = \left\{ (x, y, z, w) \mid x^2 + y^2 + (z - a - c)^2 + sw^2 \le \frac{b(a+c)^2}{\min(a, b, d, r)} \right\} \quad (18)$$

as the globally exponential attractive set and positive invariant set of system (2) according to Theorem 2.

When $a = 19.42, b = 1.91, c = 29.45, d = 2.86, r = 0.23, s = 9.64$, we can get that

$$\Omega_{1,1} = \left\{ (x, y, z, w) \mid x^2 + y^2 + (z - 48.87)^2 + 9.64w^2 \le (140.8)^2 \right\} \quad (19)$$

as the globally exponential attractive set and positive invariant set of system (2).

Figure 5 shows hyperchaotic attractor of system (2) in the $xOyz$ space defined by $\Omega_{1,1}$. Figure 6 shows hyperchaotic attractor of system (2) in the $xOyw$ space defined by $\Omega_{1,1}$. Figure 7 shows hyperchaotic attractor of system (2) in the $xOzw$ space defined by $\Omega_{1,1}$. Figure 8 shows hyperchaotic attractor of system (2) in the $yOzw$ space defined by $\Omega_{1,1}$.

3. Conclusions

In this paper, we have investigated some global dynamics of a generalized Lorenz–Stenflo system describing the evolution of finite amplitude acoustic gravity waves in a rotating atmosphere. Based on the Lyapunov method, the globally attractive sets were formulated combining simple inequalities. Finally, numerical examples were presented to show the effectiveness of the proposed method.

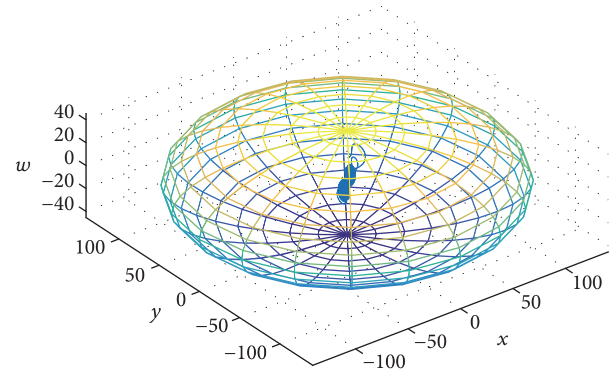

Figure 6: Localization of hyperchaotic attractor of system (2) in the $xOyw$ space defined by $\Omega_{1,1}$.

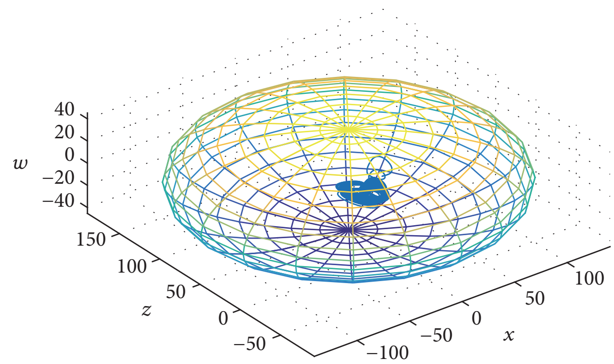

Figure 7: Localization of hyperchaotic attractor of system (2) in the $xOzw$ space defined by $\Omega_{1,1}$.

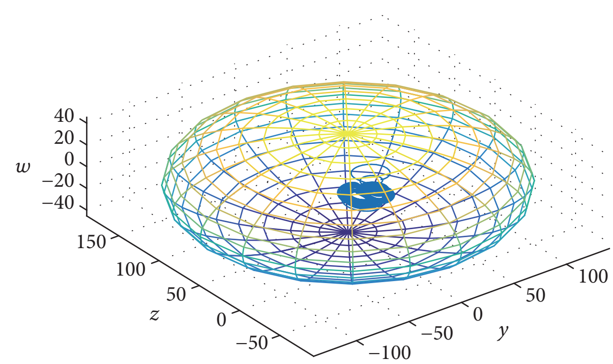

Figure 8: Localization of hyperchaotic attractor of system (2) in the $yOzw$ space defined by $\Omega_{1,1}$.

Disclosure

All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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