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Believability Relations for Select-Direct Sentential Revision

Abstract. A set of sentential revision operations can be generated in a select-direct way within a new framework for belief change named descriptor revision firstly introduced in Hansson [8]. In this paper, we adopt another constructive approach to these operations, based on a relation $\preceq$ on sentences named believability relation. Intuitively, $\varphi \preceq \psi$ means that the subject is at least as prone to believe or accept φ as to believe or accept ψ. We demonstrate that so called H-believability relations and basic believability relations, the second of which the is axiomatically characterized with a set of weak postulates, are faithful alternative models for two typical select-direct sentential revision operations. Then we investigate additional postulates on believability relations that correlate with properties of the generated revision operations. Finally, we show that traditional AGM revision operations can be reconstructed from a strengthened variant of the basic believability relation and there is a close connection between this relation and the standard epistemic entrenchment relation.

Keywords: Sentential revision, Belief change, Select-direct, Descriptor revision, Believability relation, AGM revision, Epistemic entrenchment relation.

1. Introduction

In Hansson [8] it is argued that in the classical literatures (such as Alchourrón et al. [1] and Grove [6]) on logic of belief change which mainly focus the operations of contraction and revision, we can find a standard methodology which can be summarized as "select-and-intersect": Select the most plausible sets that satisfy the *success condition* (for example, removing a specified sentence from original beliefs in the case of contraction, or adding a specified sentence to original beliefs in the case of revision or expansion), and then take their intersection as outcome. Hansson [7,8] further argued that this method has at least three major disadvantages. Firstly, the property of being an optimal potential outcome is not generally preserved under intersection. This point can be illustrated by an example in Sandqvist [15].

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Secondly, the fact that a success condition holds for all elements of a family of sets does not always imply that their intersection also satisfies this condition. For an instance, the success condition of "making up one's mind" about φ, i.e. adding φ or adding $\neg\varphi$ to original beliefs, is not generally preserved to be satisfied after intersection. Thirdly, the adequacy of the options available for selection and intersection is contestable. In the traditional framework, the selection is made among remainders (Alchourrón et al. [1]) or possible worlds (Grove [6]). It is not difficult to show that neither of them are plausible belief sets (Alchourrón & Makinson [2], Hansson [7]).

In order to avoid these disadvantages, a new approach to belief change named "descriptor revision" was introduced in a series of papers by Hansson [7–10]. In contrast to "select-and-intersect" among the implausible options, the methodology of this approach can be summarized as "select-direct" among the plausible options: It is assumed that there is a set of belief sets (not necessarily being remainders or possible worlds) as potential outcomes of belief change, and the mechanism of change is a direct choice among these potential outcomes. On the other hand, this is a very general framework since success conditions for various types of belief changes are described in a general way with the help of a metalinguistic belief operator $\mathfrak{B}$. For example, the success condition of contraction by φ is $\neg\mathfrak{B}\varphi$, that of revision by φ is $\mathfrak{B}\varphi$, that of making up one's mind about φ is $\mathfrak{B}\varphi \vee \mathfrak{B}\neg\varphi$. A descriptor is a set of such formulas with $\mathfrak{B}$ that encode the relevant success condition and descriptor revision on belief set K is performed with a unified operator $\circ$ which applies to all descriptors (Hansson [8]).

This new construction for belief change cannot only be used to investigate some interesting belief change patterns which cannot be represented in the "select-and-intersect" way, such as making up one's mind (Zhang & Hansson [16]), but also throw new light on the traditional operations. For example, sentential revision $\star$ can be reconstructed in a select-direct way from descriptor revision as $K \star \varphi = K \circ \mathfrak{B}\varphi$. Another interesting consequence of this new proposal is that an alternative construction with a relation $\overset{*}{\preceq}$ of epistemic proximity defined on descriptors was proved being an alternative modelling for the descriptor revision (Hansson [9]). Let Ψ and Φ be any two descriptors. Informally, $\Psi \overset{*}{\preceq} \Xi$ means that "the subject is at least disposed to perform a change in the belief system resulting in assent to Ψ as one resulting in assent to Ξ" (Hansson [9]). Just as a sentential revision $\star$ can be derived from a descriptor revision $\circ$, it seems a relation $\preceq$ on sentences can also be derived from a proximity relation $\overset{*}{\preceq}$. This kind of relation $\preceq$ is named as *believability relation* (Hansson [9]) since the intended interpretation of $\varphi \preceq \psi$ is as same as that of $\mathfrak{B}\varphi \overset{*}{\preceq} \mathfrak{B}\psi$.[1] Intuitively, $\varphi \preceq \psi$ means that the subject is at least as prone to believing or accepting φ as incorporating ψ into her beliefs.

As it was shown that descriptor revision can be reconstructed from the relations of epistemic proximity (Hansson [9]), the question naturally arises whether the believability relation derived from the relations of epistemic proximity can be used as an alternative modelling for the sentential revision derived from descriptor revision. The main purpose of the present contribution is to show that some revision operations generated in this way indeed can be reconstructed by believability relations satisfying some suitable conditions too. It should be noted that the idea of using relations on sentences reflecting degree of acceptance to construct revision operations also appeared in models in Cantwell [3] and Rott [4]. However, in both of them the relations reflecting degree of acceptance were not proposed in their own right, but determined by or generalized from the *epistemic entrenchment relation* (Gärdenfors & Makinson [5]). So these models are essentially different from the approach investigated here in both respects of the way of representing the degree of acceptance and the way of constructing the revision operations.

The structure of this paper will be organized in this way: in the next section, we will introduce some formal preliminaries, in particular, we will differentiate two different kinds of select-direct sentential revision derived from descriptor revision, i.e. dependent revision and independent revision. In Section 3, we will specify the believability relations that can generate the *dependent* revision, which are derived from so called *standard* epistemic proximity relations. Section 4 will be devoted to the case of the *independent* revision, which is proved to be obtainable through a set of believability relations axiomatically characterized with four weak postulates. In Section 5, more potential properties on believability relations and their impact on the properties of the relevant revision operations will be studied. As it has been shown that traditional AGM revision operations (Alchourrón et al. [1]) can be reconstructed in the select-direct way (Hansson [10]), we will investigate in Section 6 whether they can also be reconstructed using believability relations as well as the possible connection between the believability relations and the epistemic entrenchment relations. Section 7 concludes.

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[1] In this sense, the believability relation is the same as a restricted variant of epistemic proximity relation on descriptors of the most simple form (such as $\mathfrak{B}\varphi$, $\mathfrak{B}\psi$, etc.). The reason why we define it on sentences but not on descriptors is mainly for simplicity in expression and convenience in comparison with the epistemic entrenchment relation (Gärdenfors & Makinson [5]), which is also defined on sentences.

2. Preliminaries

The object language $\mathcal{L}$ is defined inductively by a set of propositional variables and the truth functional operations $\neg, \wedge, \vee$ and $\rightarrow$ in the usual way. Sentences in $\mathcal{L}$ will be denoted by lower-case Greek letters in the second half of the alphabet and sets of such sentences by upper-case Roman letters. Cn is a consequence operation for $\mathcal{L}$ satisfying supraclassicality (if φ can be derived from A by classical truth-functional logic, then $\varphi \in \mathrm{Cn}(A)$), compactness (if $\varphi \in \mathrm{Cn}(A)$, then there exists some finite $B \subseteq A$ such that $\varphi \in \mathrm{Cn}(B)$) and deduction property ($\varphi \in \mathrm{Cn}(A \cup \{\psi\})$ if and only if $\psi \rightarrow \varphi \in \mathrm{Cn}(A)$). $X \vdash \varphi$ is an alternative notation for $\varphi \in \mathrm{Cn}(X)$. $\{\varphi\} \vdash \psi$ will be simply written as $\varphi \vdash \psi$. $\varphi \equiv \psi$ means $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

The beliefs of an agent are represented by a *belief set*, which is a set $X \subseteq \mathcal{L}$ such that $X = \mathrm{Cn}(X)$. K is fixed to denote the set of the original beliefs of the agent. We assume that K is consistent unless stated otherwise.

An *atomic belief descriptor* is a sentence $\mathfrak{B}\varphi$ with $\varphi \in \mathcal{L}$. The symbol $\mathfrak{B}$ is not part of the object language $\mathcal{L}$ by which the agent's beliefs are expressed. A *molecular belief descriptor* is a truth-functional combination of atomic descriptors. A *composite belief descriptor* (in short: descriptor; denoted by upper-case Greek letters) is a set[2] of molecular descriptors. $\mathfrak{B}\varphi$ is *satisfied* by a belief set X if and only if $\varphi \in X$. Conditions of satisfaction for molecular descriptors are defined inductively, hence (letting α and β stand for molecular descriptors) X satisfies $\neg\alpha$ if and only if it does not satisfy α, it satisfies $\alpha \vee \beta$ if and only if it satisfies either α or β, etc. It satisfies a composite descriptor Φ if and only if it satisfies all its elements. $X \Vdash \Phi$ denotes that X satisfies Φ and $\Phi \Vdash \Xi$ that all belief sets satisfying Φ also satisfy Ξ (Hansson [8]).

We use $⫫$ to denote the descriptor $\mathfrak{B}\top \wedge \neg\mathfrak{B}\top$. It is easy to see that there is no belief set satisfying $⫫$.

Descriptor revision on belief set K is performed with a unified operator $\circ$ such that $K \circ \Phi$ is an operation with Φ as its success condition. Several constructions for concrete descriptor revision operations can be found in Hansson [8], of which the relational model defined as follows is of special importance.

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[2]We usually omit the brackets if the descriptor is a singleton.

[3]The definition presented here is more general than the original one in Hansson [8] as we introduce one more parameter $\mathfrak{S}$.

Definition 2.1 (*Hansson* [8], *modified*[3]). Let $\mathfrak{S}$ be any set of descriptors, $(\mathbb{X}, \leqq)_{\mathfrak{S}}$ is a relational select-direct model (in short: relational model) with respect to K if and only if it satisfies:[4]

($\mathbb{X}$1) $\mathbb{X}$ is a set of belief sets.

($\mathbb{X}$2) $K \in \mathbb{X}$.

($\leqq$1) $K \leqq X$ for every $X \in \mathbb{X}$.

($\leqq$2) For any $\Phi \in \mathfrak{S}$, if $\{X \in \mathbb{X} \mid X \Vdash \Phi\}$ (we denote it as $\mathbb{X}^{\Phi}$) is not empty, then it has a unique $\leqq$-minimal element denoted by $\mathbb{X}^{\Phi}_{<}$.

A descriptor revision $\circ$ on K is *based on* (or *determined by*) some relational model $(\mathbb{X}, \leqq)_{\mathfrak{S}}$ with respect to K if and only if for any Φ,

$$\langle \leqq \text{ to } \circ \rangle^{5} \quad K \circ \Phi = \begin{cases} \mathbb{X}^{\Phi}_{<} & \text{if } \Phi \in \mathfrak{S} \text{ and } \mathbb{X}^{\Phi} \text{ is not empty,} \\ K & \text{otherwise.} \end{cases}$$

The intuitive meaning of $\mathbb{X}$ is an outcome set which represents all the potential outcomes under several belief change patterns. $\leqq$ (with the strict part $<$) is an ordering used to select the best among candidates satisfying a certain success condition. Condition ($\leqq$2) guarantees that this kind of direct-selection is achievable for any success condition which is satisfiable in $\mathbb{X}$. To some extent, descriptor revision is in a more abstract level than the AGM revision. Consider the Definition 2.1, for any descriptor revision $\circ$, it assumes that there exists an outcome set which contains all the potential outcomes of the operation $\circ$, but it says little about what these outcomes should be like. In contrast, in the AGM framework, the belief change is supposed to perform in a way of preserving consistence and reserving information as much as possible. Therefore, the intersection in the AGM framework as a useful tool to construct the intended outcome becomes dispensable in the context of descriptor revision.

Sentential revision operation on K can be reconstructed from descriptor revision in the following way:

Definition 2.2 (*Hansson* [8]). Let $\circ$ be some descriptor revision on K. A sentential revision $\star$ on K is based on (or determined by) $\circ$ if and only if $K \star \varphi = K \circ \mathfrak{B}\varphi$ for all $\varphi \in \mathcal{L}$.

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[4] We will drop the subscript $\mathfrak{S}$ and phrase "with respect to K" if this does not affect the understanding, and use $\mathbb{X}^{\varphi}$ and $\mathbb{X}^{\varphi}_{<}$ to denote $\mathbb{X}^{\mathfrak{B}\varphi}$ and $\mathbb{X}^{\mathfrak{B}\varphi}_{<}$ for simplicity.

[5] Note that given that $(\mathbb{X}, \leqq)$ is a relational model, since $K \in \mathbb{X}$ and $K \leqq X$ for all $X \in \mathbb{X}$, $\mathbb{X}$ is equivalent to the domain of $\leqq$. So $(\mathbb{X}, \leqq)$ can be represented faithfully by $\leqq$ only.

Let $(\mathbb{X}, \leqq)_{\mathfrak{S}}$ be any relational model, we name it *global relational model* if $\mathfrak{S}$ contains all descriptors, and *sentential relational model* if $\mathfrak{S} = \{\mathfrak{B}\varphi \mid \varphi \in \mathcal{L}\}$. Accordingly, $\circ$ will be called *global descriptor revision* if it is based on some global relational model, and *sentential descriptor revision* if it is determined by some sentential relational model. With these two kinds of descriptor revision operations, at least two types of sentential revision operations can be constructed.[6]

Definition 2.3. 1. A sentential operation $\star$ on K is a dependent select-direct sentential revision (in short: dependent revision) if and only if it is based on a global descriptor revision $\circ$.

2. An operator $\star$ on K is a independent select-direct sentential revision (in short: independent revision) if and only if it is determined by a sentential descriptor revision.

It is easy to see that the dependent revision operations are special cases of independent ones. Compared with the independent, properties of dependent revision are more subtle since they are sensitive to those of other change patterns. How to axiomatically characterize the dependent revision is still an open problem. But the axiomatization of the independent revision can be obtained as an immediate corollary of the Theorem 1 in Zhang & Hansson [16]. In the next two sections, we will show that both of these operations can be reconstructed by the believability relations satisfying certain suitable conditions.

3. Believability Relations for Dependent Revision

As we have mentioned in Section 1, a relation $\overset{*}{\preceq}$ (with the strict part $\overset{*}{\prec}$ and the symmetric part $\overset{*}{\simeq}$) on descriptors called *epistemic proximity relation* was introduced in Hansson [9]. $\Psi \overset{*}{\preceq} \Xi$ intuitively means that Ψ is at least as epistemically proximal as Ξ. It is assumed that a *standard* epistemic proximity relation should satisfy the following five postulates:[7]

[6] These two kinds of sentential operations (and the associated descriptor revision operations) were not explicitly differentiated in previous work, though some studies on the same operations of different names have been separately done in Hansson [8, 10] and Zhang & Hansson [16].

[7] In Hansson [9], *epistemic proximity relation* was just referred to the relations satisfying the five postulates. Here we use this term in a more general way and call relations satisfying these postulates *standard* epistemic proximity relations instead.

transitivity: If $\Phi \overset{*}{\preceq} \Psi$ and $\Psi \overset{*}{\preceq} \Xi$, then $\Phi \overset{*}{\preceq} \Xi$

counter-dominance: If $\Phi \Vdash \Psi$, then $\Psi \overset{*}{\preceq} \Phi$

coupling: If $\Phi \overset{*}{\simeq} \Psi$ then $\Phi \overset{*}{\simeq} \Phi \cup \Psi$

amplification: Either $\Phi \cup \{\mathfrak{B}\varphi\} \overset{*}{\preceq} \Psi$ or $\Phi \cup \{\neg\mathfrak{B}\varphi\} \overset{*}{\preceq} \Psi$

absurdity avoidance: $\Phi \overset{*}{\preceq} \{\mathfrak{B}\varphi, \neg\mathfrak{B}\varphi\}$ for some Φ

It was also shown that these relations can be used to reconstruct the global descriptor revision.

PROPOSITION 3.1 (Hansson [9]). *Let $\circ$ be any descriptor revision operation on K. Then, the following two propositions are equivalent.*

1. *$\circ$ is a global descriptor revision operation on K.*
2. *$\circ$ is derived from some standard epistemic proximity relation with respect to K[8] in the following way:*

$$\langle \overset{*}{\preceq} \text{ to } \circ \rangle \quad K \circ \Phi = \begin{cases} \{\psi \mid \Phi \overset{*}{\simeq} \Phi \cup \{\mathfrak{B}\psi\}\} & \textit{if } \Phi \overset{*}{\prec} \Perp, \\ K & \textit{otherwise.} \end{cases}$$

Hansson further proposed that a set of believability relations can be derived from the standard epistemic proximity relation in the following way:

$$\langle \overset{*}{\preceq} \text{ to } \preceq \rangle^{-} \quad \varphi \preceq \psi \text{ if and only if } \mathfrak{B}\varphi \overset{*}{\preceq} \mathfrak{B}\psi.$$

However, some information contained in the original epistemic proximity relation will be lost after this kind of restriction. Consider two relations of epistemic proximity $\overset{*}{\preceq}_1$ and $\overset{*}{\preceq}_2$ that respectively satisfies $\mathfrak{B}\varphi \overset{*}{\preceq}_1 \Perp \overset{*}{\preceq}_1 \mathfrak{B}\psi$ and $\mathfrak{B}\varphi \overset{*}{\preceq}_2 \mathfrak{B}\psi \overset{*}{\preceq}_2 \Perp$. If the agent is in the belief status represented by $\overset{*}{\preceq}_1$, she will not accept ψ as her new belief in any case. The situation is different if she is in the belief status represented by $\overset{*}{\preceq}_2$. However, this difference cannot be expressed by the believability relations derived from $\overset{*}{\preceq}_1$ and $\overset{*}{\preceq}_2$ through the way of $\langle \overset{*}{\preceq} \text{ to } \preceq \rangle^{-}$. As a result, the dependent revision derived from the global descriptor revision is not able to be reconstructed by these believability (see the Observation 5 in Hansson [9]).

However, as we will show in what follows, this limitation can be resolved by a simple modification on the derivation method $\langle \overset{*}{\preceq} \text{ to } \preceq \rangle^{-}$. Consider the believability relations derived from the standard epistemic proximity relation in this way:

$$\langle \overset{*}{\preceq} \text{ to } \preceq \rangle \quad \varphi \preceq \psi \text{ if and only if } \mathfrak{B}\psi \overset{*}{\prec} \Perp \text{ and } \mathfrak{B}\varphi \overset{*}{\preceq} \mathfrak{B}\psi.$$

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[8] $\overset{*}{\preceq}$ is with respect to K if and only if $K \Vdash \bigcup\{\Phi \mid \Phi \overset{*}{\simeq} \mathfrak{B}\top\}$ (See Hansson [9], p. 78).

Although $\langle \overset{*}{\preceq} \text{ to } \preceq \rangle$ is a little different from $\langle \overset{*}{\preceq} \text{ to } \preceq \rangle^{-}$ proposed in Hansson [9], still let us call relations obtained through this way *H-believability relations*. It is easy to see that in this way if $\perp\!\!\!\perp \overset{*}{\preceq} \varphi$ then φ will not be in the domain of the derived $\preceq$. This modification is crucial for the main result of this section as follows.

Note that $\mathrm{Ref}(\preceq)$ denote the domain of $\preceq$.

Theorem 3.2. *Let $\star$ be any sentential revision operation on K. Then, the following two propositions are equivalent.*

1. *$\star$ is a dependent revision operation on K.*
2. *$\star$ is derived from some H-believability relation $\preceq$ in the following way:*

$$\langle \preceq \text{ to } \star \rangle \quad K \star \varphi = \begin{cases} \{\psi \mid \varphi \simeq \varphi \wedge \psi\} & \textit{if } \varphi \in \mathrm{Ref}(\preceq), \\ K & \textit{otherwise.} \end{cases}$$

Theorem 3.2 shows that H-believability relations are faithful alternative models for dependent revision operations, though the axiomatization of this kind of believability relations has not been settled. In the next section, we will study the believability relations for constructing independent revision operations. A positive result is that an axiomatic characterization of these relations can be obtained.

4. Believability Relations for Independent Revision

In this section, we will prove that believability relations which exactly generate all the independent revision operations on K through the way of $\langle \preceq \text{ to } \star \rangle$ can be axiomatically characterized by following postulates:

weak transitivity: Let $\varphi \simeq \psi$ and $\psi \vdash \varphi$, then (i) $\varphi \preceq \xi$ if and only if $\psi \preceq \xi$, and (ii) $\xi \preceq \varphi$ if and only if $\xi \preceq \psi$.

weak coupling: If $\varphi \simeq \varphi \wedge \psi$ and $\varphi \simeq \varphi \wedge \xi$, then $\varphi \simeq \varphi \wedge (\psi \wedge \xi)$.

relative counter-dominance: If $\varphi \in \mathrm{Ref}(\preceq)$ and $\varphi \vdash \psi$, then $\psi \preceq \varphi$.

relative minimality: $\varphi \in K$ if and only if $\varphi \in \mathrm{Ref}(\preceq)$ and $\varphi \preceq \psi$ for all $\psi \in \mathrm{Ref}(\preceq)$.

It is arguable that these postulates represent a minimum set of conditions a believability relation should satisfy. Weak transitivity is just a very weak version of transitivity which is assumed for almost all orderings. The rationale for weak coupling is that if the agent will consequently add ψ and

ξ to her beliefs in case of accepting φ, then she also add $\psi \wedge \xi$ to her beliefs in this case. This is reasonable if we assume that the beliefs of the agent are represented by a belief set. The justification of relative counter-dominance is that if φ logically entails ψ, and K must be revised to incorporate either φ or ψ, then it will be a smaller change to accept φ rather than to accept ψ, because then ψ must be added too, if we assume that the beliefs of the agent is closed under consequence operation. Relative minimality is justifiable since it needs to do nothing to add φ to K if it is already in K. So we call relations characterized by these postulates *basic believability relations* with respect to K. It is easy to see that the H-believability relations satisfy these four conditions.

In what follows, we prove the equivalence between the sentential relational models and the basic believability relations in terms of their expressive power for constructing revision operations. The work can be divided into the following three lemmas.

Note that we use $[\varphi]_{\preceq}$ to denote the set $\{\psi \mid \varphi \simeq \varphi \wedge \psi\}$ for simplicity

LEMMA 4.1. *Let* $(\mathbb{X}, \leqq)$ *be any sentential relational model and* $\preceq$ *a relation on* $\mathcal{L}$ *retrieved from* $(\mathbb{X}, \leqq)$ *in the following way:*

$\langle\leqq \text{ to } \preceq\rangle$ $\quad \varphi \preceq \psi$ *if and only if* $\mathbb{X}^{\varphi}_{<}$ *and* $\mathbb{X}^{\psi}_{<}$ *exist and* $\mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\psi}_{<}$.

Then, $\preceq$ *is a basic believability relation.*

LEMMA 4.2. *Let* $\preceq$ *be any basic believability relation. Then, a binary relation* $\leqq$ *on the power set of* $\mathcal{L}$ *can be constructed from* $\preceq$ *in the following way:*

$\langle\preceq \text{ to } \leqq\rangle$ $\quad X \leqq Y$ *if and only if* $X = [\varphi]_{\preceq}$, $Y = [\psi]_{\preceq}$ *and* $\varphi \preceq \psi$.

Moreover, $(\mathbb{X}, \leqq)$ *with* $\mathbb{X} = \mathrm{Ref}(\leqq)$ *is a sentential relational model for revision and* $\preceq$ *can be retrieved from* $(\mathbb{X}, \leqq)$ *through* $\langle\leqq \text{ to } \preceq\rangle$.

We say that a sentential revision $\star$ is based on (or determined by) some sentential (global) relational model if it is based on the sentential (global) descriptor revision determined by this model, and $\star$ is based on (or determined by) some believability relation if it is generated from this relation in the way of $\langle\preceq \text{ to } \star\rangle$. Now we can state the third lemma as follows.

LEMMA 4.3. *Let* $(\mathbb{X}, \leqq)$ *be any sentential relational model and* $\preceq$ *a believability relation retrieved from* $(\mathbb{X}, \leqq)$ *through* $\langle\leqq \text{ to } \preceq\rangle$*. Then, a revision operation is based on* $(\mathbb{X}, \leqq)$ *if and only if it is based on* $\preceq$.

Lemmas 4.1 and 4.2 jointly show that all basic believability relations can be derived from the sentential relational models. Moreover, Lemma 4.3 says that the sentential revision based on certain sentential relational model is the same as the revision determined by the believability relation that the model gives rise to. Hence, the set of the sentential revision operations generated from basic believability relation coincide with the independent revision operations. Moreover, as we have mentioned in the last paragraph of the Section 2, the independent revision can be axiomatically characterized with several postulates on the operators. We present these results together in the following representation theorem.

THEOREM 4.4. *Let $\star$ be any operation on K. Then, the following three propositions are equivalent:*

1. *$\star$ is an independent revision.*
2. *$\star$ is generated from some basic believability relation $\preceq$ through $\langle \preceq \text{ to } \star \rangle$.*
3. *$\star$ satisfies the following conditions:*

   $\mathrm{Cn}(K \star \varphi) = K \star \varphi$ *(closure)*
   If $K \star \varphi \neq K$, *then* $\varphi \in K \star \varphi$ *(relative success)*
   If $\varphi \in K$, *then* $K \star \varphi = K$ *(confirmation)*
   If $\psi \in K \star \varphi$, *then* $\psi \in K \star \psi$ *(regularity)*
   If $\psi \in K \star \varphi$ *and* $\varphi \in K \star \psi$, *then* $K \star \varphi = K \star \psi$ *(reciprocity)*

Although the postulates in Theorem 4.4 are so weak that they actually characterize a broad set of operations, the maxi-choice revision proposed in Alchourrón & Makinson [2] is not covered by them since postulate "reciprocity" cannot be derived from the first six AGM postulates (Alchourrón & Makinson [2]). However, there exist other constructions for descriptor revision (and hence for the derived select-direct sentential revision), in one of which the direct choice among the outcome set is specified by a selecting function instead of the ordering $\leqq$ in Definition 1. The resulting revision operation from this setting is even weaker and hence maxi-choice revision is a special case of it. (For more details on the construction for descriptor revision using selecting function, see Hansson [8], p. 958)

Besides, Lemma 4.2 tells us that $\langle \preceq \text{ to } \leqq \rangle$ is an injection from basic believability relations to the sentential relational model. But it is easy to find an example in which two different sentential relational models generate the same believability relation through $\langle \leqq \text{ to } \preceq \rangle$. So it is not a bijection. However, with some restriction, one-to-one correspondences between these two kinds of constructions are obtainable. We will present some this kind of results in the next section.

5. More Properties on Believability Relations

The properties of the basic believability relations introduced and studied in the previous section does not necessarily cover all plausible properties of the believability relations. In this section, we will impose some additional properties on the basic believability relations and investigate their consequences for properties of the equivalent sentential relational models and derived revision operations.

5.1. Transitivity

Given that $\varphi \preceq \psi$ is explained as meaning that it is not more difficult for the subject to become to believe φ than ψ, the postulates characterizing the basic believability relations appear to be a bit too weak. It seems that even in a very general setting a suitable believability relation at least needs to satisfy:

transitivity: If $\varphi \preceq \psi$ and $\psi \preceq \xi$, then $\varphi \preceq \xi$.

We call the basic believability relations satisfying this additional condition *transitive believability relations.*

The main goal of this subsection is just to argue for the plausibility and generality of the transitive believability relation through a formal result demonstrating that there is no distinction between the transitive and general basic believability relations from the angle of expressive power.

As a preparation, we prove that $\langle \leqq \text{ to } \preceq \rangle$ (or $\langle \preceq \text{ to } \leqq \rangle$) is a bijection between the transitive believability relations and a special subset of sentential relational model which is defined as follows.

DEFINITION 5.1. $(\mathbb{X}, \leqq)$ is a canonical sentential model with respect to K if and only if it is a sentential relational model additionally satisfying:

($\leqq$3) $\leqq$ is reflexive, i.e. $X \leqq X$ for all $X \in \mathbb{X}$, and transitive.

($*$1) For any $X \in \mathbb{X}$, there exists φ such that $X = \mathbb{X}^{\varphi}_{<}$.

It is easy to see that there exist sentential relational models which are not canonical sentential models. However, the following lemma shows that these two kinds of models are equivalent in expressive power. That is the reason why we call models defined in this way *canonical.*

LEMMA 5.2. *Let* $(\mathbb{X}, \leqq)$ *be any sentential relational model. Then, a canonical sentential model* $(\mathbb{Y}, \leqq')$ *can be constructed as follows:*

$\mathbb{Y} = \{X \in \mathbb{X} \mid$ there is no $\mathbb{D} \subseteq \mathbb{X}$ such that for any $Y \in \mathbb{D}$, $Y < X$ and $X \subseteq \bigcup \mathbb{D}\}$.

$\leqq'$ (with the strict part $<'$) is the transitive closure of $\leqq \cap (\mathbb{X} \times \mathbb{X})$.

Moreover, A revision operator is determined by $(\mathbb{X}, \leqq)$ if and only if it is based on $(\mathbb{Y}, \leqq')$.

Now we demonstrate that a bijection between the transitive believability relations and the canonical relational models can be obtained by $\langle \preceq$ to $\leqq\rangle$ and $\langle \leqq$ to $\preceq\rangle$.

LEMMA 5.3. *1. Let $\preceq$ be any transitive believability relation and let $(\mathbb{X}, \leqq)$ be obtained by $\langle \preceq$ to $\leqq\rangle$ from $\preceq$. Then, $(\mathbb{X}, \leqq)$ is a canonical sentential model and $\preceq$ can be retrieved from $(\mathbb{X}, \leqq)$ by means of $\langle \leqq$ to $\preceq\rangle$.*

2. Let $(\mathbb{X}, \leqq)$ be any canonical sentential model and $\preceq$ retrieved from $\leqq$ through $\langle \leqq$ to $\preceq\rangle$. Then, $\preceq$ is a transitive believability relation and $(\mathbb{X}, \leqq)$ can be reconstructed from $\preceq$ by means of $\langle \preceq$ to $\leqq\rangle$.

From these two lemmas and the Lemma 4.3 in previous section, the result we claimed previously follows immediately.

THEOREM 5.4. *A revision operator is based on some basic believability relation if and only if it is determined by some transitive believability relation.*

Hence, we can focus on the transitive believability relation or the subsets of them in the following part of this paper without loss of generality. And we will see that correspondence between the transitive believability relations and the canonical sentential models offers us a useful tool to investigate properties of the revision operations based on the believability relations.

5.2. Exhaustiveness, Maximality, Coupling and Completeness

Another natural strengthening of basic believability relations is to require their domain to contain all sentences from $\mathcal{L}$, i.e. every $\preceq$ should satisfy:

exhaustiveness : $\text{Ref}(\preceq) = \mathcal{L}$.

Given that $\preceq$ is a transitive believability relation, it is obvious that $\preceq$ satisfies exhaustiveness if and only if it satisfies:

counter-dominance: If $\varphi \vdash \psi$, then $\psi \preceq \varphi$.

minimality: $\varphi \in K$ if and only if $\varphi \preceq \psi$ for all ψ.

Moreover, that a believability relation $\preceq$ satisfies exhaustiveness means that every φ from $\mathcal{L}$ is possible to be accepted by the agent. It is plausible to suppose that in this situation it is strictly more difficult for a rational agent to accept or believe $\bot$ than to believe any non-falsum. In other words, $\preceq$ should satisfy:

maximality: If $\psi \preceq \varphi$ for all ψ, then $\varphi \equiv \bot$.

Weak coupling has been the only untouched postulate of those the characterizing basic believability relations as yet. All the same, there is a natural strengthening of it as follows:

coupling: If $\varphi \simeq \psi$ and $\varphi \simeq \xi$, then $\varphi \simeq \psi \wedge \xi$.

We call believability relations characterized by **transitivity**, **coupling**, **counter-dominance**, **maximality** and **minimality** *strengthened believability relations*. Let $K = \mathrm{Cn}(\{\top\})$ and let the relation $\preceq$ on $\mathcal{L}$ be defined as $\varphi \preceq \psi$ if and only if $\psi \vdash \varphi$. It is easy to see that $\preceq$ is a strengthened believability relation with respect to K but does not satisfy the condition:

completeness: $\varphi \preceq \psi$ or $\psi \preceq \varphi$.

We call strengthened believability relation additionally satisfying completeness *quasi-linear believability relation*, since it is easy to see that a linear order can be obtained on equivalence classes generated by the symmetric part of a quasi-linear believability relation. Moreover, the corresponding sentential relational models obtained by $\langle \preceq \text{ to } \leqq \rangle$ from these two types of believability relations are called *strengthened sentential models* and *linear strengthened sentential models* respectively. The following theorem explains why they are named in this way.

Theorem 5.5. *Let $\preceq$ be a transitive believability relation and $(\mathbb{X}, \leqq)$ a canonical sentential model obtained by $\langle \preceq \text{ to } \leqq \rangle$ from $\preceq$. Then,*

1. *$\preceq$ in addition satisfies exhaustiveness and maximality if and only if $(\mathbb{X}, \leqq)$ additionally satisfies*

   $(*2)$*: For any $\varphi \not\equiv \bot$, $\mathbb{X}^{\varphi}_{<} < \mathrm{Cn}(\{\bot\})$.*

2. *$\preceq$ in addition satisfies coupling if and only if $(\mathbb{X}, \leqq)$ additionally satisfies*

   $(\leqq 4)$*: $\leqq$ is anti-symmetric, i.e. for any $X, Y \in \mathbb{X}$, if $X \leqq Y$ and $Y \leqq X$, then $X = Y$.*

3. *$\preceq$ in addition satisfies completeness if and only if $(\mathbb{X}, \leqq)$ additionally satisfies*

   $(\leqq 5)$*: $\leqq$ is complete, i.e. $X \leqq Y$ or $Y \leqq X$ for all $X, Y \in \mathbb{X}$.*

Theorem 5.5 summarises the impact of strengthening the basic believability relations on the properties of the correspondent relational model. One the other hand, the correspondent impact on the properties of the generated sentential revision operations is demonstrated in the following two representation theorems.

Theorem 5.6. *Let $\star$ be any sentential revision on K. Then, $\star$ is determined by a transitive believability relation satisfying exhaustiveness and maximality if and only if it satisfies closure, confirmation, reciprocity and the following two postulates:*

$\varphi \in K \star \varphi$ *for all* $\varphi \in \mathcal{L}$. *(success)*

If $\neg\varphi \notin \mathrm{Cn}(\varnothing)$, *then* $K \star \varphi$ *is consistent. (consistency)*

Theorem 5.7. *Let $\star$ be any sentential revision operation on K, then the following propositions are equivalent:*

1. *$\star$ can be constructed from some strengthened believability relation through* $\langle \preceq \text{ to } \star \rangle$.
2. *$\star$ can be constructed from some quasi-linear believability relation through* $\langle \preceq \text{ to } \star \rangle$.
3. *$\star$ satisfies closure, confirmation, success, consistency and following postulate:*

   (8) *Given $n \in \mathbb{N}$, if for every $0 \leqslant i < n$, $\varphi_i \in K \star \varphi_{i+1}$ and $\varphi_n \in K \star \varphi_0$, then $K \star \varphi_0 = K \star \varphi_2 = \cdots = K \star \varphi_n$. (strong reciprocity)*

These two representation theorems show that sentential revision operations generated from the strengthened versions of believability relations can also be axiomatically characterized by certain plausible postulates, of which the strong reciprocity is closely relative to a non-monotonic reasoning rule named as “loop” introduced in Klaus et al. [12]. For more discussion on this, see Makinson & Gärdenfors [13].

Moreover, although the quasi-linear believability relations are a proper subset of strengthened believability relations, Theorem 5.7 shows that they have the same expressive power. It was proved in Hansson [8] that if $(\mathbb{X}, \leqq)$ is a global relational model, then $\leqq$ is linear. So as a consequence of Theorem 5.7, if a revision operation $\star$ is a dependent revision based on some global relational model satisfying ($\star$2), then it is also an independent revision derived from some sentential relational model $(\mathbb{X}, \leqq)$ with $\leqq$ is linear. But it is still unknown whether the opposite implication also holds. If so, with the results obtained in this paper, the axiomatic characterizations of dependent revision operations and H-believability relations can be obtained immediately.

6. Relationship to AGM Revision and Entrenchment Relations

6.1. Believability Relations for AGM Revision

The classic AGM revision operation was introduced in Alchourrón, et al. [1] as follows:

DEFINITION 6.1 (*Alchourrón et al.* [**1**]). For any belief set K, an operation $\star$ on K is called AGM revision operation if and only if it satisfies the following eight AGM postulates:

closure: $\mathrm{Cn}(K \star \varphi) = K \star \varphi$

success: $\varphi \in K \star \varphi$

inclusion: $K \star \varphi \subseteq \mathrm{Cn}(K \cup \{\varphi\})$

vacuity: If $\neg\varphi \notin K$, then $\mathrm{Cn}(K \cup \{\varphi\}) \subseteq K \star \varphi$

consistency: If $\neg\varphi \notin \mathrm{Cn}(\varnothing)$, then $K \star \varphi$ is consistent under Cn

extensionality: If $\varphi \leftrightarrow \psi \in \mathrm{Cn}(\varnothing)$, then $K \star \varphi = K \star \psi$

superexpansion: $K \star (\varphi \wedge \psi) \subseteq \mathrm{Cn}(K \star \varphi \cup \{\psi\})$

subexpansion: If $\neg\psi \notin K \star \varphi$, then $\mathrm{Cn}(K \star \varphi \cup \{\psi\}) \subseteq K \star (\varphi \wedge \psi)$.

Note that we assume that K is consistent. It is easy to see that confirmation can be derived from inclusion and vacuity under this assumption. And it has been proved in Makinson & Gärdenfors [13] that strong reciprocity holds for all operations which satisfy those eight AGM postulates. So, by Theorem 5.7, all AGM revision operations on K can be generated from strengthened believability relation with respect to K.

A closely related result was given in Hansson [10] from the point of view of model construction. In that paper a set of conditions on relational models was specified and it was shown that a revision operation is based on some linear strengthened sentential model satisfying those conditions if and only if it satisfies the eight AGM postulates.

Can we similarly find the conditions characterizing the believability relations that exactly give rise to all the AGM revision operations? There is a positive answer.

Consider following two conditions:

($\triangleright$1) If $\top \prec \varphi \to \psi$, then $\varphi \prec \varphi \wedge \psi$.

(▷2) If $\top \prec \neg\varphi$ and $\top \simeq \varphi \to \psi$, then $\varphi \simeq \varphi \wedge \psi$.

And their "iteration" versions:

(◁1) If $\varphi \prec \varphi \wedge (\psi \to \xi)$, then $\varphi \wedge \psi \prec \varphi \wedge (\psi \wedge \xi)$.

(◁2) If $\varphi \prec \varphi \wedge \neg\psi$ and $\varphi \simeq \varphi \wedge (\psi \to \xi)$, then $\varphi \wedge \psi \simeq \varphi \wedge (\psi \wedge \xi)$.

Given $\preceq$ satisfies transitivity and counter-dominance, it is easy to see that the (◁1) and (◁2) imply the (▷1) and (▷2). The following theorem shows that if we further strengthen the believability relations with (◁1) and (◁2), we can construct exactly the AGM belief revision operations from these strengthened relations.

THEOREM 6.2. *1. If a revision operation is based on some strengthened believability relation satisfying* (▷1) *and* (▷2)*, then it satisfies the first six AGM postulates.*

2. A revision operation is determined by some strengthened believability relation satisfying (◁1) *and* (◁2) *if and only if it satisfies all eight AGM postulates.*

The fact that AGM revision can be reconstructed in the select-direct way to some extent indicates that it is hard to draw a clear line between "select-and-intersect" and "select-direct" revision operations. It is possibly not suitable to regard the "select-direct" approach as an opposite to the "select-and-intersect" one, instead they are more like two perspectives at different levels.

6.2. Epistemic Entrenchment Relation

In what follows, we discuss the connection between the strengthened believability relation and the so called *standard epistemic entrenchment relation* which was firstly introduced in Gärdenfors & Makinson [5] in the following way:

DEFINITION 6.3 (*Gärdenfors* & *Makinson* [**5**]). A binary relation $\leqq$ (with the symmetric part $\doteq$) is a standard epistemic entrenchment relation with respect to K if and only if it satisfies:

Transitivity: If $\varphi \leqq \psi$ and $\psi \leqq \xi$, then $\varphi \leqq \xi$.

Dominance: If $\varphi \vdash \psi$, then $\varphi \leqq \psi$.

Conjunctiveness: $\varphi \leqq \varphi \wedge \psi$ or $\psi \leqq \varphi \wedge \psi$.

$*$-Minimality[9]: $\varphi \notin K$ if and only if $\varphi \leqslant \psi$ for all ψ.

$*$-Maximality: If $\psi \leqslant \varphi$ for all ψ, then $\vdash \varphi$.

Consider a variant of strengthened believability relations which is defined as follows.

Definition 6.4. A relation $\preceq$ on $\mathcal{L}$ is a maximal outcome believability relation if and only if it is a strengthened believability relation satisfying

maxi-outcome: For every $\varphi, \psi \in \mathcal{L}$, $\varphi \simeq \varphi \wedge \psi$ or $\varphi \simeq \varphi \wedge \neg\psi$.

The intuitive meaning of maximal outcome believability relations is that the possible outcomes (including K) of revision based on these relations with inputs of consistent formulas are all maximal consistent sets. In other words, in these cases, believing or accepting $\neg\varphi$ is equivalent to giving up φ. So a bridge can be built from maximal outcome believability relations to standard entrenchment relations in the following way:

$\langle \preceq$ to $\leqslant \rangle$ $\quad \varphi \leqslant \psi$ if and only if $\neg\varphi \preceq \neg\psi$.

In fact, the following theorem shows that from the maximal outcome believability relations we can exactly derive through $\langle \preceq$ to $\leqslant \rangle$ all the standard epistemic entrenchment relations satisfying the following property:

($\triangle$) If $\varphi \doteq \psi$, then $\varphi \doteq \phi \vee \psi$.

Theorem 6.5. *Let K be any maximal consistent belief set.*

1. *Let $\preceq$ be a maximal outcome believability relation with respect to K and $\leqslant$ obtained by $\langle \preceq$ to $\leqslant \rangle$ from $\preceq$, then $\leqslant$ is a standard epistemic entrenchment relation with respect to K satisfying ($\triangle$).*
2. *Let $\leqslant$ be a standard epistemic entrenchment relation with respect to K satisfying ($\triangle$) and $\preceq$ obtained from $\leqslant$ by means of the following definition:*

   $\langle \leqslant$ to $\preceq \rangle$ $\quad \varphi \preceq \psi$ *if and only if* $\neg\varphi \leqslant \neg\psi$.

   Then, $\preceq$ is a maximal outcome believability relation with respect to K and $\leqslant$ can be reconstructed from $\preceq$ by $\langle \preceq$ to $\leqslant \rangle$.

It is interesting to make a further comparison between the strengthened believability relations and the standard epistemic entrenchment relations. A direct translation between these two orderings is expected as both of

[9] We use prefix "$*$-" to distinct these versions of *minimality* and *maximality* from those for the believability relations.

them can exactly generate the AGM revision.[10] Moreover, weakened version of believability relation (such as the basic believability relations) and its connections with the *basic entrenchment* proposed in Rott [14] are also worth being studied. We left all these as future work.

7. Conclusion

The main purpose of this paper is to show that select-direct revision operations generated by descriptor revisions can be reconstructed from a type of relation on sentences called believability relations. More specifically, so called *H-believability relations* and *basic believability relations* were proved to be faithful alternative models for the dependent select-direct revision and the independent select-direct revision respectively. Particularly, an axiomatic characterization of the basic believability relation was obtained. In Section 5, we investigated more potential properties of the believability relations except those characterizing the basic believability relations and obtained a strengthened variant of believability relations. This type of relations is of special interest since independent revision operations constructed from relational models of the most well-ordered form, i.e. $\leqq$ in $(\mathbb{X}, \leqq)$ is a linear ordering, can be exactly derived from them, and all traditional AGM revision operation can be exactly generated from certain subset of them. Moreover, we showed that there is a close connection between the standard epistemic entrenchment relation and the strengthened believability relation. All these facts together confirm the importance and plausibility of the believability relation as a construction for revision operations.

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[10] Thank an anonymous reviewer for pointing this out to us.

Appendix: Proofs

PROOF (for Theorem 3.2). Let $\overset{*}{\preceq}$ be any proximity relation and $\preceq$ a H-believability relation derived from it through $\langle \overset{*}{\preceq}$ to $\preceq\rangle$. Furthermore, let $\circ$ be descriptor revision derived from $\overset{*}{\preceq}$ through $\langle \overset{*}{\preceq}$ to $\circ\rangle$ and $\star$ the sentential revision from $\preceq$ through$\langle \preceq$ to $\star\rangle$. By Proposition 3.2, $\circ$ is a global descriptor revision. So according to the definition of dependent revision operation, we only need to prove that $K \star \varphi = K \circ \mathfrak{B}\varphi$ for every φ. There are only two cases.
Case 1: Let $\varphi \in \mathrm{Ref}(\preceq)$, then it follows that $\mathfrak{B}\varphi \overset{*}{\prec} \Perp$ by $\langle \overset{*}{\preceq}$ to $\preceq\rangle$. So $K \circ \mathfrak{B}\varphi = \{\psi \mid \mathfrak{B}\varphi \overset{*}{\simeq} \mathfrak{B}\varphi \wedge \mathfrak{B}\psi\}$. By counter-dominance of $\overset{*}{\preceq}$, $\mathfrak{B}\varphi \wedge \mathfrak{B}\psi \overset{*}{\simeq} \mathfrak{B}(\varphi \wedge \psi)$. Moreover, it follows from $\mathfrak{B}\varphi \overset{*}{\prec} \Perp$ that $\mathfrak{B}(\varphi \wedge \psi) \overset{*}{\prec} \Perp$ since $\overset{*}{\preceq}$ satisfies transitivity and counter-dominance. So, due to $\langle \overset{*}{\preceq}$ to $\preceq\rangle$, $\varphi \simeq \varphi \wedge \psi$ if and only if $\mathfrak{B}\varphi \overset{*}{\simeq} \mathfrak{B}(\varphi \wedge \psi)$. Thus, $K \star \varphi = \{\psi \mid \varphi \simeq \varphi \wedge \psi\} = \{\psi \mid \mathfrak{B}\varphi \overset{*}{\simeq} \mathfrak{B}(\varphi \wedge \psi)\} = \{\psi \mid \mathfrak{B}\varphi \overset{*}{\simeq} \mathfrak{B}\varphi \wedge \mathfrak{B}\psi\} = K \circ \mathfrak{B}\varphi$.
Case 2: Let $\varphi \notin \mathrm{Ref}(\preceq)$. By $\langle \overset{*}{\preceq}$ to $\preceq\rangle$, it follows that $\mathfrak{B}\varphi \not\overset{*}{\prec} \Perp$ since $\mathfrak{B}\varphi \overset{*}{\preceq} \mathfrak{B}\varphi$ due to counter-dominance of $\overset{*}{\preceq}$. Thus, $K \star \varphi = K = K \circ \mathfrak{B}\varphi$. ■

In order to prove Lemmas 4.1 and 4.2, we first prove the following two observations on sentential relational models and relations, which are also useful in some other proofs.

OBSERVATION 7.1. *Let $\preceq$ be any binary relation on $\mathcal{L}$ satisfying weak transitivity and relative counter-dominance. Then,*

1. *These three propositions are equivalent: (i) $\varphi \in [\varphi]_{\preceq}$, (ii) $[\varphi]_{\preceq} \neq \varnothing$ and (iii) $\varphi \in \mathrm{Ref}(\preceq)$.*
2. *Let $[\varphi]_{\preceq} = [\psi]_{\preceq}$. (i) If $\varphi \preceq \xi$, then $\psi \preceq \xi$, and (ii) if $\xi \preceq \varphi$, then $\xi \preceq \psi$.*

PROOF. *1. From (i) to (ii):* Let $\varphi \in [\varphi]_{\preceq}$, then $[\varphi]_{\preceq} \neq \varnothing$. *From (ii) to (iii):* Let $[\varphi]_{\preceq} \neq \varnothing$, then there exists ψ such that $\varphi \simeq \varphi \wedge \psi$. So $\varphi \in \mathrm{Ref}(\preceq)$. *From (iii) to (i):* Let $\varphi \in \mathrm{Ref}(\preceq)$. Since $\varphi \vdash \varphi \wedge \varphi$, by relative counter-dominance, $\varphi \wedge \varphi \preceq \varphi$. So $\varphi \wedge \varphi \in \mathrm{Ref}(\preceq)$ and hence $\varphi \preceq \varphi \wedge \varphi$ by relative counter-dominance again. Thus, $\varphi \simeq \varphi \wedge \varphi$, i.e. $\varphi \in [\varphi]_{\preceq}$.
2. Let $[\varphi]_{\preceq} = [\psi]_{\preceq}$. *(i)*: Let $\varphi \preceq \xi$, then $\varphi \in \mathrm{Ref}(\preceq)$ and hence $\varphi \in [\psi]_{\preceq}$ and $\psi \in [\psi]_{\preceq}$ due to a result in the first part of this observation. It follows that $\varphi \simeq \varphi \wedge \psi$ and $\psi \simeq \psi \wedge \varphi$. So $\varphi \wedge \psi \preceq \xi$ due to weak transitivity. Moreover, we can conclude $\psi \simeq \varphi \wedge \psi$ from $\psi \simeq \psi \wedge \varphi$ due to relative counter-dominance and weak transitivity. Thus, $\xi \preceq \psi$ due to weak transitivity. *(ii)*: The proof is similar to that for *(i)*. ■

Observation 7.2. *Let* $(\mathbb{X}, \leqq)$ *be any sentential relational model and* $\psi \vdash \varphi$*, then the following three propositions are equivalent: (i)* $\psi \in \mathbb{X}^{\varphi}_{<}$*, (ii)* $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$ *and (iii)* $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\psi}_{<}$*.*

Proof. *From (i) to (ii):* Let $\psi \in \mathbb{X}^{\varphi}_{<}$, then $\mathbb{X}^{\psi} \neq \varnothing$ and $\mathbb{X}^{\psi}_{<}$ exists. Moreover, it follows from $\psi \in \mathbb{X}^{\varphi}_{<}$ that $\mathbb{X}^{\varphi}_{<} \in \mathbb{X}^{\psi}$. Thus, $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$. *From (ii) to (iii):* Let $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$. Since $\psi \vdash \varphi$, $\varphi \in \mathbb{X}^{\psi}_{<}$, i.e. $\mathbb{X}^{\psi}_{<} \in \mathbb{X}^{\varphi}$. Thus, $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\psi}_{<}$ since $\mathbb{X}^{\varphi}_{<}$ is the unique minimal element in $\mathbb{X}^{\varphi}$. *From (iii) to (i):* Obvious. ■

Proof (for Lemma 4.1). We only need to check that $\preceq$ satisfies the four postulates for the basic believability relation.

Weak transitivity: Let $\psi \vdash \varphi$ and $\varphi \simeq \psi$, then $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$ due to $\langle \leqq \text{ to } \preceq \rangle$. So $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\psi}_{<}$ due to observation 7.2. Hence, $\xi \preceq \varphi$ if and only if $\mathbb{X}^{\xi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$ due to $\langle \leqq \text{ to } \preceq \rangle$, and if and only if $\mathbb{X}^{\xi}_{<} \leqq \mathbb{X}^{\psi}_{<}$ since $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\psi}_{<}$, and if and only if $\xi \preceq \psi$ due to $\langle \leqq \text{ to } \preceq \rangle$. And by a similar argument, we can also prove that $\varphi \preceq \xi$ if and only if $\psi \preceq \xi$. Thus, weak transitivity holds for $\preceq$.

Weak coupling: Let $\varphi \simeq \varphi \wedge \psi$ and $\varphi \simeq \varphi \wedge \xi$, then $\mathbb{X}^{\varphi \wedge \psi}_{<} = \mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\phi \wedge \xi}_{<}$ by $\langle \leqq \text{ to } \preceq \rangle$ and observation 7.2. So $\psi \wedge \xi \in \mathbb{X}^{\varphi}_{<}$, and hence $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\varphi \wedge (\psi \wedge \xi)}_{<}$ due to observation 7.2. Thus, $\varphi \simeq \varphi \wedge (\psi \wedge \xi)$ by $\langle \leqq \text{ to } \preceq \rangle$.

Relative counter-dominance: Let $\varphi \in \mathrm{Ref}(\preceq)$, then $\mathbb{X}^{\varphi}_{<}$ exists due to $\langle \leqq \text{ to } \preceq \rangle$. Furthermore, let $\varphi \vdash \psi$, then $\psi \in \mathbb{X}^{\varphi}_{<}$ since $\varphi \in \mathbb{X}^{\varphi}_{<}$ and $\mathrm{Cn}(\mathbb{X}^{\varphi}_{<}) = \mathbb{X}^{\varphi}_{<}$. It follows that $\mathbb{X}^{\psi}_{<}$ exists and $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$ due to observation 7.2. Thus, $\psi \preceq \varphi$ due to $\langle \leqq \text{ to } \preceq \rangle$.

Relative minimality: $\varphi \in K$ if and only if $\mathbb{X}^{\varphi}_{<} = K$ due to $K = \mathbb{X}^{\top}_{<}$ and observation 7.2, and if and only if $\mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\psi}_{<}$ for all ψ such that $\mathbb{X}^{\psi}_{<}$ exists, and if and only if $\varphi \in \mathrm{Ref}(\preceq)$ and $\varphi \preceq \psi$ for all $\psi \in \mathrm{Ref}(\preceq)$ due to $\langle \leqq \text{ to } \preceq \rangle$. ■

Proof (for Lemma 4.2). *Part 1:* The relative minimality of $\preceq$ guarantees that $\mathbb{X}$ is not empty and it follows from observation 7.1 that the map $\langle \preceq \text{ to } \leqq \rangle$ is well-defined. Hence, there exists $(\mathbb{X}, \leqq)$ which can be constructed from $\preceq$ by $\langle \preceq \text{ to } \leqq \rangle$.

Part 2: Now we check that $(\mathbb{X}, \leqq)$ meets the requirements for the sentential relational model.

$(\mathbb{X}1)$*:* Let$[\varphi]_{\preceq} \in \mathbb{X}$,then $\varphi \in \mathrm{Ref}(\preceq)$ by $\langle \preceq \text{ to } \leqq \rangle$ and hence $[\varphi]_{\preceq} \neq \varnothing$ due to observation 7.1. Moreover, let $\psi \in [\varphi]_{\preceq}$ and $\xi \in [\varphi]_{\preceq}$, then $\varphi \simeq \varphi \wedge \psi$ and $\varphi \simeq \varphi \wedge \xi$ by definition of $[\varphi]_{\preceq}$. It follows that $\varphi \simeq \varphi \wedge (\psi \wedge \xi)$ by weak coupling. Furthermore, let $\psi \wedge \xi \vdash \chi$, then $\varphi \wedge (\psi \wedge \xi) \vdash \varphi \wedge \chi$. So $\varphi \wedge \chi \preceq \varphi \wedge (\psi \wedge \xi)$ by relative counter-dominance and hence $\varphi \wedge \chi \preceq \varphi$ by weak transitivity. Moreover, $\varphi \preceq \varphi \wedge \chi$ by relative counter-dominance. So,

$\varphi \simeq \varphi \wedge \chi$, i.e. $\chi \in [\varphi]_{\preceq}$. Hence, $\mathrm{Cn}([\varphi]_{\preceq}) = [\varphi]_{\preceq}$. Thus, $\mathbb{X}$ is a set of belief sets.
($\mathbb{X}2$)*:* Since $\top \in K$, $\top \in \mathrm{Ref}(\preceq)$ by the relative minimality of $\preceq$. Hence, $\varphi \in K$ if and only if $\varphi \in \mathrm{Ref}(\preceq)$ and $\varphi \preceq \psi$ for all φ due to relative minimality of $\preceq$, and if and only if $\varphi \preceq \top$ due to relative counter-dominance and weak transitivity of $\preceq$, and if and only if $\top \simeq \top \wedge \varphi$, i.e. $\varphi \in [\top]_{\preceq}$, due to relative counter-dominance and weak transitivity of $\preceq$. Hence, $K = [\top]_{\preceq}$. Moreover, $\top \in \mathrm{Ref}(\preceq)$ yields $[\top]_{\preceq} \in \mathbb{X}$ by $\langle \preceq$ to $\leqq \rangle$. Thus, $K \in \mathbb{X}$.
($\leqq 1$)*:* $\top \preceq \varphi$ for all $\varphi \in \mathrm{Ref}(\preceq)$ due to the relative counter-dominance of $\preceq$. Moreover, $K = [\top]_{\preceq}$ as we have showed. Thus, $K \leqq X$ for all $X \in \mathbb{X}$ due to $\langle \preceq$ to $\leqq \rangle$.
($\leqq 2$)*:* Let $\mathbb{X}^{\varphi} \neq \varnothing$, then there exists $[\psi]_{\preceq} \in \mathbb{X}$ such that $\varphi \in [\psi]_{\preceq}$. It follows that $\varphi \in \mathrm{Ref}(\preceq)$ and hence $[\varphi]_{\preceq} \in \mathbb{X}^{\varphi}$ due to $\langle \preceq$ to $\leqq \rangle$ and observation 7.1. Now we show that $[\varphi]_{\preceq} = \mathbb{X}^{\varphi}_{<}$. (i) For any ξ such that $[\xi]_{\preceq} \in \mathbb{X}^{\varphi}$, it holds that $\varphi \in [\xi]_{\preceq}$, i.e. $\xi \simeq \xi \wedge \varphi$. It follows that $\varphi \preceq \xi$ by relative counter-dominance and weak transitivity of $\preceq$. So $[\varphi]_{\preceq} \leqq [\xi]_{\preceq}$ due to $\langle \preceq$ to $\leqq \rangle$. Moreover, suppose for contradiction that there exits $[\xi]_{\preceq} \in \mathbb{X}^{\varphi}$ such that $[\xi]_{\preceq} \neq [\varphi]_{\preceq}$ and $[\xi]_{\preceq} \leqq [\varphi]_{\preceq}$. It follows by $\langle \preceq$ to $\leqq \rangle$ that $\varphi \simeq \xi$ and $\xi \simeq \xi \wedge \varphi$ since $\varphi \in [\xi]_{\preceq}$ and $[\varphi]_{\preceq} \leqq [\xi]_{\preceq}$. Let χ be any sentence in $[\xi]_{\preceq}$, then it follows from $\varphi \in [\xi]_{\preceq}$ and $\mathrm{Cn}([\xi]_{\preceq}) = [\xi]_{\preceq}$ that $\xi \simeq \xi \wedge (\varphi \wedge \chi)$. So $\varphi \simeq \xi \wedge (\varphi \wedge \chi)$ due to $\varphi \simeq \xi$ and weak transitivity. Moreover, $\xi \wedge (\varphi \wedge \chi) \vdash \varphi \wedge \chi$ and $\varphi \wedge \chi \vdash \varphi$. Hence, by relative counter-dominance and weak transitivity, $\varphi \simeq \varphi \wedge \chi$, i.e. $\chi \in [\varphi]_{\preceq}$. We can prove that if $\chi \in [\varphi]_{\preceq}$ then $\chi \in [\xi]_{\preceq}$ by a similar argument. So $[\varphi]_{\preceq} = [\xi]_{\preceq}$ which contradicts the hypothesis. Thus, $[\varphi]_{\preceq}$ is the unique minimal element in $\mathbb{X}^{\varphi}$, i.e. $\mathbb{X}^{\varphi}_{<} = [\varphi]_{\preceq}$.
Part 3: Let $\preceq'$ be relation retrieved from $(\mathbb{X}, \leqq)$ by $\langle \leqq$ to $\preceq \rangle$. Now we prove $\preceq = \preceq'$. $\varphi \preceq \psi$ if and only if $[\varphi]_{\preceq} \in \mathbb{X}$, $[\psi]_{\preceq} \in \mathbb{X}$ and $[\varphi]_{\preceq} \leqq [\psi]_{\preceq}$ due to $\langle \preceq$ to $\leqq \rangle$, and if and only if $\mathbb{X}^{\varphi}_{<}$ exists, $\mathbb{X}^{\psi}_{<}$ exists and $\mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\psi}_{<}$ as we have proved $[\varphi]_{\preceq} = \mathbb{X}^{\varphi}_{<}$ when $[\varphi]_{\preceq} \in \mathbb{X}$, and if and only if $\varphi \preceq' \psi$ since $\preceq'$ is constructed through$\langle \leqq$ to $\preceq \rangle$ from $(\mathbb{X}, \leqq)$. Thus, $\preceq = \preceq'$. ■

Proof (for Lemma 4.3). We only need to prove (i) $\mathbb{X}^{\varphi}_{<}$ exists if and only if $\varphi \in \mathrm{Ref}(\preceq)$, and (ii) $\mathbb{X}^{\varphi}_{<} = [\varphi]_{\preceq}$. *(i)*: Let $\mathbb{X}^{\varphi}_{<}$ exist. Since $K = \mathbb{X}^{\top}_{<}$ and $K \leqq X$ for all $X \in \mathbb{X}$, then $\mathbb{X}^{\top}_{<} \preceq \mathbb{X}^{\varphi}_{<}$. So $\top \preceq \varphi$ due to $\langle \leqq$ to $\preceq \rangle$ and hence $\varphi \in \mathrm{Ref}(\preceq)$. On the other hand, let $\varphi \in \mathrm{Ref}(\preceq)$, then there exists ψ such that $\varphi \preceq \psi$ or $\psi \preceq \varphi$. In both cases it follows that there exists $\mathbb{X}^{\varphi}_{<}$ due to $\langle \leqq$ to $\preceq \rangle$. Thus, $\mathbb{X}^{\varphi}_{<}$ exists if and only if $\varphi \in \mathrm{Ref}(\preceq)$. *(ii)*: For any ψ, $\psi \in \mathbb{X}^{\varphi}_{<}$ if and only if $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\varphi \wedge \psi}_{<}$ by observation 7.2, and if and only if $\varphi \simeq \varphi \wedge \psi$ due to $\langle \leqq$ to $\preceq \rangle$, i.e. $\psi \in [\varphi]_{\preceq}$. Thus, $\mathbb{X}^{\varphi}_{<} = [\varphi]_{\preceq}$. ■

Proof (for Theorem 4.4). The equivalence between propositions 1 and 2 follows from Lemmas 4.1, 4.2 and 4.3. For the equivalence between propositions 1 and 3 (i.e. axiomatization of independent revision), see Zhang & Hansson [16]. ■

Proof (for Lemma 5.2). Firstly, we show that $(\mathbb{Y}, \leqq')$ is a canonical sentential model.
($\mathbb{Y}$1): It follows immediately that $\mathbb{Y}$ is a set of belief sets with $\mathbb{Y} \subseteq \mathbb{X}$ and that $(\mathbb{X}, \leqq)$ is a relational model.
($\mathbb{Y}$2): K is the unique $\leqq$-minimal element in $\mathbb{X}$, so $K \in \mathbb{Y}$ by the definition of $\mathbb{Y}$.
($\leqq'$ 1): $K \leqq X$ for all $X \in \mathbb{Y} \subseteq \mathbb{X}$. Moreover, $\leqq \cap(\mathbb{X} \times \mathbb{X}) \subseteq \leqq'$ by the definition of $\leqq'$. Thus, $K \leqq' X$ for all $X \in \mathbb{Y}$.
($\leqq'$ 2): Let $\mathbb{Y}^{\varphi} \neq \varnothing$, then $\mathbb{X}^{\varphi} \neq \varnothing$ since $\mathbb{Y} \subseteq \mathbb{X}$ and hence $\mathbb{X}^{\varphi}_{<}$ exists. Now we prove that in this case, $\mathbb{X}^{\varphi}_{<} \in \mathbb{Y}^{\varphi}$ and $\mathbb{X}^{\varphi}_{<} = \mathbb{Y}^{\varphi}_{<'}$. Assume for contradiction that $\mathbb{X}^{\varphi}_{<} \notin \mathbb{Y}^{\varphi}$. It follows that $\mathbb{X}^{\varphi}_{<} \notin \mathbb{Y}$, i.e. there is $\mathbb{D} \subseteq \mathbb{X}$ such that for any $Y \in \mathbb{D}$, $Y < \mathbb{X}^{\varphi}_{<}$ and $\mathbb{X}^{\varphi}_{<} \subseteq \bigcup \mathbb{D}$. So there exists $Y \in \mathbb{D} \subseteq \mathbb{X}$ such that $\varphi \in Y$, i.e. $Y \in \mathbb{X}^{\varphi}$, and $Y < \mathbb{X}^{\varphi}_{<}$. This shows that $\mathbb{X}^{\varphi}_{<}$ is not the unique $\leqq$-minimal element in $\mathbb{X}^{\varphi}$. So $\mathbb{X}^{\varphi}_{<} \in \mathbb{Y}^{\varphi}$. Moreover, for any $Y \in \mathbb{Y}^{\varphi} \subseteq \mathbb{X}^{\varphi}$, $\mathbb{X}^{\varphi}_{<} \leqq' Y$ by the definition of $\leqq'$. Suppose for contradiction that there exists some $Y \neq \mathbb{X}^{\varphi}_{<}$ such that $Y \in \mathbb{Y}^{\varphi}$ and $Y <' \mathbb{X}^{\varphi}_{<}$. It follows from the definition of $\leqq'$ that there exists $X \neq \mathbb{X}^{\varphi}_{<}$ such that $X \in \mathbb{Y}^{\varphi} \subseteq \mathbb{X}^{\varphi}$ and $Y \leqq' X < \mathbb{X}^{\varphi}_{<}$. This shows that $\mathbb{X}^{\varphi}_{<}$ is not the unique $\leqq$-element in $\mathbb{X}^{\varphi}$. Thus, $\mathbb{Y}^{\varphi}_{<'}$ exists and $\mathbb{Y}^{\varphi}_{<'} = \mathbb{X}^{\varphi}_{<}$.
($*'$1): Suppose for contradiction that there exists some $X \in \mathbb{Y}$ such that there is no φ satisfying $X = \mathbb{Y}^{\varphi}_{<'}$. This means that for any $\varphi \in X$, there exists some $Y \in \mathbb{Y} \subseteq \mathbb{X}$ such that $\varphi \in Y$ and $Y <' X$. Due to the definition of $\leqq'$, it follows that for any $\varphi \in X$, there exists some $Y \in \mathbb{X}$ such that $\varphi \in Y$ and $Y < X$. It follows from the definition of $\mathbb{Y}$ that $X \notin \mathbb{Y}$.
($\leqq'$3): It follows immediately from the definition of $\leqq'$ that it is transitive. Moreover, ($*'$1) yields that $\leqq'$ is reflexive. Thus, $\leqq'$ is a pre-order on $\mathbb{Y}$.
Then, we prove that a revision operator is based on $(\mathbb{X}, \leqq)$ if and only if it is based on $(\mathbb{Y}, \leqq')$. Note that we have shown that when $\mathbb{X}^{\varphi}_{<}$ exists, $\mathbb{Y}^{\varphi}_{<'}$ exists as well and $\mathbb{Y}^{\varphi}_{<'} = \mathbb{X}^{\varphi}_{<}$. Moreover, let $\mathbb{Y}^{\varphi}_{<'}$ exist, then $\mathbb{Y}^{\varphi}_{<'} \in \mathbb{Y}^{\varphi} \subseteq \mathbb{X}^{\varphi} \neq \varnothing$. So $\mathbb{X}^{\varphi}_{<}$ exists and hence $\mathbb{X}^{\varphi}_{<} = \mathbb{Y}^{\varphi}_{<'}$ as we have showed. Thus, by the definition of $\langle \leqq \text{ to } \star \rangle$, a revision operator is based on $(\mathbb{X}, \leqq)$ if and only if it is based on $(\mathbb{Y}, \leqq')$. ■

Proof (for Lemma 5.3). *1. Part 1:* By Lemma 4.2, $(\mathbb{X}, \leqq)$ is a sentential relational model. Now we check that it is also a canonical model.
($*$1): $X \in \mathbb{X}$ if and only if there exists some φ such that $\varphi \in \mathrm{Ref}(\preceq)$ and

$X = [\varphi]_{\preceq}$ due to $\langle \preceq$ to $\leqq \rangle$, and if and only if $X = [\varphi]_{\preceq} = \mathbb{X}^{\varphi}_{<}$ as we have showed in proof of Lemma 4.2.
($\leqq$3): Let $X \leqq Y$ and $Y \leqq Z$, then there exist φ, ψ and ξ such that $X = [\varphi]_{\preceq}$, $Y = [\psi]_{\preceq}$ and $Z = [\xi]_{\preceq}$. So $\varphi \preceq \psi$ and $\psi \preceq \xi$ due to $\langle \preceq$ to $\leqq \rangle$ and hence $\varphi \preceq \xi$ by the transitivity of $\preceq$. So, $X \leqq Z$ due to $\langle \preceq$ to $\leqq \rangle$. Moreover, for any $\varphi \in \mathrm{Ref}(\preceq)$, $[\varphi]_{\preceq} \leqq [\varphi]_{\preceq}$ due to $\langle \preceq$ to $\leqq \rangle$ and the relative counter-dominance of $\preceq$. Thus, $\leqq$ is a pre-order.
Part 2: Since $\preceq$ is a basic believability relation, it follows from Lemma 4.2 that $\preceq$ can be retrieved from $(\mathbb{X}, \leqq)$ by means of $\langle \leqq$ to $\preceq \rangle$.
2. Part 1: Since Lemma 4.1 shows that $\preceq$ is a basic believability relation, we only need to check that $\preceq$ satisfies *transitivity:* Let $\varphi \preceq \psi$ and $\psi \preceq \xi$, then there exist $\mathbb{X}^{\varphi}_{<}$, $\mathbb{X}^{\psi}_{<}$ and $\mathbb{X}^{\xi}_{<}$ with $\mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\psi}_{<}$ and $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\xi}_{<}$. So $\mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\xi}_{<}$ since $(\mathbb{X}, \leqq)$ is canonical model. Thus, $\varphi \preceq \xi$ due to $\langle \leqq$ to $\preceq \rangle$.
Part 2: Let $\leqq'$ be constructed from $\preceq$ by $\langle \preceq$ to $\leqq \rangle$. Now we prove that $\leqq = \leqq'$. $X \leqq Y$ if and only if $X = \mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\psi}_{<} = Y$ due to the definition of a canonical model, and if and only if $\varphi \preceq \psi$ due to $\langle \leqq$ to $\preceq \rangle$, and if and only if $[\varphi]_{\preceq} \leqq' [\psi]_{\preceq}$ due to$\langle \preceq$ to $\leqq \rangle$, and if and only if $X \leqq' Y$ since $\mathbb{X}^{\varphi}_{<} = [\psi]_{\preceq}$ and $\mathbb{X}^{\psi}_{<} = [\psi]_{\preceq}$ as we have proved. Thus, $\leqq = \leqq'$ and hence $(\mathbb{X}, \leqq)$ can be reconstructed from $\preceq$ by means of $\langle \preceq$ to $\leqq \rangle$. ∎

Proof (for Theorem 5.4). It follows immediately from Lemmas 4.3, 5.2 and 5.3. ∎

Proof (for Theorem 5.5). *1. From left to right:* Since $\preceq$ satisfies exhaustiveness we have $\perp \in \mathrm{Ref}(\preceq)$, and hence $[\perp]_{\preceq} \in \mathbb{X}$. Moreover, by the relative counter-dominance of $\preceq$, $\perp \simeq \perp \wedge \varphi$ for every φ. Hence, $[\perp]_{\preceq} = \mathrm{Cn}(\{\perp\}) \in \mathbb{X}$. For any φ, if $\mathbb{X}^{\varphi}_{<}$ exists, then $\mathbb{X}^{\varphi}_{<} = [\varphi]_{\preceq}$ as we have shown. So for any φ, $\mathbb{X}^{\varphi}_{<} = [\varphi]_{\preceq} \leqq [\perp]_{\preceq} = \mathrm{Cn}(\{\perp\})$ since $\varphi \preceq \perp$ due to the relative counter-dominance of $\preceq$. If there exists ψ such that $\mathrm{Cn}(\{\perp\}) \leqq \mathbb{X}^{\psi}_{<}$, then $\perp \preceq \psi$ and hence $\xi \preceq \psi$ for all ξ due to the relative counter-dominance and transitivity of $\preceq$. Then $\psi \equiv \perp$ by maximality. Thus, for any $\varphi \not\equiv \perp$, $\mathbb{X}^{\varphi}_{<} < \mathrm{Cn}(\{\perp\})$.
From right to left: Suppose there exists some φ such that $\psi \preceq \varphi$ for all ψ and $\varphi \not\equiv \perp$. It follows that $\perp \preceq \varphi$ and hence $\mathrm{Cn}(\{\perp\}) = [\perp]_{\preceq} \leqq [\varphi]_{\preceq} = \mathbb{X}^{\varphi}_{<}$. By Observation 7.2, $[\perp]_{\preceq} = [\varphi]_{\preceq}$ and hence $\varphi \equiv \perp$ which contradicts the hypothesis. Moreover, $\mathrm{Cn}(\{\perp\}) \in \mathbb{X}$ yields $\perp \in \mathrm{Ref}(\preceq)$. It follows that $\varphi \preceq \perp$ for every φ due to relative counter-dominance of $\preceq$. Thus, $\preceq$ satisfies exhaustiveness and maximality.
2. From left to right: Let $\mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\psi}_{<}$ and $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$, then $\varphi \simeq \psi$ by $\langle \preceq$ to $\leqq \rangle$. So $\varphi \simeq \varphi \wedge \psi$ and $\psi \simeq \psi \wedge \varphi$ due to $\varphi \simeq \varphi$, $\psi \simeq \psi$ and coupling of $\preceq$. Hence, $\mathbb{X}^{\varphi \wedge \psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$ and $\mathbb{X}^{\varphi \wedge \psi}_{<} \leqq \mathbb{X}^{\psi}_{<}$ by $\langle \preceq$ to $\leqq \rangle$. It follows from Observation 7.2 that $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\varphi \wedge \psi}_{<} = \mathbb{X}^{\psi}_{<}$. Thus, $\leqq$ is anti-symmetric. *From right to left:* Let

$\varphi \simeq \psi$ and $\varphi \simeq \xi$, then $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\psi}_{<} = \mathbb{X}^{\xi}_{<}$ due to $\langle\preceq$ to $\leqq\rangle$ and the anti-symmetry of $\leqq$. So $\psi \in \mathbb{X}^{\xi}_{<}$ and hence $\mathbb{X}^{\varphi}_{<} = \mathbb{X}^{\psi}_{<} = \mathbb{X}^{\psi\wedge\xi}_{<}$ due to observation 7.2. Thus, by $\langle\preceq$ to $\leqq\rangle$, $\varphi \simeq \psi \wedge \xi$.
3. $\varphi \preceq \psi$ or $\psi \preceq \phi$ holds if and only if $\mathbb{X}^{\varphi}_{<} \leqq \mathbb{X}^{\psi}_{<}$ or $\mathbb{X}^{\psi}_{<} \leqq \mathbb{X}^{\varphi}_{<}$ due to $\langle\preceq$ to $\leqq\rangle$. Note that $(\mathbb{X}, \leqq)$ is a canonical sentential model, i.e. for any $X \in \mathbb{X}$, $X = \mathbb{X}^{\xi}_{<}$ for some ξ. Thus, $\preceq$ satisfies completeness if and only if $\leqq$ does so. ■

Proof (for Theorem 5.6). It has been proved in Zhang & Hansson [16] that revision operations based on sentential relational models satisfying $(\star 2)$ can be characterized by the first five postulates listed in the proposition. So by Lemma 5.2, revision operations based on canonical sentential models satisfying that condition can also be represented by those postulates. Moreover, Lemmas 4.3 and 5.3 and Theorem 5.5 jointly imply that a revision operation is based on this kind of canonical sentential model if and only if it is based on a transitive believability relation satisfying exhaustiveness and maximality. Thus, this theorem holds. ■

Proof (for Theorem 5.7). It has been proved in Zhang & Hansson [16] that a revision operation is based on some strengthened sentential model if and only if it satisfies closure, confirmation, success, consistency and strong reciprocity. Next we will prove that a revision operation is based on a strengthened sentential model if and only if it is based on a linear strengthened sentential model.
Let $(\mathbb{X}, \leqq)$ be any strengthened sentential model. Let $\mathbb{Y} = \mathbb{X} \setminus \mathrm{Cn}(\{\bot\})$, as we have shown in Theorem 5.5, $\leqq \cap (\mathbb{Y} \times \mathbb{Y})$ is a partial order. Given the axiom of choice, there is a linear order $\leqq^*$ on $\mathbb{X}$ which extends the partial order $\leqq$.[11] Let $\leqq' = \leqq^* \cup \{(X, \mathrm{Cn}(\bot)) \mid X \in \mathbb{X}\}$. Now we prove that $(\mathbb{X}, \leqq')$ is a linear strengthened sentential model. It is easy to see that $(\mathbb{X}, \leqq')$ satisfies $(\mathbb{X}1)$, $(\mathbb{X}2)$, $(\star' 2)$, $(\leqq' 1)$ and $(\leqq' 3)-(\leqq' 5)$. For $(\star' 1)$ and $(\leqq' 2)$, we only need to show that for every φ, if $\mathbb{X}^{\varphi} \neq \varnothing$, then $\mathbb{X}^{\varphi}_{<'} = \mathbb{X}^{\varphi}_{<}$.
Let $\mathbb{X}^{\varphi} \neq \varnothing$, then $\mathbb{X}^{\varphi}_{<}$ exists and $\mathbb{X}^{\varphi}_{<} \leqq' X$ for every $X \in \mathbb{X}^{\varphi}$ since $(\mathbb{X}, \leqq)$ satisfies $(\leqq 2)$ and $\leqq'$ extends $\leqq$. And for any $Y \in \mathbb{X}^{\varphi}$, if $Y \leqq' \mathbb{X}^{\varphi}_{<}$, then $Y = \mathbb{X}^{\varphi}_{<}$ since $\leqq'$ is anti-symmetric. So $\mathbb{X}^{\varphi}_{<'}$ exists and $\mathbb{X}^{\varphi}_{<'} = \mathbb{X}^{\varphi}_{<}$. This result also means that a revision operation $\star$ is based on $(\mathbb{X}, \leqq)$ if and only if it is determined by $(\mathbb{X}, \leqq')$.
Thus, by Lemmas 4.3, 5.3 and Theorem 5.5, the propositions 1 and 2 in this theorem are equivalent. ■

[11] For more details on this result, see Jech [11]. And that $\leqq^*$ extends $\leqq$ means that if $X \leqq Y$, then $X \leqq^* Y$.

Proof (for Theorem 6.2). Conditions ($\rhd$1), ($\rhd$2), ($\lhd$1) and ($\lhd$2) are just "translations" of inclusion, vacuity, superexpansion and subexpansion through following rules:

1. $\psi \in K \star \varphi$ if and only if $\varphi \simeq \varphi \wedge \psi$
2. $\varphi \in K$ if and only if $\top \simeq \varphi$
3. $\psi \notin K \star \varphi$ if and only if $\varphi \prec \varphi \wedge \psi$
4. $\varphi \notin K$ if and only if $\top \prec \varphi$

Let $\preceq$ be a strengthened believability relation and $\star$ determined by this relation. Then it is obvious that $\preceq$ and $\star$ satisfy the above four rules. It immediately follows that the proposition 1 and the *from left to right* part of the proposition 2 in this theorem hold.
For the *from right to left* part of the proposition 2, let $\star$ be a AGM revision operation, then $\star$ can be determined by some strengthened believability relation $\preceq$ as we have mentioned. So $\star$ and $\preceq$ also satisfy those translation rules. Thus, $\preceq$ satisfies ($\lhd$1) and ($\lhd$2). ■

Proof (for Theorem 6.5). *1.* We first check that $\leqq$ is a standard entrenchment relations with respect to K.
Transitivity: Let $\varphi \leqq \psi$ and $\psi \leqq \xi$, then $\neg\varphi \preceq \neg\psi$ and $\neg\psi \preceq \neg\xi$ due to $\langle \preceq \text{ to } \leqq \rangle$. So $\neg\varphi \preceq \neg\xi$ by the transitivity of $\preceq$. Thus, $\varphi \leqq \xi$ due to$\langle \preceq \text{ to } \leqq \rangle$.
Dominance: Let $\varphi \vdash \psi$, then $\neg\psi \vdash \neg\varphi$ and hence $\neg\varphi \preceq \neg\psi$ by the counter-dominance of $\preceq$. Thus, $\varphi \leqq \psi$ due to $\langle \preceq \text{ to } \leqq \rangle$.
Conjunctiveness: By maxi-outcome of $\preceq$, $\varphi \wedge \neg\psi \preceq \varphi$ or $\varphi \wedge \psi \preceq \varphi$. So, by substituting $\neg(\varphi \wedge \psi)$ for φ, (i) $\neg(\varphi \wedge \psi) \wedge \neg\psi \preceq \neg(\varphi \wedge \psi)$ or (ii) $\neg(\varphi \wedge \psi) \wedge \psi \preceq \neg(\varphi \wedge \psi)$ holds. Assume that (i) holds, since $\neg\psi \equiv \neg(\varphi \wedge \psi) \wedge \neg\psi$, then $\neg\psi \preceq \neg(\varphi \wedge \psi)$ by the counter-dominance and transitivity of $\preceq$. Hence, $\psi \leqq \varphi \wedge \psi$ due to $\langle \preceq \text{ to } \leqq \rangle$. Assume that (ii) holds, then since $\neg(\varphi \wedge \psi) \wedge \psi \vdash \neg\varphi$, $\neg\varphi \preceq \neg(\varphi \wedge \psi)$ by counter-dominance and transitivity of $\preceq$. Hence, $\varphi \leqq \varphi \wedge \psi$ due to $\langle \preceq \text{ to } \leqq \rangle$. Thus, $\psi \leqq \varphi \wedge \psi$ or $\varphi \leqq \varphi \wedge \psi$.
$*$*-Minimality*: $\varphi \notin K$ if and only if $\neg\varphi \in K$ since K is a maximal consistent belief set, and if and only if $\neg\varphi \preceq \psi$ for all ψ due to the minimality of $\preceq$, and if and only if $\neg\varphi \preceq \neg\psi$ for all ψ, and if and only if $\varphi \leqq \psi$ for all ψ due to $\langle \preceq \text{ to } \leqq \rangle$.
$*$*-Maximality*: Let $\psi \leqq \varphi$ for all ψ, then $\neg\psi \preceq \neg\varphi$ for all ψ due to $\langle \preceq \text{ to } \leqq \rangle$. It follows that $\psi \preceq \neg\varphi$ for all ψ. So $\neg\varphi \equiv \bot$ by the maximality of $\preceq$. Thus, $\vdash \varphi$.
Then we show that $\leqq$ satisfies ($\triangle$): Let $\varphi \doteq \psi$, then $\neg\varphi \simeq \neg\psi$ due to $\langle \preceq \text{ to } \leqq \rangle$. Moreover, $\neg\varphi \simeq \neg\varphi$ by the counter-dominance for $\preceq$. So $\neg\varphi \simeq \neg\varphi \wedge \neg\psi$ by

coupling for $\preceq$, and hence $\neg\varphi \simeq \neg(\varphi \vee \psi)$ by counter-dominance and transitivity for $\preceq$. Thus, $\varphi \doteq \varphi \vee \psi$ due to $\langle \preceq$ to $\leqslant\rangle$.

2. Part 1: we need to check that $\preceq$ satisfies the six conditions characterizing maximal outcome believability relation. It is easy to see that transitivity, coupling, counter-dominance, minimality and maximality are all satisfied. We only give a detailed proof for maxi-outcome.

Maxi-outcome: By the conjunctiveness and dominance of $\leqslant$, $\varphi \doteq \varphi \wedge \psi$ or $\psi \doteq \varphi \wedge \psi$. So, by substituting $\neg(\varphi \wedge \psi)$ for φ and $\neg(\varphi \wedge \neg\psi)$ for ψ, $\neg(\varphi \wedge \psi) \doteq \neg(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \neg\psi)$ or $\neg(\varphi \wedge \neg\psi) \doteq \neg(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \neg\psi)$ holds. Since $\neg(\varphi \wedge \psi) \wedge \neg(\varphi \wedge \neg\psi) \equiv \neg\varphi$, $\neg(\varphi \wedge \psi) \doteq \neg\varphi$ or $\neg(\varphi \wedge \neg\psi) \doteq \neg\varphi$ holds by the dominance and transitivity of $\leqslant$. Hence, $\varphi \simeq \varphi \wedge \psi$ or $\varphi \simeq \varphi \wedge \neg\psi$ holds due to $\langle\leqslant$ to $\preceq\rangle$.

Part 2: $\varphi \leqslant \psi$ if and only if $\neg\neg\varphi \leqslant \neg\neg\psi$ by dominance and transitive for $\leqslant$, and if only if $\neg\varphi \preceq \neg\psi$ due to $\langle\leqslant$ to $\preceq\rangle$. Thus, $\leqslant$ can be reconstructed from $\preceq$ by $\langle\preceq$ to $\leqslant\rangle$. ■

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