The Nature of External Representations in Problem Solving[1]

Jiajie Zhang
Department of Psychology
The Ohio State University

ABSTRACT

This article proposes a theoretical framework for external representation based problem solving. The Tic-Tac-Toe and its isomorphs are used to illustrate the procedures of the framework as a methodology and test the predictions of the framework as a functional model. Experimental results show that the behavior in the Tic-Tac-Toe is determined by the directly available information in external and internal representations in terms of perceptual and cognitive biases, regardless of whether the biases are consistent with, inconsistent with, or irrelevant to the task. It is shown that external representations are not merely inputs and stimuli to the internal mind and that they have much more important functions than mere memory aids. A representational determinism is suggested--the form of a representation determines what information can be perceived, what processes can be activated, and what structures can be discovered from the specific representation.

External representations are involved in many cognitive tasks, such as multiplication with paper and pencil, grocery shopping with a written list, geometrical problem solving, graph understanding, diagrammatic reasoning, chess playing, and so on. Few would deny that external representations play certain roles in these tasks. However, in comparison with internal representations, relatively little research has been directed towards the nature of external representations in cognition. This might be due to the belief that very little knowledge about the internal mind can be gained by studying external representations, or due to the view that external representations are nothing but inputs and stimuli to the internal mind, or simply due to the lack of a suitable methodology for studying external representations.

This article explores the functions of external representations, using problem solving as the task domain and test bed. It takes the position that much can be learned about the internal mind by studying external representations because much of the structure of the internal mind is a reflection of the structure of the external environment (e.g., Anderson, 1993; Shepard, 1984; Simon, 1981). It argues that external representations are not simply inputs and stimuli to the internal mind; rather, they are so intrin-

sic to many cognitive tasks that they guide, constrain, and even determine cognitive behavior. By focusing on what information in external representations can be perceived and how the information in external representations affects problem solving behavior, this article develops a theoretical framework for external representation based (henceforth, ER-based) problem solving. This framework is not only a functional model that can make specific empirical predictions but also a methodology that can be used to systematically analyze ER-based problem solving tasks.

This article is divided into five parts. The first part introduces the theoretical background, including a definition of external representations, a discussion on the relationship between internal and external representations, and a brief review of the important roles of external representations in cognition. The second part proposes the theoretical framework for ER-based problem solving. The third part uses the framework as a methodology to analyze the structure of the Tic-Tac-Toe and as a functional model to make specific predictions about the behavior in the Tic-Tac-Toe. The fourth part reports three experiments designed to test the predictions of the framework and examine the general properties of external representations. The last part summarizes the experimental results, evaluates the theoretical framework, and suggests a representational determinism.

THEORETICAL BACKGROUND

A Definition of External Representations

In the present study, external representations are defined as the knowledge and structure in the environment, as physical symbols, objects, or dimensions (e.g., written symbols, beads of abacuses, dimensions of a graph, etc.), and as external rules, constraints, or relations embedded in physical configurations (e.g., spatial relations of written digits, visual and spatial layouts of diagrams, physical constraints in abacuses, etc.). The information in external representations can be picked up, analyzed, and processed by perceptual systems alone, although the top-down participation of conceptual knowledge from internal

representations can sometimes facilitate or inhibit the perceptual processes. In contrast, internal representations are the knowledge and structure in memory, as propositions, productions, schemas, neural networks, or other forms. The information in internal representations has to be retrieved from memory by cognitive processes, although the cues in external representations can sometimes trigger the retrieval processes. Let us consider multiplying 735 by 278 using paper and pencil. The internal representations are the meanings of individual symbols (e.g., the numerical value of the arbitrary symbol "7" is seven), the addition and multiplication tables, arithmetic procedures, etc., which have to be retrieved from memory; the external representations are the shapes and positions of the symbols, the spatial relations of partial products, etc., which can be perceptually inspected from the environment (see Zhang & Norman, 1995). To perform this task, people need to process the information perceived from external representations and the information retrieved from internal representations in an interwoven, integrative, and dynamic manner.

External representations can be transformed into internal representations by memorization. But this internalization is not necessary if external representations are always available, and not possible if external representations are too complex. Internal representations can also be transformed into external representations by externalization. Externalization can be beneficial if the benefit of using external representations can offset the cost associated with the externalization process.

The Relationship between
Internal and External Representations

The importance of explicitly distinguishing external representations from internal ones has not been seriously considered until recently. In traditional cognitive science, most studies either exclusively focused on internal representations or, when taking external representations into account, often failed to separate them from internal ones. Thus, these studies often mistakenly equate external representations to internal representations, or equate representations having both internal and external components to internal representations. As noted by Kirlik, Plamondon, Lytton, and Jagacinski (1993a, 1993b) and Suchman (1987), this confusion often leads one to postulate unnecessary complex internal mechanisms to explain the complex structure of the wrongly identified internal representation, much of which is merely a reflection of the structure of the external representation.

When people do acknowledge the difference between internal and external representations, they usually have different views on their relations. One view is that external representations are merely inputs and stimuli to the internal mind. In this view, even if

it is the case that many cognitive tasks involve interactions with the environment, all cognitive processing only occurs in the internal model of the external environment. Thus, when an agent is faced with a task that requires interactions with the environment, the agent first has to create an internal model of the environment through some encoding processes, then performs mental computations on the contents (symbols, subsymbols, or other forms) in this constructed internal model, and then externalizes the products of the internal processing to the environment through some decoding processes. This is a common view in traditional AI and other fields of cognitive science (e.g., see Newell, 1990, pp. 57).

A radically different view, offered by Gibson (1966, 1979), is that the environment is highly structured—full of invariant information in the extended spatial and temporal patterns of optic arrays. The invariant information in the environment can be directly picked up without the mediation of memory, inference, deliberation, or any other mental processes that involve internal representations. To Gibson, the information in the environment is sufficient to specify all objects and events in the environment, and thus it is sufficient for perception and action. In addition, the end product of perception is not an internal representation of the environment; rather, it is the invariant directly picked up from the environment.

There are a few recent approaches that emphasize the structures of the environment and people's interactions with them without denying the important roles of internal representations. The situated cognition approach, for example, argues that people's activities in concrete situations are guided, constrained, and to some extent, determined by the physical and social context in which they are situated (e.g., Barwise & Perry, 1983; Clancey 1993; Greeno, 1989; Greeno & Moore 1993; Lave, 1988; Lewis, 1991; Suchman, 1987). In this view, it is not necessary to construct an internal model of the environment to mediate actions: people can directly access the situational information in their environment and act upon it in an adaptive manner. As another example, the distributed cognition approach explores how cognitive activity is distributed across internal human minds, external cognitive artifacts, and groups of people, and across space and time (e.g., Hutchins, 1990, 1995a, 1995b; Hutchins & Norman, 1988; Norman, 1988, 1991, 1993b; Zhang & Norman, 1994). In this view, much of a person's intelligent behavior results from interactions with external objects and with other people. For example, Hutchins (1990, 1995a, 1995b) has shown that the cognitive properties of a distributed cognitive system consisting of a group of people interacting with complex cognitive artifacts (e.g., the cockpit of a commercial airplane or the control room of a military ship) can differ radically from the cognitive properties of the

individuals, and they cannot be inferred from the properties of the individuals alone, no matter how detailed the knowledge of the properties of those individuals may be. Zhang & Norman (1994), focusing on distributed cognitive tasks that involve interactions between internal and external representations, also argue that the representation of a distributed cognitive task is neither solely internal nor solely external, but distributed as a system of distributed representations with internal and external representations as two indispensable parts.

The Important Roles of External Representations

External representations are not simply inputs and stimuli to the internal mind. They have many important properties. The most obvious one is that they can serve as memory aids: extend working memory, form permanent archives, allow memory to be shared, etc. However, the properties that truly make external representations crucial are not memory aids. For many tasks, external representations are intrinsic components, without which the tasks either cease to exist or completely change in nature. The following discussion briefly reviews some of the properties of external representations that cannot be simply considered as memory aids.

Diagrams, graphs, and pictures are a few typical types of external representations. They are used in many cognitive tasks such as problem solving, reasoning, and decision making. In the studies of the relationship between mental images and external pictures, Chambers & Reisberg (1985; Reisberg, 1987) showed that external pictures can give people access to knowledge and skills that are unavailable from internal representations. In the studies of diagrammatic problem solving, Larkin & Simon (1987; Larkin, 1989), for example, argue that diagrammatic representations support operators that can recognize features easily and make inferences directly. In the studies of logical reasoning with diagrams, Stenning & Oberlander (1995) argue that diagrammatic representations such as Euler circles limit abstraction and thereby aid processibility, that is, graphical representations can make some information interpretable and transparent in a specialized form at the expense of limiting abstraction in general forms. The representation, perception, and comprehension of graphs have been extensively studied since last century (for a few integrative studies, see Bertin, 1983; Cleveland, 1985; Schmid, 1983; Tufte, 1990). It is well-known that different forms of graphic displays have different representational efficiencies for different tasks and can cause different cognitive behaviors. For example, Kleinmuntz & Schkade (1993) showed that different representations (graphs, tables, and lists) of the same information can dramatically change decision making strategies. Zhang (1996)

suggested that all graphs can be systematically studied under a representational taxonomy based on the properties of external representations.

The studies on literacy also show the important functions of external representations. The classical view on writing, originally developed by Aristotle (1938) and restated in our own time by Bloomfield (1993) and Saussure (1959), is that writing merely transcribes or re-represents speech from one external representation in auditory form to another external representation in visual form. For some people, however, it is not a simple transcription because writing supports reflective thought (Norman, 1993b) without which the logical, analytic, rational, and scientific modes of modern thought are impossible (e.g., Goody, 1977; Ong, 1982). For example, Goody argues that the shifts from the so-called prelogical to more and more rational mode of thought resulted from the shifts from orality to various stages of literacy, that is, writing systems are not only the products of the mind but also part of the determining features of the mind. Without writing, the human mind was so occupied by the participation in dynamic utterance of speech that it could not organize and elaborate logical relations in the analytic form of linear sequences. Rational mode of thought was possible only because certain procedures were made available by the technology of writing. Ong also argues that writing has reconstructed cognition: writing systems are not mere external aids but also internal transformations of cognition. In a recent paper, Olson (1996) has made a convincing argument that writing does not merely transcribe but rather brings structural properties of speech into consciousness, that is, the development of writing was also the discovery of the representable structures of speech. From an evolutionary perspective, Donald (1991) also illustrated the important roles of external representations in the emergence of the modern mind. According to Donald, the changes in cognitive architecture mediated by external representations were no less fundamental than those mediated by biological changes in the brain: the external symbolic system, especially writing, is the most important representational system responsible for much of the virtually unlimited cognitive capacity of the modern mind.

In the study of the representational properties of distributed cognitive tasks, Zhang & Norman (1994) also identified several properties of external representations. First, they provide information that can be directly perceived and used without being interpreted and formulated explicitly. Second, they can anchor cognitive behavior. That is, the physical structures in external representations constrain the range of possible cognitive actions in the sense that some actions are allowed and others prohibited. Third, they change the nature of tasks: tasks with and without external representations are completely dif-

ferent tasks from a task performer' point of view, even if the abstract structures of the tasks are the same (see also Norman, 1991).

The above brief review clearly demonstrates that external representations are not simply inputs and stimuli to the internal mind, and they are much more than memory aids. For many tasks, external representations are so intrinsic to the tasks that they guide, constrain, and even determine the pattern of cognitive behavior and the way the mind functions. Given that external representations are so important, they need to be considered seriously, not as something trivial; and they need to be studied on their own right, not as something peripheral to internal representations. The present study is a serious attempt to study external representations in a systematic manner. A particular area, ER-based problem solving, is selected as the task domain and test bed.

A THEORETICAL FRAMEWORK

The behavior in ER-based problem solving is constrained both by the complexity of the environment and by the limitations of the mind. On one hand, the environment is complex because of too much information, real time requirement, unpredictable outcomes, etc. On the other hand, the mind is limited because of the limited bandwidth of information processing, the limited capacity of working memory and attention, the limited speed of mental operations and learning, etc. (e.g., Norman, 1993a). The complexity of the environment and the limitations of the mind, taken together, suggest that the determining factors of the behavior in ER-based problem solving are not just the structures of the mind but also the structures of the environment.

This section develops a theoretical framework for ER-based problem solving. This framework is not only a functional model that can make specific empirical predictions but also a methodology that can specify the important components that need be analyzed for any ER-based problem solving task. The major components of the framework are shown in Figure 1 and their details are described as follows.

Components of The Framework
Abstract Structures

Each ER-based problem solving task has an abstract structure. It specifies the properties of the task that are independent of specific representations and implementations. Abstract structures are usually only conceivable to theorists because task performers usually only deal with the specific representational and implementational contents in which the abstract structures are only implicitly embedded.

Representations

The abstract structure of a ER-based problem

solving task can be implemented by different isomorphic representations (not shown in Figure 1 but in Figure 4). Each isomorphic representation is a distributed representation decomposed into an internal and an external representation, which are the knowledge and structure of the task in memory and in the environment, respectively. The present framework demands such a decomposition because the representation of a ER-based problem solving task is neither solely internal nor solely external but distributed (see Zhang & Norman, 1994).

Operations and Information

Different representations activate different operations, not vice versa. It follows that operations are representation-specific. External representations activate perceptual operations, such as searching for objects that have a common shape and inspecting whether three objects lie on a straight line. In addition, external representations may have invariant information that can be directly perceived without the mediation of deliberate inferences or computations, such as whether several objects are spatially symmetrical to each other and whether one group has the same number of objects as another group. Internal representations activate cognitive operations, such as adding several numbers to get the sum. In addition, internal representations may have information that can be directly retrieved, such as the relative magnitudes of single-digit numbers.

Lookahead and Biases

The basic unit in problem solving is an action. If there are more than one alternative actions under a certain problem state, one needs to make a decision on which action to take. The decision on actions can be based on lookahead, or biases, or learned knowledge, or any combination of them.

Lookahead is the activity of mentally imagining and evaluating alternative sequences of actions before actually selecting an action. For some tasks, it is possible to lookahead all alternatives of complete sequences and then take the best sequence of actions that leads to the goal. For many other tasks, however, it is impossible to do complete lookahead, due to the complexity of the tasks and the limited resources of attention and working memory. Perceptual and cognitive operations demand different amount of attentional and working memory resources, therefore they affect the amount of lookahead that can be performed. Generally speaking, perceptual operations require less attentional and working memory resources than cognitive operations. Different perceptual operations (e.g., sequential vs. parallel search) may also have different effects on lookahead, so do different cognitive operations.
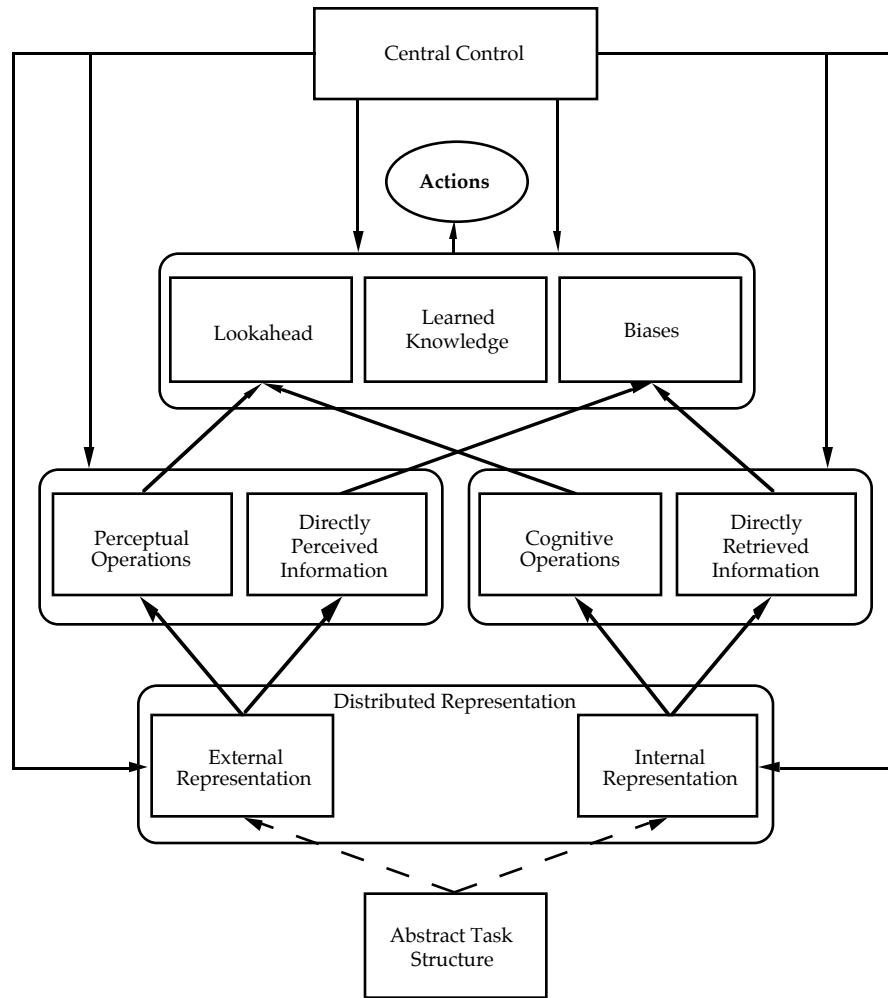
Figure 1. The theoretical framework of ER-based problem solving. See text for details.

The directly perceived information from external representations and the directly retrieved information from internal representations may elicit perceptual and cognitive biases, respectively, on the selection of actions. If the biases are consistent with the task, they can guide actions towards the goal. If the biases are inconsistent with the task, however, they can also misguide actions away from the goal. Learning effect can occur if a task is performed more than once. Thus, the decision on actions can also be affected by learned knowledge.

Central Control

The central control is the most complex but least specified component of the framework. It consists of the mechanisms of working memory and attention, interpreting and understanding, learning, deliberation and decision making, memory retrieval, and so on. Although the central control is crucial for problem solving, the current framework does not attempt to specify its details because it is not the current focus. The most important function of the central control

that is unique to ER-based problem solving is the coordination of the interplay between perception and cognition: allocating and switching attention between internal and external representations, integrating internal and external information, and coordinating perceptual and cognitive operations (for a few studies on these issues, see Carlson, Wenger, & Sullivan, 1993; Dark, 1990; Weber, Byrd, & Noll, 1986).

The Key Assumption of the Framework

The key assumption of the framework is that external representations need not be re-represented as an internal model in order to be involved in problem solving activities: they can directly activate perceptual operations and directly provide perceptual information that, in conjunction with the memorial information and cognitive operations provided by internal representations, determine problem solving behavior. There are three points that need to be elaborated about this key assumption.

First, the key assumption denies the necessity of the internal model of external representations. Per-

ception is not a peripheral device to ER-based problem solving. Rather, it is a central component, for the following reasons. First, perception is not the simple encoding processes that re-represent external representations as an internal model. The end product of perception no longer models and mirrors the external representations because the end product is the information and structure either directly picked up or already highly analyzed, processed, and transformed by perceptual systems. Second, the end product of perception is not necessarily the intermediate data prepared for high-level cognitive mechanisms. Rather, the end product of perception is often the end product of the whole problem solving process. Third, the perceptual operations activated by external representations are central mechanisms of ER-based problem solving: they are no less fundamental than the cognitive operations activated by internal representations. For these three reasons, ER-based problem solving can also be called perceptual problem solving. Because perceptual processes are central components of ER-based problem solving and because they directly operate on external representations, external representations are also central components of ER-based problem solving. Thus, there is nothing to lose when the internal model is not available. The behavior in ER-based problem solving is simply the interwoven, integrative, and dynamic processing of the information perceived from external representations and that retrieved from internal representations.

Second, the key assumption does not mean that external representations can function independently without the support of anything internal or mental. External representations have to be processed by perceptual mechanisms, which are of course internal. The end product of these perceptual mechanisms is also internal. However, this end product is not equivalent to the internal representation of the task as defined in the present study. It is not an internal model of the external representation of the task, either. As defined earlier, the internal representation of an ER-based task is the knowledge and structure of the task in memory; and the external representation is the knowledge and structure of the task in the environment. The end product of perception is merely the situational information in working memory that usually only reflects a fraction (usually crucial) of the external representation.

Third, the key assumption does not mean that external representations can always be directly and automatically used without being interpreted and controlled by high-level cognitive mechanisms. If the end product of perception is not the end product of problem solving, it has to be interpreted by high-level cognitive mechanisms for further processing. Because not everything in external representations is always relevant to a task, high-level cognitive mechanisms need to use internal task knowledge (usually supplied by task instructions) to direct attention and perceptual processes to the relevant features of external representations.

Test and Justification of the Framework

The theoretical framework is not only a functional model that can be tested empirically but also a methodology that can be used to analyze ER-based problem solving tasks. The Tic-Tac-Toe (henceforth, TTT) and its isomorphs are selected to demonstrate the procedures of the framework as a methodology and test the predictions of the framework as a functional model. TTT is selected for these dual purposes for the following reasons. First, it is simple enough to allow well-controlled laboratory studies. Second, it is complex enough to mimic real world complex problems. For example, complete mental lookahead is impossible for TTT, just like for most real world complex tasks. Third, since the focus of the present study is on external representations, the task should have several isomorphs that have rich structures in external representations such that the properties of external representations and their effects on problem solving behavior can be examined. TTT is a task that has just this property. In next section, the theoretical framework is first used as a methodology to analyze the TTT isomorphs, then it is used to make specific predictions on the problem solving behavior in the TTT.

TIC-TAC-TOE

The TTT is a well-known two-player game. A minor variation of the original TTT is shown in Figure 2A (*Line* version). The task for the two players is to select the circles in turn by coloring the circles with different colors, one at a time. The one who first gets three circles on a straight line (horizontal, vertical, or diagonal) wins the game. The TTT is a draw game, i.e., when both players use optimum strategies, neither can win.

Figures 2B, 2C, and 2D show three more isomorphs of the TTT. In the *Number* version (Figure 2B), the task is to select the numbers in turn by coloring the numbers, one at a time. The one who first gets three numbers that exactly add to 15 wins the game. In the *Shape* version (Figure 2C), the task is to select the big circles in turn by coloring the objects inside a big circle, one big circle at a time. The one who first gets three big circles that contain a common shape wins the game. In the *Color* version (Figure 2D), the task is to select the big circles in turn by drawing different background textures. The one who first gets three big circles that contain the same colored small circle wins the game.

The equivalence of the four isomorphs in Figure 2 is shown in Figure 3. To make the mappings easy to understand, the nine elements in *Number*, *Shape*, and *Color* were arranged in the same spatial relations

as in *Line*. The center, corners, and sides in *Line* (Figure 3A) correspond to the number five, even numbers, and odd numbers in *Number* (Figure 3B), the 4-object, 3-object, and 2-object big circles in *Shape* (Figure 3C), and the 4-object, 3-object, and 2-object big circles in *Color* (Figure 3D), respectively.

Theoretical Analyses

In this section, the four TTT isomorphs are analyzed in terms of the components of the theoretical framework in Figure 1.

Abstract Structures

The TTT has four abstract properties. First, it has nine elements. Second, it has eight winning triplets, each of which is a group of three elements that constitute a win. Third, the nine elements are divided into three symmetry categories. The elements in a symmetry category are identical to each other, i.e., to make a move, selecting one element is not different from selecting another element in the same category. Fourth, the elements within a symmetry category share a common invariant property—*winning invariant*, which is the number of winning triplets in which an element is part of. The winning invariants of the three symmetry categories are 2, 3, and 4.

For example, for *Line* in Figure 2A, the nine elements are the nine circles. The eight winning triplets are the 3-circle groups that lie on the 3 horizontal, 3 vertical, and 2 diagonal lines. The three symmetry categories are the center, 4 corners, and 4 sides. The winning invariants of the center, corners, and sides are 4, 3, and 2, respectively. For example, the center is an element of 4 winning triplets: 1 horizontal, 1 vertical, and 2 diagonal lines.
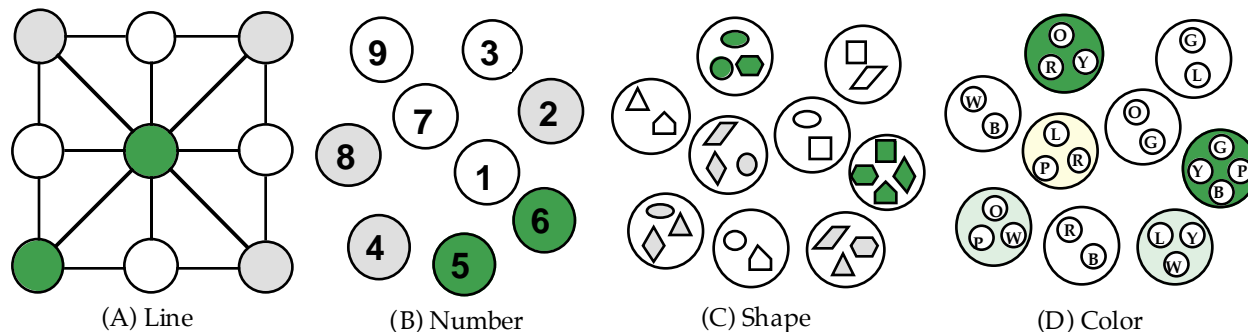

Figure 2. Four TTT isomorphs. (A) *Line*. Getting three circles on a straight line is a win. (B) *Number*. Getting three numbers that exactly add to 15 is a win. (C) *Shape*. Getting three big circles that contain a common shape is a win. (D) *Color*. Getting three big circles that contain the same colored small circle is a win. The letters inside the circles indicate the colors used in the experiments: B = Blue, G = Green, L = Light Blue, O = Orange, P = Pink, R = Red, Y = Yellow, W = Brown.
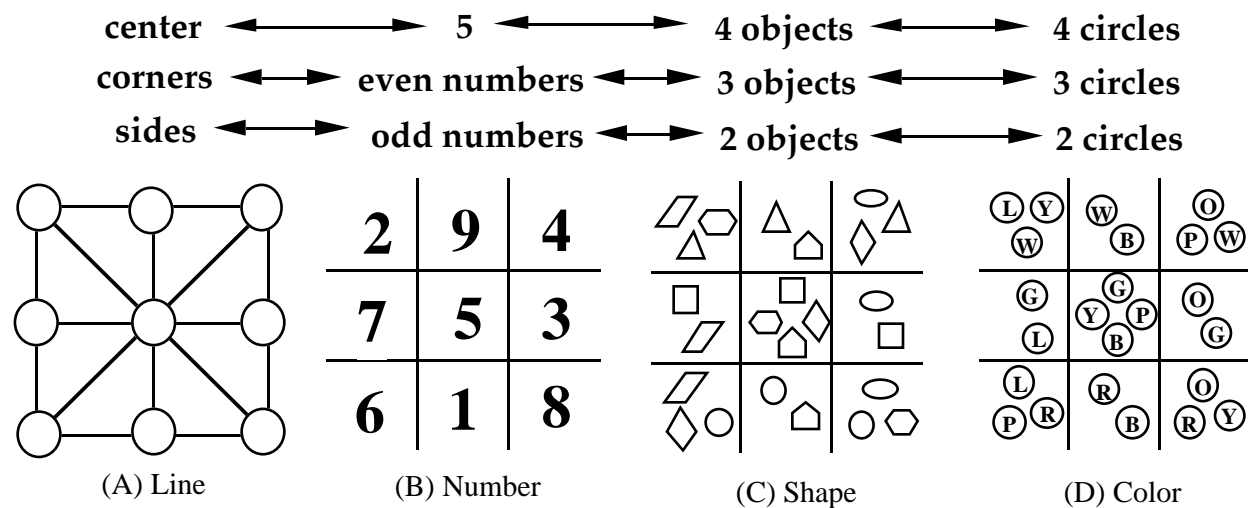

Figure 3. The mappings among the four TTT isomorphs. The center, corners, and sides in (A) correspond to five, even numbers, and odd numbers in (B), 4-object, 3-object, and 2-object big circles in (C), and 4-object, 3-object, and 2-object big circles in (D), respectively.

Representations, Operations, and Information

Each of the four formal properties of the TTT is represented differently in the four TTT isomorphs in Figure 2. Different representations activate different operations and provide different types of information.

Elements. The nine elements in *Line*, *Shape*, and *Color* are represented externally. They all correspond to nine distinct spatially-spread physical entities, which can be perceptually separated and identified. In *Number*, the nine elements are represented both externally and internally: externally because they can be perceptually separated, and internally because the meanings (numerical values) of the nine digits have to be retrieved from memory.

Winning Triplets. In *Line*, the eight winning triplets are represented externally by the 3 horizontal, 3 vertical, and 2 diagonal straight lines. To identify a wining triplet is to search for three circles lying on a straight line. Whether three circles lie on a straight line can be perceptually inspected. In *Number*, the eight winning triplets are represented internally by eight number triplets, each of which has three numbers that add to 15. To identify a winning triplet is to search for three numbers that add to 15. Whether three numbers add to 15 has to be mentally computed. In both *Shape* and *Color*, the eight winning triplets are represented externally, the former by the eight different shapes of the small objects and the latter by eight different colors of the small circles. To identify a wining triplet is to search for three big circles that contain a common shape or a common color. Both shapes and colors can be identified and searched perceptually. However, the search for shapes in *Shape* is sequential whereas the search for colors in *Color* can be parallel due to the pop-out effect of the specific colors used in the isomorph.

Symmetry Categories. In *Line*, the three symmetry categories are represented externally by spatial symmetry: the center, 4 corners, and 4 sides. For example, the 4 corners are spatially symmetrical to each other and therefore equivalent to each other. Spatial symmetry can be directly perceived. In *Number*, the three symmetry categories are represented internally by parity: the number five, 4 even numbers, and 4 odd numbers. (Though 5 is an odd number, for convenience, it is considered to form a symmetry category by itself.) The symmetry information in this version is not directly available because two numbers having the same parity does not necessarily mean that they are equivalent to each other: they have many other numerical properties such as dividable by three, prime numbers, magnitudes, etc. In both *Shape* and *Color*, the three symmetry categories are represented externally by the quantity of objects in a big circle: 4-object, 3-object, and 2-object big circles. However, the symmetry information in this case can not be directly perceived because two big circles having the same number of objects does not necessi-

tate that they are equivalent to each other.

Winning Invariants. In *Line*, the winning invariant of a symmetry category is represented externally by the number of straight lines connecting a circle: 4, 3, and 2 for the center, the corners, and the sides, respectively. The number of straight lines connecting a circle can be directly perceived. In *Number*, the winning invariants are represented internally: 4 for the number five, 3 for even numbers, and 2 for odd numbers. They are not directly represented and thus are not directly available. To get the winning invariant of a number, it must be grouped with all possible pairs of other numbers to form number triplets and the sums of the three numbers in all number triplets have to be mentally computed to see whether each sum is 15. Even with extensive mental computations, this task is very difficult if not impossible. In both *Shape* and *Color*, the winning invariant in a symmetry category is represented externally by the quantity of objects in a big circle: 4, 3, and 2, which can be perceptually identified.

Lookahead and Biases

Due to the complexity of the TTT problem space and the limited capacity of working memory, complete mental lookahead is nearly impossible for the TTT isomorphs. For example, the task in Experiment has forty-seven 8-step lookahead sequences for first moves even with symmetry considered. Thus, it is impossible to do complete lookahead for first moves for any TTT isomorphs. For second moves, there are three and eight 6-step lookahead sequences, respectively, with and without symmetry being considered. Thus, complete lookahead is still very difficult, if not impossible, for second moves for the TTT isomorphs.

When complete lookahead is difficult or impossible, people may use perceptual or cognitive biases to make decisions. The winning invariants of the TTT isomorphs, if they can be perceived, can elicit a *more-is-better* bias: if an element is involved in more winning triplets, that is, its winning invariant is larger, it should be preferred, because it may block more pieces of the opponent and create more opportunities for oneself. The more-is-better bias can be elicited in *Line*, *Shape*, and *Color* because the winning invariants in these three versions can be directly perceived from their external representations. Thus, for *Line*, *Shape*, and *Color*, the center and 4-object circles should be most preferred, corners and 3-object circles should be the next, and sides and 2-object circles should be least preferred. The more-is-better bias can not be elicited in *Number* because the winning invariants are not directly available. Alternatively, the internal representation of numerical facts in *Number* may elicit a *larger-is-better* bias: the larger a number is, the more quickly it may contribute to the sum of 15. If this bias is elicited, then larger numbers should be preferred.
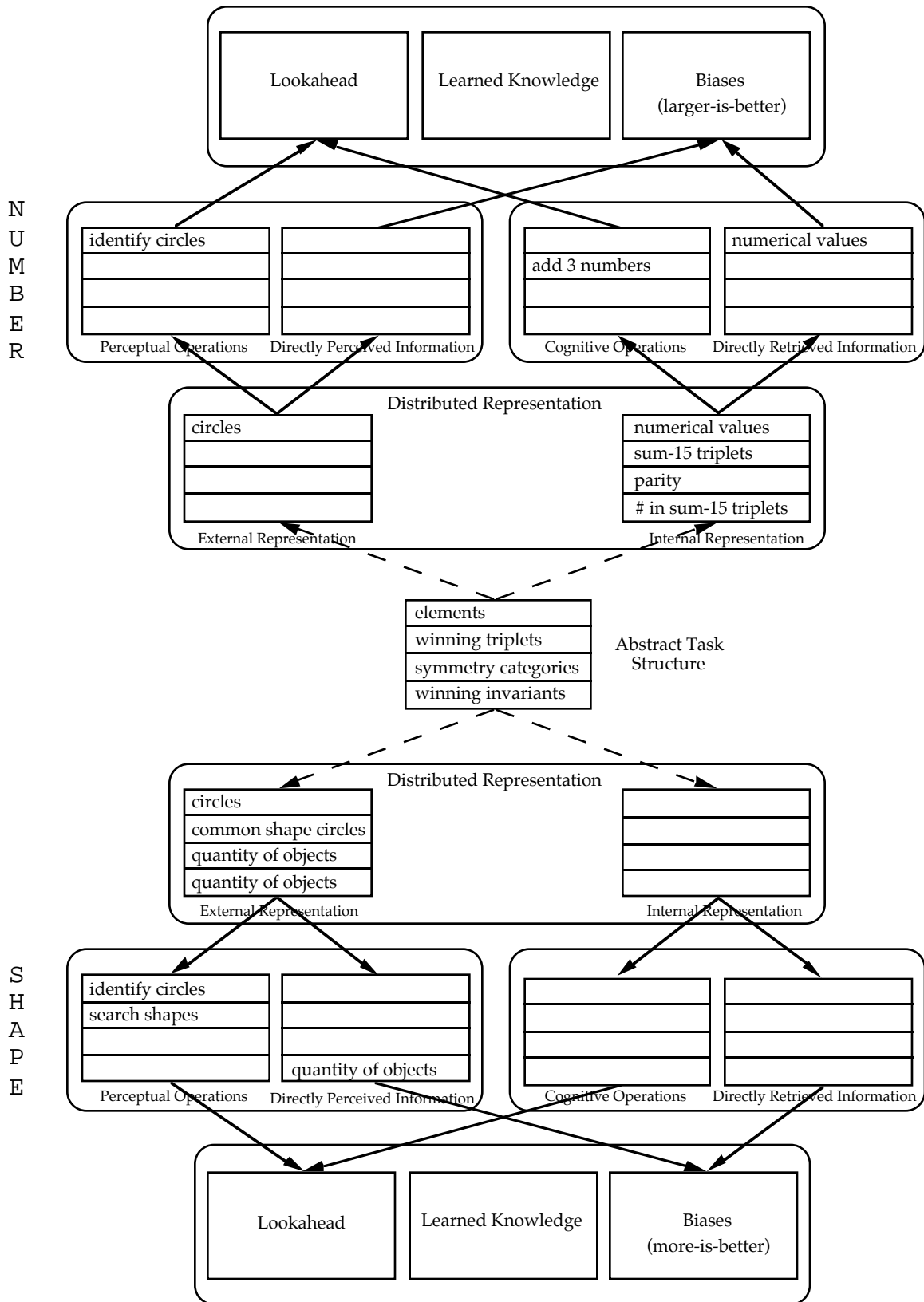
Figure 4. The components of the theoretical framework in Figure 1 are mapped onto the corresponding components in the *Number* and *Shape* isomorphs. See text for details.

The *Number* and *Shape* Isomorphs under the Framework

Based on the above analyses, the *Number* and *Shape* isomorphs are selected to show how the components of the theoretical framework in Figure 1 are mapped onto the corresponding components in these two isomorphs. The mappings are shown in Figure 4. The abstract task structure contains the four formal properties of the TTT, which are represented differently in *Number* and *Shape*. For *Number*, the four formal properties are represented internally. In addition, the elements are also partially represented externally. The external representation of the elements activates a perceptual operation (identify circles) and the internal representation of the elements activates directly retrievable information (numerical values) that elicits the larger-is-better bias. In addition, the internal representation of the winning triplets activates a cognitive operation (add 3 numbers). However, the internal representations of symmetry categories and winning invariants do not activate any operations or directly retrievable information.

For *Shape*, the four formal properties are all represented externally. The external representations of the elements and winning triplets activate perceptual operations (identify circles and search for shapes), and the external representation of the winning invariants provides directly perceivable information (winning invariants) that elicits the more-is-better bias. However, the external representation of the symmetry categories does not provide directly perceivable symmetry information.

Predictions of the Framework and Overview of the Experiments

In the theoretical framework, the decision on actions is based on lookahead, biases, and learned knowledge. Because complete mental lookahead is practically impossible for the TTT tasks, the behavior in the TTT tasks should be only determined by biases and learned knowledge. Therefore, before learning, the behavior in the TTT tasks should be solely determined by biases; during learning, it should be jointly determined by biases and learned knowledge; and after learning, it should be solely determined by learned knowledge in the form of a routine task with memorized and compiled procedures. Specifically, we have the following predictions.

First, if the biases are inconsistent with the task, they can make the task more difficult by misguiding actions away from the goal. This prediction was tested in Experiment 1. Second, if the biases are consistent with the task, they can make the task easier by guiding actions towards the goal. This prediction was tested in Experiment 2. Third, if the biases are irrelevant to the task, they should have no effects on the decision of actions. This prediction was also tested in Experiment 2. Because these three predic-

tions are based on the assumption of biases, the nature of the biases were examined in both experiments. The more-is-better bias was assumed to be the perceptual bias elicited by the external representations in *Line*, *Shape*, and *Color*, and the larger-is-better bias was assumed to be the cognitive bias elicited by the internal representation in *Number*.

In addition to the above two experiments designed to test the specific predictions about biases, Experiment 3 was designed to examine the perception of symmetry information and its effects on problem solving behavior. In all three experiments, several other properties of the theoretical framework were also examined, such as the difference between perceptual and cognitive operations and the difference between sequential and parallel perceptual operations.

EXPERIMENT 1: INCONSISTENT MAPPING

Experiment 1 focuses on the effect of inconsistent biases on problem solving behavior. In addition, it also examines how the different operations activated by different representations affect problem solving behavior. The four TTT isomorphs in Figure 2 were the four conditions of this experiment. Because all TTT isomorphs have the same formal structures, for convenience, we use the three symmetry categories of *Number*, i.e., five, even numbers, and odd numbers, to refer to the three symmetry categories of all TTT isomorphs for the rest of this article. For example, when we talk about even numbers in *Shape*, we actually refer to the 3-object big circles (see Figure 3).

In this experiment, subjects played games against a perfect computer opponent. In all conditions, the computer opponent always played first by randomly selecting an even number as its first move. The computer's strategy (see the Appendix for details) was carefully designed such that the computer could never lose and in order for the subjects to get draws, they had to strictly follow the *5-Odd* strategy:

- Always select five as the first move.
- Always select any odd number as the second move.
- For all other moves, simply block the piece that can lead to an immediate win for the computer.

The 5-Odd strategy is necessary and sufficient for subjects to get draws. The first and the second moves are crucial: if either or both are made incorrectly, then subjects always lose, regardless of how other moves are made. Under this strategy, subjects only have to make decisions for the first and second moves because for all other situations, the subjects only have one choice—blocking the piece that can lead to an immediate win for the computer. In addi-

tion, subjects' first and second moves only depend on the symmetry categories (i.e., five, even numbers, and odd numbers), not on specific elements.

As analyzed earlier, the winning invariants in the external versions of the TTT (*Line*, *Shape*, and *Color*) were assumed to elicit the more-is-better bias. Therefore, the preference order is, from most to least preferred: center > corners > sides for *Line* and 4-object > 3-object > 2-object circles for *Shape* and *Color*. Since the 5-Odd strategy requires that the first move be always five and the second move always any odd number, the more-is-better bias should lead to more correct first moves but meanwhile also more incorrect second moves in *Line*, *Shape*, and *Color*. Thus, the more-is-better bias is inconsistent with the 5-Odd strategy. In the internal version of the TTT (*Number*), the numerical facts of the nine numbers were assumed to elicit the larger-is-better bias. Therefore, larger numbers should be preferred in *Number*. The larger-is-better bias is also inconsistent with the 5-Odd strategy: it can make the correct selection of the first move (five) more difficult because the bias towards larger numbers reduces the chance of selecting five. However, the larger-is-better bias is irrelevant to the selection of second moves because there is no correlation between larger and smaller numbers and even and odd numbers.

Another issue this experiment examines is how the different operations activated by different representations affect problem solving behavior. In *Line* the winning triplets are represented externally by straight lines, whereas in *Number* they are represented internally by sums of numbers. Whether three circles lie on a straight line can be perceptually inspected, whereas whether three numbers add to 15 has to be mentally computed. These two different identification processes for winning triplets may have different effects on problem solving behavior. Different external representations can also activate different perceptual operations. The winning triplets in *Shape* and *Color* are both represented externally, the former by shapes and the latter by colors. However, the shapes in *Shape* have to be searched sequentially whereas the colors in *Color* can be searched in parallel. Thus, the difference between these two identification process for winning triplets may also affect problem solving behavior.

Method
Subjects
80 undergraduate students enrolled in introductory psychology courses at The Ohio State University participated in the experiment to earn course credit.

Stimuli
The four TTT isomorphs in Figure 2 were the four conditions of this experiment. They were programmed in SuperCard on Macintosh computers.

The four TTT isomorphs were controlled by the same program because they have the same formal structure. The computer always made the first move in all games. Its strategy was designed such that the subjects had to discover the 5-Odd strategy to get draws (see the Appendix). Subjects made moves by clicking the pieces with a mouse. The pieces selected by the computer and subjects were in different colors or background patterns such that they could be distinguished.

Design and Procedure
Each subject was randomly assigned to one of the four conditions. There were 20 subjects for each condition. The instructions were given to the subjects verbally. The first part of the instructions was different for different conditions but the second part was the same for all conditions. The first and second parts of the instructions for *Number* are as follows:

Part 1. There are nine numbers on the screen. You and the computer select numbers in turn by clicking the numbers, one at a time. Whoever first gets any three numbers that exactly add to 15 wins the game.

Part 2. Due to the specific design of this game, you can not beat the computer. So your task is to prevent the computer from winning, that is, to get draws. The computer always starts first. There is a strategy. If you can figure it out, you can always get a draw. You need to play this game over and over again until you get 10 draws in a row.

If a subject could get 10 draws in a row within 50 games, the experiment was over. Otherwise the experiment was over at the fiftieth game. Complete move sequences and time stamps for all games were recorded by the computer.

Results
Overall Performance
Figure 5 shows the percentage of subjects who got 10 draws in a row within 50 games, the number of games needed to get 10 draws in a row within 50 games, and the number of games needed to get the first draw for the four conditions. If a subject could not solve a problem within 50 games, the value was considered to be 50 games. In terms of the percentage of successful subjects, the difficulty order was: *Line* < *Color* < *Shape* ≈ *Number*. Except of the difference between *Number* and *Shape* ($\chi^2 = 0.00, p = 1.0$), all other differences were significant (smallest $\chi^2 = 4.44$, $p = 0.04$). In terms of the number of games to 10 draws, the difficulty order was: *Line* < *Color* < *Shape* ≈ *Number*. Except of the difference between *Number* and *Shape* ($p = 0.99$, Tukey HSD), all other differences were significant (largest p =

0.03, Tukey HSD). In terms of the number of games to first draw, the difficulty order was: *Line < Color ≈ Shape ≈ Number*. The differences between *Line* and the other three conditions were significant (largest p = 0.02, Tukey HSD), but all other differences were not significant (smallest p = 0.22, Tukey HSD).
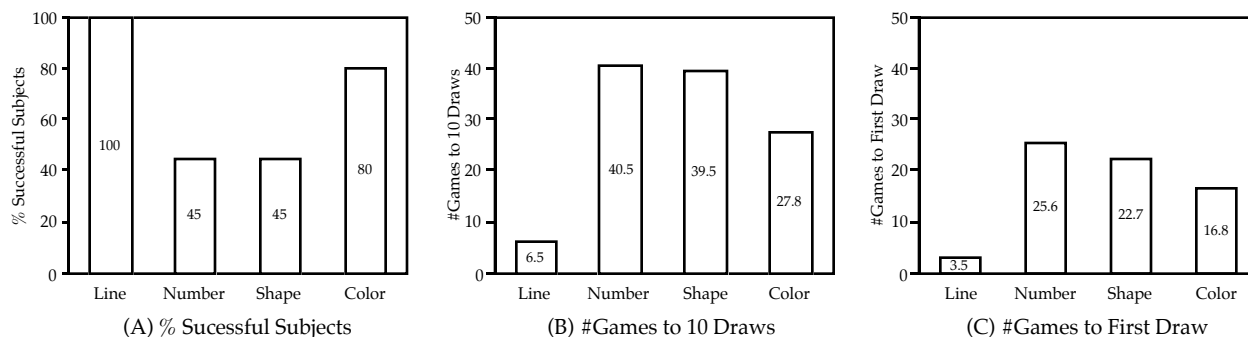

Figure 5. Overall performance in Experiment 1. (A) Percentage of subjects who got 10 draws in a row within 50 games. (B) The number of games needed to get 10 draws in a row within 50 games (excluding the 10 draws). (C) The number of games needed to get the first draw (including the first draw).
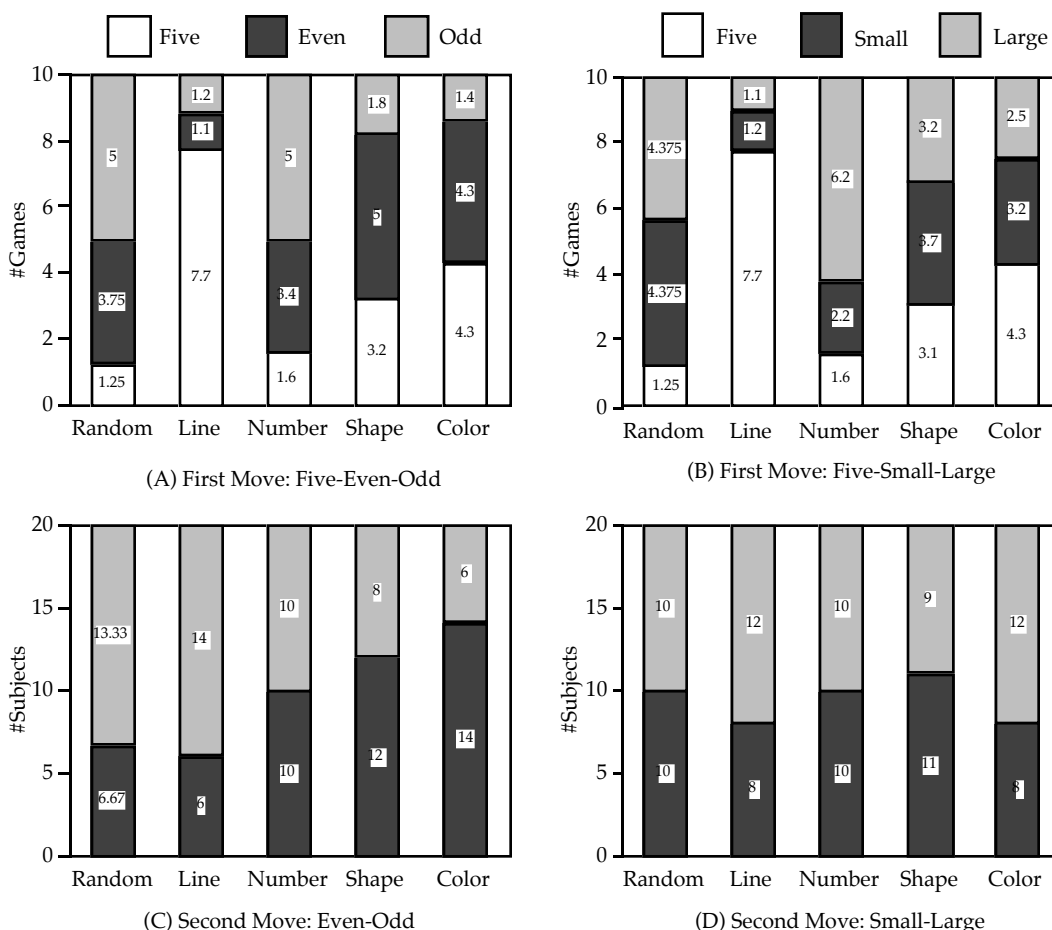

Figure 6. Move patterns in Experiment 1. (A) The average frequencies of five, even numbers, and odd numbers selected by each subject as first moves for the initial 10 games. (B) The average frequencies of five, small numbers, and large numbers selected by each subject as first moves for the initial 10 games. (C) Frequency distributions of subjects who selected even and odd numbers as their second moves in the first game in which five was selected as the first move. (D) Frequency distributions of subjects who selected small and large numbers as their second moves in the first game in which five was selected as the first move.

The Pattern of Subjects' First Moves

Subjects' first moves were divided into the three symmetry categories of five, even numbers, and odd numbers to examine the perception of the winning invariants and the use of the more-is-better bias, and into the three categories of five, small numbers (1 to 4), and large numbers (6 to 9) to examine the activation of numerical information and the use of the larger-is-better bias. (Note that we use *five*, *even* and *odd* numbers to refer to the three symmetry categories of all isomorphs.) Because the number of games played by each subject varied but all subjects had to play at least 10 games, only the first moves of the initial 10 games for each subject were analyzed.

Distribution across Five, Even and Odd Numbers. The frequencies of five, even numbers, and odd numbers selected by each subject as the first moves in the initial 10 games were counted and averaged across the 20 subjects in each condition. The results are shown in Figure 6A. The leftmost column shows the expected frequencies for random moves with replacement. The first analysis was the comparisons among the four conditions on the selection of five's in the initial 10 games. The main effect was significant ($F(3, 76) = 22.54$, $p < 0.001$). Tukey HSD pairwise comparisons showed that subjects selected five more often in *Line* than in all other three conditions (largest $p = 0.003$), more often in *Color* than in *Number* ($p < 0.005$), but there was no significant difference between *Number* and *Shape* ($p = 0.24$) and between *Shape* and *Color* ($p = 0.40$). The second analysis was the comparison between the observed and expected frequencies of five within each condition. Subjects selected significantly more five's than the expected value in all (smallest $t(19) = 4.447$, $p < 0.001$) but the *Number* condition ($t(19) = 1.018$, $p = 0.20$). The third analysis was the comparison between the frequencies of even and odd numbers within each condition. Because after the computer's first move (always an even number) the 8 remaining numbers contained 1 five, 3 even numbers, and 4 odd numbers, the number of even numbers was weighted by 8/3 and that of odd numbers was weighted by 8/4. For the weighted values, Wilcoxon signed rank tests showed that there was no significant difference between even and odd numbers within *Line* ($p = 0.97$) and within *Number* ($p = 0.36$). However, there was a strong bias toward even numbers within *Shape* ($p < 0.001$) and within *Color* ($p < 0.001$).

Taken together, these three analyses show that the bias towards five as first moves was strongest in *Line*, next in *Shape* and *Color*, and not different from random selections in *Number*. In addition, there was a bias towards even numbers over odd numbers in *Shape* and *Color*, but not in *Line* and *Number*.

Distribution across Five, Small (1 to 4) and Large (6 to 9) Numbers. The frequencies of five, small and large numbers selected by each subject as the first moves across the initial 10 games were counted and averaged across the 20 subjects in each condition. The results are shown in Figure 6B. The leftmost column shows the expected distribution for random moves with replacement. Wilcoxon signed rank test showed that there was a strong bias towards large numbers as first moves in *Number* ($p < 0.001$) but not in the other three conditions (smallest $p = 0.12$).

The Pattern of Subjects' Second Moves

After five (the correct first move) was selected as the first move for a game, a subject could select either an even number (incorrect) or an odd number (correct) as the second move. For the first game in which five was selected as the first move, the distribution of the 20 subjects who selected even and odd numbers as their second moves are shown in Figure 6C. If subjects had no bias towards even or odd numbers as their second moves after the first five, the expected frequency of subjects who select even numbers is 6.7 (2/6 of 20) and the expected frequency of subjects who select odd numbers is 13.3 (4/6 of 20), because after the first and second moves of the computer (two even numbers) and the first move of a subject (five), the remaining numbers contain 2 even numbers and 4 odd numbers. The distribution across even and odd numbers in each condition was compared with the expected distribution (indicated as Random in Figure 6C). One-tail Chi-Square tests showed that the bias towards even numbers was significant in *Shape* ($\chi^2 = 2.718$, $p = 0.05$) and *Color* ($\chi^2 = 5.189$, $p = 0.01$) but not significant in *Line* ($\chi^2 = 0.0565$, $p = 0.41$) and *Number* ($\chi^2 = 1.055$, $p = 0.15$).

After five was selected as the first move, subjects' second moves could also be divided into small and large numbers. If subjects had no bias towards small or large numbers as their second moves after the first five, the expected frequencies of the 20 subjects who select small and large numbers are both 10 (10/20 of 20). The distribution across small and large numbers for each condition is shown in Figure 6D, which was compared with the expected distribution. None of the four conditions showed bias towards small or large numbers (largest one-tail $\chi^2 = 0.404$, $p = 0.26$).

Solution Times

Because there were too many types of moves, solution times were only analyzed for the first moves of the initial 10 games. The times to make the first moves for the initial 10 games were shown in Figure 7. An ANOVA with the four conditions as the between-subject factor and the 10 games as the within-subject factor showed that the main effect of conditions was significant ($F(3, 76) = 6.61$, $p < 0.001$), but

that of game positions was not significant (F(9, 684) = 1.32, p = 0.22), nor was the interaction between conditions and game positions (F(27, 684) = 173.64, p = 0.88). Simple comparisons between pairs of conditions were significant for all pairs (smallest F(1, 38) = 4.95, p = 0.03) except the *Shape* and *Color* pair (F(1, 38) = 1.11, p = 0.30).
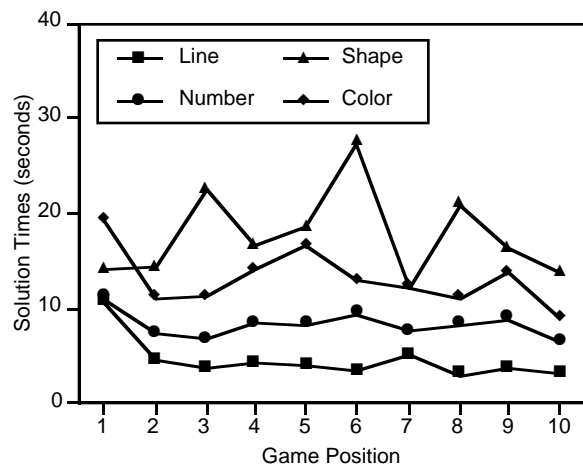

Figure 7. Solution times of the first moves for the initial 10 games in Experiment 1.

Discussion

This experiment showed that the winning invariants in external representations (*Line*, *Shape*, and *Color*) were indeed the directly perceivable information that strongly guided subjects' selections of moves by eliciting the more-is-better bias. The winning invariants in internal representations (*Number*) were not perceived by subjects and had no effect on problem solving behavior. Instead, the larger-is-better bias based on numerical values was elicited. The different difficulty levels of the four TTT isomorphs were mainly caused by the perception of the winning invariants and the use of the more-is-better bias in *Line*, *Shape*, and *Color* and the activation of numerical facts and the use of the larger-is-better bias in *Number*.

Difficulty Factors

Line was the easiest among the four isomorphs. Subjects had a strong bias to select five (correct) as the first move, but did not treat even (incorrect) and odd (correct) numbers differently for the second move. That is, subjects used the more-is-better bias for the first moves but not the second moves. This might be due to the following two reasons. First, for second moves, the value of a winning invariant was changed because three of the nine circles were already occupied: the center by subjects and two opposite corners by the computer. Under this configuration, a corner or a side only provided one potential win for subjects. Thus, corners and sides did not ap-

pear to be different for subjects. Second, as discussed earlier, it was possible, though still very difficult, to do complete lookahead for second moves for *Line* under the most symmetrical configuration. If subjects relied on lookahead, the effect of lookahead might have overridden the effect of the more-is-better bias for second moves. Two other factors might also have contributed to the ease of *Line*. First, symmetry in *Line* could be directly perceived, which could reduce the size of the problem space. Symmetry will be examined in Experiment 3. Second, since most subjects realized that *Line* is a variation of the original TTT, which they played in childhood, familiarity with the game might also have reduced problem difficulty. The familiarity effect will be examined in Experiment 2.

The difficulty of *Number* was mainly caused by the difficulty of selecting the correct first move (five). The larger-is-better bias made subjects prefer large numbers for their first moves, which indirectly made the selection of five more difficult because five is in the middle of the magnitude continuum. The larger-is-better bias had no effect on the selection of the second moves because there was no correlation between small and large numbers and even and odd numbers.

The difficulty of *Shape* and *Color* was due to the incorrect selections of second moves, caused by the more-is-better bias. Using the more-is-better bias, subjects made more correct first moves (five, i.e., 4-object circle) but at the same time also made more incorrect second moves (even numbers, i.e., 3-object circles). Although for second moves the values of winning invariants depend on the computer's first and second moves and the subjects' first move, the winning invariants, unlike those in *Line*, are still perceived as the quantity of objects contained in a big circle. This is because checking the potential wins of a 2-object or a 3-object circle requires an exhaustive search across all of the 24 small objects in different shapes or colors. *Color* was easier than *Shape* in terms of the percentage of subjects who got 10 draws in a row within 50 games and the number of games needed to get 10 draws in a row. This difference in difficulty between *Shape* and *Color*, though expected, can not be easily explained because no single factor was responsible for it. The bias towards five as the first move was stronger in *Color* than in *Shape*, but it was not reliable. The solution times for first moves of the initial 10 games were shorter in *Color* than in *Shape*, but it was not reliable, either. Maybe some other factors related to sequential and parallel search were responsible for the different difficulty levels of *Shape* and *Color*.

The results of solution times showed that it took longer to select first moves in *Shape* and *Color* than in *Line* and *Number*. This might be due to the representations of the nine elements: in *Line* and *Number*

the nine elements correspond to nine single objects whereas in *Shape* and *Color* they correspond to nine clusters of objects with a total of 24 objects. The times to make decisions for first moves in *Line* was shorter than those in *Number*, implying that the external representation of *Line* and the internal representation of *Number* were processed differently. Although the different operations activated by different representations had direct effects on solution times, they were not correlated with the selections on moves and the difficulty levels of the tasks. Generally speaking, solution times were not as informative as the patterns of subjects' moves. For the two experiments that follow, solution times were not reported.

Strategy Discovery

To get 10 draws in a row, subjects had to discover the 5-Odd strategy, which could take different forms under different representations. The different forms of the 5-Odd strategy were only determined by second moves because the first move was always five. The first form is the *fixed* strategy: the second move is fixed at one specific odd number. For example, a subject might always select 1 as the second moves for the 10 consecutive draws, regardless of the computer's first moves. This is the most specific strategy, reflecting only a specific structure and a specific portion of the TTT problem space. The second form is the *divided* strategy: the second move alternates between two odd numbers, depending on the first moves of the computer. For example, if the computer's first move is 2 or 8, a subject always selects 1 as the second move, and if the computer's first move is 4 or 6, the subject always selects 9 as the second move. The third form is the *symmetric* strategy: the second move is any randomly selected odd number. To get this strategy, a subject needs to perceive the symmetry information of the symmetry categories. This is the 5-Odd strategy in its most general form. It reflects the general structure and the whole problem space of the TTT.

Table 1 shows the distribution of subjects across these three forms for the four isomorphs. In *Line*, 10% of the subjects used the fixed strategy and 45% used the divided strategy. When interviewed after the experiment, all of them could generalize their strategies to the symmetric strategy, and they said they used the fixed and the divided strategies just for convenience and for making faster responses. Thus, the symmetry information in *Line* was perceived by all subjects. In *Number*, the strategies discovered by most subjects were the fixed strategy (67%) and the divided strategy (22%). Among these subjects, none could generalize their strategies to the symmetric strategy when interviewed after the experiment. Thus, for *Number*, most subjects could not realize the symmetry. In both *Shape* and *Color*, most subjects used the fixed and the divided strategies. Among

these subjects, few could generalize their strategies to the symmetric strategy. Thus, even if the symmetry categories of *Shape* and *Color* were represented externally, the symmetry information could not be easily perceived by most subjects. Because different forms of the 5-Odd strategy reflect different structures and different portions of the TTT problem space, the finding that the different isomorphs elicited different forms of the strategy is an indication that representational formats of the isomorphs determine what structure can be discovered and what portion of the problem space can be explored.

Table 1: Different Forms of the 5-Odd Strategy

Different Forms	Conditions			
	Line	Number	Shape	Color
Fixed strategy	10%	67%	44%	44%
Divided strategy	45%	22%	34%	44%
Symmetric strategy	45%	11%	22%	12%

A Control Experiment

Experiment 1 showed that subjects had a strong bias towards large numbers as their first moves in *Number*. This bias is counterintuitive because although a large number could lead to 15 faster, it could also overshoot 15. Thus, it is reasonable to examine whether the bias was merely an artifact of experimental design. The positions of the nine numbers on the computer screen were fixed. From Figure 2B we can see that the large numbers 9, 8, and 7 are all at the top left corner. Since subjects might have no prior knowledge about which number should be selected, they might simply start with a number at the top left corner, just like what they normally do in reading English text. If the bias towards large numbers as first moves in *Number* was an artifact caused by the specific positions of the large numbers, then it should disappear when the positions of the nine numbers are randomized for each new game. In *Shape* and *Color*, the positions of the nine big circles were also fixed. It is possible that the biases towards five and even numbers were also artifacts caused by the specific positions of the nine big circles. If these biases are still present with randomization of positions, then they were not artifacts. A separate experiment was carried out to test these possibilities. Since this is only a control experiment to rule out possible artifacts in experimental design, the results are only reported in a summary format as follows to save space.

This control experiment had four conditions. *Number-Fixed* and *Shape-Fixed* were identical to *Number* and *Shape* in Experiment 1: the positions of the nine numbers and the positions of the nine circles remained fixed across all games. *Number-Random* was the same as *Number-Fixed* except that the positions of the nine numbers were randomized for each new game. *Shape-Random* was the same as *Shape-*

Fixed except that the positions of the nine circles were also randomized for each new game. The computer's strategy in this experiment was identical to that in Experiment 1. Thus, the 5-Odd strategy was necessary and sufficient for subjects to get draws. There were 20 subjects for each of the four conditions. The procedure was exactly the same as in Experiment 1. The results show that there was significant bias towards large numbers in *Number-Random*, suggesting that the bias in *Number* of Experiment 1 was not an artifact caused by the specific positions of the nine numbers but indeed a real bias used by subjects. There were also significant biases towards five and even numbers in *Shape-Random*, suggesting that the same biases in *Shape* of Experiment 1 were not artifacts, either.

EXPERIMENT 2:
CONSISTENT MAPPING

In Experiment 1, the 5-Odd strategy was inconsistent with the more-is-better and the larger-is-better biases. The current experiment uses a strategy that is consistent with the more-is-better bias but irrelevant to the larger-is-better bias.

The same four TTT isomorphs in Experiment 1 (Figure 2) were used in this experiment. However, the computer's strategy was changed. The computer always started first by selecting five as the first move. The computer's strategy (see the Appendix for details) was carefully designed such that the computer could never lose and in order for the subjects to get draws, they had to strictly follow the *Even-Even* strategy:

- Always select any even number as the first move;
- Always select any even number as the second move;
- For all other moves, simply block the piece that can lead to an immediate win for the computer.

Similar to the 5-Odd strategy in Experiments 1 and 2, this Even-Even strategy is necessary and sufficient for subjects to get draws and subjects only have to make decisions for first and second moves. In addition, subjects' first and second moves only depend on the three symmetry categories (i.e., five, even numbers, and odd numbers), not on specific elements.

Unlike the 5-Odd strategy, the Even-Even strategy is consistent with the more-is-better bias. Thus, we predict that the more-is-better bias should make easier the correct selections of both first moves (even numbers) and second moves (even numbers) in *Line*, *Shape*, and *Color*, in which the winning invariants can be perceived. In addition, if the selection of

moves is solely based on biases, then *Shape* and *Color* should be no harder than *Line*, even if *Line* is familiar to most subjects. In *Number*, the same larger-is-better bias found in Experiment 1 might be used, that is, large numbers should be preferred to small numbers. Because the larger-is-better bias is irrelevant to the selection of correct first and second moves (both are even numbers), the selections in *Number* might be random and thus this version should be harder than the other three versions.

Method

The method of this experiment was exactly the same as that of Experiment 1, except that there were 15 subjects in each condition and the Even-Even strategy was necessary and sufficient for subjects to get draws.

Results
Overall Performance

All subjects in all four conditions succeeded to get 10 draws in a row within 50 games. The average number of games needed to get 10 draws in a row for each condition is shown in Figure 8A. The difficulty order was: *Line* ≈ *Color* ≈ *Shape* < *Number*. Tukey HSD pairwise comparisons showed that it took significantly more games to get 10 draws in a row in *Number* than in *Line* ($p = 0.0006$), *Color* ($p = 0.003$), and *Shape* ($p = 0.09$, marginal). All other pairs did not significantly differ from each other (smallest $p = 0.25$). The average number of games needed to get the first draw in each condition is shown in Figure 8B. The difficulty order was: *Line* ≈ *Color* ≈ *Shape* < *Number*. Tukey HSD pairwise comparisons showed that it took significantly more games to get the first draw in *Number* than in all other three conditions (largest $p = 0.009$), which did not differ from each other significantly (smallest $p = 0.60$).

The Pattern of Subjects' First Moves

Figure 9A shows the average frequencies of even and odd numbers selected by subjects as the first moves for the initial 10 games and the expected distribution for random moves with replacement. First, subjects selected even numbers less often in *Number* than in all other three conditions (largest $p = 0.002$, Tukey HSD test), which did not differ from each other significantly (smallest $p = 0.95$). Second, subjects selected significantly more even numbers than the expected value in all four conditions (smallest $t(14) = 2.63$, $p = 0.02$). Third, there was significant difference between even and odd numbers within all four conditions (largest $p = 0.02$, Wilcoxon signed rank test). Since the four conditions in this experiment were much easier than the four conditions in Experiment 1, the averaging across the initial 10 games included a substantial learning effect, which
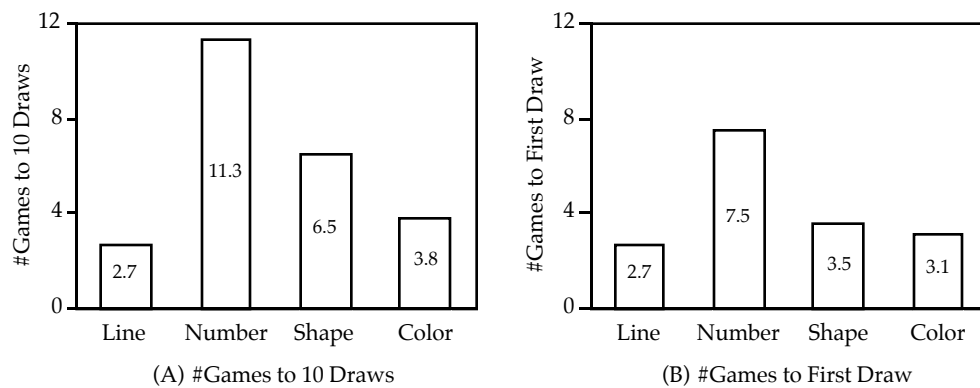
Figure 8. Overall performance in Experiment 2. (A) The number of games needed to get 10 draws in a row (excluding the 10 games) (B) The number of games needed to get the first draw (including the first draw).
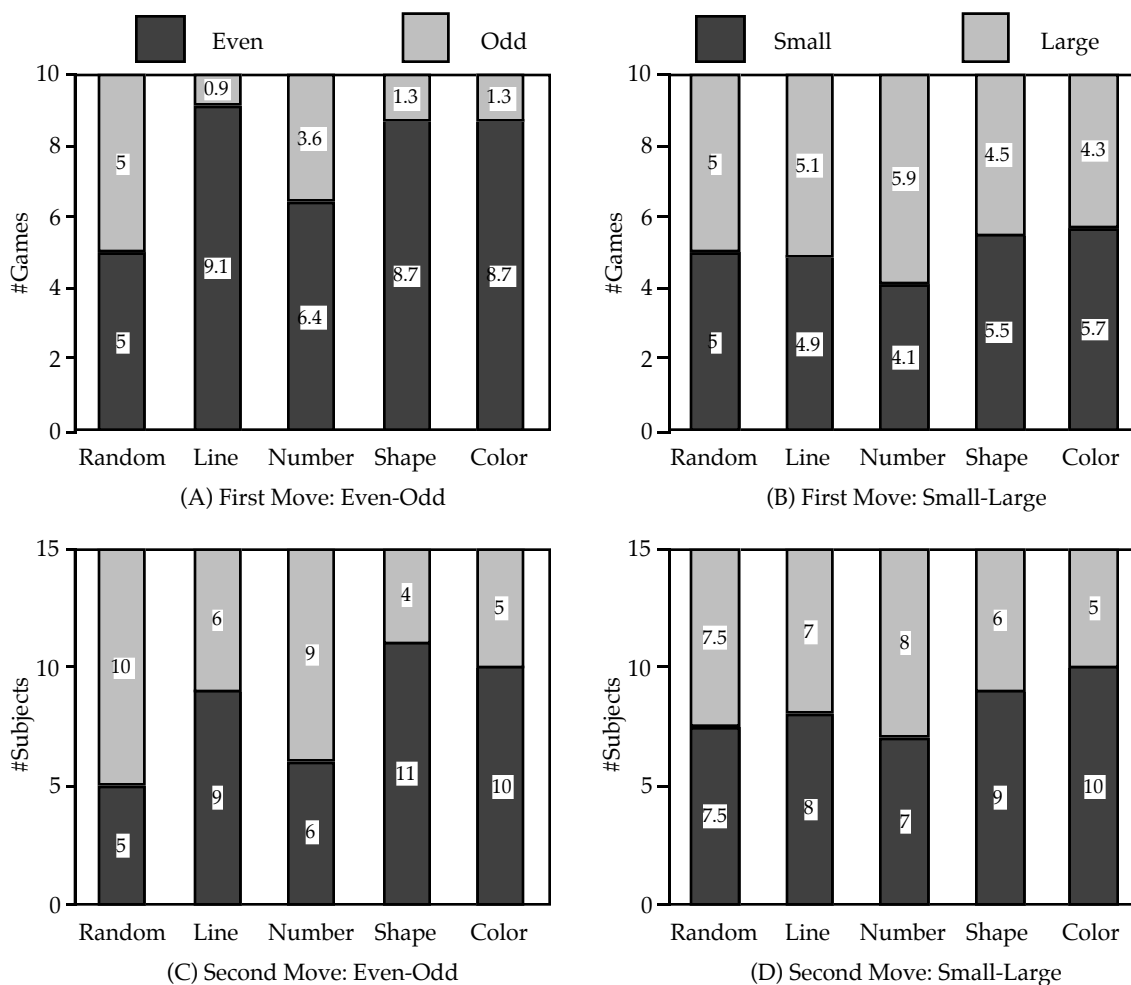

Figure 9. Move patterns in Experiment 2. (A) The average frequencies of even and odd numbers selected by each subject as first moves for the initial 10 games. (B) The average frequencies of small and large numbers selected by each subject as first moves for the initial 10 games. (C) Frequency distributions of subjects who selected even and odd numbers for second moves in the first games in which even numbers were selected as the first moves. (D) Frequency distributions of subjects who selected small and large numbers for second moves in the first games in which even numbers were selected as the first moves.

contributed to the bias towards even numbers in all four conditions, including *Number*. One-tail Chi-Square tests with a criterion of $p = 0.05$ for each game position of the initial 10 games produced the following results: there was a significant bias towards even numbers for all but the second game in *Line*, for all but the first and second games in *Shape*, and for all but the first game in *Color*. In *Number*, the bias towards even numbers was only significant for the fifth, seventh, eighth, and ninth games. Thus, with or without learning, even numbers were more preferred in *Line*, *Shape*, and *Color* than in *Number*.

Figure 9B shows the average frequencies of small and large numbers selected by the subjects across the initial 10 games and the expected distribution for random moves with replacement. Wilcoxon signed rank test showed that there was no significant difference between small and large numbers in any of the four conditions (smallest $p = 0.16$). Similar to the above analysis of even and odd numbers, the averaging across the initial 10 games also included a large learning effect, which resulted in no biases towards large numbers in any of the four conditions, including *Number*. One-tail Chi-Square tests with a criterion of $p = 0.05$ for each game position of the initial 10 games showed that there was a bias towards large numbers only for the first game in *Number* ($p = 0.005$).

The Pattern of Subjects' Second Moves

Figure 9C shows the distribution of subjects selecting even and odd numbers as their second moves for the first game in which an even number was selected as the first move; and Figure 9D shows similar distribution across small and large numbers. One-tail Chi-Square tests showed that there was a significant bias towards even numbers in *Shape* ($\chi^2 = 4.28$, $p = 0.01$) and *Color* ($\chi^2 = 3.33$, $p = 0.03$), but not in *Line* ($\chi^2 = 2.14$, $p = 0.07$) and *Number* ($\chi^2 = 0.14$, $p = 0.35$). None of the four conditions showed bias towards small or large numbers (largest one-tail $\chi^2 = 0.83$, $p = 0.18$).

Discussion

Unlike Experiment 1, in this experiment the Even-Even strategy was consistent with the more-is-better bias. This resulted in a different pattern of problem solving behavior. There was no difference in difficulty among *Line*, *Shape*, and *Color*, which were all easier than *Number*. *Line* was easy because subjects had a stronger bias towards even numbers (corners) for first moves, and *Shape* and *Color* were easy because subjects had a bias towards even numbers (i.e., 3-object circles) for both the first and second moves. The same difficulty level of these three versions implies that subjects' familiarity with *Line* could not be a major factor for the ease of *Line* in Experiment 1. Though *Color* was a bit easier than *Shape*, the differ-

ence was not reliable. *Number* was hard because subjects had no bias towards even numbers for either first or second moves. The bias towards even numbers shown in the average of the initial 10 games was due to learning since the bias did not exist for the initial four games. Subjects had a bias towards large numbers for first moves. However, this bias was significant only for the first game and was soon overcome by learning.

The four conditions in this experiment were all easier than their corresponding ones in Experiment 1, implying that the Even-Even strategy was easier to acquire than the 5-Odd strategy. This was due to the fact that in this experiment four of the eight numbers were correct first moves (even numbers) for the subjects whereas in Experiment 1 only one of the eight numbers was the correct first move (five).

In Experiment 1, there was no difference in difficulty between *Shape* and *Number*. In the current experiment, *Shape* was easier than *Number* because *Shape* was made easier by the more-is-better bias whereas *Number* was not affected by either the more-is-better or the larger-is-better bias. These results suggest problem solving behavior in the TTT isomorphs depended on not only biases but also on the structures of the tasks.

**EXPERIMENT 3:
SYMMETRY INFORMATION**

The elements in a symmetry category of the TTT are symmetrical to each other, that is, to make a move, selecting one element is not different from selecting another one in the same symmetry category. If the symmetry information can be perceived, then fewer trials are needed to find a strategy to get draws. This experiment was designed to test the perception and use of symmetry information in TTT isomorphs.

Figure 10 shows the four TTT isomorphs designed for this experiment. In all four versions, the nine elements are represented by nine circles. In *Line-Symmetry* and *Line-Asymmetry* the eight winning triplets are represented by eight straight lines whereas in *Color-Symmetry* and *Color-Asymmetry* they are represented by eight different colored lines. The winning invariant of a symmetry category is represented by the number of straight lines connecting a circle in *Line-Symmetry* and *Line-Asymmetry*, and by the number of different colored lines connecting a circle in *Color-Symmetry* and *Color-Asymmetry*. The critical factor, symmetry information, is represented in *Line-Symmetry* and *Color-Symmetry* by spatial symmetry, which can be directly perceived. For example, the four sides in *Line-Symmetry* are spatially symmetrical to each other, and the four inner circles in *Color-Symmetry* are also spatially symmetrical to each other. In *Line-Asymmetry* and *Color-Asymmetry*, the symmetry information is not directly represented and thus can not be directly perceived.
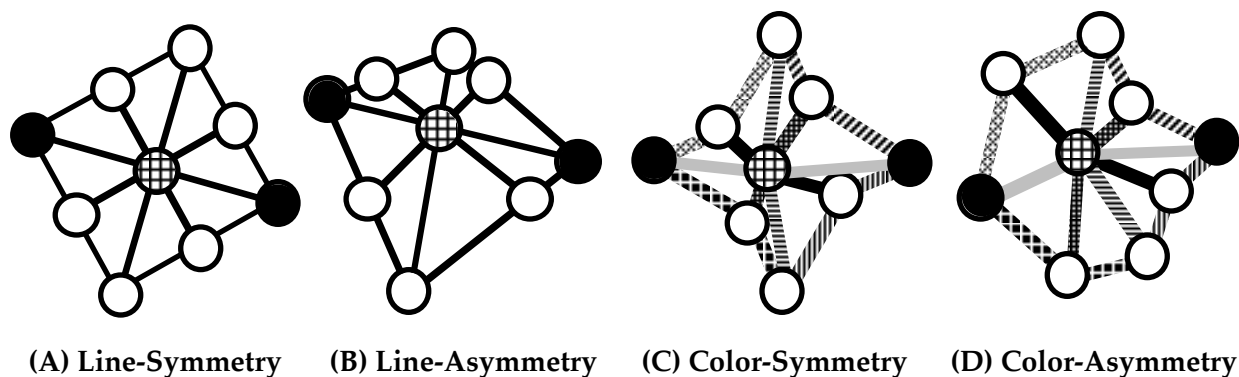
(A) Line-Symmetry   (B) Line-Asymmetry   (C) Color-Symmetry   (D) Color-Asymmetry

Figure 10. Four more TTT isomorphs for Experiment 3. In (C) and (D), the eight different colored lines used in the real experiment are shown in the figures in eight different textured lines because of the inconvenience of color graphics. (A) *Line-Symmetry*. This is identical to *Line* in Figure 2A except of a spatial rotation. (B) *Line-Asymmetry*. This is identical to *Line-Symmetry* except that the spatial symmetry is broken. (C) *Color-Symmetry*. The symmetry information is represented by spatial symmetry. The eight winning triplets are represented by eight different colors. Getting any three circles connected by the same colored line is a win. (D) *Color-Asymmetry*. This is identical to *Color-Symmetry* except that the spatial symmetry is broken.

For *Line-Symmetry* and *Line-Asymmetry*, getting three circles that lie on a straight line is a win, whereas for *Color-Symmetry* and *Color-Asymmetry*, getting three circles that are connected by the same colored line is a win. The computer's strategy was the same as the one used in Experiment 1, that is, the 5-Odd strategy is necessary and sufficient for subjects to get draws.

Since the symmetry information could be directly perceived in *Line-Symmetry* and *Color-Symmetry* but not in *Line-Asymmetry* and *Color-Asymmetry*, subjects should play fewer games to get the first draw and to get 10 draws in a row in *Line-Symmetry* than in *Line-Asymmetry*, and in *Color-Symmetry* than in *Color-Asymmetry*. In addition, if the identification of winning triplets based on straight lines is not different from that based on same colored lines, there should be no difference between *Line-Symmetry* and *Color-Symmetry* and between *Line-Asymmetry* and *Color-Asymmetry*.

Method

The design and procedure were exactly the same as in Experiment 1 except that the four TTT isomorphs in Figure 10 were the four conditions for the present experiment. There were 20 subjects in each condition. The 5-Odd strategy was necessary and sufficient for subjects to get draws.

Results
Overall Performance

All subjects got 10 draws in a row within 50 games for all four conditions. The number of games needed to get 10 draws in a row in each condition is shown in Figure 11A. The main effect was significant (F(3, 76) = 4.61, p = 0.005). Fisher LSD tests showed that it took significantly more games to get 10 draws in a row in *Line-Asymmetry* and *Color-*

Asymmetry than in *Line-Symmetry* and *Color-Symmetry* ($0.003 \leq p \leq 0.03$). There was no significant difference between *Line-Symmetry* and *Color-Symmetry* (p = 0.66) and between *Line-Asymmetry* and *Color-Asymmetry* (p = 0.68).

The number of games needed to get the first draw in each condition is shown in Figure 11B. The main effect was significant (F(3, 76) = 4.61, p = 0.005). Fisher LSD tests showed that it took marginally more games to get first draws in *Line-Asymmetry* and *Color-Asymmetry* than in *Line-Symmetry* and *Color-Symmetry* ($0.05 \leq p \leq 0.09$). There was no significant difference between *Line-Symmetry* and *Color-Symmetry* (p = 0.84) and between *Line-Asymmetry* and *Color-Asymmetry* (p = 0.93).

The Pattern of Subjects' First and Second Moves

Figure 11C shows the average frequencies of five, even and odd numbers selected by the subjects for the first moves of the initial 10 games and the expected distribution for random moves with replacement. First, Fisher LSD tests showed that subjects selected five less often in *Line-Asymmetry* than in *Color-Symmetry* (p = 0.03) and in *Line-Symmetry* (p = 0.03). All other comparisons were not significant (smallest p = 0.16). Second, subjects selected significantly more five's than the expected value in all four conditions (smallest t(19) = 10.65, p < 0.001). Third, there was no significant difference between even and odd numbers within any of the four conditions (smallest p = 0.18, Wilcoxon Sign Rank test).

Figure 11D shows the distribution of the 20 subjects selecting even and odd numbers as their second moves for the first game in which five was selected as the first move and the expected distribution for random moves. There was no bias towards even numbers in any of the four conditions (largest χ^2 = 2.81, p = 0.10).
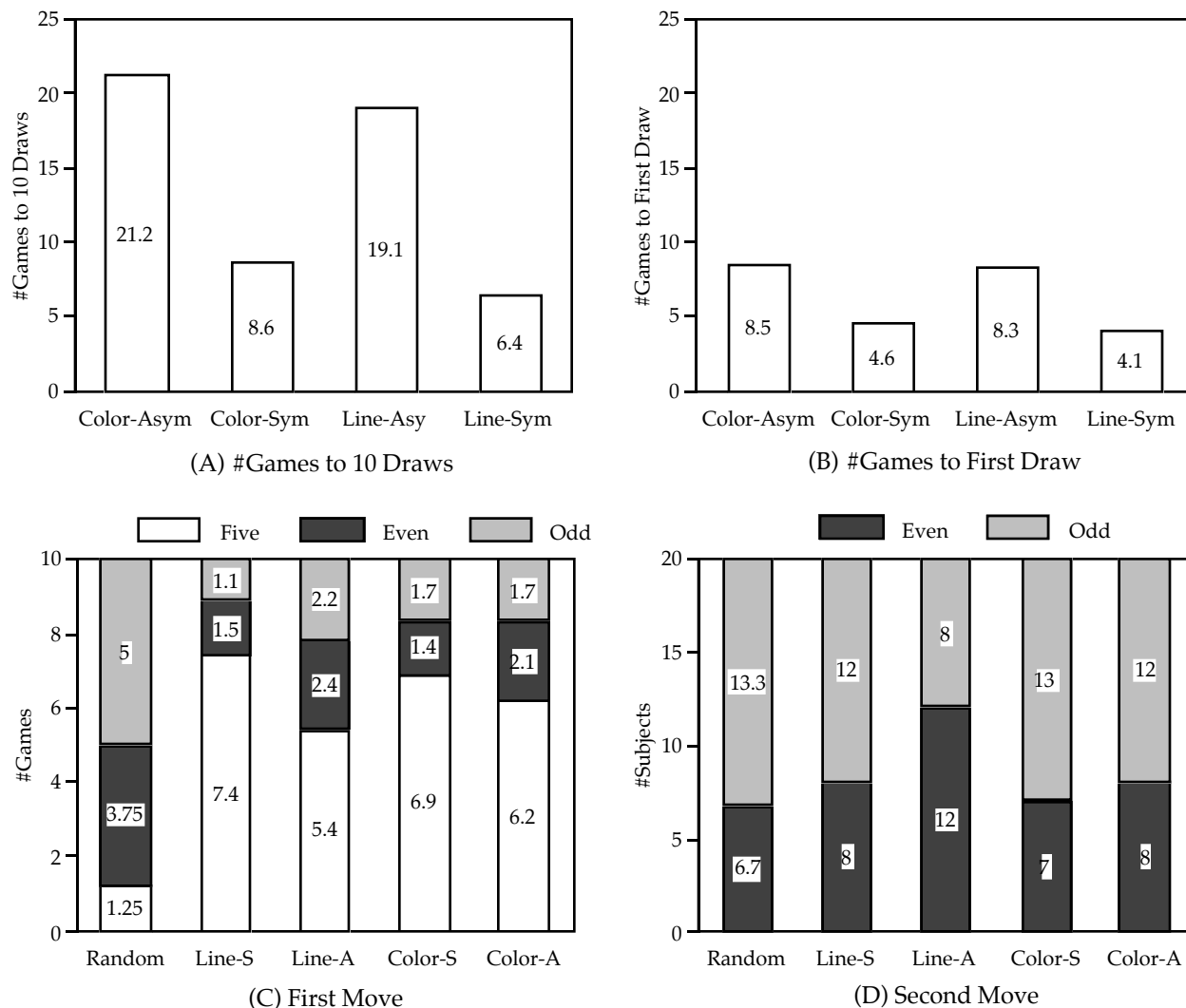
Figure 11. Results of Experiment 3. (A) The number of games needed to get 10 draws in a row (excluding the 10 draws). (B) The number of games needed to get the first draw (including the first draw). (C) The average frequencies of five, even and odd numbers selected by each subject as first moves for the initial 10 games. (D) Frequency distributions of subjects who selected even and odd numbers as their second moves in the first game in which five was selected as the first move.

The Frequencies of 5-Even and 5-Odd Games Before 10 Draws in a Row

Before discovering the 5-Odd strategy to get 10 draws in a row, subjects usually needed to play several 5-Even games (first move = 5, second move = even number) and several 5-Odd games (first move = 5, second move = odd number). The frequencies of 5-Even and 5-Odd games before 10 draws in a row are shown in Figure 12. The main effect of 5-Odd games across conditions was not significant (F(3, 76) = 1.72, p = 0.17), nor did any of the pairwise comparisons (0.06 ≤ p ≤ 0.95, Fisher LSD test). 5-Odd games are draw games with positive feedback. The insignificant results of 5-Odd games suggest that the number of positive games needed to discover the 5-

Odd strategy is the same for different TTT isomorphs, that is, independent of problem representations.

The main effect of 5-Even games across conditions was significant (F(3, 76) = 5.44, p = 0.002). Fisher LSD tests showed that there were more 5-Even games in *Line-Asymmetry* and *Color-Asymmetry* than in *Line-Symmetry* and *Color-Symmetry* (0.002 ≤ p ≤ 0.02). However, there was no significant difference between *Line-Asymmetry* and *Color-Asymmetry* (p = 0.40) and between *Line-Symmetry* and *Color-Symmetry* (p = 0.95). Because none of the four conditions had bias towards either even or odd numbers for first and second moves, the frequencies of 5-Even games reflect the influence of

the symmetry information. There should be more 5-Even games if the 2 even numbers for second moves are perceived to be different than if they are perceived to be equivalent. The finding that there were more 5-Even games in *Line-Asymmetry* and *Color-Asymmetry* than in te *Line-Symmetry* and *Color-Symmetry* suggests that symmetry information was perceived in the latter two conditions but not in the former two conditions.
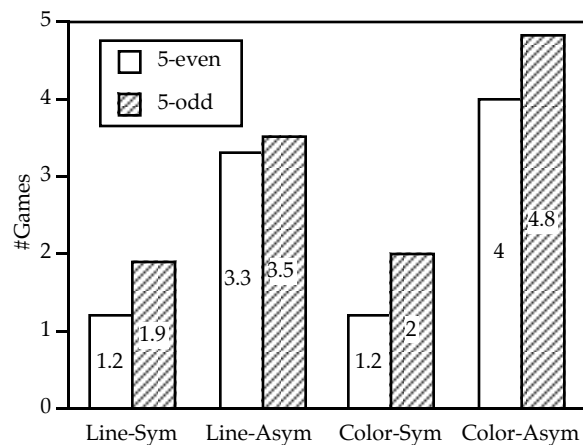

Figure 12. Frequencies of 5-Even and 5-Odd games before 10 draws in a row in Experiment 3.

Discussion

This experiment showed that symmetry could be perceived in some external representations but not in others. When it could be perceived, problems became easier. In *Line-Symmetry* and *Color-Symmetry*, the symmetry information was represented by spatial symmetry, which could be perceived by subjects. In *Line-Asymmetry* and *Color-Asymmetry*, it was not directly represented and could not be perceived. The perception of the symmetry information can reduce the number of alternatives that have to be tried before a consistent strategy can be found because the elements in a symmetry category are equivalent to each other. This experiment showed that subjects needed fewer games to get the first draw and to get 10 draws in a row in *Line-Symmetry* and *Color-Symmetry* than in *Line-Asymmetry* and *Color-Asymmetry*. The insignificant difference between *Line-Symmetry* and *Color-Symmetry* and between *Line-Asymmetry* and *Color-Asymmetry* suggest that the representations of winning invariants by straight lines and those by colored lines did not have different effects on problem solving behavior.

GENERAL DISCUSSION

Summary of Experiments

The experiments in the present study showed that the problem solving behavior in the TTT tasks was de-termined by the biases elicited by the directly available information in external or internal representations, regardless of whether biases were consistent with, inconsistent with, or irrelevant to the task.

In external representations (*Line*, *Shape*, and *Color*), the directly available information was invariant information--the winning invariants and symmetry. Winning invariants in external representations were perceived by subjects, and they determined subjects' problem solving behavior by eliciting the more-is-better bias. When the more-is-better bias was inconsistent with the task structure (the 5-Odd strategy), as in Experiment 1, it made problems more difficult. When it was consistent with the task structure (the Even-Even strategy), as in Experiment 2, it made problems easier. Symmetry can reduce the number of alternatives and thereby make problems easier. Experiment 3 showed that symmetry information could be directly perceived when it was represented by spatial symmetry.

In internal representations (*Number*), the directly available information was not the invariant information because neither winning invariants nor symmetry could be perceived by subjects. Thus, the invariant information had no effect on problem solving behavior in internal representations. Instead, the directly available information was the magnitude information of the nine numbers. It was directly retrieved from internal representations, and it guided subjects' behavior by eliciting the larger-is-better bias. The larger-is-better bias was inconsistent with the 5-Odd strategy, thus it made more difficult the *Number* version with the 5-Odd strategy; and it was irrelevant to the Even-Even strategy, thus it had no effect on *Number* with the Even-Even strategy.

Evaluation of the Theoretical Framework

The theoretical framework developed in the present study is not a process model with explicitly specified mechanisms. However, as a functional model, it has specific theoretical predications that can be tested empirically. In addition, as a methodology, it specifies the major components that should be analyzed for any ER-based problem solving task.

On the methodological side, the potential of the framework was illustrated in the representational analysis of the major components of the TTT isomorphs. First, the abstract structure of the TTT was analyzed. Then, the abstract structure was implemented by different isomorphic representations, each of which was a distributed representation that was decomposed into an internal and an external component. Finally, different operations activated by different representations were specified and different types of information provided by different representations were examined. With the identifications of these properties, we can analyze the common and different properties of isomorphs and their roles in

problem solving.

On the empirical side, based on the theoretical analysis, assumptions were made on what information in external and internal representations was directly available and specific predictions were made on how the directly available information affected problem solving behavior in terms of perceptual and cognitive biases. The assumptions on directly available information were supported by the experimental results. The directly available information in the external representations of the TTT was invariant information (winning invariants and symmetry), and the directly available information in internal representations was directly retrievable information (magnitude information). Furthermore, the major prediction of the framework for the TTT tasks was confirmed by the experiments: the problem solving behavior in the TTT tasks was determined by the biases elicited by the directly available information in external or internal representations, regardless of whether biases were consistent with, inconsistent with, or irrelevant to the task.

The theoretical framework developed in the present study needs to be further developed and tested. It leaves many things unspecified. For example, it does not give any details on how perceptual and cognitive operations are coordinated, which is an interesting issue worth of further research (for a few studies on this issue, see Carlson, Wenger, & Sullivan, 1993; Dark, 1990; Weber, Byrd, & Noll, 1986).

Invariant Information and Affordances

Affordances, as defined by Gibson (1979), are the invariant information in the environment that provides action opportunities and guides the behavior of perceivers. Affordances can be directly picked up by perceivers. In addition, those affordances which an environment supports depend on not only the properties of the environment but also the properties of the perceiver, that is, the environment and the perceiver are mutually constraining and complementary.

The winning invariants in the external versions of TTT isomorphs are concrete examples of affordances in problem solving. First, they are the invariant information in the external representations: the winning invariants of all elements in a symmetry category have the same value. Second, the winning invariant of an element specifies the number of potential wins afforded by the element, that is, action opportunities. Third, the winning invariants can be directly picked up by perceivers and can guide their behavior by eliciting the more-is-better bias. Fourth, the functions afforded by winning invariants are jointly determined by the properties of external representations and the properties of the tasks. For example, the winning invariants in *Shape* are the quantities of objects inside big circles. They are perceived as winning opportunities because the goal of the TTT is

to generate as many potential wins as possible for oneself and block as many potential wins as possible for the opponent. The goal of the TTT is part of the internal representations of the task as task instructions.

The effects of winning invariants on problem solving behavior revealed in the present experiments suggest an interesting property of affordances: affordances do not always provide positive functions. If they are consistent with task structures, they can make tasks easier. However, if they are inconsistent with task structures, they can also make tasks harder. It is usually considered that problem solving is primarily a higher-level task independent of perceptual factors. The strong effect of perceptual information on problem solving shown in the present study suggests that problem solving can also be driven by perceptual factors: due to the limitations of the mind such as the limited capacity of working memory and attentional resources, mental operations (planning, computation, deliberation, etc.) are difficult or impossible to perform to overcome perceptual biases.

Representational Determinism

All of the TTT isomorphs in the present study (Figures 2 and 10) have the same abstract structure—the same TTT problem space. However, they produced different cognitive behaviors. This is an example of the representational effect—the effect that different representations of a common abstract structure can cause dramatically different cognitive behaviors (e.g., Zhang & Norman, 1994). The formulation of isomorphic problems in terms of a common abstract structure and a set of different representations is very useful. By considering alternative representations of a common abstract structure, we can identify the factors that affect cognitive behavior and the processing mechanisms in cognitive tasks. In addition, we can compare the representational efficiencies of isomorphic representations, which are crucial in the design of effective representations.

However, abstract structures are mathematical constructs by theorists: they are usually unavailable to task performers. For many tasks, as illustrated in the TTT tasks of the present study, task performers simply perform the operations activated and process the information provided by specific representations. Therefore, the representational effect is not just a matter of different efficiencies and different behaviors. For learned isomorphic representations, such as Arabic and Roman numerals, we can compare their relative efficiencies in learned tasks such as addition and multiplication. However, for novel and discovery tasks, whose abstract structures are not known, the format of a representation can determine what information can be perceived, what processes can be activated, and what structures can be discovered from the specific representation. This is called *represen-*

tational determinism. Without the change of representational forms, some portion of the task space may never be explored and some structures of the task may never be discovered, due to various constraints such as the complexity of the environment and the limitations of the mind. For example, in Experiment 1 of the present study, different forms of the 5-Odd strategy were discovered by subjects for different representations of the same task, ranging from a very specific fixed strategy to a very general symmetric strategy. These different forms of the 5-Odd strategy indicate that different structures were discovered and different portions of the TTT problem space were explored by the subjects. The structures such as symmetry and winning invariants were much more difficult to discover in *Number* than in *Line*. If we do not re-represent *Number* as *Line* or do not even know the existence of an isomorphism between *Number* and *Line*, those invariant structures in *Number* would be very difficult to be discovered. Thus, under internal and external constraints, cognitive behavior is much like constraint satisfaction, with many local minima, some of which may never be overcome without a change in representational forms. The following are two interesting observations of the representational determinism in the real world.

The first observation is the impact of writing on cognition. Goody (1977) and Ong (1982), among many others, argue that without writing, the logical, analytical, rational, and scientific mode of modern thought was impossible. Writing made available certain knowledge, skills, and procedures essential for the rational mode of thought, such as organizing, manipulating, elaborating, and reflecting upon logical relations in the analytic form of linear sequences. Speech, which is the external representation of language in spoken form, is constrained by the transient and dynamic nature of utterance and the limited capacity of work memory. These constraints resulted in some unique properties of spoken language, such as the schemas and formula in oral poetry developed for memorization and recital (see Ong, 1982). However, these properties of spoken language did not support the emergence of the rational mode of thought, which is only made possible by writing. Olson (1996) has recently argued that writing does not merely transcribe but rather brings structural properties of speech into consciousness. In his view, the development of writing was the discovery of the representable structures of speech. Thus, according to him, the inventors of writing systems did not already know about language and its structures--sentences, words, phonemes, and the like, which were discovered only with the existence of writing systems.

The second observation is the development of numeration systems. Very few things in the world are as universal as the Arabic numeration system. It has been argued (e.g., Dantzig, 1939) that the invention of Arabic numerals was instrumental in the emergence and development of algebra. Algebra deals with operations upon symbolic forms, including numerals, unknowns, and arithmetic operators. The analysis in Zhang & Norman (1995) shows that what makes the Arabic system so special and widely accepted is that it integrates representation and calculation into a single system, along with its nice features of efficient information encoding, extendibility, spatial representation, small base, effectiveness of calculation, and especially important, ease of writing. Greeks, who were highly advanced in geometry, did not develop an algebra in the modern sense. This is probably because Greek alphabetical numerals, though easy to manipulate, were neither efficient for representation nor for calculation.

Conclusion

This article explored the nature of external representations in problem solving, using the Tic-Tac-Toe and its isomorphs to manipulate the information in external representations to examine what information in external representations can be perceived and how the information in external representations affects problem solving behavior. Although the Tic-Tac-Toe is a simple task, it reflected the properties of real world complex tasks and revealed the important roles of external representations in cognition. External representations are neither mere inputs and stimuli to nor mere memory aids to the internal mind. They are intrinsic components of many cognitive tasks; they guide, constrain, and even determine cognitive behavior. For complex tasks requiring interactions with the environment, the complexity of the environment and the limitations of the mind suggest that cognitive behavior is much like constraint satisfaction through the execution of the operations directly activated by external and internal representations and the processing of the information directly available from external and internal representations. External representations have different properties from those of internal ones. They need to be studied on their own right, not as something peripheral to internal representations.

REFERENCES

Aristotle (1938). De Interpretatione (H. P. Cook, Trans.). London: Loeb Classical Library.

Anderson, J. R. (1993). *Rules of the mind*. Hillsdale, NJ: Erlbaum.

Barwise, J. & Perry, J. (1983). *Situations and attitudes*. Cambridge, MA: The MIT Press.

Bertin, J. (1983). *Semiology of graphics*. Madison, WI: University of Wisconsin Press.

Bloomfield, L. (1993). *Language*. New York: Holt, Rinehart & Winston.

Carlson, R. A., Wenger, J. L., & Sullivan, M. A. (1993). Coordinating information from percep-

tion and working memory. *Journal of Experimental Psychology: Human Perception and Performance, 19* (3), 531-548.

Chambers, D., & Reisberg, D. (1985). Can mental images be ambiguous? *Journal of Experimental Psychology: Human Perception & Performance, 11,* 317-328.

Clancey, W. J. (1993). Situated action: A neuropsychological interpretation (Response to Vera and Simon). *Cognitive Science, 17*, 87-116.

Cleveland, W. S. (1985). *The elements of graphing data*. Monterey, CA: Wadsworth.

Dark, V. J. (1990). Switching between memory and perception: Moving attention or memory retrieval? *Memory & Cognition, 18* (2), 119-127.

Donald, M. (1991). *Origins of the modern mind: Three stages in the evolution of culture and cognition*. Cambridge, MA: Harvard University Press.

Gibson, J. J. (1966). The senses considered as perceptual systems. New York: Houghton Mifflin.

Gibson, J. J. (1979). *The Ecological Approach to Visual Perception*. New York: Houghton Mifflin.

Goody, J. (1977). *The Domestication of the Savage Mind*. New York: Cambridge University Press.

Greeno, J. G. (1989). Situations, mental models, and generative knowledge. In D. Klahr & K. Kotovsky (Eds.), *Complex information processing: The impact of Herbert A. Simon.* Hillsdale, NJ: Erlbaum .

Greeno, J. G., & L., M. J. (1993). Situativity and symbols. (Response to Vera and Simon). *Cognitive Science, 17*, 49-59.

Hutchins, E. (1990). The technology of team navigation. In J. Galegher, R. Kraut, & C. Egido (Ed.), *Intellectual teamwork: Social and technical bases of collaborative work.* Hillsdale, NJ: Erlbaum.

Hutchins, E. (1995a). *Cognition in the wild*. Cambridge, MA: The MIT Press.

Hutchins, E. (1995b). How a cockpit remembers its speed. *Cognitive Science, 19*, 265-288.

Hutchins, E., & Norman, D. A. (1988). *Distributed cognition in aviation: A concept paper for NASA* (Grant No. NCC 2-591). San Diego: University of California, Department of Cognitive Science.

Kleinmuntz, D. N., & Schkade, D. A. (1993). Information displays and decision processes. *Psychological Science, 4* (4), 221-227.

Kirlik, A., Plamondon, B. D., Lytton, L. and Jagacinski, R. J. (1993a). Supervisory control in a dynamic and uncertain environment: A process model of skilled human-environment interaction. *IEEE Transactions on Systems, Man, and Cybernetics, 23* (4), 929-952.

Kirlik, A., Plamondon, B. D., Lytton, L. and Jagacinski, R. J. (1993b). Supervisory control in a dynamic and uncertain environment: Laboratory task and crew performance. *IEEE Transactions on Systems, Man, and Cybernetics, 23* (4), 1130-1138.

Larkin, J. H. (1989). Display-based problem solving. In D. Klahr & K. Kotovsky (Eds.), *Complex Information Processing: The Impact of Herbert A. Simon.* Hillsdale, NJ: Erlbaum.

Larkin, J. H., & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science , 11*, 65-99.

Lave, J. H. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life*. New York: Cambridge University Press.

Lewis, C. M. (1991). *Representational aiding*. Unpublished manuscript. Pittsburgh, PA: University of Pittsburgh, Department of Information Science.

Newell, A. (1990). *Unified theories of cognition*. Cambridge, MA: Harvard University Press.

Norman, D. A. (1988). *The psychology of everyday things*. New York: Basic Books.

Norman, D. A. (1991). Cognitive Artifacts. In J. M. Carroll (Ed.), *Designing interaction: Psychology at the human-computer interface.* New York: Cambridge University Press.

Norman, D. A. (1993a). Knowledge in the head and in the world: An introduction to the special issue on situated action. *Cognitive Science, 17,* 1-6.

Norman, D. A. (1993b). *Things that make us smart*. Reading, MA: Addision-Wesley.

Olson, D. R. (1996). Towards a psychology of literacy: On the relations between speech and writing. *Cognition, 60*, 83-104.

Ong, W. J. (1982). *Orality and literacy: The technologizing of the word*. London: Methuen.

Reisberg, D. (1987). External representations and the advantages of externalizing one's thoughts. *Proceedings of the Eighth Annual Conference of the Cognitive Science Society*. Hillsdale, NJ: Erlbaum.

Saussure, F. (1983). *Course in general linguistics.* London: Duckworth.

Schmid, C. F. (1983). *Statistical Graphics*. New York: John Wiley & Sons.

Shepard, R. N. (1984). Ecological constraints on internal representation: resonant kinematics of perceiving, imaging, thinking, and dreaming. *Psychological Review, 91* (4), 417-447.

Simon, H. A. (1981). *The sciences of the artificial* (2nd ed.). Cambridge, MA: The MIT Press.

Stenning, K., & Oberlander, J. (1994). A cognitive theory of graphical and linguistic reasoning: Logic and implementation. *Cognitive Science 19*, 97-140.

Suchman, L. A. (1987). *Plans and situated action: The problem of human-machine communication*. New York: Cambridge University Press.

Tufte, E. R. (1990). *Envisioning information*. Cheshire, Connecticut: Graphics Press.

Weber, R. J., Burt, D. B., & Noll, N. C. (1986). Attention switching between perception and memory. *Memory & Cognition, 14* (3), 238-245.

Zhang, J. (1996). A representational analysis of relational information displays. *International Journal of Human-Computer Studies, 45*, 59-74.

Zhang, J., & Norman, D. A. (1994). Representations in distributed cognitive tasks. *Cognitive Science, 18*, 87-122.

Zhang, J., & Norman, D. A. (1995). A representational analysis of numeration systems. *Cognition 57*, 271-295.

APPENDIX: COMPUTER STRATEGIES

The following are the two strategies of the computer opponent. Strategy 1 was used in Experiments 1 and 3, which forced the subjects to discover the 5-Odd strategy to get draws. Strategy 2 was used in Experiment 2, which forced the subjects to discover the Even-Even strategy to get draws. The computer opponent always played first. C1-C5 are computer's five moves and S1-S4 are subjects' four moves.

Strategy 1

Step 1: C1 = any even number.
Step 2: S1 = subject's first move.
Step 3: If S1 = 5, then C2 = 15 - C1 - S1;
        else if S1 = odd, then C2 = 5;
                else if S1 = even & C1 + S1 + 5 ≠ 15, then C2 = 15 - C1 - 5;
                        else if S1 = even & C1 + S1 + 5 = 15, then C2 = any even number.
Step 4: S2 = subject's second move.
Step 5: If x leads to a win for the computer, then C3 = x and stop;
        else if x leads to a win for the person, then C3 = x;
                else if S1 = odd & S1 + C1 + any number = 15 then
                    let E3 & E4 = the two remaining even numbers;
                    if S1 + C1 + E3 = 15, then C3 = E4, else C3 = E3.
Step 6: S3 = subject's third move.
Step 7: If x leads to a win for the computer, then C4 = x and stop.
        else if x leads to a win for the person, then C4 = x;
                else C4 = any remaining number.
Step 8: S4 = subject's fourth move.
Step 9: If x leads to a win for the computer, then C5 = x and stop.
        else if x leads to a win for the person, then C5 = x;
                else C5 = any remaining number.

Strategy 2

Step 1: C1 = 5
Step 2: S1 = subject's first move
Step 3: If S1 = even, then C2 = 15 - C1 - S1;
                else if S1 = odd, then C2 = any even number.
Step 4: S2 = subject's second move.
Step 5: If x leads to a win for the computer, then C3 = x and stop;
        else if x leads to a win for the person, then C3 = x;
                else if S1 = odd, then C5 = 15 - C1 - (15 - S1 -C2);
                    else if S1 = even & S2 = odd, then C5 = 15 - C1 - (15 -S2 - C2).
Step 6: S3 = subject's third move.
Step 7: If x leads to a win for the computer, then C4 = x and stop.
        else if x leads to a win for the person, then C4 = x;
                else C4 = any remaining number.
Step 8: S4 = subject's fourth move.
Step 9: If x leads to a win for the computer, then C5 = x and stop.
        else if x leads to a win for the person, then C5 = x;
                else C5 = any remaining number.