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MODEL THEORY OF MODULES*

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Contents

Introduction

The model theory of abelian groups was developed by Szmielew ([28] quantifier elimination and decidability) and Eklof & Fisher [4], who observed that $\aleph_1$-saturated abelian groups are pure injective. Eklof & Fisher related the structure theory of pure injective abelian groups with their model theory.

The extension of this theory to modules over arbitrary rings became possible after the work of Baur [1], Monk [14], Fisher [6] and Warfield [30]. Baur proved that – for any fixed module M – every formula is equivalent to a boolean combination of positive primitive formulas $\varphi(x_1, \ldots, x_n)$ (which assert the solvability of

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finite systems of linear equations with parameters $x_1, \ldots, x_n$). The lattice of pp-definable subgroups $\varphi(M)$ of M gives a lot of information: The elementary type of M is determined by the indices $\varphi/\psi(M) = (\varphi(M) : \psi(M))$ ($\in\{1, 2, \ldots, \infty\}$), (Monk [14]). M is totally transcendental iff the lattice of pp-definable subgroups of M is well-founded. Therefore totally transcendental modules are 'compact' (Garavaglia and Macintyre [9]).

Warfield and Fisher defined and developed the structure theory of pure injective modules. (We use 'compact' for 'pure injective'.) Fisher proved the uniqueness of the representation of a compact module as the pure hull of a direct sum of indecomposable compact (short: 'indecomposable') modules. This was completed by a theorem of Zimmermann & Zimmermann-Huisgen [19]: The endomorphism ring of an indecomposable module is local.

In [11] Garavaglia showed that a compact module with elementary Krull dimension (i.e. there is no densely ordered chain of pp-definable subgroups) is the pure hull of a direct sum of indecomposables. In a sense our paper is a continuation of Garavaglias work. To keep our paper self-contained we reprove the results mentioned above. In Section 1 Baur's and Monk's results are proved. In Section 2 we give the characterization of totally transcendental and superstable modules by means of their lattice of pp-definable subgroups [9]. In Sections 3 and 6 we present Warfield's theory of smallness, pure hulls (this is also in [3], [11] and [20]) and prove a slight generalization of a theorem of Fisher:

6.1. *Every compact module has a unique representation as the pure hull of a direct sum of indecomposables and a compact module without indecomposable factors.*

We begin our study of indecomposable modules in Section 4 with a new proof of the theorem of Zimmermann & Zimmermann-Huisgen (4.3). Our main technical tool is based on this theorem: A syntactical characterization of indecomposable types (4.4): Let a be a non-zero element of the compact module M. There is a minimal direct factor $H(a)$ of M which contains a. ($H(a)$ is unique up to an isomorphism. $M = H(a)$ if M is indecomposable.) We have:

4.4. *$H(a)$ is indecomposable iff for all pp-definable subgroups $\psi_1(M)$, $\psi_2(M)$ not containing a there is a pp-definable subgroup $\varphi(M)$ s.t. $a \in \varphi(M)$, $a \notin \psi_1(M) \cap \varphi(M) + \psi_2(M) \cap \varphi(M)$.*

As one application of 4.4 one can define a quasi compact topology on $\mathbb{U}$, the set of all isomorphism types of indecomposable R-modules. A base for the open sets consists of all $(\varphi/\psi) = \{U \in \mathbb{U} \mid \varphi/\psi(U) > 1\}$. The closed sets are $U_M = \{U \in \mathbb{U} \mid U$ is a direct factor of a module elementarily equivalent to $M\}$ (4.9, 4.10).

As a second application one can construct a lot of indecomposables:

4.8. *If $\varphi/\psi(M) > 1$, (φ/ψ) contains an element of $\mathbb{U}_M$.*

Consequences are:

6.9. *Every module is elementarily equivalent to a direct sum of indecomposables.*

6.11. (1) *the elementary type of the module M is determined by the invariants*

$$I_U(M) = \min\{^{\varphi/\psi(U)} \log \varphi/\psi(M) \mid \varphi, \psi \text{ ppf}\} \qquad (U \in \mathbb{U}).$$

For the computation of the I_U it is enough to let the pairs φ, ψ range over a base of neighbourhoods (φ/ψ) of U. For a suitable choice of bases we obtain (for $R = \mathbb{Z}$) the Szmielew invariants as a special case (cf. 6.12, 9.6).

In Section 5 we determine the indecomposables in several cases: injective indecomposables (5.10) and indecomposable R-modules for a commutative ring R whose localization $R_{\mathfrak{P}}$ at maximal ideals $\mathfrak{P}$ are valuation rings (5.2) (proofs are given in the case where all $R_{\mathfrak{P}}$ are fields or discrete valuation rings.) Here we can restrict ourselves to the case that R is a valuation ring, since for commutative rings R:

5.4. *Indecomposable R-modules are indecomposable $R_{\mathfrak{P}}$-modules for maximal ideals $\mathfrak{P}$.*

For Dedekind rings R we give an explicit description of the topological space $\mathbb{U}$ in 9.5. For effectively given Dedekind rings (e.g. $R = \mathbb{Z}$) this yields the decidability of the theory of all R-modules (9.7). This is a general theorem:

9.4. *Let R be a recursive ring and (φ_i/ψ_i), $i \in \mathbb{N}$, an effective enumeration of a base of $\mathbb{U} = \{U_j\}_{j\in\mathbb{N}}$. Then the theory of all R-modules is decidable if $\varphi_i/\psi_i(U_j)$ depends recursively on i, j.*

Similarly we reprove the theorem of Koslov & Kokorin [21, 22] (generalized to Dedekind rings): the decidability of torsion free abelian groups with a distinguished subgroup (9.10). In fact most of our general theory holds for more general structures than modules: for abelian groups with a family of additive relations.

Let M be a module. When are all compact modules elementarily equivalent to M the pure hull of a direct sum of indecomposables? By the above mentioned result of Garavaglia this is the case, if M has elementary Krull dimension. Also if the lattice of pp-definable subgroups of M is linearly ordered this is true. On the other hand the pure hull of an atomless boolean ring is not the pure hull of a direct sum of indecomposables.

In Section 7 we define the notion of 'bounded width' and show:

7.1. (1) *If M has bounded width, then every compact module elementarily equivalent to M is the pure hull of a direct sum of indecomposables.*

(2) *If R is countable, the converse is true.*

The width of a module depends only on the structure of the lattice of pp-definable subgroups. Elementary Krull dimension implies bounded width.

In Section 8 we study modules with elementary Krull dimension. (Often we say simply 'Krull dimension', but we do not mean the classical notion.)

Here we assign to every pair $\psi(M) \subset \varphi(M)$ of pp-definable subgroups of a module M an ordinal (or ∞) $\dim_M(\varphi/\psi)$, which measures the extent to which there is 'almost' a dense chain of pp-definable subgroups between $\psi(M)$ and $\varphi(M)$. Then M has Krull dimension iff $\dim_M(M/0) < \infty$. (Garavaglia has a similar 'dimension' in [11].) We prove in Section 8:

8.6. *If R is countable or* $\dim_M(\varphi/\psi) < \infty$, $\dim_M(\varphi/\psi)$ *equals the Cantor–Bendixson rank of the topological space* $(\varphi/\psi) \cap \mathbb{U}_M$.

I do not know if the countability of the ring R is necessary.

Thus, if R is countable and M has Krull dimension, $\mathbb{U}_M$ must be countable. The converse is also true:

8.1, 8.4. *Let R be countable. Then M has Krull dimension iff $\mathbb{U}_M$ is countable iff $\mathbb{U}_M$ has a Cantor–Bendixson rank, which is then* $\dim_M(M/0)$.

As an application we give for modules M with Krull dimension an explicit description of all compact N, elementarily equivalent to M (9.1). It turns out that there is a smallest such N.

This is the first step towards our solution of the (uncountable) spectrum problem for complete theories of infinite modules over a countable ring in Section 10.

Let for infinite M, $I_M(\kappa)$ denote the number of non-isomorphic modules of cardinality κ which are elementarily equivalent to M. We show in 10.1 that exactly 6 functions $\kappa \mapsto I_M(\kappa)$ $(\kappa > \aleph_0)$ occur. (The first function depends on a parameter λ, $1 \leqslant \lambda \leqslant \aleph_0$). For the superstable case we use:

10.2. *Every superstable module is the direct sum of a totally transcendental module and a module of cardinality at most* $2^{\aleph_0}$.

For $I_M(\aleph_0)$ we give some information in 10.3. E.g. that M has finite Krull dimension if $I_M(\aleph_0) < 2^{\aleph_0}$.

Finally we characterize $\aleph_0$-categorical (that is due to Baur [27]) and $\aleph_1$-categorical modules in (10.6).

We conclude this paper with an investigation of some notions of stability theory in the case of modules (this again was first done by Garavaglia in a special case [10]). Note that every module is stable. In 11.1 we characterize 'forking' by means of pp-definable subgroups. Then we determine regular and orthogonal types using indecomposables (this is in the special case when $M \oplus M \equiv M$.) It turns out that regular types correspond to certain indecomposables. (Most of the result in Section 11 were independently obtained in [25], [26].)

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Chapter I: Preliminaries

1. Elimination of quantifiers

We consider (unital) left modules over an associative ring R with 1.

R-modules are L_R-structures, where the language L_R contains 0, +, − and a unary function symbol for every $r \in R$.

(The reader is invited to follow a suggestion of G. Cherlin: Look at abelian groups, endowed not only with a family of endophisms but also with a family of additive relations. Most of our general results remain valid. We make use of it in an example: 5.7, 9.8.)

Definition. An equation is a formula

$$r_1x_1 + r_2x_2 + \cdots + r_mx_m \doteq 0.$$

A *positive primitive* formula (ppf) has the form

$$\exists \mathbf{y}\ (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n),$$

where the γ_i are equations. ($\mathbf{y}$ stands for a finite sequence $y_1, \ldots, y_n$ of variables.)

The main result of this section is the following theorem of Baur [1] and Monk [14]:

Theorem 1.1. *For every module M, every L_R-formula is equivalent to a boolean combination of positive primitive formulas.*

We list some remarks on pp-formulas.

(1) If R is a principal ideal domain or a valuation ring, the 'Elementarteilersatz' implies that every ppf is equivalent to a conjunction of formulas of type $\exists y\ ry \doteq r_1x_1 + \cdots + r_mx_m$.

(2) We assume the class of positive primitive formulas to be closed under $\wedge$.

(3) The validity of ppfs is preserved under extensions, products and homomorphisms.

(4) A ppf $\varphi(x_1, \ldots, x_n)$ defines a subgroup $\varphi(M^n)$ of M^n:

$$M \vDash \varphi(\mathbf{0}), \qquad M \vDash \varphi(\mathbf{x}) \wedge \varphi(\mathbf{y}) \rightarrow \varphi(\mathbf{x} - \mathbf{y})$$

If R is commutative $\varphi(M^n)$ is a submodule of M^n. $\varphi(M)$ is called a *pp-definable* subgroup of M. (pp-definable subgroups were introduced in [34] as 'endlich matrizielle Untergruppen'.)

Lemma 1.2. *Let $\varphi(x, \mathbf{y})$ be a pp-formula and $\boldsymbol{a} \in M$. Then $\varphi(M, \boldsymbol{a})$ is empty or a coset of $\varphi(M, \mathbf{0})$. ($\boldsymbol{a}$ stands for a finite sequence $a_1, \ldots, a_n$ of elements of M.)*

Proof. $M \models \varphi(x, \boldsymbol{a}) \to (\varphi(y, \mathbf{0}) \leftrightarrow \varphi(x+y, \boldsymbol{a}))$.

Corollary 1.3. *Let $\boldsymbol{a}, \boldsymbol{b} \in M$, $\varphi(x, \mathbf{y})$ a ppf. Then (in M) $\varphi(x, \boldsymbol{a})$ and $\varphi(x, \boldsymbol{b})$ are equivalent or contradictory.*

Note. The pp-definable subgroups are closed under $\cap$ and $+$. If $\varphi(x)$, $\psi(x)$ are ppf, we write

$$\varphi \cap \psi = \varphi \wedge \psi,$$
$$\varphi + \psi = \exists y, z\, \varphi(y) \wedge \psi(z) \wedge y + z \doteq x.$$

By $\varphi \subset \psi$ we mean that $\vdash \varphi(x) \to \psi(x)$.

For the proof of 1.1 we need two further lemmas:

Lemma 1.4 (B.H. Neumann). *Let H_i denote abelian groups. If $H_0 + a_0 \subset \bigcup_{i=1}^{n} H_i + a_i$ and $H_0/(H_0 \cap H_i)$ is infinite for $i > k$, then $H_0 + a_0 \subset \bigcup_{i=1}^{k} H_i + a_i$.*

Lemma A (for sets A_i). *If A_0 is finite, then $A_0 \subset \bigcup_{i=1}^{k} A_i$ iff*

$$\sum_{\Delta \subset \{1,\ldots,k\}} (-1)^{|\Delta|} \left| A_0 \cap \bigcap_{i \in \Delta} A_i \right| = 0. \qquad \text{(Easy)}$$

Proof of Theorem 1.1. Fix M. We have to show: If $\psi(x, \mathbf{y})$ is in M equivalent to a boolean combination of ppf, then also $\forall x\, \psi$ is. Since ppf are closed under conjunction, ψ is M-equivalent to a conjunction of formulas

$$\varphi_0(x, \mathbf{y}) \to \varphi_1(x, \mathbf{y}) \vee \cdots \vee \varphi_n(x, \mathbf{y}), \qquad \varphi_i \text{ ppf}.$$

We can assume that already ψ has this form.

Let $H_i = \varphi_i(M, \mathbf{0})$. By 1.2 the $\varphi_i(M, \mathbf{y})$ are empty or cosets of H_i. (Think of $\mathbf{y}$ as being fixed in M.) Let $H_0/(H_0 \cap H_i)$ be finite for $i = 1, \ldots, k$ and infinite for $i = k+1, \ldots, n$ ($k \geq 0$). By 1.4

$$M \models \forall x\, \psi \leftrightarrow \forall x\, (\varphi_0(x, \mathbf{y}) \to \varphi_1(x, \mathbf{y}) \vee \cdots \vee \varphi_k(x, \mathbf{y})).$$

We apply Lemma A to the sets $A_i = \varphi_i(M, \mathbf{y})/(H_0 \cap \cdots \cap H_k)$: $\varphi_0(M, \mathbf{y}) \cap \bigcap_{i \in \Delta} \varphi_i(M, \mathbf{y})$ is empty or consists of N_Δ cosets of $H_0 \cap \cdots \cap H_k$, where

$$N_\Delta = \left| H_0 \cap \bigcap_{i \in \Delta} H_i/(H_0 \cap \cdots \cap H_k) \right|.$$

Whence

$$M \models \forall x\, \psi \leftrightarrow \sum_{\Delta \in \mathcal{N}} (-1)^{|\Delta|} N_\Delta = 0,$$

where

$$\mathcal{N} = \left\{ \Delta \subset \{1, \ldots, k\} \mid \exists x \left(\varphi_0(x, \boldsymbol{y}) \wedge \bigwedge_{i \in \Delta} \varphi_i(x, \boldsymbol{y}) \right) \right\}.$$

The resulting formula depends only on the indices N_Δ. Since pp-sentences are always true, the above proof shows:

Corollary 1.5 (Monk [14]). *M_1 and M_2 are elementarily equivalent iff*

$$\varphi/\psi(M_1) = \varphi/\psi(M_2) \quad \textit{for all ppf } \psi \subset \varphi.$$

(*Notation:* $\varphi/\psi(M) = (\varphi(M) : \psi(M)) \bmod \infty$. We assume $\varphi/\psi(M)$ to be a natural number or $=\infty$. *Convention*: $n \cdot \infty = \infty \cdot n = \infty$ $(n \geqslant 1)$ etc.)

Definition. M is a *pure* submodule of N, if $M \subset N$ and

$$N \models \varphi(\boldsymbol{a}) \Leftrightarrow M \models \varphi(\boldsymbol{a}) \quad \text{for all ppf } \varphi \text{ and } \boldsymbol{a} \in M.$$

Examples. $M \prec N$, M a direct factor of N.

Corollary 1.6 (Sabbagh [29]). *M is an elementary substructure of N iff M is pure in N and elementarily equivalent to N.*

Proof. Since $M \equiv N$, every L_R-formula is – in M and in N – equivalent to the same boolean combination of ppfs.

Corollary 1.7. *Suppose $L \subset M \subset N$. If $L \prec N$ and M pure in N, then $M \prec N$.*

Proof. $\varphi/\psi(L) \leqslant \varphi/\psi(M) \leqslant \varphi/\psi(N)$ by pureness, whence $M \equiv L \equiv N$.

Corollary 1.8. *Let κ be an infinite cardinal. Denote by $\prod_{i \in I}^{\kappa} M_i$ the product of the M_i restricted to sequences with $<\kappa$ members $\neq 0$. Then $\prod_{i \in I}^{\kappa} M_i \prec \prod_{i \in I} M_i$.*

Proof. $\prod_{i \in I}^{\kappa} M_i$ is the directed union of the modules $\prod_{i \in J} M_i$, $|J| < \kappa$, which are direct factors of $\prod_{i \in I} M_i$. Whence $\prod_{i \in I}^{\kappa} M_i$ is pure $\prod_{i \in I} M_i$. One computes easily

$$\varphi\left(\prod_{i \in I}^{\kappa} M_i\right) = \prod_{i \in I}^{\kappa} \varphi(M_i).$$

Whence

$$\varphi/\psi\left(\prod_{i \in I} M_i\right) = \prod_{i \in I} \varphi/\psi(M_i) = \varphi/\psi\left(\prod_{i \in I}^{\kappa} M_i\right).$$

We conclude this section with the introduction of a notion which will ease some later computations

Definition. "$\varphi/\psi \leqslant \bar{\varphi}/\bar{\psi}$" is the smallest transitive relation between pairs $\psi \subset \varphi$, $\bar{\psi} \subseteq \bar{\varphi}$ of pp-formulas s.t. $\bar{\psi} \subset \psi \subset \varphi \subset \bar{\varphi} \Rightarrow \varphi/\psi \leqslant \bar{\varphi}/\bar{\psi}$ and $(\psi+\delta)/\psi \leqslant \delta/(\psi \cap \delta) \leqslant (\psi+\delta)/\psi$.

Clearly

Lemma 1.9. *$\varphi/\psi \leqslant \bar{\varphi}/\bar{\psi}$ implies $\varphi/\psi(M) \leqslant \bar{\varphi}/\bar{\psi}(M)$.*

2. The stability classification

We use the results of Section 1 to determine totally transcendental and superstable modules by means of their pp-definable subgroups. (See [17] for definitions.) We apply this in Sections 10 and 11.

Theorem 2.1. ((1) is due to Fisher, (2) & (3) to Macintyre and Garavaglia [9].)

(1) *All modules are stable.*

(2) *M is totally transcendental iff there is no infinite descending sequence of pp-definable subgroups of M.*

(3) *M is superstable iff there is no infinite descending sequence of definable subgroups of M, each of infinite index in its predecessor.*

Proof. Let B be a subset of M. By $S_M(B)$ we denote the set of all complete types over B which are realized in M, i.e. the set of all

$$\text{tp}(a/B) = \{\Phi(x, \boldsymbol{b}) \mid \boldsymbol{b} \in B,\ \Phi \text{ a formula},\ M \models \Phi(a, \boldsymbol{b})\}.$$

N is *stable in* λ if $|S_M(B)| \leqslant \lambda$ for all $M \equiv N$, $|B| \leqslant \lambda$.

N is *stable* if N is stable in some infinite cardinal.

N is *superstable* if there is an infinite cardinal μ s.t. N is stable in all $\lambda \geqslant \mu$.

N is *totally transcendental* if $N \upharpoonright R_0$ is stable in $\aleph_0$ for all countable subrings $R_0 \subset R$.

For fixed M every $\text{tp}(a/B)$ is axiomatized by

$$\text{tp}^{\pm}(a/b) = \text{tp}^{+}(a/B) \cup \text{tp}^{-}(a/B),$$

where

$$\text{tp}^{+}(a/b) = \{\varphi(x, \boldsymbol{b}) \mid \varphi \text{ ppf},\ M \models \varphi(a, \boldsymbol{b})\}$$

and

$$\text{tp}^{-}(a/B) = \{\neg\varphi(x, \boldsymbol{b}) \mid \varphi \text{ ppf},\ M \models \neg\varphi(a, \boldsymbol{b})\}, \qquad (1.1).$$

Clearly tp^{-} is determined by tp^{+}.

Proof of (1). By 1.3, $\text{tp}^{+}(a/B)$ contains – up to equivalence – at most one formula $\varphi(x, \boldsymbol{b})$ for every ppf $\varphi(x, \mathbf{y})$. Whence $\mathrm{tp}^+(a/B)$ is determined by a partial map F: ppf $\to B^\omega$ in the sense that it is axiomatized by $\{\varphi(x, F(\varphi)) \mid \varphi \text{ ppf}\}$. We have

$$|S_M(B)| \leq (|B| + \aleph_0)^{|R| + \aleph_0}.$$

Thus N is stable in every λ s.t. $\lambda = \lambda^{|R| + \aleph_0}$.

Proof of (2). Suppose there is no infinite descending sequence of pp-definable subgroups in M. Then every type $\mathrm{tp}^+(a/B)$ contains a formula $\varphi(a, \boldsymbol{b})$ with minimal $\varphi(M, \mathbf{0})$. It follows that

$$\mathrm{tp}^+(a/B) = \{\psi(x, \boldsymbol{b}') \mid \varphi(M, \boldsymbol{b}) \subset \psi(M, \boldsymbol{b}'), \psi \text{ ppf}\}.$$

Therefore $|S_M(B)| \leq |B| + |R| + \aleph_0$ (=number of formulas $\varphi(x, \boldsymbol{b})$).

For the converse let $\varphi_i(M)$ be a proper descending sequence of pp-definable subgroups of M. Choose $a_i \in \varphi_i(M) \backslash \varphi_{i+1}(M)$. The types

$$p_\eta(x) = \{x \in b_i^\eta + \varphi_i(M) \mid i \in \omega\}, \qquad \eta \in {}^\omega 2$$

where $b_i^\eta = \sum_{j<i} \eta(j) a_j$ are (in M) consistent and pairwisely contradictory. Whence

$$|S_{M \restriction R_0}(\{a_0, a_1, \ldots\})| \geq 2^{\aleph_0}$$

if L_{R_0} contains the φ_i.

Proof of (3). Suppose there is no infinite descending sequence as in (3). Then every type $\mathrm{tp}^+(a/B)$ contains a $\varphi(x, \boldsymbol{b})$ s.t. $\varphi(M, \mathbf{0})$ is minimal w.r.t infinite index.

$\mathrm{tp}^+(a/B)$ can be axiomatized by formulas $\psi(x, \boldsymbol{b}')$ s.t. $\psi(M, \mathbf{0}) \subset \varphi(M, \mathbf{0})$. But there is only a finite number of nonequivalent $\psi(x, \boldsymbol{b}')$ for every ppf ψ, where $\psi(M, \mathbf{0})$ is of finite index in $\varphi(M, \mathbf{0})$. Whence

$$|S_M(B)| \leq (|B| + |R| + \aleph_0) + 2^{|R| + \aleph_0},$$

for we have $|B| + |R| + \aleph_0$ many choices of $\varphi(x, \boldsymbol{b})$ and then $2^{|R| + \aleph_0}$ many choices of the $\psi(x, \boldsymbol{b}')$. Thus M is stable in all $\lambda \geq 2^{|R| + \aleph_0}$.

For the converse let $\varphi_i(M)$ be an infinite descending sequence s.t. $\varphi_{i+1}(M)$ is of infinite index in $\varphi_i(M)$. Choose $a_i^j \in \varphi_i(M)$, $j \in \omega$, pairwisely inequivalent mod $\varphi_{i+1}(M)$, $i = 0, 1, \ldots$, and define

$$p_\eta(x) = \{\varphi_i(x - b_i^\eta) \mid i \in \omega\}, \qquad \eta \in {}^\omega\omega$$

where $b_i^\eta = \sum_{j<i} a_j^{\eta(j)}$. The proof of [17; II 3.5 (5) $\Rightarrow$ (2)] shows that M is not superstable.

Corollary 2.2. (1) *Let M be a pure submodule of N. Then N is totally transcendental (superstable) if M and N/M are totally transcendental (superstable).*

(2) *If M is totally transcendental, M^κ is totally transcendental.*

(3) *If M^κ is superstable, M is totally transcendental* ($\kappa \geq \aleph_0$).

Proof. (1) First we derive two formulas:

$$\varphi(N/M) = \varphi(N) + M/M, \qquad \varphi \text{ ppf}. \tag{1}$$

'$\supset$' is clear. Let

$$\varphi(x) = \exists \mathbf{y} \bigwedge_{i<n} t_i(x, \mathbf{y}) \doteq 0,$$

where the t_i are linear expressions in x, $\mathbf{y}$. If $a + M$ is in $\varphi(N/M)$, there are $m_i \in M$ s.t. $\exists \mathbf{y} \bigwedge_{i<n} t_i(a, \mathbf{y}) \doteq m_i$ holds in N. Since M is pure in N, there is $b \in M$ s.t. $\exists \mathbf{y} \bigwedge_{i<n} t_i(b, \mathbf{y}) \doteq m_i$ holds in N. Now $a - b \in \varphi(N)$ and $a \in \varphi(N) + M$.

$$\varphi/\psi(N) = \varphi/\psi(M) \cdot \varphi/\psi(N/M), \qquad \psi \subset \varphi \text{ ppfs.} \tag{2}$$

Look at the following isomorphisms:

$$\begin{aligned}\varphi(N/M)/\psi(N/M) &\cong (\varphi(N) + M)/(\psi(N) + M) \\ &\cong \varphi(N)/((\psi(N) + M) \cap \varphi(N)) = \varphi(N)/(\psi(N) + \varphi(M)) \\ &\cong (\varphi(N)/\psi(N))/(\psi(N) + \varphi(M)/\psi(N)).\end{aligned}$$

But

$$(\psi(N) + \varphi(M))/\psi(N) \cong \varphi(M)/\psi(M).$$

We can assume that a descending sequence of pp-definable subgroups is always given by a sequence $\varphi_0 \supset \varphi_1 \supset \varphi_2 \supset \cdots$. (For otherwise we replace φ_i by $\varphi_0 \wedge \varphi_1 \wedge \cdots \wedge \varphi_i$.)

Now (2) shows, that if $\varphi_i(M)$ or $\varphi_i(N/M)$ is proper descending (with infinite indices), then also $\varphi_i(N)$ is proper descending (with infinite index). Conversely, if $\varphi_i(N)$ is proper descending (with infinite index) $\varphi_i(M)$ or $\varphi_i(N/M)$ contains such a subsequence.

(2 & 3) $\varphi_i(M)$ is proper descending iff $\varphi_i(M^\kappa)$ is proper descending iff $\varphi_i(M^\kappa)$ is proper descending with infinite indices.

Lemma 2.3. *If N is superstable and M an elementary submodule of N, N/M is totally transcendental.*

Proof. N/M is superstable. Since $\varphi/\psi(M) = \varphi/\psi(N) = \varphi/\psi(M) \cdot \varphi/\psi(N/M)$, we have $\varphi/\psi(N/M) > 1 \Rightarrow \varphi/\psi(M) = \infty$. Whence a proper descending sequence of pp-definable subgroups of N/M yields a descending sequence of pp-definable subgroups of N with infinite indices.

Chapter II: Decomposition of compact modules

3. Algebraically compact modules

Definition. A module M is algebraically compact (we say 'compact') if every homomorphism from a pure submodule N' of a module N to M can be extended to a homomorphism from N to M.

$$\begin{tikzcd} N \arrow[dr, dashed] & \\ N' \arrow[u, "\text{pure}"] \arrow[r] & M \end{tikzcd}$$

Notation. a pp-type is a type consisting of pp-formulas.

Theorem 3.1. *For every Module M the following are equivalent:*

(a) *M is a direct factor in every pure extension.*

(b) *Every consistent pp-type $p(x)$ over $A \subset M$, $|A| \leq |R| + \aleph_0$, is realized in M.*

(c) *Every consistent pp-type $p(x^I)$ over M is realized in M.*

(d) *M is compact.*

Proof. (d) $\rightarrow$ (a). Let N be a pure extension of M. Apply the definition to the diagram

$$\begin{tikzcd} N & \\ M \arrow[u] \arrow[r, "\text{id}"'] & M \end{tikzcd}$$

to obtain $h : N \rightarrow M$. Then $N = M \oplus \operatorname{Ker} h$.

(a) $\rightarrow$ (c). Let a^I realize p in an elementary extension N of M. Since M is a direct factor of N there is a projection $\pi : N \rightarrow M$, $\pi \restriction M = \mathrm{id}_M$. Then $\pi(a^I)$ realizes p in M.

(c) $\rightarrow$ (b). Clear.

(b) $\rightarrow$ (d). We note first that by 1.3, (b) implies that every consistent pp-type $p(x)$ over M is realized in M. Let N' be a pure submodule of N and g a homomorphism from N' to M. Then g is a 'partial homomorphism' from N to M in the following sense:

Definition. A partial mapping f from N to M which preserves pp-formulas (and negations of pp-formulas), i.e.

$$N \models \varphi(\boldsymbol{a}) \underset{(\Leftrightarrow)}{\Rightarrow} M \models \varphi(f(\boldsymbol{a})), \qquad \varphi \text{ ppf}, \quad \boldsymbol{a} \in \operatorname{dom} f,$$

is called a *partial homomorphism* (*isomorphism*) from N to M.

Remark 3.2. If $\operatorname{dom} f$ is a pure submodule of N, then f is a partial homomorphism (isomorphism) iff f is a homomorphism (isomorphism onto a pure submodule of M).

Now, to prove (d), let $f: A \to M$ be a maximal extension of g to a partial homomorphism from N to M. Let $b \in N$ be arbitrary. The pp-type

$$p(x) = f(\mathrm{tp}^+(b/A)) = \{\varphi(x, f(\boldsymbol{a})) \mid N \vDash \varphi(b, \boldsymbol{a}), \boldsymbol{a} \in A, \varphi \text{ pp}\}$$

is consistent in M. For if $\varphi_i(x, f(\boldsymbol{a}))$, $i < n$ are in p, then

$$N \vDash \exists x \bigwedge_{i<n} \varphi_i(x, \boldsymbol{a}) \quad \text{and therefore} \quad M \vDash \exists x \bigwedge_{i<n} \varphi_i(x, f(\boldsymbol{a})).$$

Let $c \in M$ realize p. Then $f' = f \cup \{\langle b, c\rangle\}$ extends f. Thus $b \in A$.

Corollary 3.3. (1) *M is compact iff every partial homomorphism from N to M can be extended to a homomorphism from N to M.*

(2) *$(|R| + \aleph_0)^+$-saturated modules are compact.*

Examples of compact modules are injective modules, finite modules, modules with a compact topology which is compatible with the operations.

Lemma 3.4. (1) *Direct summands of compact modules are compact.*

(2) *If the M_i are compact and* cf $\kappa > |R| + \aleph_0$, *then $\prod_{i\in I}^{\kappa} M_i$ is compact.*

Proof. (1) As the proof of 3.1 (a) $\to$ (c).

(2) Clearly $\prod_{i\in I} M_i$ is compact. But a pp-type with at most $|R| + \aleph_0$ many parameters is already defined over a direct factor $\prod_{i\in J} M_i$, $|J| < \kappa$.

Lemma 3.5 (Garavaglia [8], Zimmermann [34]). (1) *Totally transcendental modules are compact.*

(2) *M is totally transcendental iff $M^{\aleph_0}$ is compact.*

Proof. (1) Every pp-type over a totally transcendental module which is closed under conjunction is principal (see the proof of 2.1(2)).

(2) If M is totally transcendental, then $M^{\aleph_0}$ is totally transcendental (1.8, 2.2(2)), and compact.

If M is not totally transcendental, choose an infinite descending sequence $\varphi_i(M)$ of pp-definable subgroups and $a_i \in \varphi_i(M) \setminus \varphi_{i+1}(M)$.

Define $b^i \in M^{\aleph_0}$ by $b^i = (a_0, a_1, \ldots, a_{i-1}, 0, 0, 0, \ldots)$. Then $p(x) = \{\varphi_i(x - b^i) \mid i \in \omega\}$ is consistent but not realized in $M^{\aleph_0}$.

Remark. A compact module which is not totally transcendental is of power at least $2^{\aleph_0}$. (For the types constructed in the proof of 2.1(2) are pp.)

Definition. Let A be a subset of the compact module M. $H(A)$ is called a *hull* of A in M if

(a) $H(A)$ is a compact pure submodule of M containing A.

(b) If B is a compact pure submodule of M and $A \subset B \subset H(A)$, then $B = H(A)$.

The following theorem is due to Fisher (unpublished, but see [3]).

Theorem 3.6. *Let A be a subset of the compact module M. Then there is a hull $H(A)$ of A in M. $H(A)$ is unique in the following sense: Let $H(B)$ be a hull of B in N. Then any partial isomorphism $f: A \twoheadrightarrow B$ from M to N can be extended to an isomorphism from $H(A)$ to $H(B)$.*

Notation. The theorem makes the following notation possible: We call a type of the form $p(x) = \mathrm{tp}^{\pm}(a/0)$ $(a \in M)$ *pp-complete*. Then $H(a)$ is – up to isomorphy – determined by p. We write $H(a) = H(p)$.

Proof. To satisfy (b), $H(A)$ has to be small over A in the sense of the

Definition. Let $A \subset B$ be subsets of the module M. B is *small* over A if every partial homomorphism $f: B \to N$ from M to N whose restriction to A is a partial isomorphism is a partial isomorphism.

We construct small extensions using the following characterisation.

Lemma 3.7. *B is small over A iff $\mathrm{tp}^+(\boldsymbol{b}/A) \vdash_M \mathrm{tp}(\boldsymbol{b}/A)$ for every finite sequence* $\boldsymbol{b} \in B$. (See proof of 2.1 for notation.)

Proof of 3.7. Assume $\mathrm{tp}^+(\boldsymbol{b}/A) \nvdash_M \mathrm{tp}(\boldsymbol{b}/A)$. Then there is $\neg\varphi(\boldsymbol{x}, \boldsymbol{a}) \in \mathrm{tp}^-(\boldsymbol{b}/A)$ s.t. $p(x) = \mathrm{tp}^+(\boldsymbol{b}/A) \cup \{\varphi(\boldsymbol{x}, \boldsymbol{a})\}$ is consistent in M. Realize p in a compact elementary extension N of M, by $\boldsymbol{c}$. $g = \mathrm{id}_A \cup \{\langle \boldsymbol{b}, \boldsymbol{c}\rangle\}$ is a partial homomorphism from M to N, extend it to a partial homomorphism f defined on B. f is not a partial isomorphism – for $M \models \neg\varphi(\boldsymbol{b}, \boldsymbol{a})$, $N \models \varphi(\boldsymbol{c}, \boldsymbol{a})$ – but partially isomorphic on A. Thus B is not small over A.

Assume that the condition of 3.7 is satisfied. Let $f: B \twoheadrightarrow C$ be a partial homomorphism from M to N which is partially isomorphic on A. Extend the partial homomorphism $f^{-1} \upharpoonright f(A)$ to a partial homomorphism g from N to an elementary extension M' of M with $\mathrm{dom}\, g = C$.

For all $\boldsymbol{b} \in B$ we have $\mathrm{tp}^+(gf(\boldsymbol{b})/A) \supset \mathrm{tp}^+(\boldsymbol{b}/A)$ and therefore $\mathrm{tp}(gf(\boldsymbol{b})/A) = \mathrm{tp}(\boldsymbol{b}/A)$. Thus gf and f are partial isomorphisms. This shows that B is small over A.

Corollary 3.8. *$A \cup \{b_1, \ldots, b_n\}$ is small over A iff* $\mathrm{tp}^+(\boldsymbol{b}/A) \vdash_M \mathrm{tp}(\boldsymbol{b}/A)$.

Proof of existence (3.6). By 3.7 we can use Zorn's lemma to obtain a maximal small extension $H(A)$ of A inside M. Property (b) is already clear: If B is a compact pure submodule of M lying between A and $H(A)$, look at the projection π of M onto the direct factor B. π is partially isomorphic on A and therefore on $H(A)$. This is only possible if $H(A)=B$.

That $H(A)$ is a compact pure submodule of M is the same as to say that every M-consistent pp-type $p(\boldsymbol{x})$ over $H(A)$ is realized by an element of $H(A)$. Let p be given. Choose a maximal pp-type q over $H(A)$ which extends p and is consistent in M. Let $\boldsymbol{b}\in M$ be a realization of q. By maximality $q\vdash_M \mathrm{tp}(\boldsymbol{b}/H(A))$, thus $H(A)\cup\{\boldsymbol{b}\}$ is small over $H(A)$ and therefore small over A. We conclude that $\boldsymbol{b}\in H(A)$.

Since we can do the above construction inside a hull of A, we can conclude, that all hulls of A are small over A. The arguments we gave up to now prove the following

Corollary 3.10. *B is a hull of A in M iff B is a maximal small extension of A in M iff B is small over A, compact and pure in M.*

Proof of uniqueness (3.6). Let $H(B)$ a hull of B in N and $f: A \twoheadrightarrow B$ a partial isomorphism from M to N. Since $H(B)$ is pure in N, f is also a partial isomorphism from M to $H(B)$. Extend f to a partial homomorphism $g: H(A)\to H(B)$ from M to $H(B)$ (which is compact). Since $H(A)$ is small over A, g is a partial isomorphism. Thus since $H(A)$ is pure and compact in M, $g(H(A))$ must be pure and compact in $H(B)$, whence $g(H(A))=H(B)$.

Corollary 3.11. (1) $|H(A)|\leq(|A|+1)^{|R|+\aleph_0}$.

(2) *If M is totally transcendental, then* $|H(A)|\leq|A|+|R|+\aleph_0$.

Proof. (1) We find an $(|R|+\aleph_0)^+$-saturated N s.t. $A\subset N$, $(N,a)_{a\in A}\equiv(M,a)_{a\in A}$ and $|N|\leq(|A|+1)^{|R|+\aleph_0}$. But $H_M(A)\cong H_N(A)$.

(2) Choose $A\subset N\prec M$, $|N|\leq|A|+|R|+\aleph_0$. Since N is compact (3.5(1)), we find $H(A)\subset N$.

Definition. $\bar{M}$ is a *pure* hull of M if:

(a) $\bar{M}$ is a pure compact extension of M.

(b) If N is a compact pure extension of M, $\bar{M}$ is – over M – isomorphic to a pure submodule of N.

Thus, if M is compact, it is its own pure hull.

Theorem 3.12 (Warfield [30]). *Every module M has a unique pure hull $\bar{M}$.*

Proof. Let K be a compact elementary extension of M. Set $\bar{M}=H_K(M)$. M is pure in $\bar{M}$, since M is pure in K.

If N is a compact pure extension of M, id_M is a partial isomorphism from K to N, whence $\bar{M}$ is isomorphic to $H_N(M)$. This shows that $\bar{M}$ is a pure hull of M.

If N happens to be another pure hull of M, N is isomorphic to a pure submodule B of $\bar{M}$, $M \subset B$. Since B is compact, $B = \bar{M}$.

Corollary 3.13. *B is the pure hull of M iff M is pure in B, B is compact and small over M.*

Corollary 3.14 (Sabbagh [31]). *$\bar{M}$ is an elementary extension of M.*

Proof. By the proof of 3.12 and 1.7.

4. Indecomposable modules

Definition. A non-zero compact module U is *indecomposable* if U is not the direct sum of two non-zero modules.

Lemma 4.1. *Let U be non-zero and compact. Then U is indecomposable iff $U = H(a)$ for all $a \in U \setminus 0$.* ($H(A)$ is defined before 3.6.)

Proof. If $U = M \oplus N$ is a nontrivial decomposition and $a \in M \setminus 0$, then U is not the hull of a, since M is compact and pure in U.

If $a \in U \setminus 0$, $H(a)$ is a nontrivial direct factor of U.

Corollary 4.2. (1) *There are at most $2^{|R|+\aleph_0}$ non-isomorphic indecomposable R-modules.*

(2) *An indecomposable module has power at most $2^{|R|+\aleph_0}$.*

(3) *If U is indecomposable and totally transcendental, $|U| \leq |R| + \aleph_0$.*

Proof. (1) Every indecomposable is of the form $H(p)$ (see 3.6 for notation).

(2) By 3.11(1).

(3) By 3.11(2).

The following characterization of indecomposables is due to Zimmermann & Zimmermann-Huisgen. We give a new selfcontained proof.

Theorem 4.3. *A non-zero compact module U is indecomposable iff its endomorphism-ring is local, i.e. for all $f \in \mathrm{End}(U)$ $1-f$ or f is an automorphism of U.*

Proof. If U is a nontrivial direct sum, the two projections satisfy $1 = \pi_1 + \pi_2$ but neither π_1 nor π_2 is an automorphism. Whence $\mathrm{End}(U)$ is not local.

Suppose now that U is indecomposable and $f \in \mathrm{End}(U)$ s.t. $1-f$ is not an automorphism. We have to show that f is an automorphism. Choose $a \in U \setminus 0$. Then $g \in \mathrm{End}(U)$ is an automorphism of U iff $g \upharpoonright a$ is a partial isomorphism from U to U. For, if $g \upharpoonright a$ is partially isomorphic, g is a partial isomorphism (U is small over a). Then $g(U)$ is a pure compact submodule of U and must therefore equal U. Let $f(a) = b$. By our assumption $1-f$ is not isomorphic on a. Whence there is a ppf φ s.t.

$$U \nvDash \varphi(a), \qquad U \vDash \varphi(a-b).$$

Let now $p(x)$ be a pp-type which extends $\mathrm{tp}^+(b)$, is consistent with $\varphi(a-x)$ and maximal with this property. Let $c \in U$ realize $p(x) \cup \{\varphi(a-x)\}$, it follows $c \neq 0$.

Since $\mathrm{tp}^+(a) \subset \mathrm{tp}^+(b) \subset \mathrm{tp}^+(c)$, there is an endomorphism g of U which extends the partial homomorphism $a \mapsto c$. The equation

$$a - g(c) = (g+1)(a-c)$$

together with $U \vDash \varphi(a-c)$ implies $U \vDash \varphi(a - g(c))$. Therefore $g(c)$ again realizes $p(x) \cup \{\varphi(a-x)\}$.

By the maximal choice of p we have

$$\mathrm{tp}^+(c) = \mathrm{tp}^+(g(c)) = p.$$

Thus – since $U = H(c)$ – our above remark shows that g is an automorphism. But then

$$\mathrm{tp}^+(b) \subset \mathrm{tp}^+(c) = \mathrm{tp}^+(a) \subset \mathrm{tp}^+(b),$$

i.e. also f is an automorphism.

Definition. Let $p(x)$ be a pp-complete type. We call p *indecomposable* if $H(p)$ is indecomposable. (See 3.6 for notation.)

Note that every indecomposable has the form $H(p)$.

The next theorem is a translation of 4.3 to characterize indecomposable types.

Theorem 4.4. *A pp-complete type p is indecomposable iff* $(x \doteq 0) \notin p$ *and for all pp-formulas* ψ_1, ψ_2 *not in p there is a pp-formula* $\varphi \in p$ *s.t.*

$$(\psi_1 \cap \varphi) + (\psi_2 \cap \varphi) \notin p.$$

Example. If the set of pp-definable subgroups of M is linearly ordered, pp-complete types which do not contain $x \doteq 0$ are indecomposable.

Proof. Let $H(p) = H(a)$, where $\mathrm{tp}^{\pm}(a) = p$.

Assume first that p does not satisfy the above condition. If $x \doteq 0 \in p$, we have $H(a) = 0$ and p is not indecomposable. Otherwise there are $\psi_1, \psi_2 \notin p$ s.t. for all

$\varphi \in p$

$$(\psi_1 \cap \varphi) + (\psi_2 \cap \varphi) \in p.$$

Then $\{a \doteq x + y, \psi_1(x), \psi_2(y)\} \cup p^+(x) \cup p^+(y)$ is consistent in $H(a)$. Choose a realization b, c and an endomorphism f of $H(a)$ s.t. $f(a) = b$. Since $H(a) \models \psi_1(f(a))$, f is not an automorphism, since $H(a) \models \psi_2((1-f)(a))$, $1-f$ is not an automorphism. Thus $\mathrm{End}(H(a))$ is not local.

Now assume that $\mathrm{End}(H(a))$ is not local. We have $f \in \mathrm{End}(H(a))$ s.t. neither f nor $1-f$ is an automorphism. Then $f \restriction a$ and $(1-f) \restriction a$ are not partially isomorphic. Thus there are $\psi_1, \psi_2 \notin p$ s.t. $H(a) \models \psi_1(f(a))$ and $H(a) \models \psi_2((1-f)(a))$. The equation $a = f(a) + (1-f)(a)$ shows that

$$H(a) \models ((\psi_1 \cap \varphi) + (\psi_2 \cap \varphi))(a) \quad \text{for all } \varphi \in p.$$

Notation. Let $\psi(x)$, $\varphi(x)$ be a pair of pp-formulas s.t. $\psi \subset \varphi$. We say $\varphi/\psi \in p$ instead of $\varphi \in p$ & $\neg\psi \in p$.

Corollary 4.5. *Let Δ be a finite subset of the indecomposable type p. Then there is $\varphi/\psi \in p$ s.t. for all pp-formulas χ, $\chi \cap \varphi \subset \psi$ if $\neg\chi \in \Delta$ and $\varphi \subset \chi$ if $\chi \in \Delta$. Thus $\vdash \varphi \wedge \neg\psi \rightarrow \bigwedge \Delta$.*

Proof. Iterated application of 4.4. *Note*: $\varphi/\psi \leq \sigma/\chi$ for all $\sigma/\chi \in \Delta$.

Corollary 4.6. *Let p be indecomposable and $\bar{\varphi}/\bar{\psi}$, $\sigma/\chi \in p$. Then there is $\varphi/\psi \in p$ s.t. $\varphi/\psi \leq \sigma/\chi$ and $\bar{\psi} \subset \psi \subset \varphi \subset \bar{\varphi}$.*

Proof. Choose $\tilde{\varphi}/\tilde{\psi} \in p$ s.t. $\tilde{\varphi}/\tilde{\psi} \leq \sigma/\chi$ and $\bar{\psi} \cap \tilde{\varphi} \subset \tilde{\psi} \subset \tilde{\varphi} \subset \bar{\varphi}$. Set $\varphi = \bar{\psi} + \tilde{\varphi}$ and $\psi = \bar{\psi} + \tilde{\psi}$.

The next lemma shows that there are a lot of indecomposable types.

Lemma 4.7. *Let q be an M-consistent type consisting of ppfs and negations of ppfs. Suppose that $\neg x \doteq 0 \in q$ and that for all $\neg\psi_1, \neg\psi_2 \in q$ there is a $\varphi \in q$ s.t. $\neg(\psi_1 \cap \varphi + \psi_2 \cap \varphi) \in q$.*

Then we can construct an M-consistent indecomposable extension p of q as follows: Choose a maximal pp-type p^+ which is M-consistent with q. Set $p = p^+ \cup \{\neg\chi \mid \chi \notin p, \chi \text{ ppf}\}$.

Proof. Since $q \cup p^+ \vdash_M \neg\chi$ for all ppf $\chi \notin p^+$, p is an M-consistent pp-complete type. We show that p satisfies the condition of 4.4.

Let $\psi_1, \psi_2 \notin p$. Then there are $\bar{\varphi} \in p^+, \neg\bar{\psi}_1, \ldots, \neg\bar{\psi}_n \in q$ s.t.

$$M \models \bar{\varphi} \wedge \neg\bar{\psi}_1 \wedge \cdots \wedge \neg\bar{\psi}_n \rightarrow \psi_i, \quad i = 1, 2.$$

Choose $\tilde{\varphi}_i \in q$ s.t. $\neg\tilde{\psi}_i \in q$, where

$$\tilde{\psi}_1 = (\psi_1 \cap \tilde{\varphi}_1) + (\psi_2 \cap \tilde{\varphi}_1) \quad \text{and} \quad \tilde{\psi}_{i+1} = (\psi_{i+1} \cap \tilde{\varphi}_{i+1}) + (\tilde{\psi}_i \cap \tilde{\varphi}_{i+1}).$$

Set $\varphi = \bar{\varphi} \wedge \tilde{\varphi}_1 \wedge \cdots \wedge \tilde{\varphi}_n$ (ϵp). Then $M \models \varphi \wedge \neg\tilde{\psi}_n \to \neg\psi_i$. Whence $M \models \varphi \wedge \psi_i \to \tilde{\psi}_n$ and therefore $M \models (\varphi \wedge \psi_1) + (\varphi \wedge \psi_2) \to \tilde{\psi}_n$. This implies $(\varphi \wedge \psi_1) + (\varphi \wedge \psi_2) \notin p$.

Corollary 4.8. *Suppose $\varphi/\psi(M) > 1$. Then there in an M-consistent indecomposable type which contains φ/ψ.*

Proof. Take $q = \{\neg x \doteq 0, \neg\psi, \varphi\}$ in 4.7. (To be precise, close q under logical equivalence.)

Corollary 4.5 enables us to topologize the set $\mathbb{U}^R$ of all isomorphism types of indecomposable R-modules in the following manner: For every pair φ/ψ of pp-formulas set

$$(\varphi/\psi) = \{U \in \mathbb{U}^R \mid \varphi/\psi(U) > 1\}.$$

Theorem 4.9. *The sets (φ/ψ) form the basis of a topology on $\mathbb{U}^R$. $\mathbb{U}^R$ and all (φ/ψ) are quasicompact. If p is indecomposable and $\bar{\varphi}/\bar{\psi} \in p$, then the*

$$(\varphi/\psi), \qquad (\varphi/\psi \in p, \bar{\psi} \subset \psi \subset \varphi \subset \bar{\varphi})$$

form a basis for the neighbourhoods of $H(p)$.

Proof. Let $U = H(p) \in (\varphi_i/\psi_i)$ $(i = 1, 2)$, $\bar{\varphi}/\bar{\psi} \in p$. Suppose that $a \in U$ realize p and $U \models \varphi_i(a_i)$, $U \models \neg\psi_i(a_i)$. By 3.7, $\mathrm{tp}^+(a_i/a) \vdash_U \mathrm{tp}(a_i/a)$. Whence there are ppf $\rho_i(x, a) \in \mathrm{tp}^+(a_i/a)$ s.t.

$$\vdash_U \rho_i(x, a) \to \varphi_i(x) \wedge \neg\psi_i(x).$$

Thus the formulas $\exists y\ (\rho_i(y, x) \wedge \varphi_i(y))$ and $\forall y\ (\rho_i(y, x) \to \neg\psi_i(y))$ are in p. By 4.5 there is a pair $\sigma/\chi \in p$ s.t. $\sigma \wedge \neg\chi$ implies these formulas. We have then

$$U \in (\sigma/\chi) \subset (\varphi_i/\psi_i) \qquad (i = 1, 2).$$

When we choose φ/ψ as in 4.6, we have $U \in (\varphi/\psi) \subset (\sigma/\chi)$.

It remains to show that (φ/ψ) is quasicompact. Assume that no finite subset of $((\varphi_i/\psi_i))_{i \in I}$ covers (φ/ψ). Then by compactness the theory

$$\{\varphi/\psi > 1\} \cup \{\varphi_i/\psi_i = 1 \mid i \in I\} \cup \text{"}R\text{-module"}$$

is consistent. Let M be a model of it. By 4.8 there is an M-consistent indecomposable type q containing φ/ψ. Since $H(q)$ is a direct factor of an compact module elementarily equivalent to M, one sees that $H(q)$ is not contained in any of the (φ_i/ψ_i), $i \in I$. But $H(q) \in (\varphi/\psi)$.

Corollary 4.10. *The closed subsets of* $\mathbb{U}^R$ *are the sets*

$$\mathbb{U}_M = \{H(p) \mid p \ M\text{-consistent, indecomposable}\}.$$

Proof. By 4.5 and 4.9. If $\{U_i\}_{i\in I}$ is closed, set $M = \bigoplus_{i\in I} U_i$.

Corollary 4.11. *If* $\varphi/\psi(M) > 1$ *for all pairs* φ/ψ *in a base* $\mathcal{S}$ *of neighbourhoods of* U, *then* $U \in \mathbb{U}_M$.

Proof. Let $U = H(p)$, $\varphi/\psi \in p$. Choose $\sigma/\chi \in \mathcal{S}$ with $(\sigma/\chi) \subset (\varphi/\psi)$. The next remark implies $\varphi/\psi(M) > 1$. Thus p is M-consistent. We show, that $(\sigma/\chi) \subset (\varphi/\psi)$, $\sigma/\chi(M) > 1 \Rightarrow \varphi/\psi(M) > 1$: By 4.9 there is an indecomposable M-consistent type q with $\sigma/\chi \in q$. By assumption $H(q) \in (\varphi/\psi)$. Since $H(q)$ is a direct factor in a module elementarily equivalent to M, $\varphi/\psi(M) > 1$.

5. Some examples

We study indecomposable R-modules U in three special cases.

(1) R is commutative and for every maximal ideal $\mathfrak{M}$ is $R_{\mathfrak{M}}$ – the localization at $\mathfrak{M}$ – a field or a discrete valuation ring. (Examples are Dedekind rings or von Neumann regular rings.)

(2) (See the first remark in Section 1.) R is a Dedekind ring, $\mathbf{U} = (U, V)$ is an indecomposable pair consisting of a torsionfree U and a submodule V.

(3) U is injective

The following observation is well known.

Lemma 5.1. *Let A be a discrete valuation ring, $\mathfrak{M}$ its maximal ideal, K its quotient field. The indecomposable A-modules are $A/\mathfrak{M}^n$ $(n \geq 1)$, K/A, $\tilde{A}$ = the completion of A, K. These modules are pairwise non-isomorphic.*

Proof. Let $\mathfrak{M} = A \cdot p$. We note first that the 'Elementarteilersatz' implies that every pp-formula is equivalent to a conjunction of formulas

$$\exists y \, p^n y \doteq p^m x, \qquad p^m x \doteq 0.$$

(1) *The given modules are compact.* K and K/A are divisible, therefore injective and compact. $A/\mathfrak{M}^n$ has $n+1$ pp-definable subgroups and is therefore compact. The pp-definable subgroups of $\tilde{A}$ are $\tilde{A}p^n$ and 0. Thus if $\varphi_1(\tilde{A}, \boldsymbol{b}_1) \supsetneqq \varphi_2(\tilde{A}, \boldsymbol{b}_2) \supsetneqq \varphi_3(\tilde{A}, \boldsymbol{b}_3) \supsetneqq \cdots$ and $c_i \in \varphi_i(\tilde{A}, \boldsymbol{b}_i)$, (c_i) is a Cauchy sequence converging to an element which realizes $\{\varphi_i(x, \boldsymbol{b}_i) \mid i \in \omega\}$. This shows that $\tilde{A}$ is compact.

(2) *The given modules are indecomposable.* K and K/A have the following property: If x, y are non-zero, then there are $s, r \in A$ s.t. $sx = ry \neq 0$. This implies that K and K/A are indecomposable.

Let k be the residue class field $A/\mathfrak{M}$. For every A-module $M[p] =$

$\{x \in M \mid px = 0\}$ and M/pM are k-vector spaces. If M is $A/\mathfrak{M}^n$, $\dim_k(M[p]) = 1$. Thus, if $N_1 \oplus N_2$ is a decomposition of M, we have $\dim_k(N_1[p]) = 0$ (say). Then N_1 is torsionfree and must be 0.

If $M = \tilde{A}$, $\dim_k(M/pM) = 1$. Thus, if $N_1 \oplus N_2$ is a decomposition of M, we have e.g. $\dim_k(N_1/pN_1) = 0$. Then N_1 is divisible and must be 0. Thus $\tilde{A}$ is indecomposable and the pure hull of A.

(3) *All indecomposables occur.* Let U be an indecomposable A-module. Choose $a \in U \setminus 0$. We write

$$h(a) = \sup\{n \mid p^n \text{ divides } a \text{ (in } U)\} \qquad (\text{'height'})$$

and

$$\mathrm{Ann}(a) = \{r \in A \mid ra = 0\} \qquad (\text{'annihilator'}).$$

If $\mathrm{Ann}(a) = Ap^{n+1}$, we have $\mathrm{Ann}(p^n a) = \mathfrak{M}$. Whence we can assume that $\mathrm{Ann}(a) = \mathfrak{M}$ or $\mathrm{Ann}(a) = 0$.

Case 1: $h(a) = n$, $\mathrm{Ann}(a) = \mathfrak{M}$. Choose $b \in U$ s.t. $p^n b = a$. Then $Ab \cong A/\mathfrak{M}^{n+1}$ via $b \mapsto 1 + \mathfrak{M}^{n+1}$ and Ab is pure in U. Therefore $U = Ab \cong A/\mathfrak{M}^{n+1}$.

Case 2: $h(a) = \infty$, $\mathrm{Ann}(a) = 0$. Then a is uniquely divisible (in U) by all $r \in A \setminus 0$. Ka is a well-defined submodule of U and isomorphic to K. Therefore $U = Ka \cong K$.

Case 3: $h(a) = n$, $\mathrm{Ann}(a) = 0$. Choose $b \in U$ s.t. $p^n b = a$. $Ab \cong A$ is then pure in U. Whence $U = H(Ab) = \overline{Ab} \cong \tilde{A}$.

Case 4: $h(a) = \infty$, $\mathrm{Ann}(a) = \mathfrak{M}$. By compactness there is a sequence $a_0 = a$, $pa_{i+1} = a_i$, in U. Let M be the submodule generated by $\{a_i\}_{i \in \omega}$. Then $M \cong K/A$ via $a_i \mapsto 1/p^{i+1} + A$. Thus $U = M \cong K/A$.

Our proof shows that every compact non-zero A-module has an indecomposable direct factor. This follows also from the fact that every A-module has Krull dimension. See 7.3, 8.2.

Remark. Let R be a valuation ring, K its quotient field. A fractional ideal is an additive subgroup of K which is closed under multiplication with elements of R. The indecomposable R-modules are

$$\overline{A/B} = \text{pure hull of } A/B,$$

where $B \subsetneqq A$ are fractional ideals.

$$\overline{A/B} \text{ and } \overline{C/D} \text{ are isomorphic}$$

iff A/B and C/D are isomorphic iff $A = xC$, $B = xD$ $(x \in K \setminus 0)$.

Sketch of proof. Up to logical equivalence the pp-formulas $\varphi(x)$ are conjunction of formulas $\varphi_{a,b}(x) = \exists y\ ay \doteq bx$ $(a \in R,\ b \in R \setminus aR)$.

The only relations between these formulas are

$$\vdash \varphi_{a,b}(x) \to \varphi_{c,db}(x) \qquad (c \text{ divides } da).$$

Using 4.4 one sees that indecomposable types p correspond to pairs $\boldsymbol{a}$, $\boldsymbol{h}$ of proper ideals via

$$p(x)=\{\varphi_{0,b}(x)\mid b\in\boldsymbol{a}\}\cup\{\neg\varphi_{0,b}(x)\mid b\notin\boldsymbol{a}\}$$
$$\cup\{\varphi_{a,1}(x)\mid a\notin\boldsymbol{h}\}\cup\{\neg\varphi_{da,b}(x)\mid a\in\boldsymbol{h}\}.$$

It is easy to check that these types are just the types of the non-zero elements in the modules A/B. These modules are clearly small over every non-zero element. This proves the first assertion.

Let $\tilde{K}$ be an immediate maximally valued extension of the valued field K. To every fractional ideal A of K there corresponds, a fractional $\tilde{A}=\tilde{R}A$ of $\tilde{K}$. A/B is a pure submodule of the compact module $\tilde{A}/\tilde{B}$. A type which is realized in $\overline{A/B}$ is realized in $\tilde{A}/\tilde{B}$ and therefore in A/B (for $y+B$ and $z+\tilde{B}$ are of the same type, if they have the same value). But if A/B and C/D realize a common non-zero type, they are isomorphic. This proves the second assertion.

Note, that – using 5.4 – we have also determined the indecomposables for all commutative rings R where all localizations at maximal ideals are valuation rings (e.g. for Prüfer rings).

A local–global principle (5.4) allows us to transfer the result of 5.1 to more general rings.

Theorem 5.2. *Let R be a commutative ring for which all the localizations $R_{\mathfrak{M}}$ at maximal ideals are fields or discrete valuation rings.*

The indecomposable R-modules are $R_{\mathfrak{M}}$ if $R_{\mathfrak{M}}$ is a field, and $R_{\mathfrak{M}}/R_{\mathfrak{M}}\mathfrak{M}^n$ $(n\geq 1)$, $\mathrm{Quot}(R_{\mathfrak{M}})/R_{\mathfrak{M}}$, $\tilde{R}_{\mathfrak{M}}$, $\mathrm{Quot}(R_{\mathfrak{M}})$ if $R_{\mathfrak{M}}$ is a discrete valuation ring, where $\mathfrak{M}$ is any maximal ideal (all modules are viewed as R-modules.)

Examples are:

Dedekind rings, e.g. $\mathbb{Z}$. The indecomposable abelian groups are $Z(p^n)=$ cyclic group of order p^n $(n\geq 1)$, $Z(p^\infty)=$ the Pruefer group, $\mathbb{Z}_p=$ the p-adic integers and $\mathbb{Q}$, (p a prime number).

Regular von Neumann rings (commutative, x^2 divides x). Here all localizations $R_{\mathfrak{M}}\cong R/\mathfrak{M}$ are fields. Thus e.g. if R is a boolan ring the indecomposable R-modules are $Z(2)$ with R-module structure coming from a ring homomorphism $R\to Z(2)$. Thus the indecomposables correspond to the ultrafilters.

Proof. The given modules are indecomposable as $R_{\mathfrak{M}}$-modules. That they are also indecomposable as R-modules is a general fact: Let S be a multiplicative subset of the commutative ring R. An $S^{-1}R$-module is then nothing else than an R-module with unique division by elements of S. (See Bourbaki, 'Algebre commutative'). This is reflected by the trivial fact that we can translate every $L_{S^{-1}R}$ pp-formula into an equivalent pp-formula of L_R (multiply the coefficients by elements of S). Therefore the following lemma is easy to prove:

Lemma 5.3. *Let S be multiplicative subset of the commutative ring R, M, N be $S^{-1}R$-modules and ${}_RM$, ${}_RN$ the same modules regarded as R-modules. Then*

(1) *$M \equiv N$ iff ${}_RM \equiv {}_RN$.*

(2) *M is compact (injective, indecomposable) iff ${}_RM$ is compact (injective, indecomposable).*

(3) *M is small over the subset A iff ${}_RM$ is small over A.*

(4) *N is a pure submodule (direct factor, pure hull) of M iff ${}_RN$ is a pure submodule (direct factor, pure hull) of ${}_RM$.*

That the modules given in 5.2 are the only indecomposable R-modules follows from 5.1 and the next theorem.

Theorem 5.4. *Let R be a commutative ring. Then every indecomposable R-module is an $R_{\mathfrak{M}}$-module for some maximal ideal $\mathfrak{M}$.*

Proof. Since R is commutative, multiplication by an element of R is an endomorphism of every R-module. But the endomorphism ring of an indecomposable U is local (4.3). Thus

$$\mathfrak{M} = \{r \in R \mid x \mapsto rx \text{ is not an automorphism of } U\}$$

is a maximal ideal of R and U is an $R_{\mathfrak{M}}$-module.

We add another application of localization. First we note the following computational fact:

Lemma 5.5. *Let S be a multiplicative subset of the commutative ring R, M an S-module and $\varphi(x)$ a pp-formula of L_R. Then*

$$\varphi(S^{-1}M) = S^{-1}(\varphi(M)),$$

and therefore

$$\frac{a}{1} \in \varphi(S^{-1}M) \quad \textit{iff} \quad sa \in \varphi(M) \quad \textit{for some } s \in S.$$

Theorem 5.6 (Garavaglia). *Let M be an R-module, R commutative. Then $M \equiv \bigoplus_{\mathfrak{M}} M_{\mathfrak{M}}$, where $\mathfrak{M}$ ranges over all maximal ideals of R.*

Proof. It is well known that $|N| = \prod_{\mathfrak{M}} |N_{\mathfrak{M}}| \bmod \infty$. Now by 5.5

$$\varphi/\psi(M) = \prod_{\mathfrak{M}} |(\varphi(M)/\psi(M))_{\mathfrak{M}}| = \prod_{\mathfrak{M}} |\varphi(M_{\mathfrak{M}})/\psi(M_{\mathfrak{M}})| = \varphi/\psi\Big(\bigoplus_{\mathfrak{M}} M_{\mathfrak{M}}\Big).$$

Next we treat 'indecomposable pairs'. First note that most of our general theory holds for more general structures:

Let L be any language containing $0, +, -$. An additive L-structure $\mathfrak{A}$ is an L-structure which is an abelian group (w.r.t. 0, $+$, $-$) and where

$f^{\mathfrak{A}}: A^n \to A$ is a homomorphism (f an n-place function symbol, in L),

$R^{\mathfrak{A}} \subset A^n$ is a subgroup (R an n-place relation symbol in L).

pp-formulas are of the form $\exists \boldsymbol{x}\, \varphi$ where φ is a conjunction of atomic formulas. It is now clear how to define "pure, compact, indecomposable, small, pure hull, etc." for additive structures.

R-modules are additive L_R-structures which satisfy $r(sx) = (rs)x$ and $rx + sx = (r+s)x$ and $1x = x$.

Now let R be a ring and $L = L_R \cup \{P\}$, P a unary predicate. A pair (U, V) where V is a submodule of the R-module U, is a special additive L-structure.

Theorem 5.7. *Let R be a Dedekind ring and V a submodule of the torsion-free R-module U. The pair (U, V) is indecomposable iff it is of the form* $(\mathrm{Quot}(R), 0)$, $(\mathrm{Quot}(R), \mathrm{Quot}(R))$, $(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}} \cdot \mathfrak{P}^n)$, $(\tilde{R}_{\mathfrak{P}}, 0)$ or $(\mathrm{Quot}(\tilde{R}_{\mathfrak{P}}), \tilde{R}_{\mathfrak{P}})$, $\mathfrak{P}$ *a prime ideal,* $n \geq 0$.

Proof. By 5.4 (generalized version) we can assume that R is a discrete valuation ring. Let Rp be the maximal ideal, K the quotient field of R and $\tilde{K}$ the quotient field of $\tilde{R}$.

The pairs $(K, 0)$, (K, K), $(\tilde{R}, \tilde{R}p^n)$, $(\tilde{R}, 0)$ are indecomposable since U is indecomposable as an R-module and V is a pp-definable submodule.

Assume $(\tilde{K}, \tilde{R}) = (U_1, V_1) \oplus (U_2, V_2)$, a non-trivial decomposition. Then – since $\tilde{R}$ and $\tilde{K}/\tilde{R} \cong K/R$ are indecomposable – we have e.g. $V_1 = 0$ and $V_2 = U_2$. But then $\tilde{K}$ cannot be the divisible hull of $\tilde{R}$.

It remains to show that $(\tilde{K}, \tilde{R})$ is compact. An adaptation of the Elementarteilersatz yields that – for all pairs of R-modules – every ppf $\varphi(x_1, \ldots, x_m)$ is equivalent to a ppf

$$
\begin{aligned}
0 \doteq Q_1(\boldsymbol{x}) \wedge \cdots \wedge 0 \doteq Q_a(\boldsymbol{x}) \wedge \exists y_1, \ldots, y_b, z_1, \ldots, z_c \\
&(p^{n_1} y_1 \doteq R_1(\boldsymbol{x}) \\
&\quad \vdots \\
&\wedge p^{n_b} y_b \doteq R_b(\boldsymbol{x}) \\
&\wedge s_{1,1} y_1 + \cdots + s_{1,b} y_b + p^{m_1} z_1 \equiv S_1(\boldsymbol{x}) \bmod P \\
&\quad \vdots \\
&\wedge s_{c,1} y_1 + \cdots + s_{c,b} y_b + p^{m_c} z_c \equiv S_c(\boldsymbol{x}) \bmod P \\
&\wedge t_{1,1} y_1 + \cdots + t_{1,b} y_b \equiv T_1(\boldsymbol{x}) \bmod P \\
&\quad \vdots \\
&\wedge t_{d,1} y_1 + \cdots + t_{d,b} y_b \equiv T_d(\boldsymbol{x}) \bmod P)
\end{aligned}
$$

where Q_i, R_i, S_i, T_i are R-linear combinations of the $\boldsymbol{x}$, n_i, $m_i \geq 0$ and $s_{i,j}$, $t_{i,j} \in R$.

Since in the case of $(\tilde{K}, \tilde{R})$ the y_i, z_i exist uniquely, the pp-definable subgroups of $(\tilde{K}, \tilde{R})$ are the $\tilde{R}a$ $(a \in \tilde{K})$. The completeness of the valued field $\tilde{K}$ entails now the compactness of $(\tilde{K}, \tilde{R})$.

Finally we show that all indecomposable torsionfree pairs are in our list. For this we note that by the above Elementarteilersatz a substructure (W, X) of (U, V) – U *torsionfree* – is pure iff for all p^n

$$W \cap p^n U = p^n W, \qquad W \cap (V + p^n U) = X + p^n W$$

and W, X are R-submodules.

One sees that $(\tilde{R}, \tilde{R}p^n)$, $(\tilde{R}, 0)$, $(\tilde{K}, \tilde{R})$ are the pure hulls of (R, Rp^n), $(R, 0)$ resp. (K, R).

Let (U, V) be indecomposable, V a submodule of the torsionfree U. Then U/V is compact and, by the proof of 5.1, U/V is zero or contains a direct factor of the form R/Rp^n $(n \geq 1)$, $\tilde{R}$, K or K/R.

Case 1: $U/V = 0$. Then $U = V$ is indecomposable and $(U, V) = (K, K)$ or $= (\tilde{R}, \tilde{R})$.

Case 2: U/V has a direct factor R/Rp^n $(n \geq 1)$. Let $a + V$ correspond to $1 + Rp^n$. Then $p^m a$ is divisible by p^k iff k divides m and $p^m a$ is divisible by p^k mod V iff k divides m mod n. Therefore $(Ra, Rp^n a) = (Ra, V \cap Ra)$ is a pure substructure of (U, V). Thus (U, V) is the pure hull of $(Ra, Rp^n a) \cong (R, Rp^n)$ and $(U, V) \cong (\tilde{R}, \tilde{R}p^n)$.

Case 3: U/V has a direct factor $\tilde{R}$. Let $a + V$ correspond to $1 \in R$. Then $p^m a$ is divisible by p^k mod V iff k divides m. Therefore $(Ra, 0) = (Ra, V \cap Ra) \cong (R, 0)$ is pure in (U, V) and $(U, V) \cong (\tilde{R}, 0)$.

Case 4: U/V has a direct factor K. Let $a + V$ correspond to $1 \in K$. For every n there is $v \in V$ s.t. $a + v$ is divisible by p^n. By compactness there is a $v \in V$ s.t. $b = a + v$ is divisible by all p^n. Then $(Kb, 0) = (Kb, V \cap Kb) \cong (K, 0)$ is pure in (U, V) and $(U, V) \cong (K, 0)$.

Case 5: U/V has a direct factor K/R. Let $a + V$ correspond to $1 + R$. As in Case 4 one can assume that a is divisible by all p^n. then $(Ka, V \cap Ka) \cong (K, R)$ is pure in (U, V) and $(U, V) = (K, R)$.

Remark. A theorem similar to 5.7 is true for Prüferrings.

Now we turn to *injective modules*. Let

$$\varphi(x_1, \ldots, x_n) = \exists y_1 \cdots y_m \bigwedge_{j<k} r_{j,1} y_1 + r_{j,2} y_2 + \cdots = s_{j,1} x_1 + \cdots + s_{j,n} x_n$$

be a pp-formula. If M is injective and $\boldsymbol{a} \in M$, then $\varphi(\boldsymbol{a})$ holds in M iff $\varphi(\boldsymbol{a})$ holds in some extension of M. But the existence of an extension in which $\varphi(\boldsymbol{a})$ is true is for all modules M equivalent to the condition

$$\sum_j t_j (s_{j,1} a_1 + \cdots + s_{j,n} a_n) = 0,$$

whenever

$$\sum_j t_j r_{j,1} = \sum_j t_j r_{j,2} = \cdot = 0 \qquad (t_j \in R).$$

Thus, if we define (in the case $n = 1$) the left ideal

$$\mathfrak{A}_\varphi = \left\{ \sum_j t_j s_{j,1} \;\middle|\; \sum_j t_j r_{j,1} = \cdots = \sum_j t_j r_{j,m} = 0 \right\},$$

we have for injective M

$$M \models \varphi(a) \quad \text{iff} \quad \mathfrak{A}_\varphi a = 0.$$

Since every left ideal $\mathfrak{A}$ is the annihilator of some element of a suitable injective module (of $1 + \mathfrak{A}$ in the injective hull of $R/\mathfrak{A}$) we have

$$\mathfrak{A}_\varphi = \{ r \in R \mid \text{for all injective } M,\ M \models \varphi(x) \to rx \doteq 0 \}.$$

This helps to see that the following lemma is true.

Lemma 5.8 (see [5]). (1) *For every pp-formula $\varphi(x)$ there is an unique left ideal $\mathfrak{A}_\varphi$ s.t. $M \models \varphi(a)$ iff $\mathfrak{A}_\varphi a = 0$ in all injective M. The left ideals $\mathfrak{A}_\varphi$ are just the pp-definable subgroups of the right R-module R_R.*

(2) $\mathfrak{A}_{\varphi \cap \psi} = \mathfrak{A}_\varphi + \mathfrak{A}_\psi$, $\mathfrak{A}_{\varphi + \psi} = \mathfrak{A}_\varphi \cap \mathfrak{A}_\psi$.

(3) *The pp-complete types realized in injective modules are in 1–1 correspondence with the left ideals of R via*

$$p(x) \mapsto \text{“Ann}(x)\text{”} = \{ r \in R \mid rx \doteq 0 \in p \}$$

and

$$\mathfrak{A} \mapsto \{ \varphi \mid \mathfrak{A}_\varphi \subset \mathfrak{A} \} \cup \{ \neg\chi \mid \mathfrak{A}_\psi \not\subset \mathfrak{A} \} = p_{\mathfrak{A}}.$$

Corollary 5.9 (Garavaglia). *If R is left noetherian, every injective R-module is totally transcendental.*

In the next theorem, which is essentially due to Matlis [13] (see also [32]), we determine all injective indecomposable modules. We refer the reader to [18].

Notation. We write $H(\mathfrak{A})$ for $H(p_{\mathfrak{A}})$, where $p_{\mathfrak{A}}$ is the pp-complete type which corresponds to the left ideal $\mathfrak{A}$.

Theorem 5.10. (1) *$H(\mathfrak{A})$ is indecomposable iff $\mathfrak{A}$ is irreducible, i.e. $\mathfrak{A} \neq R$ and $\mathfrak{A} \subsetneqq \mathfrak{B}$, $\mathfrak{A} \subsetneqq \mathfrak{C}$ implies $\mathfrak{A} \subsetneqq \mathfrak{B} \cap \mathfrak{C}$ for all left ideals $\mathfrak{B}$, $\mathfrak{C}$.*

(2) *Let $\mathfrak{A}$, $\mathfrak{B}$ be irreducible. Then $H(\mathfrak{A}) \cong H(\mathfrak{B})$ iff $\mathfrak{A}$ and $\mathfrak{B}$ have a common quotient, i.e. $(\mathfrak{A} : r) = (\mathfrak{B} : s)$ for some $r \notin \mathfrak{A}$, $s \notin \mathfrak{B}$.*

Proof. (1) Let $\mathfrak{A}$ be irreducible. If ψ_1, ψ_2 are pp-formulas not in p, we have $\mathfrak{A}_{\psi_i} \not\subset \mathfrak{A}$. Since $\mathfrak{A} \subsetneqq (\mathfrak{A} + \mathfrak{A}_{\psi_1}) \cap (\mathfrak{A} + \mathfrak{A}_{\psi_2})$, there are $r_i \in \mathfrak{A}$ s.t.

$(Rr_1+\mathfrak{A}_{\psi_1})\cap(Rr_2+\mathfrak{A}_{\psi_2})\not\subset\mathfrak{A}$. Set $\varphi(x)=r_1x\doteq 0\wedge r_2x\doteq 0$. Then

$$\varphi\in p_{\mathfrak{A}} \quad\text{and}\quad (\psi_1\cap\varphi)+(\psi_2\cap\varphi)\notin p_{\mathfrak{A}}.$$

By 4.4, $p_{\mathfrak{A}}$ is indecomposable.

Let now $p_{\mathfrak{A}}$ be indecomposable. Clearly $\mathfrak{A}\neq R$. Assume $\mathfrak{A}\subsetneqq\mathfrak{B}_1$, $\mathfrak{A}\subsetneqq\mathfrak{B}_2$. Pick $r_i\in\mathfrak{B}_i\setminus\mathfrak{A}$, and set $\psi_i(x)=r_ix\doteq 0$. Since $p_{\mathfrak{A}}$ is indecomposable there is a pp-formula $\varphi\in p_{\mathfrak{A}}$ s.t.

$$(\psi_1\cap\varphi)+(\psi_2\cap\varphi)\notin p_{\mathfrak{A}},\quad\text{i.e.}\quad (\mathfrak{A}_{\varphi}+Rr_1)\cap(\mathfrak{A}_{\varphi}+Rr_2)\not\subset\mathfrak{A}.$$

Therefore $\mathfrak{A}\subsetneqq\mathfrak{B}_1\cap\mathfrak{B}_2$, which shows that $\mathfrak{A}$ is irreducible.

(2) *Note.* If M is *injective*, N is a pure compact submodule of M iff N is a direct factor of M iff N is an injective submodule of M.

This shows that the hull (in M) of a submodule N of M is the injective hull of N. Therefore a submodule L of M is small over N iff every non-zero submodule of L has non-zero intersection with N. (See e.g. [18]).

To prove 5.10.2, let $a\in H(\mathfrak{A})$, $\mathrm{Ann}(a)=\mathfrak{A}$ and $b\in H(\mathfrak{B})$, $\mathrm{Ann}(b)=\mathfrak{B}$. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ have the common quotient $(\mathfrak{A}:r)=(\mathfrak{B}:s)$. Since $r\notin\mathfrak{A}$, we have $ra\neq 0$ and therefore $H(\mathfrak{A})=H(ra)=H(\mathrm{Ann}(ra))=H(\mathfrak{A}:r)$. Since also $H(\mathfrak{B})=H(\mathfrak{B}:s)$, we conclude $H(\mathfrak{A})=H(\mathfrak{B})$.

For the converse assume $U=H(\mathfrak{A})=H(\mathfrak{B})$. Since U is small over Ra, $Rb\cap Ra\neq 0$. Let $ra=sb\neq 0$. Then $(\mathfrak{A}:r)=\mathrm{Ann}(ra)=(\mathfrak{B}:s)$.

Remark. M is called absolutely pure if it is pure in every extension. This is the same as to say that, for all ppf φ, $M\models\varphi(\boldsymbol{a})$ iff $\boldsymbol{a}$ satisfy the set of equations given before 5.8. It is easy to see that M is absolutely pure iff the pure hull of M is injective. A pure submodule of an absolutely pure module is again absolute pure. If R is left coherent (e.g. left noetherian) absolute purity is elementarily expressible [5].

Example. Let R be a boolean ring. Since all $R_{\mathfrak{M}}$ are fields, all $M_{\mathfrak{M}}$ are absolutely pure. Therefore all R-modules are absolutely pure (see Bourbaki [2]).

If R is atomless (= no principal prime ideals), the injective hull (= the pure hull) of R has no indecomposable factors. For: if $a\in\bar{R}\setminus 0$, we have $ra\in R\setminus 0$ for some $r\in R$. And if $H(a)$ were an indecomposable factor of R, ra would realize an indecomposable type in R. But this is impossible.

Corollary 5.11 [13]. *Let R be a noetherian commutative ring.*

(1) *The injective indecomposable R-modules are in* 1–1 *correspondence with the prime ideals $\mathfrak{P}$ via $\mathfrak{P}\mapsto H(\mathfrak{P})$.*

(2) *If $\mathfrak{A}$ is irreducible, $\{a\mid\exists b\notin\mathfrak{A}\, ab\in\mathfrak{A}\}=\mathfrak{P}$ is the unique prime ideal $\mathfrak{P}$ s.t. $H(\mathfrak{A})=H(\mathfrak{P})$.*

Proof. Clearly prime ideals are irreducible. Since $(\mathfrak{P}:r)=\mathfrak{P}$ $(r\notin\mathfrak{P})$, we have $H(\mathfrak{P})=H(\mathfrak{Q})\Rightarrow\mathfrak{P}=\mathfrak{Q}$ for prime ideals $\mathfrak{P}$ and $\mathfrak{Q}$.

Let $\mathfrak{A}$ be irreducible. If $r, s \notin \mathfrak{A}$, choose $t \in (\mathfrak{A}+Rr) \cap (\mathfrak{A}+Rs) \setminus \mathfrak{A}$. Then

$$(\mathfrak{A}:r)+(\mathfrak{A}:s) \subset (\mathfrak{A}:t).$$

This shows that there is a largest quotient $(\mathfrak{A}:t)$ (R is noetherian), which must be $\{r \mid \exists s \notin \mathfrak{A}\ rs \in \mathfrak{A}\}$. This description gives immediately the primeness off $(\mathfrak{A}:t)$.

Note. It is well known, that a maximal quotient $(\mathfrak{A}:r)$, $r \notin \mathfrak{A}$, is prime. Irreducible ideals are primary. We constructed the associated prime ideal.

6. The Krull–Remak–Schmidt theorem

We are going to prove:

Theorem 6.1. *Every compact module M has a unique (up to isomorphism of the factors) decomposition*

$$M = \overline{\bigoplus_{i \in I} U_i} \oplus E$$

where the U_i are indecomposable and E has no indecomposable direct factors.

The theorem is essentially due to Fisher (unpublished). Our uniqueness proof covers also the case of the Krull–Remak–Schmidt–Azumaya theorem which states the uniqueness of the representation of a module as a direct sum of modules with local endomorphism ring.

The *existence* is easy to prove: Let $(U_i)_{i \in I}$ be a family of indecomposable submodules of M, s.t. the sum $\sum_{i \in I} U_i$ is direct and pure in M, and maximal with this property. Then

$$H\Big(\sum_{i \in I} U_i\Big) \cong \overline{\bigoplus_{i \in I} U_i}$$

is a direct summand in

$$M = H\Big(\sum_{i \in I} U_i\Big) \oplus E.$$

Because of the maximality of (U_i), E has no indecomposable factor.

For the *uniqueness* proof we develop dimension theory in modules. Let us drop the assumption that M is compact.

We define a dependence relation on the set of direct factors of M as follows: K depends on $\mathbb{F}$ ($\mathbb{F}$ a set of direct factors) if there is a finite subset $\mathbb{F}_0$ of $\mathbb{F}$ s.t. no decomposition $M = K \oplus K'$, $\bigcup \mathbb{F}_0 \subset K'$ exists.

The following axioms are satisfied:

(D_0) K depends on $\{K\}$ (if $K \neq 0$).

(D_1) K depends on $\mathbb{F}$ iff K depends on a finite subset of $\mathbb{F}$.

(D_2) If K depends on $\mathbb{F} \cup \{L\}$ but not on $\mathbb{F}$, then L depends on $\mathbb{F} \cup \{K\}$.

Proof of (D_2). Suppose that K does not depend on $\mathbb{F}$, L not on $\mathbb{F}\cup\{K\}$ and let $\mathbb{F}_0$ be a finite subset of $\mathbb{F}$. Then there are decompositions $M=K\oplus K'=L\oplus L'$ where $\bigcup\mathbb{F}_0\subset K'$ and $\{K\}\cup\bigcup\mathbb{F}_0\subset L'$. Since $K\subset L'$, we have $M=K\oplus L\oplus (K'\cap L')$. This shows that K does not depend on $\mathbb{F}\cup\{L\}$.

The next lemma show that, if we restrict ourself to factors with local endomorphism ring, also a weak transitivity axiom holds.

Lemma 6.2. *Let* U_i, U, L, $\mathbb{F}$ be direct factors of M, $\mathrm{End}(U_i)$ *local. Assume that* (a) $\{L\}\cup\{U_i\}_{i\in I}$ *is independent,* (b) *every* U_i *which is isomorphic to a direct factor of* U *depends on* $\{L\}\cup\mathbb{F}$, (c) U *depends on* $\{L\}\cup\{U_i\}_{i\in I}$.

Then U *depends on* $\{L\}\cup\mathbb{F}$. ($\mathbb{F}$ *is independent if no* $U\in\mathbb{F}$ *depends on* $\mathbb{F}\setminus\{U\}$.)

Proof. We can assume that $I=\{1,\ldots,n\}$ and $\mathbb{F}$ is finite. If U does not depend on $\{L\}\cup\mathbb{F}$, we write $M=U\oplus K$, $\{L\}\cup\bigcup\mathbb{F}\subset K$. Let $g:M\to U$ and $f:M\to K$ be the corresponding projections.

We will use the following general fact: $M=U'\oplus K$ iff g induces an isomorphism from U' to U. Since $\{L,U_1,\ldots,U_n\}$ is independent, we have (cf. the proof of (D_2) above)

$$M=L\oplus U_1\oplus\cdots\oplus U_n\oplus M'.$$

Let $\pi:M\to U_1$ be the second projection. Since $\mathrm{End}(U_1)$ is local and $\pi(f+g)\restriction U_1=\mathrm{id}\restriction U_1$ there are two cases:

Case 1: $\pi g\restriction U_1$ is an automorphism of U_1. Then π induces an isomorphism from $g(U_1)$ to U_1. Whence

$$M=L\oplus g(U_1)\oplus U_2\oplus\cdots\oplus M'.$$

Set $B=L\oplus U_2\oplus\cdots\oplus M'$. Then $U=g(U_1)\oplus U\cap B$ and g induces an isomorphism from $U_1\oplus U\cap B$ to U. Therefore $M=U_1\oplus U\cap B\oplus K$.

We have: U_1 is isomorphic to a direct factor of U, U_1 does not depend on $\{L\}\cup\mathbb{F}$. Thus Case 1 cannot occur.

Case 2: $\pi f\restriction U_1$ is an automorphism of U_1. As above we conclude that

$$M=L\oplus f(U_1)\oplus U_2\oplus\cdots\oplus M'$$

and $f\restriction U_1:U_1\to f(U_1)$ is an isomorphism. Proceeding in this manner, we finally arrive at

$$M=L\oplus f(U_1)\oplus f(U_2)\oplus\cdots\oplus M'$$

and isomorphisms $f\restriction U_i:U_i\to f(U_i)$. We have

$$K=f(U_1)\oplus\cdots\oplus f(U_n)\oplus(L\oplus M')\cap K$$

and f induces an isomorphism from

$$U_1\oplus\cdots\oplus U_n\oplus(L\oplus M')\cap K$$

to K. Therefore

$$M = U \oplus U_1 \oplus \cdots \oplus U_n \oplus (L \oplus M') \cap K.$$

This contradicts the assumption that U depends on $\{L\} \cup \{U_i\}_{i \in I}$.

If we restrict our dependence relation to direct factors of M with local endomorphism ring we have by 6.2 (take $L = 0$):

(D_3) If $\{U_i\}_{i \in I}$ is independent, all U_i depend on $\mathbb{F}$, U depends on $\{U_i\}_{i \in I}$, then U depends on $\mathbb{F}$.

The axioms D_0–D_3 are enough to conclude that a basis exists (that is a maximal independent set) and that all basis' have the same cardinality: the *dimension* of the dependence structure.

Let U be a module with local endomorphism ring (e.g. an indecomposable). We define U-dim(M) to be the dimension of the dependence structure whose underlying set is the set of all direct factors of M isomorphic to U.

We are now in a position to prove the classical K–R–S–A theorem. It is enough to show that, if the End(U_i), End(U) are local,

$$U\text{-dim}\left(\bigoplus_{i \in I} U_i\right) = |\{i \in I \mid U_i \cong U\}|.$$

Thus, let U be a direct factor of $\bigoplus_{i \in I} U_i$. Since U depends on $\{U_i\}_{i \in I_0}$ if $\sum_{i \in I_0} U_i \cap U \neq 0$ (I_0 finite), U depends on $\{U_i\}_{i \in I}$. By 6.2, U depends on $\{U_i \mid U_i \cong U\} = \mathbb{F}$. Therefore $\mathbb{F}$ is an U-basis of $\bigoplus_{i \in I} U_i$.

To prove 6.1 we will prove that for indecomposable U

$$U\text{-dim}(M) = |\{i \in I \mid U_i \cong U\}|.$$

where M is compact and decomposed as in 6.1.

Here the above argument does not work, because a non-zero direct factor of M has not to intersect non-trivially with $\bigoplus_{i \in I} U_i$. But for *compact M* we can develop our dimension theory a little further:

Definition. Let A, B two subsets of the compact module M. A and B are *independent* if the following equivalent conditions are satisfied:

(a) There is a decomposition $K = K \oplus L$, $A \subset K$, $B \subset L$.

(b) For all pp-formulas $\varphi(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{a} \in A$, $\boldsymbol{b} \in B$ $M \models \varphi(\boldsymbol{a}, \boldsymbol{b}) \Rightarrow M \models \varphi(\mathbf{0}, \boldsymbol{b})$ (and therefore also $M \models \varphi(\boldsymbol{a}, \mathbf{0})$).

(c) There is a partial homomorphism f from M to N s.t. $f \restriction A = 0$, $f \restriction B$ partially isomorphic.

By (a, b) our two notions of dependence agree for direct factors of M:

Corollary 6.3. *Let U, $\mathbb{F}$ be direct factors of M. Then U depends on $\mathbb{F}$ iff U and $\sum \mathbb{F}$ are dependent (in the sense of the last definition).*

Proof (equivalence of (a), (b), (c)). (a) $\to$ (b) Let $\pi: M \to L$ be the second projection. If $M \models \varphi(\boldsymbol{a}, \boldsymbol{b})$, we have $M \models \varphi(\pi(\boldsymbol{a}), \pi(\boldsymbol{b}))$.

(b) $\to$ (c) If (b) holds, $\mathrm{id}_B \cup 0_A$ is a partial homomorphism.

(c) $\to$ (a) follows from the next lemma.

Lemma 6.4. *Let A, B independent (in the sense of (c)), B' small over B. Then*

(1) *A and B' are independent.*

(2) *If $H(A)$, $H(B)$ are hulls of A, B in M, then $M = H(A) \oplus H(B) \oplus M'$.*

Proof. (1) If $f \restriction A = 0$ and $f \restriction B$ is partially isomorphic, extend f to a homomorphism g defined on M. Then g is partially isomorphic on B'.

(2) If $f \restriction A = 0$, $f \restriction B$ is partially isomorphic, then we extend $f^{-1} \restriction f(B)$ to a homomorphism g from N to M. Then $\mathrm{id}_A \cup 0_B$ is a partial homomorphism from M to M, since it is extended by $1 - gf$. This shows that we can apply 1 twice to conclude that $H(A)$ and $H(B)$ are independent. And furthermore that then $\mathrm{id}_{H(A)} \cup 0_{H(B)}$ is partially homomorphic. Let $h: M \to H(A)$ extend this partial homomorphism. Then $M = H(A) \oplus \ker h$. But $H(B)$ is a direct factor of $\ker h$.

Corollary 6.5. (1) $\overline{L \oplus K} = \bar{L} \oplus \bar{K}$ (recall that $\overline{}$ denotes the pure hull.)

(2) *B' is small over its subset B iff A, B independent $\Rightarrow$ A, B' independent for all $A \subset M$.*

Proof. (1) Let $M = \overline{L \oplus K}$. Then $M = H(L) \oplus H(K) \oplus M'$. Since M is small over $L \oplus K$, $M' = 0$.

(2) Let the condition of (2) be satisfied. Write $M = A \oplus H(B)$. Since A, B' are independent, we have $M = A \oplus B''$, $B' \subset B''$. The canonical isomorphism $H(B) \cong M/A \cong B''$ fixes B. Whence B'' is small over B as $H(B)$ is.

We return to the *proof of* 6.1. Let M be decomposed as in 6.1 and U indecomposable. We want to show that $\mathbb{F} = \{U_i \mid U_i \cong U\}$ is a U-basis of M. This implies then that $|\{i \in I \mid U_i \cong U\}|$ equals U-dim(M) and is therefore independent of the particular decomposition of M. Thus "the U_i are uniquely determined."

That $\mathbb{F}$ is independent is clear.

Let U be a direct factor of M. U and $\bigoplus_{i \in I} U_i + E$ are dependent. Whence, by 6.3, U depends on $\{E\} \cup \{U_i\}_{i \in I}$. Suppose that $\{U\} \cup \{U_i\}_{i \in I}$ is independent. Then, by 6.2, it is impossible that E depends on $\{U\} \cup \{U_i\}_{i \in I}$. Whence U depends on $\{U_i\}_{i \in I}$ and therefore – by 6.2 – on $\mathbb{F}$. Thus $\mathbb{F}$ is a U-basis.

Finally we show that E is uniquely determined up to isomorphism. Assume

$$M = \overline{\bigoplus_{i \in I} U_i'} \oplus E'.$$

By 6.2, E cannot depend on $\{U_i'\}_{i \in I}$. By 6.3 and 6.4,

$$M = \overline{\bigoplus_{i \in I} U_i'} \oplus E \oplus K.$$

We conclude $E' \cong E \oplus K$ and similarly $E \cong E' \oplus L$. $E \cong E'$ by the next lemma.

Lemma 6.6 (Fisher). *Let E be compact and $E \cong E \oplus K \oplus L$. Then $E \cong E \oplus K$.*

Proof. Let $E = E_0 \oplus M_0$, where $E_0 \cong E$ and $M_0 \cong K \oplus L$. We can proceed: $E_i = E_{i+1} \oplus M_{i+1}$, where $E_{i+1} \cong E$ and $M_{i+1} \cong K \oplus L$, $i = 0, 1, 2, \ldots$. Since all $M_0 \oplus M_1 \oplus \cdots \oplus M_i$ are direct factors in E, $\overline{\bigoplus_{i\in\omega} M_i}$ is pure in E. Therefore $E = \overline{\bigoplus_{i\in\omega} M_i} \oplus M'$. But

$$\overline{\bigoplus_{i\in\omega} M_i} = \overline{K^{\aleph_0} \oplus L^{\aleph_0}},$$

therefore by 6.5(1)

$$\overline{\bigoplus_{i\in\omega} M_i} \oplus K \cong \overline{\bigoplus_{i\in\omega} M_i}.$$

This implies $E \cong E \oplus K$.

Remark 6.7. *If $M_k = \overline{\bigoplus_{i\in I_k} U_i} \oplus E_k$ $(k = 1, 2)$ are decompositions as in* 6.1. *Then also*

$$M_1 \oplus M_2 = \overline{\bigoplus_{i\in I_1 \cup I_2} U_i} \oplus (E_1 \oplus E_2)$$

is a decomposition as in 6.1, *i.e. $E_1 \oplus E_2$ has no indecomposable factors.*

Proof. Set $M = E_1 \oplus E_2$ and let U be an indecomposable factor of M. By 6.2, E_1 does not depend on U and again, by 6.2, E_2 does not depend on $\{E_1, U\}$ (for E_2 does not depend on E_1). Whence $\{E_1, E_2, U\}$ is independent. This is impossible.

In general one cannot expect a compact module to be the pure hull of a direct sum of indecomposables. But up to elementary equivalence this is true:

Definition. M is weakly saturated if every M-consistent type $p(x_1, \ldots, x_n)$ is realized in M.

Theorem 6.8. *Let M be weakly saturated and compact. If*

$$M = \overline{\bigoplus_{i\in I} U_i} \oplus E$$

is a decomposition as in 6.1, *then M and $\overline{\bigoplus_{i\in I} U_i}$ are elementarily equivalent.*

Proof. For μ a cardinal and p an indecomposable type we define

$$p_\mu(\boldsymbol{x}) = \bigcup_{i<\mu} p(x_i) \cup \{\varphi(x_{i_1}, \ldots, x_{i_n})$$
$$\to \varphi(0, x_{i_2}, \ldots, x_{i_n}) \mid \varphi \text{ ppf}, i_1, \ldots, i_n < \mu \text{ all different.}\}$$

By 6.4, $p_\mu(\boldsymbol{x})$ is realized in the compact module N iff

$$U\text{-dim}(N) \geq \mu, \quad \text{where } U = H(p).$$

Thus U-dim$(M) \geq n$ iff $p_n(\boldsymbol{x})$ is M-consistent, and we can conclude that for compact N

$$N \equiv M \Rightarrow U\text{-dim}(N) \leq U\text{-dim}(M) \mod \infty.$$

6.8 follows from 1.5 and

$$(*) \qquad \varphi/\psi(E) > 1 \Rightarrow \varphi/\psi\Big(\overline{\bigoplus_{i \in I} U_i}\Big) = \infty.$$

To prove (*) assume that $\varphi/\psi(E) > 1$. By 4.8, φ/ψ is contained in an E-consistent indecomposable type. Thus there is a compact $N \equiv E$ and an indecomposable U s.t. $\varphi/\psi(U) > 1$ and U-dim$(N) > 0$. *Now* $\overline{\bigoplus_{i \in I} U_i} \oplus N \equiv M$ and therefore

$$U\text{-dim}\Big(\overline{\bigoplus_{i \in I} U_i}\Big) + U\text{-dim}(N) \leq U\text{-dim}(M) = U\text{-dim}\Big(\overline{\bigoplus_{i \in I} U_i}\Big) \mod \infty.$$

This implies

$$U\text{-dim}\Big(\overline{\bigoplus_{i \in I} U_i}\Big) = \infty \quad \text{and} \quad \varphi/\psi\Big(\overline{\bigoplus_{i \in I} U_i}\Big) = \infty.$$

Corollary 6.9. *Every module is elementarily equivalent to a direct sum of indecomposables.*

Corollary 6.10. *Two weakly saturated and compact modules M, N are elementarily equivalent iff* U-dim$(M) = U$-dim$(N) \mod \infty$ *for all indecomposable U.*

If we define $I_U(M)$ to be the largest $n \in \{0, 1, \ldots, \infty\}$ s.t. $p_n(\boldsymbol{x})$ is M-consistent ($U = H(p)$ indecomposable), we have by 6.10 for any M, N

$$M \equiv N \quad \text{iff} \quad I_U(M) = I_U(N) \quad \text{for all indecomposable } U.$$

The elementary invariants $I_U(\cdots)$ contain less redundancy than the invariants $\varphi/\psi(\cdots)$ (see next page). In the case of abelian groups the I_U are just the Szmielew-invariants (see 9.6).

We give now an alternative description of the $I_U(\cdots)$:

Definition. For every module M and every indecomposable U define

$$I_U(M) = \min\{{}^{\varphi/\psi(U)}\log(\varphi/\psi(M)) \mid \psi \subset \varphi \text{ ppf}\}.$$

(By convention ${}^1\log(\cdots) = {}^{\cdots}\log(\infty) = \infty$ and ${}^{\alpha}\log(1) = {}^{\infty}\log(\beta) = 0$ $(\alpha > 1,\ \beta < \infty)$.)

Theorem 6.11. (1) *$M \equiv N$ iff $I_U(M) = I_U(N)$ for all indecomposable U.*

(2) *If M is weakly saturated and compact,* U-dim$(M) = I_U(M)$.

Proof. (1) follows from (2) and 6.8.

(2) follows from the following two claims. We assume M to be weakly saturated and compact.

Claim 1. U-dim$(M) \leq I_U(M)$.

Proof. For all ppfs $\psi \subset \varphi$ we have $\varphi/\psi(M) \geq (\varphi/\psi(U))^{U\text{-dim}(M)}$ and therefore ${}^{\varphi/\psi(U)}\log(\varphi/\psi(M)) \geq U\text{-dim}(M)$.

Claim 2. $n < I_U(M) \Rightarrow n < U\text{-dim}(M)$.

Proof. Choose p s.t. $U = H(p)$. There are two cases.

Case 1: $U \oplus U \equiv U$. $0 < I_U(M)$ implies that $\varphi/\psi(M) > 1$ for all $\varphi/\psi \in p$. By 4.5, p is M-consistent. Thus $U\text{-dim}(M) > 0$ and therefore $M \equiv M \oplus U^m$ for all m. We have $U\text{-dim}(M) = \infty$.

Case 2: $U \oplus U \not\equiv U$. We can assume that p contains a pair $\bar{\varphi}/\bar{\psi}$ s.t. $\bar{\varphi}/\bar{\psi}(U) < \infty$. We prove Claim 2 by induction on n. Assume $n < I_U(M)$. By induction we have $M = U^n \oplus N$. Let $\varphi/\psi \in p$ be arbitrary. We show that $\varphi/\psi(N) > 1$. By 4.5, we can assume that $\varphi/\psi \leq \bar{\varphi}/\bar{\psi}$. Then $\varphi/\psi(U) = m$ is finite. Now $m^{I_U(M)} \leq \varphi/\psi(M) = m^n \cdot \varphi/\psi(N)$. This implies $\varphi/\psi(N) > 1$. Thus p_{n+1} is M consistent and we have $n + 1 \leq U\text{-dim}(M)$.

Corollary 6.12. *Let* $\{(\varphi/\psi) \mid \varphi/\psi \in \mathcal{S}\}$ *be a basis for the neighbourhoods of* U *(cf.* 4.9). *Then*

$$I_U(M) = \min\{{}^{\varphi/\psi(U)}\log(\varphi/\psi(M)) \mid \varphi/\psi \in \mathcal{S}\}.$$

Proof. This follows from the above proof of 6.11 and 4.11. All that we have to know is, that

(*) $\quad U \oplus U \not\equiv U \Rightarrow \varphi/\psi(U)$ finite for some $\varphi/\psi \in \mathcal{S}$.

Proof of (*). Let $\varphi_0/\psi_0(U)$ be finite and >1. There is $\varphi/\psi \in \mathcal{S}$ s.t. (φ/ψ) is contained in (φ_0/ψ_0). We claim that $\varphi/\psi(U)$ is finite. If not, there is an elementary extension $N = U \oplus M$ of U s.t. the index of $\psi(N)$ in $\varphi(N)$ is – say – larger than $|U|$. It follows that $\varphi/\psi(M) > 1$. But then $\varphi_0/\psi_0(M) > 1$ by 4.11 (proof). This contradicts the finiteness of $\varphi_0/\psi_0(U) = \varphi_0/\psi_0(N)$.

Note that for (*) we used only that $\mathcal{S}$ is a basis for the neighbourhoods of U in $\mathbb{U}_U$.

Corollary 6.13. *If* $\{(\varphi/\psi) \mid \varphi/\psi \in \mathcal{S}\}$ *is a basis of* $\mathbb{U}^R$, *M and N are elementarily equivalent iff* $\varphi/\psi(M) = \varphi/\psi(N)$ *for all* $\varphi/\psi \in \mathcal{S}$.

Proof. By 6.11.1 and 6.12.

Remark. Look at the topological space $\mathbb{U}^R$ defined in 4.9. First we note that the closed subsets $\mathbb{U}_M$ (see 4.10) can be described as $\{U \mid I_U(M) > 0\}$.

Now let $(m_U)_{U \in \mathbb{U}^R}$ be a family of numbers $0, 1, \ldots, \infty$. Define $\mathbb{U}_m$ to be the set of all U s.t. $m_U > 0$. Then there is a module M s.t. $I_U(M) = m_U$ $(U \in \mathbb{U}^R)$ *iff* $m_U = \infty$, whenever U is in the closure of $\mathbb{U}_m \setminus \{U\}$ or when $U \oplus U \equiv U \in \mathbb{U}_m$. (*Proof.* Note that $I_U(U) = \infty$, if $U \oplus U \equiv U$, and $= 1$ otherwise. Take $\bigoplus_{i \in I} U_i^{m_i}$ for M.)

We conclude this section by an explicit description of saturated modules.

Theorem 6.14. *Let* $(U_i)_{i\in I}$ *be a family of indecomposables s.t.*

$$U\text{-dim}\left(\overline{\bigoplus_{i\in I} U_i}\right)=\begin{cases} n, & \text{if } I_U(M)=n \text{ is finite,}\\ \lambda, & \text{if } I_U(M)=\infty.\end{cases}$$

If $|R|+\aleph_0<\operatorname{cf}\kappa$, $\kappa\leq\lambda$, *then* $\prod_{i\in I}^{\kappa} U_i$ *is* λ*-saturated and elementarily equivalent to* M.

Proof. $M\equiv\prod_{i\in I}^{\kappa} U_i$ follows from 1.8, 6.8, 6.11. Let p be a type with parameters from $A\subset\prod_{i\in I} U_i$, $|A|<\lambda$, which is realized in an elementary extension N of $\prod_{i\in I}^{\kappa} U_i$, say by b. Choose $I_0\subset I$, s.t.

$$A\subset\prod_{i\in I_0}^{\kappa} U_i=K\equiv M \quad\text{and}\quad U\text{-dim}\left(\overline{\bigoplus_{i\in I_0} U_i}\right)<\lambda.$$

By 3.4, K is compact. We write $N=K\oplus L$ and $b=a+c$, $a\in K$, $c\in L$. Let $\neg\psi\in\operatorname{tp}^-(c)$. By 4.7 there is an L-consistent indecomposable q containing $\operatorname{tp}^+(c)\cup\{\neg\psi\}$. Since $I_{H(q)}(L)>0$, $I_{H(q)}(M)=\infty$. Thus for λ-many $i\in I\setminus I_0$, $U_i=H(q)$. This shows that we can choose a sequence $d\in\prod_{i\in I\setminus I_0}^{\kappa} U_i$ s.t. for every $\neg\psi\in\operatorname{tp}^-(c)$ there is an $i\in I\setminus I_0$ s.t. d_i satisfies $\neg\psi$ and all d_i realize $\operatorname{tp}^+(c)$. Whence $\operatorname{tp}^\pm(d)=\operatorname{tp}^\pm(c)$. It follows that $\operatorname{tp}^\pm(b/A)=\operatorname{tp}^\pm(a+d/A)$. $a+d$ realizes p inside $\prod_{i\in I}^{\kappa} U_i$.

Corollary 6.15. *If* $\lambda^{|R|+\aleph_0}=\lambda$, *then every infinite module is elementarily equivalent to a saturated module of power* λ, *which is of the form*

$$\prod_{i\in I}^{(|R|+\aleph_0)^+} U_i,\qquad U_i \text{ indecomposable.}$$

Remarks. (1) A special case of 6.1 is: Every injective module has a unique decomposition into a direct sum of a module without indecomposable direct factors and of the injective hull of a direct sum of indecomposable injectives.

(2) Let M be as in 6.1 and $J\subset I$. Then the following are equivalent:

(a) If $M=\overline{\bigoplus_{i\in I} V_i}\oplus F$, $V_i\cong U_i$, $F\cong E$, then $\overline{\bigoplus_{i\in J} V_i}=\overline{\bigoplus_{i\in J} U_i}$.

(b) If $j\in J$ and there is a non-zero homomorphism from U_j to U_i, then $i\in J$.

7. Modules of bounded width

In this section we give a sufficient condition on a compact module to be the pure hull of a direct sum of indecomposables. For countable R this condition is – in a sense – also necessary.

Definition. Let M be a module. We define $w_M(\varphi/\psi)$ – the *width* – for every pair

$\psi \subset \varphi$ of pp-formulas by recursion on the ordinal α.

$$\begin{aligned} w_M(\varphi/\psi) = 0 \quad &\text{iff} \quad \psi(M) = \varphi(M), \\ w_M(\varphi/\psi) = \alpha \quad &\text{iff} \quad w_M(\varphi/\psi) \not< \alpha \text{ and for all } \psi \subset \chi_i \subset \varphi \ (i = 1, 2) \\ &\qquad w_M(\chi_1/\chi_1 \cap \chi_2) < \alpha \text{ or } w_M(\chi_2/\chi_1 \cap \chi_2) < \alpha \ (\alpha > 0), \\ w_M(\varphi/\psi) = \infty \quad &\text{iff} \quad w_M(\varphi/\psi) \neq \alpha \text{ for all ordinals } \alpha. \end{aligned}$$

M is of *bounded width* if $w_M(x = x/x = 0) < \infty$.

Note that $w_M(\varphi/\psi) \leq 1$ iff the set of all pp-definable subgroups of M between $\psi(M)$ and $\varphi(M)$ is linearly ordered by inclusion. (*Added in proof:* M. Prest has a nicer definition of width.)

We will prove

Theorem 7.1. (1) *If M has bounded width, then every compact module elementarily equivalent to M is the pure hull of direct sum of indecomposables.*

(2) *If R is countable, then the converse it true: If every compact module elementarily equivalent to M is the pure hull of a direct sum of indecomposables, then M is of bounded width.*

Problem. Is 7.1(2) true for arbitrary R?

7.1(1) is a generalization of a theorem of Garavaglia:

Definition (Garavaglia). M has *elementary Krull dimension* (we say Krull dimension), if there is no dense chain of pp-definable subgroups of M.

By 2.1 superstable modules have Krull dimension. By 5.8 all injective R-modules have Krull dimension iff R has Krull dimension as a right R-module.

Lemma 7.2. *Every module with Krull dimension is of bounded width.*

Proof. If $w_M(\varphi/\psi) = \infty$, we find χ_1, χ_2 between ψ and φ s.t. $w_M(\chi_i/(\chi_1 \cap \chi_2)) = \infty$ $(i = 1, 2)$. By 7.4(2) below $w_M(\varphi/\chi_1) = w_M(\chi_1/\psi) = \infty$. If we continue in this way, we construct a dense chain of definable subgroups between $\psi(M)$ and $\varphi(M)$.

Corollary 7.3 (Garavaglia). (1) *Every compact module with Krull dimension is the pure hull of direct sum of indecomposables.*

(2) *Every totally transcendental module is the direct sum of indecomposables.*

7.3(2) follows from 7.3(1), since every totally transcendental module is compact. We give an independent proof of 7.3 at the end of this section. By 5.9 we have as a corollary a theorem of Matlis [13]: If R is left noetherian, every injective R-module is the direct sum of indecomposables.

Examples. (1) If R is an valuation ring, $w_M(x=x/x=0)\leqslant 2$ for every M. This follows from our description of all pp-formulas in Section 5. Thus every compact module over a valuation ring is the pure hull of a direct sum of indecomposables.

(2) From 5.8(1) and 7.1 follows: If R_R is of bounded width, all injective modules are the injective hull of a direct sum of indecomposable injectives. If R is countable, the converse is true. (All injectives are the injective hull of a direct sum of indecomposables iff every left ideal of R has an irreducible quotient.)

(3) If R is an atomless boolean ring, ${}_RR$ has unbounded width. Indeed, in the example preceding 5.11 we showed that $\bar{R}$ has no indecomposable factors.

We start our proof of 7.1 with some observations on w_M. (The proof of 7.1 will be completed after 7.7.)

Lemma 7.4. (1) $M\equiv N$ *implies* $w_M(\cdots)=w_N(\cdots)$. *We have also* $w_M(\cdots)=w_{M^\kappa}(\cdots)$.

(2) $\varphi/\psi\leqslant\bar{\varphi}/\bar{\psi}$ *(in the sense of* 1.9) *implies* $w_M(\varphi/\psi)\leqslant w_M(\bar{\varphi}/\bar{\psi})$.

(3) *Suppose that* $\psi\subset\sigma\subset\varphi$. *Then* $w_M(\varphi/\psi)\leqslant w_M(\varphi/\sigma)+w_M(\sigma/\psi)$.

(4) $w_M(\varphi/\psi)$ *depends only on* $\varphi(M)$, $\psi(M)$ *and* M.

(5) $w_N(\cdots)\leqslant w_M(\cdots)$, *if* N *is a pure submodule of* M.

Proof. (1) and (4) follow by an easy induction on $w_M(\cdots)$. Also it is clear that $w_M(\varphi/\psi)$ depends only on the isomorphism type of the lattice of definable subgroups between $\psi(M)$ and $\varphi(M)$. Therefore 2 is true (again by induction).

We prove (3) by induction on $\alpha=w_M(\sigma/\psi)$: the case $\alpha=0$ is clear. So let $0<\alpha$ and $\beta=w_M(\varphi/\sigma)$. If $\psi\subset\chi_i\subset\varphi$ $(i=1,2)$, we have $\psi\subset\chi_i\cap\sigma\subset\sigma$ and therefore e.g.

$$w_M((\chi_1\cap\sigma)/(\chi_1\cap\chi_2\cap\sigma))=\gamma<\alpha.$$

By (2) we have

$$w_M((\chi_1\cap\sigma+\chi_1\cap\chi_2)/(\chi_1\cap\chi_2))=\gamma.$$

Now $\chi_1/(\chi_1\cap\sigma+\chi_1\cap\chi_2)\leqslant\varphi/\sigma$ implies $w_M(\chi_1/(\chi_1\cap\sigma+\chi_1\cap\chi_2))\leqslant\beta$ and by induction

$$w_M(\chi_1/(\chi_1\cap\chi_2))\leqslant\beta+\gamma<\beta+\alpha.$$

(A similar proof shows that also $w_M(\varphi/\psi)\leqslant\alpha+\beta$.)

We will not use (5), the proof is left to the reader.

Definition. Let p be a pp-complete type.

(1) A pp-formula ψ is *large* in p, if $\psi\notin p$ and for all $\psi\subset\psi_i\notin p$ there is a $\varphi\in p$ s.t. $\psi\subset\varphi$ and $(\psi_1\cap\varphi)+(\psi_2\cap\varphi)\notin p$.

(2) A complete pp-type q is *associated* to p via ψ, if $\psi\notin p$ and $\varphi\in p$ iff $\varphi\in q$ for all ppf φ which lie above ψ.

Examples 7.5. (1) p is indecomposable iff every ppf $\psi \notin p$ is large in p.

Proof. If p is indecomposable and $\psi \subset \psi_i \notin p$, there is $\varphi_0 \in p$ s.t. $(\psi_1 \cap \varphi_0) + (\psi_2 \cap \varphi_0) \notin p$. If we set $\varphi = \varphi_0 + \psi$, we have $\psi \subset \varphi$ and $(\psi_1 \cap \varphi) + (\psi_2 \cap \varphi) \notin p$.

(2) If q is indecomposable and associated to p via ψ, then ψ is large in p.

(3) If p is M-consistent and $\varphi/\psi \in p$ is an *M-minimal pair* (i.e. $\psi(M) = \chi(M)$ iff $\psi(M) \subset \chi(M) \subsetneqq \varphi(M)$), then ψ is large in p.

(4) Two M-consistent types containing the same M-minimal pair φ/ψ are associated via ψ.

Proof. If p is M-consistent and contains φ/ψ, then $\chi \in p$ iff $\varphi(M) \subset \chi(M)$, for all χ above ψ.

Theorem 7.6. *Let p be a pp-complete type. For every ψ large in p there is an indecomposable type q associated to p via ψ. $H(q)$ is isomorphic to a direct factor of $H(p)$ and – up to isomorphy – uniquely determined by ψ. All direct indecomposable factors of $H(p)$ are obtained in this way.*

Proof. Let ψ be large in p. Choose q^+ as a set of pp-formulas, closed under conjunction, with $\psi + \varphi \in p$ for all $\varphi \in q^+$, and maximal with these properties. Set $q = q^+ \cup \{\neg\varphi \mid \varphi \notin q^+\}$. We will see below that q is consistent. If $\varphi \in q$ is above ψ, then clearly $\varphi \in p$. If conversely $\varphi \in p$ is above ψ, we have $(\psi + \bar{\varphi}) \cap \varphi = \psi + (\bar{\varphi} \cap \varphi) \in p$ for all $\bar{\varphi} \in q$. this shows $\varphi \in q$. It remains to show that q is indecomposable: For this assume $\psi_i \notin q$ $(i = 1, 2)$. Then there is a $\varphi \in q$ s.t. $\psi + (\psi_i \cap \varphi) \notin p$. Since ψ is large in p, there is a $\bar{\varphi} \in p$ s.t. $\psi \subset \bar{\varphi}$ (whence $\bar{\varphi} \in q$) and

$$(\psi + \psi_1 \cap \varphi) \cap \bar{\varphi} + (\psi + \psi_2 \cap \varphi) \cap \bar{\varphi} \notin p.$$

But the last expression equals

$$\psi + (\psi_1 \cap \varphi \cap \bar{\varphi}) + (\psi_2 \cap \varphi \cap \bar{\varphi}) \notin p.$$

Therefore

$$\psi_1 \cap \varphi \cap \bar{\varphi} + \psi_2 \cap \varphi \cap \bar{\varphi} \notin q.$$

Now let q be indecomposable (but we do not assume consistency) and associated to p via ψ. Let p be realized by $a \in H(p)$. Since a satisfies $\psi + \varphi$ for every $\varphi \in q^+$, $q^+ \cup \{\psi(x - a)\}$ is consistent in $H(p)$. Let $b \in H(p)$ be a realization of $q^+ \cup \{\psi(x - a)\}$. We show that b realizes q. Thus, let $\bar{\psi} \notin q$. Since q is indecomposable, there is $\varphi \in q$ s.t. $(\psi \cap \varphi) + (\bar{\psi} \cap \varphi) \notin q$. Then also $\psi + (\bar{\psi} \cap \varphi) \notin q$. But now we can conclude that $\psi + (\bar{\psi} \cap \varphi) \notin p$. This implies that b does not satisfy $\bar{\psi} \cap \varphi$. We have shown that $H(q) \cong H(b)$, which is a direct factor of $H(p)$.

If q_i $(i = 1, 2)$ are indecomposable and associated to p via ψ, then q_1 is associated to q_2. Therefore $H(q_1)$ is isomorphic to a direct factor of $H(q_2)$, whence isomorphic to $H(q_2)$.

Finally we show that every indecomposable direct factor U of $H(p)$ comes from a type which is associated to p. For this let a realize p in $H(p) = U \oplus C$ and let π

be the projection from $H(p)$ onto C. Since π is no partial isomorphism, there is a ppf ψ s.t. $H(p) \nvDash \psi(a)$ and $H(p) \vDash \psi(\pi(a))$.

Thus, if q is the type of $b = a - \pi(a)$, we have $H(q) \cong H(b) = U$ and q is associated to p via ψ.

Corollary 7.7. *The following two properties of M are equivalent:*

(a) *Every compact module elementarily equivalent to M is the pure hull of a direct sum of indecomposables.*

(b) *Every M-consistent pp-complete type has a large formula.*

Theorem 7.1 follows from the next lemma ($\varphi/\psi = x \doteq x / x \doteq 0$).

Lemma 7.8. *Let M be a module, φ/ψ a pair of pp-formulas.*

(1) *If $w_M(\varphi/\psi) < \infty$, then every M-consistent pp-complete containing φ/ψ has a large formula.*

(2) *If R is countable and $w_M(\varphi/\psi) = \infty$, then φ/ψ belongs to an M-consistent pp-complete type, which has no large formula.*

Examples. If R is an atomless boolean ring, the type of 1 in R has no large formula. (See 5.8(3)): $p_{\mathfrak{A}}$ has a large formula iff $\mathfrak{A}$ has an irreducible quotient.

Proof. (1) Let p be M-consistent and $\varphi/\varphi \in p$. Choose $\varphi_0/\psi_0 \in p$ of minimal width α. Then ψ_0 is large in p. For if $\psi_0 \subset \psi_i \notin p$ $(i = 1, 2)$ are given, we have e.g. $w_M((\psi_1 \cap \varphi_0)/(\psi_1 \cap \psi_2 \cap \varphi_0)) < \alpha$. By 7.4(2) also $w_M((\psi_1 \cap \varphi_0 + \psi_2 \cap \varphi_0))/(\psi_2 \cap \varphi_0)) < \alpha$. The minimal choice of α implies $\psi_1 \cap \varphi_0 + \chi_2 \cap \varphi_0 \notin p$.

(2) Let $w_M(\varphi/\psi) = \infty$ and $\{\bar{\psi}_i\}$ an enumeration of all pp-formulas. We construct a tree T of pairs of pp-formulas such that

(i) T is finitely branched, has length ω and no endpoints.

(ii) All pairs in T have width ∞.

(iii) If $\bar{\sigma}/\bar{\chi} \in T$ is above $\sigma/\chi \in T$, then (a) $\bar{\sigma} \subset \sigma$ and (b) $\chi \cap \bar{\sigma} \subset \bar{\chi}$.

We construct the layers T_n of T recursively: $T_0 = \{\varphi/\psi\}$. Let T_n be defined. We choose for every $\sigma/\chi \in T_n$ one or two immediate successors in T_{n+1}:

Case 1: $w_M((\sigma \cap \bar{\psi}_n)/(\chi \cap \bar{\psi}_n)) = \infty$. Then σ/χ has the successor $(\sigma \cap \bar{\psi}_n)/(\chi \cap \bar{\psi}_n)$.

Case 2: $w_M((\sigma \cap \bar{\psi}_n)/(\chi \cap \bar{\psi}_n)) < \infty$. Then by 7.4(2) we have also $w_M((\sigma \cap \bar{\psi}_n + \chi)/\chi) < \infty$ and by 7.4(3), $w_M(\sigma/(\sigma \cap \bar{\psi}_n + \chi)) = \infty$. Whence there are $\sigma \cap \bar{\psi}_n + \chi \subset \tau_i \subset \sigma$ s.t. $w_M(\tau_i/(\tau_1 \cap \tau_2)) = \infty$. We define $\tau_1/(\tau_1 \cap \tau_2)$ and $\tau_2/(\tau_1 \cap \tau_2)$ as the two immediate successors of σ/χ in T_{n+1}.

Now set $p^+ = \{\sigma \mid \text{for some } n \text{ and all } \bar{\sigma}/\bar{\chi} \in T_n\ \bar{\sigma}(M) \subset \sigma(M)\}$.

Claim. $\rho \notin p^+$ iff $\rho \cap \sigma \subset \chi$ for some $\sigma/\chi \in T$.

Proof. If $\rho \cap \sigma \subset \chi$ for some $\sigma/\chi \in T$, we have by property (iii) $\rho \cap \bar{\sigma} \subset \bar{\chi}$ for all

$\bar{\sigma}/\bar{\chi}$ above σ/χ. Whence $\bar{\sigma}(M) \not\subset \rho(M)$ for all such $\bar{\sigma}/\bar{\chi} \in T$, and ρ cannot belong to p^+.

If conversely $\rho = \bar{\psi}_n \notin p^+$, then in the construction of T case 2 must occur for some $\sigma/\chi \in T_n$. But then $\tau_1 \cap \rho \subset \sigma \cap \bar{\psi}_n \subset \tau_1 \cap \tau_2$.

Set $p = p^+ \cup \{\neg\sigma \mid \sigma \notin p^+\}$. We show first that p is M-consistent. Since – by property (iiia) – p^+ is closed under conjunction, it is enough to show, that $\sigma(M) \subset \chi_1(M) \cup \cdots \cup \chi_n(M)$ and $\sigma \in p$ implies that some χ_i belongs to p. But by 1.4 one of the groups $\sigma \cap \chi_i(M)$ is of finite index in $\sigma(M)$. Now the above claim allows us to conclude that $\sigma \cap \chi_i \in p$, for $w_M(\sigma/(\sigma \cap \chi_i))$ must be finite. Note, that the claim also implies that $\varphi/\psi \in p$.

Finally we show that no $\bar{\psi}_n$ is large in p. Assume $\bar{\psi}_n \notin p$. Let $\sigma_1/\chi_1, \ldots, \sigma_m/\chi_m$ be the elements of T_{n+1}, which are constructed in case 2. We have $m \geq 1$, since $\bar{\psi}_n \notin p$. Set $\psi_i = \bar{\psi}_n + \sigma_i$. We show $\psi_i \notin p$ and $(\psi_1 \cap \rho) + \cdots + (\psi_m \cap \rho) \in p$ for all $\rho \in p$. This implies, as one easily sees, that $\bar{\psi}_n$ is not large in p.

Now let σ_i/χ_i be constructed as $\tau_1/(\tau_1 \cap \tau_2)$ as immediate successor of $\sigma/\chi \in T_n$. Then

$$\psi_i \cap \tau_2 = (\bar{\psi}_n + \tau_1) \cap \sigma \cap \tau_2 = (\bar{\psi}_n \cap \sigma + \tau_1) \cap \tau_2 = \tau_1 \cap \tau_2.$$

The above claim yields $\psi_i \notin p$.

Finally assume that $\rho \in p$, and that $\bar{\sigma}(M) \subset \rho(M)$ for all $\bar{\sigma}/\bar{\chi} \in T_k$. W.l.o.g. $k > n$. Look at $\bar{\sigma}/\bar{\chi} \in T_k$, let $\bar{\sigma}/\bar{\chi}$ lie above $\sigma/\chi \in T_n$. If in our construction case 1 applies to σ/χ, then $\bar{\sigma} \subset \bar{\psi}_n$. If in the construction case 2 occurred, $\bar{\sigma}/\bar{\chi}$ lies above some σ_i/χ_i, whence $\bar{\sigma} \subset \psi_i$. In both cases we have $\bar{\sigma}(M) \subset (\psi_1 \cap \rho) + \cdots + (\psi_m \cap \rho)(M)$. Whence $(\psi_1 \cap \rho) + \cdots + (\psi_m \cap \rho) \in p$.

This proves 7.7(2) and Theorem 7.1.

We indicate a more direct proof of 7.3(1), which resembles the original proof of Garavaglia's:

Lemma 7.9. *A pure submodule M of a module N with Krull dimension has also Krull dimension.*

Proof (7.9 follows also from 8.5(2)). Let $(\varphi_r)_{r \in \mathbb{Q}}$ define a dense chain in M s.t. $r \leq s \Leftrightarrow \varphi_r(M) \subset \varphi_s(M)$. Let $(r_i)_{i \in \mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Define

$$\psi_{r_i} = (\varphi_{r_i} \cap \bigcap \{\psi_{r_j} \mid j < i, r_i < r_j\}) + \sum \{\psi_{r_j} \mid j < i, r_j < r_i\}.$$

Then $\varphi_r(M) = \psi_r(M)$ and $r \leq s \Rightarrow \psi_r \subset \psi_s$. Whence $(\psi_r)_{r \in \mathbb{Q}}$ defines a dense chain of pp-definable subgroups in N.

To prove 7.3(1), we assume that M is compact and has Krull dimension. Look at the decomposition 6.1 of M into a direct sum of the pure hull of a direct sum of indecomposable and a module E without indecomposable factors. We want to prove that $E = 0$.

Now by 7.9, E has Krull dimension. Then, if $E \neq 0$, there is an E-minimal pair φ/ψ. (Otherwise one constructs easily a dense chain in E.) By 4.8 there is an E-consistent indecomposable type which contains φ/ψ. Choose $a \in \varphi(M) \setminus \psi(M)$. Then by 7.5(3) and 7.6, $H(p)$ is isomorphic to a direct factor of $H(a)$. Whence E has an indecomposable direct factor, isomorphic to $H(p)$. Contradiction.

7.5(4) and 7.6 imply, that two M-consistent indecomposable types which contain the same M-minimal pair determine isomorphic indecomposable modules. The next result strengthen this observation.

Lemma 7.10. *Let p, q be two indecomposable types, containing φ/ψ. If $H(p)$ and $H(q)$ are not isomorphic, then there is a ppf χ s.t. $\psi \subset \chi \subset \varphi$ and either $\varphi/\chi \in p$, $\chi/\psi \in q$ or $\varphi/\chi \in q$, $\chi/\psi \in p$.*

Proof. We prove first that $p(x) \cup q(y) \cup \{\psi(x-y)\}$ is inconsistent. Otherwise there is a compact module M, $a \in M$ realizing p and $b \in M$ realizing q s.t. $M \models \psi(a-b)$. But then $H(a)$ and $H(b)$ are non-isomorphic dependent indecomposables. This is impossible by 6.2.

By 4.5 there is $\bar{\varphi}/\bar{\psi} \in p$ s.t.

$$(*) \qquad \vdash \bar{\varphi}(x) \wedge \psi(x-y) \wedge q(y) \to \bar{\psi}(x) \quad \text{and} \quad \psi \cap \bar{\varphi} \subset \bar{\psi} \subset \bar{\varphi} \subset \varphi.$$

Case 1: $\bar{\psi}(x) + \psi(x) \in q$. Then we set $\chi = \bar{\psi} + \psi$. Clearly $\psi \subset \chi \subset \varphi$ and $\chi/\psi \in q$. We have $\varphi/\chi \in p$ since $\chi \cap \bar{\varphi} = \bar{\psi}$.

Case 2: $\bar{\psi}(x) + \psi(x) \notin q$. Now set $\chi = \bar{\varphi} + \psi$. Clearly $\psi \subset \chi \subset \varphi$ and $\chi/\psi \in p$. It remains to show that $\varphi/\chi \in q$ i.e. $\chi \notin q$. But otherwise we have

$$\exists x\, (\bar{\varphi}(x) \wedge \psi(x-y)) \in q(y).$$

This together with $(*)$ implies $\exists x\, (\bar{\psi}(x) \wedge \psi(x-y)) \in q(y)$, i.e. $\bar{\psi} + \psi \in q$. Contradiction.

Chapter III: Applications

8. Ranks of indecomposable modules

We use a rank analysis of indecomposable modules to prove that (for countable R) a module M has Krull dimension iff there are only countably many isomorphism types of indecomposable factors in modules elementary equivalent to M (8.1).

In 8.6 we compare this rank with the dimension of modules with Krull dimension.

Definition. For a module M we denote by $\mathbb{U}_M$ the class of all indecomposable modules which occur as direct factors in modules elementarily equivalent to M.

Note. $\mathbb{U}_M = \{H(p) \mid p$ indecomposable and M-consistent$\}$
$= \{U$ indecomposable $\mid I_U(M) > 0\}$ (see 6.11).

Theorem 8.1. *Let R be a countable ring. An R-module M has Krull dimension iff $\mathbb{U}_M$ contains only countably many isomorphism types.*

First we prove

Corollary 8.2. *Let R be a commutative noetherian ring, where all localizations $R_{\mathfrak{M}}$ are fields or discrete valuation rings. Then every R-module has Krull dimension.*

Proof. If R is countable, we have countably many maximal ideals and the claim follows from 5.2.

Now let R be uncountable. If the R-module M has no Krull dimension, there is a countable $S_0 \subset R$ s.t. all ${}_SM$, where S is a ring between S_0 and R, have no Krull dimension. If S is an elementary substructure of R, S is again noetherian. (For an ascending sequence $\mathfrak{A}_0 \subsetneqq \mathfrak{A}_1 \subsetneqq \cdots$ of finitely generated ideals of S leads to a sequence $R\mathfrak{A}_1 \subsetneqq R\mathfrak{A}_2 \subsetneqq \cdots$ of ideals of R.) Since in noetherian rings the fact that all $R_{\mathfrak{M}}$ are valuation rings is expressible by an $L_{\omega_1\omega}$-sentence, we find an S with this property. But now ${}_SM$ has Krull dimension.

One half of 8.1 follows from

Lemma 8.3. *Let R be countable, M an R-module. If there is a dense chain of pp-definable subgroups between $\psi(M)$ and $\varphi(M)$, there are $2^{\aleph_0}$ non-isomorphic $U \in \mathbb{U}_M$ s.t. $\varphi/\psi(U) > 1$.*

Proof. Let a dense family be defined by pp-formulas $(\varphi_r)_{r\in\mathbb{Q}}$ s.t. $r < s$ implies

$$\psi(M) \subset \varphi_r(M) \subsetneqq \varphi_s(M) \subset \varphi(M).$$

Let $\{\psi_i\}_{i\in\omega}$ be an enumeration of all pp-formulas. For every real α let

$$p_\alpha = \{\varphi_r \mid r \geqslant \alpha\} \cup \{\neg\varphi_s \mid s < \alpha\}.$$

Define $I_\alpha \subset \omega$ inductively ($\alpha \in \mathbb{R}$)

$$i \in I_\alpha \quad \text{iff} \quad p_\alpha \cup \{\psi_j \mid j \in I_\alpha, j < i\} \cup \{\psi_i\} \text{ is } M\text{-consistent.}$$

Set

$$q_\alpha = \{\psi_j \mid j \in I_\alpha\} \cup \{\neg\psi_j \mid j \notin I_\alpha\}.$$

Using 4.7 we can see that the q_α are M-consistent indecomposable types.

We will show that $2^{\aleph_0}$ of the $H(q_\alpha)$ are non-isomorphic. Since q_α is realized in $H(q_\beta)$ iff $H(q_\alpha) \cong H(q_\beta)$, it is enough to show that in each $U \in \mathbb{U}_M$ at most countably many q_α are realized.

Assume that at the contrary $O_0 = \{\alpha \mid q_\alpha$ is realized in $U\}$ is uncountable. Let q_α be realized by $a_\alpha \in U$ ($\alpha \in O_0$). Fix $a \in U \setminus 0$. For each $\alpha \in O_0$ there is a pp-formula $\chi_\alpha(x, y)$ s.t.

$$U \vDash \chi_\alpha(a, a_\alpha) \wedge \neg\chi_\alpha(0, a_\alpha).$$

Since there are only countably many formulas, there is an uncountable $O_1 \subset O_0$ s.t. $\bar{\varphi} = \chi_\alpha$ for all $\alpha \in O_1$. Let $\psi_n = \bar{\varphi}(0, x)$. Since there are only finitely many $I_\alpha \cap n$, there are reals $\beta < \alpha$ in O_1 with $I_\alpha \cap n = I_\beta \cap n = J$. We have $M \vDash \neg\chi_\alpha(0, a_\alpha)$ and therefore $\neg\psi_n \in q_\alpha$. By construction

$$M \vDash (p_\alpha \cup \{\psi_j \mid j \in J\}) \rightarrow \neg\psi_n$$

and, since $U \in \mathbb{U}_M$,

$$U \vDash (p_\alpha \cup \{\psi_j \mid j \in J\}) \rightarrow \neg\psi_n. \tag{1}$$

$U \vDash \psi_j(a_\alpha) \wedge \psi_j(a_\beta)$ $(j \in J)$ implies $U \vDash \psi_j(a_\alpha - a_\beta)$ $(j \in J)$. This together with (1) and $U \vDash p_\alpha(a_\alpha - a_\beta)$ yields $U \vDash \neg\psi_n(a_\alpha - a_\beta)$. But on the other hand $U \vDash \bar{\varphi}(a, a_\alpha) \wedge \bar{\varphi}(a, a_\beta)$ implies $U \vDash \psi_n(a_\alpha - a_\beta)$. Contradiction.

For the proof of the other half of 8.1 we give every $U \in \mathbb{U}_M$ a rank.

Definition. Let M be an R-module. Let $\mathrm{rk}_M(U) = \mathrm{rk}(U)$ be the Cantor–Bendixon rank of U in the topological space $\mathbb{U}_M \subset \mathbb{U}^R$ (see 4.9). I.e. for all ordinals α $\mathrm{rk}(U) = \alpha$ *iff* there is a pair $\psi \subset \varphi$ of pp-formulas s.t. $V \cong U \Leftrightarrow \mathrm{rk}(V) \not< \alpha$ & $\varphi/\psi(V) > 1$ for all $V \in \mathbb{U}_M$. (We say: *φ/ψ isolates U.*)

If $\mathrm{rk}(U) = \alpha$ for some ordinal α, we say 'U has a rank'. Otherwise $\mathrm{rk}(U) = \infty$.

The definition of rank depends on M insofar M determines the class $\mathbb{U}_M$. If we want to deal with all indecomposable R-modules simultaneously our definition yields this as a special case. For there are M s.t. $\mathbb{U}_M = \mathbb{U}^R$.

Lemma 8.4. (1) *Up to isomorphism there are at most $|R| + \aleph_0$ many $U \in \mathbb{U}_M$ with a rank.*

(2) *If M has Krull dimension, every $U \in \mathbb{U}_M$ has a rank.*

It is clear that if α occurs as a rank also all smaller ordinals are ranks. If we define $\mathrm{rank}(M) = \sup\{\mathrm{rk}(U) \mid U \in \mathbb{U}_M\}$, 8.4(1) implies that $\mathrm{rank}(M) = \infty$ or is $< (|R| + \aleph_0)^+$.

Proof. (1) If φ/ψ isolates U, then φ/ψ isolates no other $V \in \mathbb{U}_M$, $V \not\cong U$. Therefore there are at most as many U with a rank as there are pairs of pp-formulas.

(2) Let $\{U_i\}_{i \in I}$ represent the isomorphism types of all $U \in \mathbb{U}_M$ without rank. Let N be the pure hull of the direct sum of the U_i. N is a direct factor of a module elementarily equivalent to M (since the U_i are assumed to be pairwisely non-isomorphic). Whence by 7.9 N has Krull dimension.

Claim. $\mathbb{U}_N = \{U_i\}_{i \in I}$. (This is 4.10.)

Proof. Clearly $\{U_i\}_{i\in I}\subset \mathbb{U}_N \subset \mathbb{U}_M$. If V has a rank and is isolated by φ/ψ, we have $\varphi/\psi(U_i)=1$ $(i\in I)$ and therefore $\varphi/\psi(N)=1$. Whence $V\notin \mathbb{U}_N$.

Assume now for contradiction that $N\neq 0$. Then there is a N-minimal pair φ/ψ. By 4.8 there is an $U_i\in\mathbb{U}_N$ s.t. $\varphi/\psi(U_i)>1$. By 7.10, U_i is uniquely determined. Let β be greater than all ranks. Since $\mathrm{rk}(U_i)\not<\beta$, φ/ψ isolates U_i. Contradiction.

This completes the proof of Theorem 8.1.

In a module with Krull dimension, we can attach an ordinal to every interval φ/ψ which measures the extent to which there is 'almost a dense chain' in φ/ψ (Garavaglia). We are going to compare this with our rank analysis of indecomposables. (Note that the dimension defined below grows faster than Garavaglias dimension [11]).

Definition. Let M be a module. We define for pairs $\psi\subset\varphi$ of pp-formulas $\dim_M(\varphi/\psi)=\alpha$ (the 'dimension') by induction on α.

$\dim(\varphi/\psi)=-1$ iff $\psi(M)=\varphi(M)$,

$\dim(\varphi/\psi)=\alpha$ iff (a) $\dim(\varphi/\psi)\not<\alpha$,

(b) there is no infinite sequence $\psi\subset\varphi_0\subset\varphi_1\subset\cdots\varphi$ with $\dim(\varphi_{i+1}/\varphi_i)\not<\alpha$,

(c) there is no infinite sequence $\varphi\supset\varphi_0\supset\varphi_1\cdots\supset\psi$ with $\dim(\varphi_i/\varphi_{i+1})\not<\alpha$.

If $\dim(\varphi/\psi)=\alpha$ is defined for some α, we say that φ/ψ has a dimension. Otherwise we write $\dim(\varphi/\psi)=\infty$. $\dim(M)=\dim(x\doteq x/x\doteq 0)$.

Clearly $\dim(\varphi/\psi)\leqslant\dim(\bar\varphi/\bar\psi)$ if $\bar\psi\subset\psi\subset\varphi\subset\bar\varphi$. This shows that in the above definition (b), (c) it is enough to have $\dim(\varphi_{i+1}/\varphi_i)\not<\alpha$, $\dim(\varphi_i/\varphi_{i+1})\not<\alpha$ for infinitely many i.

The following lemma is in [11].

Lemma 8.5. (1) $\dim(\varphi/\psi)=\infty$ *iff there is a dense chain of pp-definable subgroups of M between ψ and φ. Whence* $\dim(M)<\infty$ *iff M has Krull dimension.*

(2) *If M is pure in N, then*

$$\dim_N(\varphi/\psi)=\max(\dim_M(\varphi/\psi),\dim_{N/M}(\varphi/\psi)).$$

(3) *If $N\equiv M$, then*

$$\dim_N(\varphi/\psi)=\dim_M(\varphi/\psi)=\dim_{M^\kappa}(\varphi/\psi)\qquad(\kappa>0).$$

(4) *If $\varphi/\psi\leqslant\bar\varphi/\bar\psi$ (in the sense of* 1.9), *then*

$$\dim(\varphi/\psi)\leqslant\dim(\bar\varphi/\bar\psi).$$

(5) *If $\psi\subset\chi\subset\varphi$, then*

$$\dim(\varphi/\psi)=\max(\dim(\varphi/\chi),\dim(\chi/\psi)).$$

Proof. (1) One shows immediately by induction on α: If there is a dense chain between ψ and φ, $\dim(\varphi/\psi)\neq\alpha$. On the other hand, if $\dim(\varphi/\psi)=\infty$, there is e.g. a sequence $\psi\subset\varphi_0\subset\varphi_1\subset\cdots\subset\varphi$ s.t. $\dim(\varphi^{i+1}/\varphi_i)=\infty$. Thus $\dim(\varphi/\varphi_1)=\dim(\varphi_1/\psi)=\infty$. Proceeding in this manner one constructs a dense chain in φ/ψ.

(3) This is immediate, since $\varphi(M)\mapsto\varphi(N)\mapsto\varphi(M^\kappa)$ yields an isomorphism of the lattices of pp-definable subgroups of M, N and M^κ.

(4) Noethers isomorphism theorem implies: the lattice of pp-definable subgroups between ψ and $\psi+\varphi$ is isomorphic to the lattice of $\psi\cap\varphi$ and φ. Whence $\dim(\psi+\varphi/\psi)=\dim(\varphi/\psi\cap\varphi)$.

(2) Suppose $N=M\oplus L$ (by (3) we can do this). We show by induction on α that

$$\dim_N(\varphi/\psi)>\alpha \quad \text{iff} \quad \dim_M(\varphi/\psi)>\alpha \text{ or } \dim_L(\varphi/\psi)>\alpha.$$

$\alpha=-1$ is clear. $\dim_M(\varphi/\psi)>\alpha$ iff (e.g.) there are $\psi\subset\varphi_0\subset\varphi_1\subset\cdots\subset\varphi$ s.t. $\dim_M(\varphi_{i+1}/\varphi_i)\geqslant\alpha$ for infinitely many i iff (by induction) there are $\psi\subset\varphi_0\subset\varphi_1\subset\cdots\subset\varphi$ s.t. $\dim_M(\varphi_{i+1}/\varphi_i)\geqslant\alpha$ for infinitely many i or $\dim_L(\varphi_{i+1}/\varphi_i)\geqslant\alpha$ for infinitely many i iff $\dim_M(\varphi/\psi)\geqslant\alpha$ or $\dim_L(\varphi/\psi)\geqslant\alpha$.

(5) Clearly $\dim(\varphi/\psi)\geqslant\max(\cdots)$. We show by induction on α that $\dim(\varphi/\psi)>\alpha$ implies $\dim(\varphi/\chi)>\alpha$ or $\dim(\chi/\psi)>\alpha$; If $\dim(\varphi/\psi)>\alpha$, there is (e.g.) a chain $\psi\subset\varphi_0\subset\varphi_1\subset\cdots\subset\varphi$ s.t. $\dim(\varphi_{i+1}/\varphi_i)\geqslant\alpha$. Our two chains (the other is $\psi\subset\chi\subset\varphi$) have refinements

$$\psi\subset\varphi_0\subset\sigma_0\subset\varphi_1\subset\sigma_1\subset\cdots\subset\varphi,$$

$$\psi\subset\tau_0\subset\tau_1\subset\cdots\subset\chi\subset\rho_0\subset\rho_1\subset\cdots\subset\varphi$$

s.t.

$$\sigma_i/\varphi_i\leqslant\tau_{i+1}/\tau_i \quad \text{and} \quad \varphi_{i+1}/\sigma_i\leqslant\rho_{i+1}/\rho_i.$$

(Look at any proof of the Jordan–Hölder–Schreier theorem). By induction $\dim(\varphi_{i+1}/\sigma_i)\geqslant\alpha$ for infinitely many i or $\dim(\sigma_i/\varphi_i)\geqslant\alpha$ for infinitely many i. Whence by (4), $\dim(\rho_{i+1}/\rho_i)\geqslant\alpha$ for infinitely many i or $\dim(\tau_{i+1}/\tau_i)\geqslant\alpha$ for infinitely many i. That means $\dim(\varphi/\chi)>\alpha$ or $\dim(\chi/\psi)>\alpha$.

If $\dim(\varphi/\psi)=\alpha$, it is easy to see that there is a sequence $\psi=\chi_0\subset\chi_1\subset\cdots\subset\chi_n=\varphi$ s.t. the pairs χ_{i+1}/χ_i are α-minimal, i.e. $\dim(\chi_{i+1}/\chi_i)=\alpha$ and for all $\chi_i\subset\sigma\subset\chi_{i+1}$ either $\dim(\chi_{i+1}/\sigma)<\alpha$ or $\dim(\sigma/\chi_i)<\alpha$.

We call n – which is uniquely determined (Jordan–Hölder argument) – the multiplicity $\mu(\varphi/\psi)$. The 0-minimal pairs are just the minimal pairs of M.

One sees immediately that $\varphi/\psi\leqslant\bar\varphi/\bar\psi$ and $\dim(\varphi/\psi)=\dim(\bar\varphi/\bar\psi)$ imply $\mu(\varphi/\psi)\leqslant\mu(\bar\varphi/\bar\psi)$.

Definition. Let M be a fixed module. We call a pair φ/ψ of pp-formulas *small*, if every M-consistent pp-complete type which contains φ/ψ has a large formula (see 7.5).

Remarks. (1) φ/ψ is small, iff $\varphi/\psi(E)=1$ for every compact E which has no indecomposable factor and is a direct factor of a module elementarily equivalent to M.

The proof of 4.11 shows that φ/ψ is small, if $\bar{\varphi}/\bar{\psi}$ is small and $(\varphi/\psi)\cap \mathbb{U}_M \subset (\bar{\varphi}/\bar{\psi})\cap \mathbb{U}_M$.

(2) $\dim_M(\varphi/\psi)<\infty \Rightarrow w_M(\varphi/\omega)<\infty \Rightarrow \varphi/\psi$ is small (7.8(1)).

(3) If R is countable, then φ/ψ small $\Leftrightarrow w_M(\varphi/\psi)<\infty$ (7.8(2)).

The following is our main theorem on ranks:

Theorem 8.6. *Let M be an R-module. (Dimension, rank and smallness are defined w.r.t. M. The U range over $\mathbb{U}_M$.* $\max\emptyset=-1$.) *If R is countable or φ/ψ is small, then*

$$\dim(\varphi/\psi)=\max\{\mathrm{rk}(U)\mid \varphi/\psi(U)>1\}.$$

Problem. Do the two equation hold without the assumption "R countable...?" E.g. is it true that M has Krull dimension if all U have ranks?

Corollary 8.7. *If R is countable or U has a small neighbourhood, and if $\mathcal{S}$ is a base of neighbourhoods of U in $\mathbb{U}_M$, then*

$$\mathrm{rk}(U)=\min\{\dim(\varphi/\psi)\mid \varphi/\psi\in\mathcal{S}\}.$$

Proof. "$\leqslant$" follows immediately from 8.6. If $\mathrm{rk}(U)=\alpha$ $(<\infty)$, we find $\varphi/\psi\in\mathcal{S}$ which isolates U. The above remark shows that we can assume – if R is uncountable – that φ/ψ is small. 8.6 yields $\dim(\varphi/\psi)=\alpha$.

Corollary 8.8. (1) *If R is countable or M of bounded width, then*

$$\dim(M)=\max\{\mathrm{rk}(U)\mid U\in\mathbb{U}_M\}.$$

(2) *M has Krull dimension iff every $U\in\mathbb{U}_M$ has a rank and every compact module elementarily equivalent to M is the pure hull of a direct sum of indecomposables.*

We begin the proof of 8.6 with two lemmas (8.10 and 8.11) which are special cases of 8.6. the following 8.9 is used in their proofs.

Definition. Let $\psi\subset\varphi$ be a pair of pp-formulas, M a module. By $[\psi,\varphi]_M$ we denote the interval of all pp-definable subgroups of M between $\psi(M)$ and $\varphi(M)$. If $a\in M$ and an interval is given, we denote by $F(a)$ the filter of all groups in the interval, which contain a.

Lemma 8.9 ('Goursat's theorem', cf. [29, p. 171]). *Let $a, b\in M$ be dependent i.e. there is a pp-formula $\vartheta(x,y)$ s.t. $M\models\vartheta(a,b)$ and $M\not\models\vartheta(a,0)$. Then the two* *structures*

$$([\vartheta(x,0), \exists y\, \vartheta(x,y)]_M, \subset, F(a)) \quad \textit{and} \quad ([\vartheta(0,y), \exists x\, \vartheta(x,y)]_M, \subset, F(b))$$

are isomorphic.

Proof. We define the isomorphism and its inverse by $\psi(M) \mapsto \psi^*(M)$ and $\psi(M) \mapsto \psi^+(M)$, where $\psi^*(y) = \exists x\, (\vartheta(x,y) \wedge \psi(x))$ and $\psi^+(x) = \exists y\, (\vartheta(x,y) \wedge \psi(y))$. We have to prove:

(1) $\psi^*(M) \in [\vartheta(0,y), \exists x\, \vartheta(x,y)]_M$.
(2) $\psi^+(M) \in [\vartheta(x,0), \exists y\, \vartheta(x,y)]_M$.
(3) $\psi(M) \subset \varphi(M)$ implies $\psi^*(M) \subset \varphi^*(M)$ and $\psi^+(M) \subset \varphi^+(M)$.
(4) $a \in \psi(M)$ implies $b \in \psi^*(M)$.
(5) $b \in \psi(M)$ implies $a \in \psi^+(M)$.
(6) If $\psi(M) \in [\vartheta(x,0), \exists y\, \vartheta(x,y)]_M$, then $\psi(M) = \psi^{*+}(M)$.
(7) If $\psi(M) \in [\vartheta(0,y), \exists x\, \vartheta(x,y)]_M$, then $\psi(M) = \psi^{+*}(M)$.

Only (6) and (7) require a proof. We prove (6): If $c \in \psi(M)$, then, since $\psi(M) \subset \exists y\, \vartheta(x,y)(M)$, there is $d \in M$ s.t. $M \models \vartheta(c,d)$. Clearly $d \in \psi^*(M)$ and therefore $c \in \psi^{*+}(M)$.

If $c \in \psi^{*+}(M)$, there is $d \in \psi^+(M)$ s.t. $M \models \vartheta(c,d)$ and $e \in \psi(M)$ s.t. $M \models \vartheta(e,d)$. We obtain $M \models \vartheta(c-e, 0)$ and – since $\vartheta(M,0) \subset \psi(M)$ – $c - e \in \psi(M)$. This gives $c \in \psi(M)$.

Lemma 8.10. *Suppose $H(p) \cong H(q) \in \mathbb{U}_M$. If p contains a pair of pp-formulas of dimension $\leqslant \alpha$, then also q contains such a pair.*

Proof. We can assume that p and q are realized in M by dependent elements a, b (cf. 8.5(3)), 8.9 gives $\varphi/\psi \in p$ and $\sigma/\chi \in q$ s.t. $([\psi, \varphi]_M, \subset, F(a))$ and $([\chi, \sigma]_M, \subset, F(b))$ are isomorphic. Suppose $\rho/\tau \in p$ and $\dim(\rho/\tau) \leqslant \alpha$. By 4.6 there is $\bar{\varphi}/\bar{\psi} \in p$ s.t. $\psi \subset \bar{\psi} \subset \bar{\varphi} \subset \varphi$ and $\bar{\varphi}/\bar{\psi} \leqslant \rho/\tau$. We have $\dim(\bar{\varphi}/\bar{\psi}) \leqslant \alpha$ by 8.5(4). Let $\tilde{\psi}(M)$ and $\tilde{\varphi}(M) \in [\chi, \sigma]_M$ correspond to $\bar{\psi}(M)$ and $\bar{\varphi}(M)$. We can assume that $\tilde{\psi} \subset \tilde{\varphi}$. Then $\tilde{\psi}/\tilde{\varphi} \in q$ and $\dim(\tilde{\psi}/\tilde{\varphi}) = \dim(\bar{\psi}/\bar{\varphi}) \leqslant \alpha$.

Note. Since every interval of dimension 0 decomposes into finitely many M-minimal pairs, we have: If p contains an M-minimal pair, then q too.

Lemma 8.11. *Suppose R is countable or φ/ψ is small. If (φ/ψ) contains – up to isomorphy – exactly one $U \in \mathbb{U}_M$, then there is an M-minimal pair between ψ and φ.*

Proof. If there is no M-minimal pair between ψ and φ, one can construct a dense chain of pp-definable subgroups between ψ and φ. If R is countable, 8.3 gives the contradiction.

Now suppose that φ/ψ is small. The proof of 7.9 yields a family $(\chi_r)_{r \in \mathbb{Q}}$ of pp-formulas s.t. $\psi \subset \chi_r \subset \chi_s \subset \varphi$, $\chi_r(M) \neq \chi_s(M)$ for all $r < s$. We construct two M-consistent indecomposable types p and q s.t. $\varphi/\psi \in p, q$ and $H(p) \not\cong H(q)$. For p we choose the type constructed in 4.8. ψ is *maximal* in p, i.e. $\psi \notin p$ and for all $\bar{\psi} \notin p$ above ψ there is $\sigma \in p$ s.t. $\bar{\psi} \cap \sigma \subset \psi$. (In fact for *all* $\bar{\psi} \notin p$ there is such a $\sigma \in p$.)

On the other hand let r^+ be a set of pp-formulas, maximal with the properties: (a) r^+ is closed under conjunction, (b) $(\chi_t \cap \sigma)(M) \neq (\chi_s \cap \sigma)(M)$ for all $t < s$ and $\sigma \in r^+$. Set $r = r^+ \cup \{\neg\sigma \mid \sigma \notin r^+\}$. Clearly $\varphi/\psi \in r$.

Claim. r is M-consistent.

Proof. If not, there are $\sigma \in r$ and $\sigma_i \notin r$ s.t. $\sigma(M) \subset \sigma_1(M) \cup \cdots \cup \sigma_n(M)$. By 1.4 we have that e.g. $\sigma/\sigma \cap \sigma_1(M)$ is finite. If $\rho \in r$ is $\subset \sigma$ we have for all $t < s$

$$(\chi_s \cap \rho)/(\chi_s \cap \sigma_1 \cap \rho)(M) \cdot (\chi_s \cap \sigma_1 \cap \rho)/(\chi_t \cap \sigma_1 \cap \rho)(M) \geqslant (\chi_s \cap \rho)/(\chi_t \cap \rho)(M).$$

The right hand side is infinite, the first factor on the left side is finite, whence $(\chi_s \cap \sigma_1 \cap \rho)/(\chi_t \cap \sigma_1 \cap \rho(M)$ is infinite. This shows that $\sigma_1 \in r$. Contradiction.

Since φ/ψ is small, we have $\varphi/\psi(E) = 1$ for every factor E of $H(r)$ which has no indecomposable factor. Whence by 6.1, there is an indecomposable factor $H(q)$ of $H(r)$ s.t. $\varphi/\psi \in q$. We prove that $H(p) \not\cong H(q)$: Otherwise let a and b realize r and p in $H(r) = N$. Then 8.9 gives $\rho/\chi \in r$ and $\bar{\rho}/\bar{\chi} \in p$ s.t. $([\chi, \rho]_N, \subset, F(a))$ and $([\bar{\chi}, \bar{\rho}]_N, \subset, F(b))$ are isomorphic. Choose $\pi \in p$ s.t. $\bar{\chi} \cap \pi \subset \psi$. One checks easily that $(\psi \cap \pi + \bar{\chi}) \cap \bar{\rho}$ is maximal in p. Choose $\tilde{\psi}$ between χ and ρ s.t. $\tilde{\psi}(M)$ corresponds to $(\psi \cap \pi + \bar{\chi}) \cap \bar{\rho}$ in the above isomorphism. Clearly $\tilde{\psi}$ is maximal in r. Since $\tilde{\psi} \notin r$, there is $\bar{\sigma} \in r$ and $t < s$ s.t.

$$(\chi_t \cap \bar{\sigma} \cap \tilde{\psi})(M) = (\chi_s \cap \bar{\sigma} \cap \tilde{\psi})(M).$$

Choose $t < u < s$ and set $\bar{\psi} = \chi_u \cap \bar{\sigma} + \tilde{\psi}$. Then $\tilde{\psi} \subset \bar{\psi} \notin r$, for $(\chi_s \cap \bar{\sigma} \cap \bar{\psi})(M) = (\chi_u \cap \bar{\sigma} \cap \bar{\psi})(M)$. Since $\tilde{\psi}$ is maximal in r there is $\sigma \in r$ s.t. $\bar{\psi} \cap \sigma \subset \tilde{\psi}$. But then

$$(\chi_t \cap \bar{\sigma} \cap \bar{\psi} \cap \sigma)(M) = (\chi_s \cap \bar{\sigma} \cap \bar{\psi} \cap \sigma)(M).$$

Since $\chi_u \cap \bar{\sigma} \subset \bar{\psi}$,

$$(\chi_t \cap \bar{\sigma} \cap \chi_u \cap \sigma)(M) = (\chi_s \cap \bar{\sigma} \cap \chi_u \cap \sigma)(M).$$

This yields

$$(\chi_t \cap \bar{\sigma} \cap \sigma)(M) = (\chi_u \cap \bar{\sigma} \cap \sigma)(M)$$

contradicting $\bar{\sigma} \cap \sigma \in q$.

Proof of 8.6. The theorem follows from the first two of the following three claims:

Claim 1. $U \in (\varphi/\psi)$, $\mathrm{rk}(U) > \alpha \Rightarrow \dim(\varphi/\psi) > \alpha$.

Claim 2. If R is countable or φ/ψ is small, then $\dim(\varphi/\psi) \geqslant \alpha \geqslant 0 \Rightarrow$ there is $U \in (\varphi/\psi)$ s.t. $\mathrm{rk}(U) \geqslant \alpha$.

Claim 3. If R is countable, or U has a small neighbourhood, then $\mathrm{rk}(U)=\alpha \Rightarrow$ there is a pair s.t. $U\in(\varphi/\psi)$ and $\dim(\varphi/\psi)=\alpha$. ($\alpha$ an ordinal or $=-1$).

Proof of Claim 1: Induction on α. Suppose $\mathrm{rk}(U)>\alpha$. Then, whenever $\bar{\varphi}/\bar{\psi}(U)>1$, there is a $V\not\cong U$, $\mathrm{rk}(V)\geqslant\alpha$, $\bar{\varphi}/\bar{\psi}(V)>1$. We define a sequence $\psi=\psi_0\subset\psi_1\subset\cdots\subset\varphi_1\subset\varphi_0=\varphi$ s.t. $\varphi_i/\psi_i(U)>1$.

If φ_i/ψ_i is defined, choose V_i s.t. $V_i\not\cong U$, $\mathrm{rk}(V_i)\geqslant\alpha$, $\bar{\varphi}_i/\bar{\psi}_i(V_i)>1$. By 7.10, there is $\psi_i\subset\chi\subset\varphi_i$ s.t. $\chi/\psi_i(U)>1$, $\varphi_i/\chi(V_i)>1$ (set $\varphi_{i+1}/\psi_{i+1}=\chi/\psi_i$) *or* $\chi/\psi_i(V_i)>1$, $\varphi_i/\chi(U)>1$ (if the first case does not apply, set $\varphi_{i+1}/\psi_{i+1}=\varphi_i/\chi$.)

By induction we have for all i

$$\dim(\varphi_i/\varphi_{i+1})\geqslant\alpha \quad \text{or} \quad \dim(\psi_{i+1}/\psi_i)\geqslant\alpha.$$

Thus $\dim(\varphi/\psi)>\alpha$.

We prove Claims 2 and 3 by simultaneous induction on α.

Ad Claim 2. Suppose that R is countable or φ/ψ is small. Let $\dim(\varphi/\psi)\geqslant\alpha\geqslant 0$. Look at pp-formulas $\chi_i\supset\psi$ s.t. $\dim(\chi_i/\psi)<\alpha$. Since $\chi_1+\chi_2/\chi_1\leqslant\chi_2/\psi$, we have $\dim(\chi_1+\chi_2/\chi_1)<\alpha$ and therefore $\dim(\chi_1+\chi_2/\psi)<\alpha$. The above shows that

$$q=\{\varphi\}\cup\{\neg\chi \mid \dim(\chi/\psi)<\alpha\}\cup\{\neg x\doteq 0\}$$

is M-consistent and satisfies the condition of 4.7.

Let p be the M-consistent indecomposable type we constructed from q in Lemma 4.7. Let $U=H(p)$. We have $\varphi/\psi(U)>1$ and show $\mathrm{rk}(U)\geqslant\alpha$. Assume $\mathrm{rk}(U)=\beta<\alpha$. By induction hypothesis there is $\bar{\varphi}/\bar{\psi}$ s.t. $U\in(\bar{\varphi}/\bar{\psi})$ and $\dim(\bar{\varphi}/\bar{\psi})=\beta$ (Claim 3). By 8.10 and Claim 1 we can assume that $\bar{\varphi}/\bar{\psi}\in p$. Now the construction of 4.7 implies that there is $\sigma\in p^+$, $\neg\chi\in q^-$ s.t. $\bar{\psi}\cap\sigma\subset\chi$, $\sigma\subset\bar{\varphi}$. But then $\dim(\psi+\bar{\psi}\cap\sigma/\psi)<\alpha$. Since – as one computes easily – $\psi+\sigma/\psi+\bar{\psi}\cap\sigma\leqslant\bar{\varphi}/\bar{\psi}$. We have

$$\dim((\psi+\sigma)/(\psi+\bar{\psi}\cap\sigma))\leqslant\beta.$$

This together with $\dim((\psi+\bar{\psi}\cap\sigma)/\psi)<\alpha$ yields $\dim(\psi+\sigma/\psi)<\alpha$. This is impossible.

Ad Claim 3. Suppose R is countable or U has a small neighbourhood, and $\mathrm{rk}(U)=\alpha$. Choose a neighbourhood $(\bar{\varphi}/\bar{\psi})$ of U which isolates U. We can assume that $\bar{\varphi}/\bar{\psi}$ is small. $\mathbb{U}_U$ is the closure of $\{U\}$ in $\mathbb{U}_M$ (cf. 4.10). Whence $(\bar{\varphi}/\bar{\psi})\cap\mathbb{U}_U$ contains only U. Since $(\bar{\varphi}/\bar{\psi})$ is also small w.r.t. U, we can apply 8.11 (to U instead of M) to obtain an U-minimal pair φ/ψ between $\bar{\psi}$ and $\bar{\varphi}$. By Claim 1, $\dim(\varphi/\psi)\geqslant\alpha$. We show that φ/ψ is α-minimal: If not, there is χ between ψ and φ s.t. $\dim(\varphi/\chi), \dim(\chi/\psi)\geqslant\alpha$. By Claim 2 there are V_i s.t. $V_1\in(\varphi/\chi)$, $V_2\in(\chi/\psi)$ and $\mathrm{rk}(V_i)\geqslant\alpha$. But φ/ψ isolates U, thus $U\cong V_i$ and $\psi(U)\neq\chi(U)\neq\varphi(U)$. This contradicts the U-minimality of φ/ψ.

Corollary 8.12. *If* $\dim(\varphi/\psi)=\alpha$, (φ/ψ) *contains at most* $\mu(\varphi/\psi)$*-many non-isomorphic* U *of rank* α. (Conventions as in 8.6.)

Proof. If φ/ψ is α-minimal, 7.10 and 8.6 (we need only Claim 1) show that there is only one U or rank α in (φ/ψ). Generally (φ/ψ) is the union of $\mu(\varphi/\psi)$ α-minimal neighbourhoods (on $\mathbb{U}_M$).

Remark. The above proof shows that 8.6 remains true, if we replace 'small' by the following weaker notion: Call φ/ψ small, iff for every $U \in \mathbb{U}_M$ every U-consistent pp-complete type which contains φ/ψ has a large formula.

Example. If R is a boolean ring, every pair is small is this sense.

9. Applications

We give two applications of our methods.

First we describe the class of all compact modules which are elementarily equivalent to a fixed module with Krull dimension (9.1). It turns out that there is a smallest compact module elementarily equivalent to a fixed module with Krull dimension.

Then we show how to decide a theory of modules if one has an effective control over the indecomposables – a phenomenon one can expect in the case of Krull dimension (9.4).

We conclude the section with two examples: We study the notion of rank for modules over a Dedekind ring and for pairs of torsion free modules over a Dedekind ring. As a byproduct we reprove the decidability of pairs of torsion free modules.

Theorem 9.1. *If M has Krull dimension, there are four sets*

$$\{U_h\}_{h\in H}, \qquad \{U_i\}_{i\in I}, \qquad \{U_j\}_{j\in J}, \qquad \{U_k\}_{k\in K}$$

of pairwisely non-isomorphic indecomposable modules, and natural numbers $\neq 0$ $(m_h)_{h\in H}$ s.t. the compact modules which are elementarily equivalent to M are just the modules of the form

$$\overline{\bigoplus_{h\in H} U_h^{m_h}} \oplus \overline{\bigoplus_{i\in I} U_i^{\mu_i}} \oplus \overline{\bigoplus_{j\in J} U_j^{\mu_j}} \oplus \overline{\bigoplus_{k\in K} U_k^{\mu_k}}, \qquad \mu_i \geq \aleph_0, \quad \mu_j \geq 1.$$

Corollary 9.2. *If M has Krull dimension, there is a smallest compact module M_0 elementarily equivalent to M: M_0 is direct factor of every compact module which is elementarily equivalent to M.*

Eklof & Sabbagh proved 9.1 in the case of injective modules over commutative rings [5]. M. Prest proved 9.1 independently [20] for totally transcendental M. The following result of Garavaglia is a special case of 9.2: If $U \equiv V$ are indecomposable modules with Krull dimension, $U \cong V$.

Proof. ($\mathrm{rk}_M = \mathrm{rk}$ is defined before 8.4, $I_U(M)$ before 6.11.) Set

$$\{U_h\}_{h\in H} = \{U \in \mathbb{U}_M\} \mid I_U(M) < \infty\} \quad \text{and} \quad m_h = I_{U_h}(M).$$

There must be a pair φ/ψ s.t. $U_h \in (\varphi/\psi)$ and $\varphi/\psi(M) < \infty$. Then $\dim_M(\varphi/\psi) = 0$ and $\mathrm{rk}(U_h) = 0$ by 8.6.

$$\{U_i\}_{i\in I} = \{U \in \mathbb{U}_M \mid \mathrm{rk}(U) = 0, I_U(U) = 1, I_U(M) = \infty\},$$
$$\{U_j\}_{j\in J} = \{U \in \mathbb{U}_M \mid \mathrm{rk}(U) = 0, I_U(U) = \infty\},$$
$$\{U_k\}_{k\in K} = \{U \in \mathbb{U}_M \mid \mathrm{rk}(U) > 0\}.$$

Clearly

$$\mathbb{U}_M = \{U_g \mid g \in H \cup I \cup J \cup K\}.$$

Since every compact module elementarily equivalent to M is the pure hull of a direct sum of elements of $\mathbb{U}_M$ (7.3(1)), we have to show that a module

$$N = \overline{\bigoplus_{h\in H} U_h^{\mu_h}} \oplus \overline{\bigoplus_{i\in I} U_i^{\mu_i}} \oplus \overline{\bigoplus_{j\in J} U_j^{\mu_j}} \oplus \overline{\bigoplus_{k\in K} U_k^{\mu_k}} \tag{1}$$

is elementarily equivalent to M iff

$$\mu_h = m_h, \qquad \mu_i \geqslant \aleph_0, \qquad \mu_j \geqslant 1. \tag{2}$$

Now choose for every $g \in H \cup I \cup J$ a pair φ/ψ which isolates U_g. by 6.12 (or rather a version where $\mathbb{U}^R$ is replaced by $\mathbb{U}_M$) we have

$$I_{U_g}(N) = {}^{\varphi/\psi(U_g)}\log(\varphi/\psi)(N)) = {}^{\varphi/\psi(U_g)}\log((\varphi/\psi(U_g))^{\mu_g}) = \mu_g \cdot I_{U_g}(U_g).$$

First suppose that $N \equiv M$. Then $I_{U_g}(M) = \mu_g \cdot I_{U_g}(U_g)$. This yields (2) immediately.

If on the other hand (2) holds, we have $I_U(N) = I_U(M)$ for all $U \in \mathbb{U}_M$ of rank 0. If the rank of $U \in \mathbb{U}_M$ is non-zero, U is an accumulation point of elements U_h, U_i, U_j. Therefore also $I_U(N) = I_U(M) = \infty$. For all $U \notin \mathbb{U}_M$ we have $I_U(N) = I_U(M) = 0$. Thus 6.11 implies $M \equiv N$.

Remark 9.3. *Let M be a module and $U \in \mathbb{U}_M$.*

(1) *$\mathrm{rk}_M(U) = 0$ iff U occurs in every direct sum of indecomposables which is elementarily equivalent to M.*

(2) *There is an M-minimal pair φ/ψ s.t. $U \in (\varphi/\psi)$ iff U is a direct factor of every compact module elementarily equivalent to M.*

Proof. (1) Let φ/ψ isolate U and $\bigoplus\{V^{\mu_V} \mid V \in \mathbb{U}_M\} \equiv M$. Then $\varphi/\psi(M) = (\varphi/\psi(U))^{\mu_U}$. Whence $\mu_U > 0$.

Now suppose that $\mathrm{rk}(U) > 0$. We find a decomposition $M \equiv U^{\mu} \oplus N$, where N is a direct sum of elements of $\mathbb{U}_M \setminus \{U\}$. $\mathrm{rk}(U) > 0$ implies $U \in \mathbb{U}_N$ and we can conclude that $U^{\mu} \oplus N \equiv N$. But now $M \equiv N$ and U is not a direct factor of N.

(2) Let $U \in (\varphi/\psi)$, φ/ψ M-minimal. We find a pp-complete p s.t. $U = H(p)$ and

$\varphi/\psi \in p$. If N is compact and elementarily equivalent to M, we choose $a \in \varphi(M) \setminus \psi(M)$. Then p and $\text{tp}^{\pm}(a)$ are associated via ψ. By 7.6, $H(p)$ is a direct factor of $H(a)$.

For the converse, suppose that U is a direct factor in every compact module elementarily equivalent to M. Part (1) implies that $\text{rk}(U) = 0$. Choose a pair $\bar{\varphi}/\bar{\psi}$ which isolates U.

Claim. $\bar{\varphi}/\bar{\psi}$ is small ('small' is defined after 8.5).

Proof. Let $N \oplus E$ be compact, weakly saturated, elementarily equivalent to M, N a pure hull of a direct sum of indecomposables and E without indecomposable direct factors. Write $N = U^{\mu} \oplus K$ where K has not direct factors isomorphic to U. 6.8 implies

$$I_V(E \oplus K) = I_V(K) = I_V(M) \quad \text{for all } V \not\cong U.$$

But U is not a direct factor of $E \oplus K$, whence by our assumption $M \not\equiv E \oplus K$ and by 6.11(1), $I_U(E \oplus K) < I_U(M)$. By the first part of the proof, there are no E-minimal pairs. Therefore all indices $\varphi/\psi(E)$ are $=1$ or $=\infty$. Since $I_U(E \oplus K)$ is finite, $\bar{\varphi}/\bar{\psi}(E)$ is finite and therefore $=1$. This means that $\bar{\varphi}/\bar{\psi}$ is small.

Now we can apply 8.7 to obtain a neighbourhood (φ'/ψ') of U s.t. $\dim(\varphi'/\psi') = 0$. But then U has also an M-minimal neighbourhood.

Example (cf. the remark following 5.1). Let K be a maximal valued field with densely ordered non-trivial valuation group. Let R be the valuation ring. The pp-definable subgroups of the R-module R are the principal ideals of R.

$\mathbb{U}_M$ consists – up to isomorphy – of the non-zero ideals of R. All $V \in \mathbb{U}_M$ are elementarily equivalent. Two ideals A, B are isomorphic iff $A = xB$ $(x \in K \setminus 0)$. Thus we have at least to non-isomorphic indecomposables in $\mathbb{U}_M$ – which are elementarily equivalent.

Theorem 9.4. *Let R be a recursive ring and T an axiomatizable theory of R-modules s.t. $M \oplus N \models T$ iff $M \models T$ and $N \models T$. Suppose that (φ_i/ψ_i), $i \in \mathbb{N}$, is an effective list of a base of the topological space $\{U_i\}_{i \in \mathbb{N}}$ of all (isomorphism types of) indecomposable models of T. Then T is decidable if $\varphi_i/\psi_i(U_j)$ depends recursively on i, j.*

Proof. Note that $\{U_i\}_{i \in \mathbb{N}}$ is a closed subspace of $\mathbb{U}^R$. By a suitable adaptation of 6.13 two models M, N of T are elementarily equivalent iff $\varphi_i/\psi_i(M) = \varphi_i/\psi_i(N)$ for all i. Thus the complete theory of U_i is axiomatized by

$$T_i = T \cup \{\varphi_j/\psi_j(\cdots) = \varphi_j/\psi_j(U_i) \mid j \in \mathbb{N}\}.$$

It follows that the T_i are – uniformly in i – decidable. The Feferman–Vaught theorem yields an effective enumeration of the – decidable – complete theories of all finite direct sums of the U_i. Since every sentence which is satisfiable in a model of T is satisfied in some finite direct sum of the U_i, we get an effective enumeration of all sentences consistent with T. Since also the set of all consequences of T is effectively enumerable we obtain the decidability of T.

Remark. One can prove that T is decidable iff T is axiomatizable and there is effective list φ_i/ψ_i of a base of $\mathbb{U}_T$ s.t.

$$\text{“}\exists U \in \mathbb{U}_T \ \varphi_1/\psi_1(U) \in [n_1, m_1] \wedge \cdots \wedge \varphi_k/\psi_k(U) \in [n_k, m_k]\text{”}$$

is an r.e. relation $(n_1, \ldots, n_k \in \mathbb{N}; m_1, \ldots, m_k \in \mathbb{N} \cup \{\infty\})$.

Example 9.5. *Modules over a Dedekind ring R.* We use the following notation: K = quotient field of R. We denote maximal ideals by $\mathfrak{P}$. $a \in \mathfrak{P}^n M$ can be expressed by a pp-formula $M \models \mathfrak{P}^n \mid a$, since $\mathfrak{P}$ is finitely generated. Also $\mathfrak{P}^n x \doteq 0$ – i.e. $ax \doteq 0$ $(a \in \mathfrak{P}^n)$ – is a pp-formula.

By 8.2 every R-module has Krull dimension. We will show that the dimension is ≤ 2. (There is a similar result in [11] for $R = \mathbb{Z}$.)

5.2 gave the indecomposable R-modules as

$$R/\mathfrak{P}^n \quad (n \geq 1), \qquad K/R_{\mathfrak{P}}, \qquad \tilde{R}_{\mathfrak{P}}, \qquad K.$$

The modules $R/\mathfrak{P}^n$ are isolated by the pairs

$$A^n_{\mathfrak{P}} = (\mathfrak{P}x \doteq 0 \wedge \mathfrak{P}^{n-1} \mid x)/(\mathfrak{P}x \doteq 0 \wedge \mathfrak{P}^n \mid x).$$

Thus the $R/\mathfrak{P}^n$ have rank 0. (We compute the rank w.r.t. $\mathbb{U}^R$.) $K/R_{\mathfrak{P}}$ is isolated by every pair $B^m_{\mathfrak{P}} = (\mathfrak{P}x \doteq 0 \wedge \mathfrak{P}^m \mid x)/(x \doteq 0)$. Since the $(B^m_{\mathfrak{P}}) = \{K/R_{\mathfrak{P}}\} \cup \{R/\mathfrak{P}^n \mid n > m\}$ are quasicompact, every infinite sequence of $R/\mathfrak{P}^n$'s converges to $K/R_{\mathfrak{P}}$. Therefore $K/R_{\mathfrak{P}}$ has rank 1 and the $(B^m_{\mathfrak{P}})$ form a basis of neighbourhoods of $K/R_{\mathfrak{P}}$.

The same reasoning shows that $\tilde{R}_{\mathfrak{P}}$ has rank 1 and that the

$$(C^m_{\mathfrak{P}}) = \{\tilde{R}_{\mathfrak{P}}\} \cup \{R/\mathfrak{P}^n \mid n > m\}$$

form a basis of the neighbourhoods of $\tilde{R}_{\mathfrak{P}}$. $C^m_{\mathfrak{P}}$ is the pair $(\mathfrak{P}^{m-1} \mid x)/(\mathfrak{P}^m \mid x)$.

Since $\mathbb{U}^R$ is quasicompact, the last indecomposable K must be of rank 2. The proof of 5.1 (case 2) shows that the $\pm$-type of any non-zero element of K is axiomatized by the pairs $(r \mid x)(r \mid x \wedge rx \doteq 0)$ $(r \in R \setminus 0)$. Thus – by the proof of 4.9 – these pairs constitute a base for the neighbourhoods of K. Every such neighbourhood contains all $K/R_{\mathfrak{P}}$, $\tilde{R}_{\mathfrak{P}}$, and almost all $R/\mathfrak{P}^n$. Whence also the pairs $D^r = (x \doteq x)/(rx \doteq 0)$ defines a basis of neighbourhoods of K.

Corollary 9.6. *Let $k_{\mathfrak{P}} = |R/\mathfrak{P}|$. Then the elementary type of an R-module M is determined by the invariants $I_U(M)$, where*

$$\begin{aligned}
I_U(M) &= \log^{k_{\mathfrak{P}}}(A^n_{\mathfrak{P}}(M)) && (U = R/\mathfrak{P}^n,\ n \geq 1)\\
&= \min\{\log^{k_{\mathfrak{P}}}(B^n_{\mathfrak{P}}(M)) \mid n > 0\} && (U = K/R_{\mathfrak{P}})\\
&= \min\{\log^{k_{\mathfrak{P}}}(C^n_{\mathfrak{P}}(M)) \mid n > 0\} && (U = \tilde{R}_{\mathfrak{P}})\\
&= \min\{\log^{|K|}(D^r(M)) \mid r \in R \setminus 0\} && (U = K).
\end{aligned}$$

Proof. See 6.12.

Corollary 9.7. *Let R be a recursive Dedekind ring with an effective 1–1 list $\mathfrak{P}_i = Ra_i + Rb_i$ of all maximal ideals. Suppose that the cardinality of $k_{\mathfrak{P}_i}$ can be computed from i. Then the theory of all R-modules is decidable.*

Proof. See 9.4.

Note that in the case of 9.7, Theorems 9.4 and 9.5 yield also the decidability of the theory of torsion free or the theory of divisible R-modules.

Example 9.8. *Pairs of torsion free modules over a Dedekind ring R.* We determined the indecomposable torsion free pairs in 5.7 as

$$(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}} \cdot \mathfrak{P}^n),\ n \geqslant 1, \quad (\tilde{R}_{\mathfrak{P}}, 0), \quad (\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}), \quad (\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}), \quad (K, 0), \quad (K, K).$$

By 8.2 (also true in this case) every torsion free pair has Krull dimension (we will show $\leqslant 2$). Thus every compact torsion-free pair is the pure hull of a direct sum of the indicated indecomposables.

Notation. Let $\varphi(x)$ be a pp-formula without the new predicate symbol P. "φ mod P" is satisfied by $a \in (M, N)$ if $a + N$ satisfies $\varphi(x)$ in M/N.

We compute the rank in the topological space $\mathbb{U}$ of all torsion free indecomposables. Here (φ/ψ) is restricted to $\mathbb{U}$. Since $\tilde{R}_{\mathfrak{P}}/(\tilde{R}_{\mathfrak{P}}\mathfrak{P}^n) \cong R/\mathfrak{P}^n$, $\tilde{K}_{\mathfrak{P}}/\tilde{R}_{\mathfrak{P}} \cong K/R_{\mathfrak{P}}$ the pairs "$A^n_{\mathfrak{P}}$ mod P" isolate the $(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}\mathfrak{P}^n)$, which have therefore rank 0. Since

$$(B^m_{\mathfrak{P}} \bmod P) = \{(\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}})\} \cup \{(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}\mathfrak{P}^n) \mid n > m\},$$

these sets form a base for the neighbourhoods of $(\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}})$. $(\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}})$ has rank 1. (Use quasicompactness.) The $(\tilde{R}_{\mathfrak{P}}, 0)$ have rank 1 and a neighbourhood base

$$(C^m_{\mathfrak{P}} \bmod P) = \{(\tilde{R}_{\mathfrak{P}}, 0)\} \cup \{(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}\mathfrak{P}^n) \mid n > m\} \qquad (m \geqslant 0).$$

The $(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}})$ have rank 1 and a neighbourhood base

$$(P(x)/P(x) \wedge \mathfrak{P}^m \mid x) = \{(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}})\} \cup \{(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}\mathfrak{P}^n) \mid n \geqslant m\} \qquad (m \geqslant 0).$$

We want to show that the

$$\begin{aligned}(D^r \bmod P) = \{(K, 0)\} &\cup \{(\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}), (\tilde{R}_{\mathfrak{P}}, 0) \mid \mathfrak{P} \text{ maximal}\} \\ &\cup \{(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}\mathfrak{P}^n) \mid \mathfrak{P} \text{ maximal, } r \notin \mathfrak{P}^n\}\end{aligned}$$

form a base for the neighbourhoods of $(K, 0)$. It is already clear that $\mathrm{rk}((K, 0)) = 2$ (by quasicompactness). For this it is enough to show that the $(\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}})$, $(\tilde{R}_{\mathfrak{P}}, 0)$ lie in every neighbourhood of $(K, 0)$. Thus let (M, N) be weakly saturated and compact and elementarily equivalent to $(\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}})$ or $(R, 0)$. Then M/N is weakly saturated, compact and elementarily equivalent to $K/R_{\mathfrak{P}}$ or $\tilde{R}_{\mathfrak{P}}$. Now one of the indecomposable direct factors of M/N must be K. Since all indecomposable factors of M/N are of the form U/V, where (U, V) is an indecomposable factor of (M, N), $(K, 0)$ is an indecomposable factor of (M, N).

A similar reasoning – use N instead of M/N – shows that

$$(P(x)/x \doteq 0) = \{(K, K)\} \cup \{(\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}\mathfrak{P}^n), (\tilde{R}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}), (\tilde{K}_{\mathfrak{P}}, \tilde{R}_{\mathfrak{P}}) \mid \mathfrak{P} \text{ maximal}, n \geq 1\}$$

is the smallest neighbourhood of (K, K). We have $\mathrm{rk}((K, K)) = 2$.

Corollary 9.9. *The elementary type of a torsion free pair (M, N) is determined by the elementary type of M/N, by the fact that $N = 0$ (or $\neq 0$) and by* $\min\{(N : \mathfrak{P}^m M \cap N) \mid m \in \mathbb{N}\}$.

Proof. By 6.11 and 6.12. (Actually for 9.9 the whole picture of 9.8 is not needed.)

Corollary 9.10. *Let R be a recursive Dedekind ring as in* 9.7. *Then the theory of torsion free pairs of R-modules is decidable.*

Proof. By 9.4 and the analysis in 9.8.

In the case $R = \mathbb{Z}$, 9.10 is due to Koslov & Kokorin [21, 22]. Our proof was inspired by [23].

10. The spectrum

We fix throughout this section a countable ring R.

Let M be an infinite R-module. We are interested in the spectrum function I_M, which is defined for infinite cardinals κ as

$$I_M(\kappa) = |\{N/\cong \mid N \equiv M, |N| = \kappa\}|,$$

the number of non-isomorphic modules elementarily equivalent to M of cardinality κ.

We will get a complete description of the possible I_M with the exception of $\kappa = \aleph_0$. For $I_M(\aleph_0)$ we have only partial information.

Theorem 10.1. *Restricted to uncountable arguments $\kappa = \aleph_\alpha$, I_M is one of the following functions:*

(I^1_λ) $\quad |\alpha + 1|^\lambda - |\alpha|^\lambda \quad (1 \leq \lambda \leq \aleph_0)$,

(I^2) $\quad |\alpha + \omega|$,

(I^3) $\quad 2^{\min(\kappa, 2^{\aleph_0})}$,

(I^4) $\quad 2^{\min(\kappa, 2^{\aleph_0})} + |\alpha|$,

(I^5) $\quad 2^{\min(\kappa, 2^{\aleph_0})} + |\alpha|^{\aleph_0}$,

(I^6) $\quad 2^\kappa$.

M is totally transcendental iff I_M is one of the functions I^1, I^2. M is superstable iff I_M is not I^6.

In the case of $R=\mathbb{Z}$ all functions occur.

Notation. $|\alpha+1|^\lambda-|\alpha|^\lambda$ is not well-defined for infinite arguments. It should be the number of all functions from λ to $\alpha+1$ with $\sup\{f(i)\mid i<\lambda\}=\alpha$. Thus we set

$$|\alpha+1|^\lambda-|\alpha|^\lambda=\begin{cases}1 & \text{for } \lambda=1,\\ |\alpha+1|^\lambda & \text{for } \lambda>1, \alpha\geqslant\omega \textit{ or } \lambda\geqslant\aleph_0\end{cases}.$$

Proof. For modules M with Krull dimension, 9.1 gives us four sets $\{U_h\}$, $\{U_i\}$, $\{U_j\}$, $\{U_k\}$ of indecomposables and natural numbers $0<m_h<\omega$ s.t. the compact $N\equiv M$ are just the modules

$$N=\overline{\bigoplus_{h\in H} U_h^{m_h}}\oplus\overline{\bigoplus_{i\in I} U_i^{\mu_i}}\oplus\overline{\bigoplus_{j\in J} U_j^{\mu_j}}\oplus\overline{\bigoplus_{k\in K} U_k^{\mu_k}},\qquad \mu_i\geqslant\aleph_0,\quad \mu_j\geqslant 1. \tag{1}$$

and we know (8.4) that $H\cup I\cup J\cup K$ is countable and that $\lambda=|I\cup J\cup K|$ lies between 1 and $\aleph_0$. For there are arbitrarily large N.

Let $J_M(\kappa)$ be the number of all functions $(\mu_g)_{g\in I\cup J\cup K}$ s.t.

$$\mu_i\geqslant\aleph_0,\qquad \mu_j\geqslant 1,\qquad \sup\{\mu_g\mid g\in I\cup K\cup J\}=\kappa. \tag{2}$$

Let now M be *totally transcendental*. Then M has Krull dimension and all $N\equiv M$ are compact (3.5(1)). Furthermore all U_g are countable (4.2(3)). Whence the modules $N\equiv M$, $|N|=\kappa>\aleph_0$, are just the modules N in (1), where the (μ_g) satisfy (2). Therefore $I_M(\kappa)=J_M(\kappa)$.

But $J_M(\cdots)$ is easily computed (for uncountable κ): If $\lambda=1$ or $\lambda=\aleph_0$ or $K\cup J=\emptyset$, this is I^1_λ. If $1<\lambda<\aleph_0$ and $K\cup J\neq\emptyset$, this is I^2. This proves 10.1 in the case that M is totally transcendental.

If M is *not superstable*, we have $I_M=I^6$ for uncountable κ by a general theorem of Shelah [17, VIII 0.3].

Now let M be *superstable* and *not totally transcendental*. Then M has Krull dimension. We prove that for uncountable κ

$$I_M(\kappa)=2^{\min(\kappa,2^{\aleph_0})}+J_M(\kappa). \tag{3}$$

This yields

$$I_M=I^3\qquad(\lambda=1),$$
$$I^M=I^4\qquad(1<\lambda<\aleph_0),$$
$$I_M=I^5\qquad(\lambda=\aleph_0).$$

By a theorem of Shelah [17, VIII 1.7, 1.8] on non-totally transcendental theories we have $2^{\min(\kappa,2^{\aleph_0})}\leqslant I_M(\kappa)$ for uncountable κ. This proves (3) for $\aleph_0<\kappa\leqslant 2^{\aleph_0}$.

If $\kappa\geqslant 2^{\aleph_0}$, all the modules N in (1) – but without pure hulls – are of power κ if (2) holds. (Note that $|U_g|\leqslant 2^{\aleph_0}$ (4.2(2))). Therefore

$$2^{\min(\kappa,2^{\aleph_0})}+J_M(\kappa)\leqslant I_M(\kappa).$$

To prove the inverse inequality for $\kappa > 2^{\aleph_0}$, we need the following lemma (here R is arbitrary).

Lemma 10.2. *Every superstable R-module N is of the form $A \oplus L$, where $|A| \leqslant 2^{|R|+\aleph_0}$ and L is totally transcendental.*

We finish the proof of 10.1 and prove 10.2 later.

Let $N \equiv M$, $|N| = \kappa > 2^{\aleph_0}$. Write $N = A \oplus L$, $|A| \leqslant 2^{\aleph_0}$, L totally transcendental. Then $|L| = \kappa$ and since L is compact, it is a direct factor of $\bar{N} \equiv M$. We have

$$I_M(\kappa) \leqslant (\text{number of possible } A) + (\text{number of possible } L).$$

But the number of possible A is at most $2^{2^{\aleph_0}}$. $\bar{N}$ has the form (1), therefore L is of the form

$$\bigoplus \{U^{\underline{\mu}_g} \mid g \in H \cup I \cup J \cup K\},$$

where

$$\mu_h \leqslant m_h \quad \text{and} \quad \sup\{\mu_g \mid g \in I \cup J \cup K\} = \kappa. \tag{4}$$

The number of the (μ_g) satisfying (4) – and therefore the number of possible L – is not greater than $2^{\aleph_0} + J_M(\kappa)$. This proves (3).

Finally we give examples of abelian groups which have the several spectrum functions. We give the groups in the form (1). Thus displaying all compact elementarily equivalent groups.

(I^1_n)	$Z(p_1)^{\underline{\mu}_1} \oplus \cdots \oplus Z(p_n)^{\underline{\mu}_n}$	$(\mu_i \geqslant \aleph_0)$,
$(I^1_{\aleph_0})$	$\bigoplus_p Z(p^\infty)^{\underline{\mu}_p} \oplus \mathbb{Q}^{\underline{\nu}}$	$(\mu_p \geqslant \aleph_0)$,
(I^2)	$Z(2)^{\underline{\mu}} \oplus \mathbb{Q}^{\underline{\nu}}$	$(\mu \geqslant \aleph_0)$,
(I^3)	$\mathbb{Z}_2 \oplus \mathbb{Q}^{\underline{\nu}}$	
(I^4)	$\mathbb{Z}_2 \oplus Z(2^\infty)^{\underline{\mu}} + \mathbb{Q}^{\underline{\nu}}$	$(\mu \geqslant \aleph_0)$,
(I^5)	$\mathbb{Z}_2 \oplus \bigoplus_p Z(p^\infty)^{\underline{\mu}_p} + \mathbb{Q}^{\underline{\nu}}$	$(\mu_p \geqslant \aleph_0)$,
(I^6)	$\mathbb{Z}_2^{\underline{\mu}} \oplus \mathbb{Q}^{\underline{\nu}}$	$(\mu \geqslant \aleph_0)$.

Proof of 10.2. Let N be a superstable R-module (R arbitrary). Choose an elementary submodule K of N of cardinality at most $|R| + \aleph_0$. By 2.3, N/K is totally transcendental. Whence (3.5, 7.3)

$$N/K = \bigoplus_{i \in I} V_i, \qquad V_i \text{ indecomposable}, \qquad |V_i| \leqslant |R| + \aleph_0.$$

Let $\pi : N \to \bigoplus_{i \in I} V_i$ be the canonical projection. Set $M_i = \pi^{-1}(V_i)$. We have $|M_i| \leqslant |R| + \aleph_0$.

Since there are at most $2^{|R|+\aleph_0}$ isomorphism types of extensions of K of power $\leqslant |R| + \aleph_0$, there is a set $J \subset I$, $|J| \leqslant 2^{|R|+\aleph_0}$, s.t. for all $i \in I \setminus J$, there is $j_i \in J$ and an

K-isomorphism

$$f_i : M_i \xrightarrow[K]{\sim} M_{j_i}.$$

We set $A = \sum_{j\in J} M_j$. Since $A = \pi^{-1}(\bigoplus_{j\in J} V_j)$, π induces a homomorphism

$$\bar{\pi} : N \to \bigoplus_{i\in I\setminus J} V_i, \quad \text{with kernel } A.$$

We define a cross section $h : \bigoplus_{i\in I\setminus J} V_i \to N$, by

$$h(a_i) = x_i - f_i(x_i) \quad \text{for } a_i = \pi(x_i) \in V_i.$$

h is well-defined:

$$\pi(x_i) = \pi(y_i) \Rightarrow x_i - y_i \in K \Rightarrow x_i - y_i = f_i(x_i - y_i)$$
$$\Rightarrow x_i - f_i(x_i) = y_i - f_i(y_i).$$

h is a cross section: Because of $\bar{\pi}(x_i - f_i(x_i)) = \bar{\pi}(x_i) = \pi(x_i)$, we have $\bar{\pi}h = \mathrm{id}$. If we set $L = h(\bigoplus_{i\in I\setminus J} V_i)$, we have $N = A \oplus L$.

The next theorem contains our knowledge about the number of countable models. It is still open, if Vaught's conjecture can be verified in our case, i.e. if always $I_M(\aleph_0) \leqslant \aleph_0$ or $I_M(\aleph_0) = 2^{\aleph_0}$. For totally transcendental modules Vaught's conjecture was settled by Garavaglia [9]. 10.3(3) is due to G. Cherlin and, independently, to M. Prest.

Theorem 10.3. (1) *If M is totally transcendental, then $I_M(\aleph_0) = 1$, $= \aleph_0$ or $I_M(\aleph_0) = 2^{\aleph_0}$.*

(2) *If R is a Dedekind ring, and M not totally transcendental, then $I_M(\aleph_0) = 2^{\aleph_0}$.*

(3) *There is an R-module M – for suitable R – which is not superstable, but $I_M(\aleph_0) = \aleph_0$.*

(4) *If $I_M(\aleph_0) < 2^{\aleph_0}$, M has finite Krull dimension.*

(5) *If $I_M(\aleph_0)$ is finite, the Krull dimension of M is zero* (*cf.* 10.5).

Proof. (1) Look at (1) in the proof of 10.1. In our case the modules U_g are countable, and all $N \equiv M$ are compact. Whence the countable $N \equiv M$, are just the modules of the form (1), where $\mu_i = \aleph_0$, $1 \leqslant \mu_j \leqslant \aleph_0$ and $\mu_g \leqslant \aleph_0$. This yields

$$I_M(\aleph_0) = \begin{cases} 1, & \text{if } J \cup K = \emptyset, \\ \aleph_0, & \text{if } \emptyset \neq J \cup K \text{ is finite}, \\ 2^{\aleph_0}, & \text{if } J \cup K \text{ is infinite}. \end{cases}$$

(2) We need the following lemma, due to G. Cherlin.

Lemma 10.4. *If there is a sequence $r_i \in R$ s.t. $r_1M \supsetneqq r_2M \supsetneqq r_3M \supsetneqq \cdots$, there is a* *subsequence* $s_1M \supsetneqq s_2M \supsetneqq s_3M \supsetneqq \cdots$ *s.t.*

$$\forall n\, \exists a \in M\, \forall i < n \quad s_i a \notin s_{i+1}M.$$

Proof. Choose (s_i) as a subsequence of (r_i) s.t. $|s_iM/s_{i+1}M| \geq 2^i$. Set $H_i = \{a \in M \mid s_i a \in s_{i+1}M\}$. Then $|M/H_i| \geq 2^i$. An easy computation shows $H_1 \cup H_2 \cup \cdots \cup H_n \subsetneqq M$. Choose $a \in M \setminus (H_1 \cup \cdots \cup H_n)$.

Proof of 10.3(2) (contd.). *Case* 1: There is an infinite sequence $r_1M \supsetneqq r_2M \supsetneqq \cdots$. Let (s_i) be the subsequence in 10.4. We can assume that M is weakly saturated. Then we find $a \in M$, with $s_i a \notin s_{i+1}M$ for all i. The proof of 2.1(2) constructed $2^{\aleph_0}$ contradictory pp-types with parameters $s_i a$. Now we need only the parameter a. This yields $2^{\aleph_0}$ contradictory types with 2 variables without parameters. Thus $I_M(\aleph_0) = 2^{\aleph_0}$.

Case 2: There is no sequence as above. Then there is $r \in R \setminus 0$ s.t. rM is divisible. rM is injective and, by 5.9, totally transcendental. rM is pure in M.

$N = M/rM$ is bounded: $rN = 0$. Then $\overline{rN}$ is a finite sum of multiples of modules $R/\mathfrak{P}^n$. Therefore totally transcendental. Now M is totally transcendental by 2.2(1).

Proof of 10.3 (contd.). (3) Let k be a finite field. R is the ring obtained from the polynomial ring $k[X_1, X_2, \ldots]$ by factoring through the ideal generated by all X_iX_j. Whence R is a commutative k-algebra, with k-basis $1, T_1, T_2, \ldots$ which satisfies $T_iT_j = 0$.

Now look at the following R-modules M:

$$M = M[T_1] \supset M[T_2] \supset M[T_3] \supset \cdots,$$

$$\dim_k(M[T_i]/M[T_{i+1}]) = \infty,$$

the T_iM are contained in all $M[T_j]$ and k-linearly independent.

It is easy to see that all such modules are elementarily equivalent and that there are exactly $\aleph_0$ many countable M. (A countable M is determined up to isomorphism by $\dim_k(\bigcap \{M[T_i] \mid i \in \omega\}/\bigoplus_{i\in\omega} T_iM)$.) Clearly M is not superstable (2.1(3)).

(4) *Case* 1: M does not have Krull dimension. By 6.8, M is elementarily equivalent to a direct sum $\bigoplus_{i\in I} U_i$ of indecomposables, where all $U \in \mathbb{U}_M$ occur – up to isomorphism – among the U_i. A Loeweheim–Skolem argument shows, that there is a countable $J \subset I$ with $M \equiv \bigoplus_{j\in J} U_j$. Choose an infinite countable subset $H \subset I \setminus J$ s.t. the U_h are pairwisely non-isomorphic and not isomorphic to any U_j. (This is possible by 8.1.) Now let V_i be a countable elementary submodule of U_i. Then for every subset $K \subset H$, $M_K = \bigoplus \{V_i \mid i \in J \cup K\}$, is countable and elementarily equivalent to M (see 1.6, 1.7). Furthermore all the M_K are non-isomorphic. (Note that $\bar{M}_K =$ pure hull of $\bigoplus \{U_i \mid i \in J \cup K\}$.) Whence $I_M(\aleph_0) = 2^{\aleph_0}$.

Case 2: $\omega \leq \dim(M) < \infty$. Look at Theorem 9.1: Since there are infinitely many isomorphism types of indecomposables with rank>0, (for every $\alpha \leq \dim(M)$, there is at least one U with $\mathrm{rk}(U)=\alpha$), K is infinite. Let V_g a countable elementary submodule of U_g. For every subset $L \subset K$ define

$$M_L = \bigoplus_{h \in H} V_h^{m_h} \oplus \bigoplus_{i \in I} V_i^{\aleph_0} \oplus \bigoplus_{j \in J} V_j \oplus \bigoplus \{V_k \mid k \in L\}.$$

The M_L are countable, elementarily equivalent to M, and pairwise non-isomorphic. Thus $I_M(\aleph_0) = 2^{\aleph_0}$.

(5) By 10.3(4), M has finite Krull dimension. Whence 9.1 applies. If $K \neq \emptyset$, the argument above ((4), case 2) gives infinitely many non-isomorphic countable models $\equiv M$: Choose $V \in K$ and set

$$M_n = \bigoplus_{h \in H} V_h^{m_h} \oplus \bigoplus_{i \in I} V_i^{\aleph_0} \oplus \bigoplus_{j \in J} V_j \oplus V^n \qquad (n \in \mathbb{N}).$$

Therefore $K = \emptyset$. But then all $U \in \mathbb{U}_M$ have rank 0 and $\dim(M) = 0$ by 8.8.

Corollary 10.5 (A. Pillay [24]). $I_M(\aleph_0) < \aleph_0 \Rightarrow I_M(\aleph_0) = 1$.

Proof. By 10.3(5), $\dim(M) = 0$ if $I_M(\aleph_0) < \aleph_0$. But then M is totally transcendental and the result follows from 10.3(1).

We conclude this section with a description of $\aleph_0$- and $\aleph_1$-categorical modules. (An infinite module is κ-categorical iff the complete theory of it is κ-categorical.)

Theorem 10.6. *Let M be an infinite module.*

(1) (Baur [27]). *M is $\aleph_0$-categorical iff*

$$M = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r} \oplus W_0^{\kappa_0} \oplus \cdots \oplus W_s^{\kappa_s},$$

where the V_h, W_i are finite indecomposables, the m_h are finite and the κ_i are infinite cardinals.

In this case the modules N which are elementarily equivalent to M are the modules of the form

$$N = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r} \oplus W_0^{\lambda_0} \oplus \cdots \oplus W_s^{\lambda_s} \qquad (\lambda_i \geq \aleph_0).$$

(2) *M is $\aleph_0$-categorical and $\aleph_1$-categorical iff M is of the form as in* (1) *above, where $s = 0$.*

(3) *M is $\aleph_1$- (and not $\aleph_0$-) categorical iff one of the following cases occurs:*

(a) *$M = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r} \oplus W^{\kappa}$, where the V_h are finite indecomposables, the m_h finite, W is an infinite indecomposable, $\kappa \geq 1$ and* $\dim W = 0$.

In this case the N which are elementarily equivalent to M are the modules of the form

$$N = V_1^{m_1} \oplus \cdots \oplus V_r^{m_r} \oplus W^{\lambda} \qquad (\lambda \geq 1).$$

(b) *M is totally transcendental, $\varphi/\psi(M)$ is finite for all pairs of ppf with $\dim_M(\varphi/\psi) = 0$, there is exactly one indecomposable $W \in \mathbb{U}_M$ with M-rank > 0.*

In this case $\dim M = 1$ *and there are countably many indecomposables* U_h, *finite* m_h $(h \in H)$ *s.t. the* N *elementarily equivalent to* M *are the modules of the form*

$$N = \bigoplus_{h \in H} U_h^{m_h} \oplus W^\lambda \qquad (\lambda \geqslant 0).$$

Proof. (1) Let M be of the given form. Since $\dim(V_h) = \dim(W_i) = 0$, $\dim(M) = 0$ by 8.5. Therefore all $N \equiv M$ are compact (3.5(1)), and are therefore given by 9.1 – where the pure hulls are superfluous. We adopt the notation of 9.1.

The U_k are the elements of $\mathbb{U}_M$ with rank>0 (cf. the proof of 9.1). Thus by 8.6, $K = \emptyset$. Since the U_j are infinite, $J = \emptyset$. We conclude that

$$\{V_1, \ldots, V_r\} = \{U_h\}_{h \in H} \quad \text{and} \quad \{W_0, \ldots, W_s\} = \{U_i\}_{i \in I}.$$

By 9.1 the $N \equiv M$ are as desired. And this shows that M is $\aleph_0$-categorical. (Alternatively Ryll–Nardzewski is easily applied: there only finitely many pp-formulas $\varphi(x_1, \ldots, x_n)$ – up to M-equivalence.)

Suppose that M is $\aleph_0$-categorical. By 10.3(5), $\dim M = 0$ (or use Ryll–Nardzewski). Therefore the $N \equiv M$ are given by 9.1 (no pure hulls). In 9.1 we have $K = J = \emptyset$ (by $\aleph_0$-categoricity). By 8.8 (or Ryll–Nardzewski) $H \cup T$ is finite.

It remains to show that the U_h, U_i are finite:

(+) If U is indecomposable, $\dim U = 0$ and $U \oplus U \not\equiv U$, U is finite.

Proof. Let $0 = \varphi_0(U) \subset \varphi_1(U) \subset \cdots \subset \varphi_n(U) = U$ be a decomposition of U into U-minimal pairs. Since a U-minimal pair constitutes a base of neighbourhoods of U (in $\mathbb{U}_U$), we have that all $\varphi_{i+1}/\varphi_i(U)$ are finite (namely by (*) in the proof of 6.12). Thus U is finite.

(2) follows from (1).

(3) Let M be as in (a). By (1) – and unique decomposition (6.1) – M is not $\aleph_0$-categorical. Since $\dim M = 0$, the $N \equiv M$ are given by 9.1 (no pure hulls). Furthermore $K = \emptyset$. Since W is infinite, $|J| = 1$ and W is the only U_j (use (+) above). Thus I must be empty and $\{V_1, \ldots, V_r\} = \{U_h\}_{h \in H}$. Now the $N \equiv M$ are as claimed and therefore M is $\aleph_1$-categorical.

Now let M be as in (b). M is not $\aleph_0$-categorical, since $\dim M > 0$. Since M is totally transcendental all $N \equiv M$ are given by 9.1. There is only one U_k : W, which must be of rank 1. Whence $\dim M = 1$. Since $\varphi/\psi(M)$ is finite for all pairs φ/ψ of dimension 0, I and J are empty. This shows that the $N \equiv M$ are as we wanted, which implies that M is $\aleph_1$-categorical.

Let conversely M be $\aleph_1$-categorical. Then M is totally transcendental and, if 9.1 gives all $N \equiv M$, we have $|I \cup J \cup K| = 1$.

Case (a): $\dim M = 0$ and M is not $\aleph_0$-categorical. $\dim M = 0$ means $K = \emptyset$. By 8.8, H is finite and all U_h are finite by (+). Since M is not $\aleph_0$-categorical, $I = \emptyset$. Thus $|J| = 1$. Taken now the U_h for the V_h and the single U_j for W. Then M has the form as in (a).

Case (b): $\dim M > 0$. Then $I = J = \emptyset$ and $|K| = 1$. Whence $\dim M = 1$ and the only $W \in U_M$ of rank>0 is U_k. Let $\dim_M(\varphi/\psi) = 0$. Since φ/ψ decomposes into finitely many M-minimal pairs, it is enough to show that $\varphi/\psi(M)$ is finite if φ/ψ is M-minimal. But then (φ/ψ) is the smallest neighbourhood of some U_h in U_M. By $(*)$ (in the proof of 6.12) $\varphi/\psi(U_h)$ is finite. Since φ/ψ isolates U_h, also $\varphi/\psi(M)$ is finite.

11. Forking

We investigate the meaning of some notions of stability theory in the case of modules: forking, regular types and orthogonality. We refer the reader to Shelah [17] and Lascar & Poizat [15]. Note that modules are *stable.*

We fix a 'large' saturated module $\mathbb{M}$. All 'light face' subsets A of $\mathbb{M}$ we deal with are supposed to be of 'small' cardinality. (We need $2^{|A|+|R|+\aleph_0} < |\mathbb{M}|$).

If $A \subset \mathbb{M}$, let us denote by $S(A)$ the set of all complete 1-types which are $\mathbb{M}$-consistent and have parameters in A.

Let $p \in S(A)$, $A \subset B$, $q \in S(B)$ an extension of p. The notion "q is a non-forking extension of p" or "q does not fork over A" has the following properties (see [15]). The first two facts can be used as a definition of forking. Let $\mathbf{q} \in S(\mathbb{M})$ be an extension of p.

Fact 1. *$\mathbf{q}$ is a non-forking extension of p iff $\mathbf{q}$ has at most $2^{|R|+\aleph_0}$ many conjugates over A. (If π is an automorphism of $\mathbb{M}$ which leaves the elements of A fixed, then $\pi(\mathbf{q})$ is a conjugate of $\mathbf{q}$ over A.)*

Fact 2. *q is a nonforking extension of p iff q has an extension $\mathbf{q} \in S(\mathbb{M})$ which does not fork over A.*

Fact 3. *All non-forking extensions $\mathbf{q} \in S(\mathbb{M})$ of p are conjugate over A.*

Definition. Let $p \in S(A)$. $G(p)$ is the set of all pp-definable subgroups $\varphi(\mathbb{M}, \mathbf{0})$, where $\varphi(x, \boldsymbol{a}) \in p$ for some $\boldsymbol{a} \in A$.

Theorem 11.1. *q is a non-forking extension of p iff every $\mathbf{G} \in G(q)$ is of finite index in some $\mathbf{H} \in G(p)$.* (11.1 was independently proved in [25].)

Proof. *Claim* 1. There is an extension $\hat{\mathbf{q}}$ of p s.t. every $\mathbf{G} \in G(\hat{\mathbf{q}})$ is of finite index in some $\mathbf{H} \in G(p)$.

Proof. Let H be the set of all pp-definable subgroups of M which are of finite index in some $\mathbf{H} \in G(p)$. We show that the set

$$p \cup \{\neg\psi(x, \boldsymbol{m}) \mid \boldsymbol{m} \in \mathbb{M}, \psi(\mathbb{M}, \mathbf{0}) \notin H\}$$

is M-consistent. (Then take for $\hat{\mathbf{q}}$ any complete extension of this set.)

If the above set is inconsistent, there are $\varphi(x, \boldsymbol{a}) \in p$, $\neg\chi_i(x, \boldsymbol{a}) \in p$, $\psi_i(x, \boldsymbol{m})$ s.t. $\psi_i(\mathbb{M}, \mathbf{0}) \notin H$ and

$$\mathbb{M} \vDash \varphi(x, \boldsymbol{a}) \to (\chi_1(x, \boldsymbol{a}) \vee \cdots \vee \chi_n(x, \boldsymbol{a}) \vee \psi_1(x, \boldsymbol{m}) \vee \cdots \vee \psi_n(x, \boldsymbol{m})).$$

(φ, χ_i, ψ_i are pp-formulas). Since

$$M \nvDash \varphi(x, \boldsymbol{a}) \to (\chi_1(x, \boldsymbol{a}) \vee \cdots \vee \chi_n(x, \boldsymbol{a})),$$

some $\varphi(\mathbb{M}, \mathbf{0}) \cap \psi_j(\mathbb{M}, \mathbf{0})$ is of finite index in $\varphi(\mathbb{M}, \mathbf{0})$, (see the proof of 1.1). But then $\psi_j(\mathbb{M}, \mathbf{0})$ is of finite index in $\varphi(\mathbb{M}, \mathbf{0}) + \psi_j(\mathbb{M}, \mathbf{0})$. This is a contradiction, because $\varphi(\mathbb{M}, \mathbf{0}) + \psi_j(\mathbb{M}, \mathbf{0}) \in G(p)$.

Claim 2: If $\mathbf{q}$ has the property of Claim 1, then also every conjugate over A of $\mathbf{q}$ has this property.

Proof. $G(\pi(\mathbf{q})) = G(\mathbf{q})$.

Claim 3. There are at most $2^{|R|+\aleph_0}$ many $\mathbf{q} \in S(\mathbb{M})$ with the property of Claim 1.

Proof. $\varphi(\mathbb{M}, \mathbf{0}) \mapsto \varphi(\mathbb{M}, \boldsymbol{a})$ ($\varphi(x, \boldsymbol{m}) \in \mathbf{q}$) defines a partial map, which assigns to every pp-definable subgroup $\mathbf{G}$ of M at most one coset of $\mathbf{G}$. $\mathbf{q}$ is completely determined by this map.

But if the property of Claim 1 holds, there are always only finitely many cosets possible. Whence

$$\text{number of } \mathbf{q}\text{'s} \leqslant 2^{\text{number of pp-definable subgroups}}.$$

Conclusion: $\hat{\mathbf{q}}$ is a non-forking extension of p.

To prove 11.1, let q be a non-forking extension of p. Then there is an extension $\mathbf{q}$ of q which does not fork over A. Since $\hat{\mathbf{q}}$ and $\mathbf{q}$ are conjugate over A, every $\mathbf{G} \in G(\mathbf{q})$ is of finite index in some $\mathbf{H} \in G(p)$. But $G(q) \subset G(\mathbf{q})$.

For the converse assume that q has the property of 11.1. By Claim 1 we find an extension $\mathbf{q}$ of q s.t. every $\mathbf{G} \in G(\mathbf{q})$ is of finite index in some $\mathbf{H} \in G(q)$. But since $\mathbf{H}$ is of finite index in some $\mathbf{K} \in G(p)$, we can conclude that every $\mathbf{G} \in G(\mathbf{q})$ is of finite index in some $\mathbf{K} \in G(p)$. Thus $\mathbf{q}$ does not fork over A.

For the rest of this section we assume that there are no finite indices $\varphi/\psi(\mathbb{M}) \neq 1$, i.e.

(*) $\qquad \mathbb{M} \oplus \mathbb{M} \equiv \mathbb{M}$.

Corollary 11.2 (*). (1) (Makkai). *q is a non-forking extension of p iff $G(p) = G(q)$. There is only one non-forking extension $q \in S(B)$ of p.*

(2) (Garavaglia). $\mathrm{tp}(a/A)$ *does not fork over* 0 *iff a and A are independent.*

Proof of (2). $G(\mathrm{tp}(a/0)) = G(\mathrm{tp}(a/A))$ just expresses independence of a and A in the sense of (b) of the definition before 6.3. (Forking in injective modules was also studied in [33].)

A type $p \in S(A)$ is *regular* iff for all $B \supset A$, $a, b \in \mathbb{M}$, $\mathrm{tp}(a/B)$ non-forking extension of p, $\mathrm{tp}(b/B)$ forking extension of $p \Rightarrow \mathrm{tp}(a/B \cup \{b\})$ does not fork over A.

We assume a regular type to be non-algebraic.

Remark 11.3. Let $p \in S(A)$, q the (!) non-forking extension of p to $H(A)$. One knows that p is regular iff q is regular. q can be decomposed: If a realizes q and is written as $a_1 + a_2$ according to a decomposition $H(A) \oplus \mathbb{A}_2 = \mathbb{M}$, $\mathrm{tp}(a_2)$ is uniquely determined by p, and is regular iff p is. Therefore we will restrict ourself in the sequel to complete types over 0. We identify these types with pp-complete types.

Theorem 11.4 (*). *Let $p = \mathrm{tp}(a)$. Then the following are equivalent:*

(a) *p is regular.*

(b) *p^+ is a maximal pp-type which is satisfied by a non-zero element of $H(a)$.*

(c) *An endomorphism of $H(a)$ is an automorphism iff it does not map a to 0.*

(This was independently proved in [26].)

Proof. (a) $\to$ (b). Let $b_1 \in H(a) \setminus 0$, $p^+ \subset \mathrm{tp}^+(b_1)$. Write $\mathbb{M} = H(a) \oplus \mathbb{N}$. By (*) we find $b_2 \in \mathbb{N}$ s.t. $p = \mathrm{tp}(b_2)$. $b = b_1 + b_2$ realizes p. $\mathrm{tp}(a/\mathbb{N})$ does not fork over 0. But, since a depends on $\mathbb{N} \cup \{b\}$, $\mathrm{tp}(a, N \cup \{b\})$ forks over 0. By regularity $\mathrm{tp}(b/\mathbb{N})$ does not fork over 0. Thus b does not depend on $\mathbb{N}$. If $\mathbb{M} = K \oplus \mathbb{N}$, $b \in K$, the projection onto $H(a)$ induces an isomorphism from K onto $H(a)$. This isomorphism maps b into b_1, thus $p^+ = \mathrm{tp}^+(b) = \mathrm{tp}^+(b_1)$.

(b) $\to$ (a). Let $\mathrm{tp}(a/B)$ be a non-forking extension of p and suppose that $\mathrm{tp}(a/B \cup \{b\})$ forks over 0 and $\mathrm{tp}(b) = p$. Write $\mathbb{M} = H(a) \oplus \mathbb{N}$, $B \subset \mathbb{N}$ and accordingly $b = b_1 + b_2$. Since a depends on $B \cup \{b\}$, we have $b_1 \neq 0$. We have $p^+ \subset \mathrm{tp}^+(b_1)$, whence by assumption $p = \mathrm{tp}(b_1) = \mathrm{tp}(b)$.

Let $\boldsymbol{b} \in B$ and φ a pp-formula. Then $\mathbb{M} \models \varphi(\boldsymbol{b}, b)$, i.e. $\mathbb{M} \models \varphi(\boldsymbol{b}, b_1 + b_2)$ implies $\mathbb{M} \models \varphi(\mathbf{0}, b_1 + 0)$ and thus $\mathbb{M} \models \varphi(\mathbf{0}, b)$. This proves that $\mathrm{tp}(b/B)$ does not fork over 0.

(b) $\to$ (c). If $f \in \mathrm{End}(H(a))$ maps a into $b \neq 0$. Then $\mathrm{tp}(b) = \mathrm{tp}(a)$ and $f \restriction a$ is partially isomorphic. Thus f is an automorphism. (See the proof of 4.3.)

(c) $\to$ (b). If $b \in H(a) \setminus 0$, $\mathrm{tp}^+(b) \supset p^+$, $a \mapsto b$ is a partial endomorphism which can be extended to an endomorphism of $H(a)$. This endomorphism by assumption is an automorphism. Thus $p = \mathrm{tp}(b)$.

Corollary 11.5 (*). (1) *Regular types* (*over* 0) *are indecomposable.*

(2) (Makkai). *Let $p \in S(0)$, N be a pure submodule of $\mathbb{M}$. If p^+ is maximal among those pp-types for which $p^+ \cup \{\neg x \doteq 0\}$ is N-consistent, then p is regular.*

11.5(1) follows from 11.4: $\mathrm{End}(H(p))$ is local.

Examples. If R is a Dedekind ring, the indecomposable modules which belong to regular types are $R/\mathfrak{P}^n$, $R/R_{\mathfrak{P}}$, K.

If the indecomposable module U is totally transcendental, $U = H(p)$ for a regular type. ($p = \mathrm{tp}(a)$, $a \in \varphi(U)\setminus 0$, where $\varphi(U)$ is a minimal pp-definable subgroup.)

Injective modules (see 5.10, 5.11): If R is commutative, the regular types (which determine injective indecomposables) are in 1–1 correspondence with the prime ideals of R. (For $\mathfrak{P}$ is prime iff maximal among the irreducible ideals $\mathfrak{A}$ with $H(\mathfrak{P}) \cong H(\mathfrak{A})$. See the proof of 5.11.) For noetherian commutative rings this result is due to Kucera [12].

$p, q \in S(A)$ are *orthogonal* iff for all $B \supset A$, $a, b \in \mathbb{M}$, $\mathrm{tp}(a/B)$, resp. $\mathrm{tp}(b/B)$ is a non-forking extension of p, resp. $q \Rightarrow \mathrm{tp}(a/B \cup \{b\})$ does not fork over A.

Theorem 11.6 (*). *Assume that M has bounded width. Then $p, q \in S(0)$ are orthogonal, iff $H(p)$ and $H(q)$ have no isomorphic indecomposable factor in common.*

Proof. Let a, resp. b realize p, resp. q. If $H(a)$ and $H(b)$ have a common factor, $H(a) \cap H(b) \neq 0$. Whence a and b are not independent and p, q not orthogonal. (Take $B = 0$ in the definition above.)

Assume now, that $H(p)$ and $H(q)$ have no non-trivial common factor. Let B, a, b as in the definition of orthogonality. Since $H(b)$ has bounded width, $H(b) = \overline{\bigoplus_{i \in I} U_i}$, U_i indecomposable. $\mathrm{tp}(b/B)$ does not fork over 0, the family $H(B)$, $U_i (i \in I)$ is therefore independent. Suppose that $H(a)$ depends on $\{H(B)\} \cup \{U_i\}_{i \in I}$. Since no U_i is isomorphic to a direct factor of $H(a)$, 6.2 shows that $H(a)$ depends on $H(B)$ alone. But this contradicts our assumption that $\mathrm{tp}(a/B)$ does not fork over 0. Thus $H(a)$ does not depend on $H(B) \cup \{U_i\}$, i.e. a and $B \cup \{b\}$ are independent.

Mike Prest proved in [26] that p and q are orthogonal iff $H(p)$ and $H(q)$ have no non-zero direct factor in common. (*) and width are not needed.

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