

Real computation with least discrete advice: A complexity theory of nonuniform computability with applications to effective linear algebra

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ABSTRACT

It is folklore particularly in numerical and computer sciences that, instead of solving some general problem $f : X \ni x \mapsto f(x) \in Y$, additional structural information about the input $x \in X$ (e.g. any kind of promise that x belongs to a certain subset $X' \subseteq X$, or does not) should be taken advantage of. In several examples from real number computation, such advice even makes the difference between computability and uncomputability. We turn this into a both topological and combinatorial complexity theory of information, investigating for several practical problems how much advice is necessary and sufficient to render them computable.

Specifically, finding a nontrivial solution to a homogeneous linear equation $A \cdot \vec{x} = 0$ for a given singular real $n \times n$ -matrix A is possible when knowing $\text{rank}(A) \in \{0, 1, \dots, n-1\}$; and we show this to be best possible. Similarly, diagonalizing (i.e. finding a basis of eigenvectors to) a given real symmetric $n \times n$ -matrix A is possible when knowing the number of distinct eigenvalues: an integer between 1 and n (the latter corresponding to the nondegenerate case). And again we show that n -fold (i.e. roughly $\log n$ bits of) additional information is indeed necessary in order to render this problem (continuous and) computable; whereas for finding *some single* eigenvector of A , providing the truncated binary logarithm of the dimension of the least-dimensional eigenspace of A —i.e. $\lfloor 1 + \log_2 n \rfloor$ -fold advice—is sufficient and optimal.

Our proofs employ, in addition to topological considerations common in Recursive Analysis, also combinatorial arguments.

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1. Introduction

Recursive Analysis, that is Turing's theory of rational approximations with prescribable error bounds [49,50], is generally considered a very realistic model of real number computation [4]. Much effort has been spent in 'effectivizing' classical mathematical theorems, that is replacing mere existence claims

(i) "For all x , there exists some y such that ..."

with

(ii) "For all computable x , there exists some computable y such that ..."

or even uniformly:

(iii) "Given x , one can compute y such that ..."

Cf. e.g. the Intermediate Value Theorem in classical analysis [52, Theorem 6.3.8.1] or the Krein–Milman Theorem from convex geometry [16]. Note that Claim (ii) is nonuniform: it asserts y to be computable whenever x is; yet, there may be no way of *converting* (as in Claim (iii)) a Turing machine M computing x into a machine N computing y [52, Section 9.6]. In fact, computing a function $f : x \mapsto y$ is significantly limited by the sometimes so-called Main Theorem, requiring that any such f

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be necessarily continuous: because finite approximations to the argument x do not allow to determine the value $f(x)$ up to absolute error smaller than the ‘gap’ $\limsup_{t \rightarrow x} f(t) - \liminf_{t \rightarrow x} f(t)$ in case x is a point of discontinuity of f . In particular any non-constant discrete-valued function on the reals is uncomputable—for information-theoretic (as opposed to recursion-theoretic) reasons. Thus, Recursive Analysis is sometimes criticized for rendering uncomputable even functions as simple as Gauß’ staircase [30].

1.1. Motivating examples

On the other hand many applications do provide, in addition to approximations to the continuous argument x , also certain discrete hints or ‘advice’; e.g. a bit indicating whether x is integral or not. And such additional information can render many otherwise uncomputable problems computable:

Example 1. The Gauß staircase is discontinuous, hence uncomputable. Restricted to integers, however, it is simply the identity, thus computable. And restricted to *non*-integers, it is computable as well; cf. [52, Exercise 4.3.2]. Thus, one bit of additional advice (“*integer or not*”) suffices to make $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ computable.

It is long known [50] that a sequence of rational approximations to some $x \in \mathbb{R}$ with error bounds cannot continuously be converted into a binary expansion of x . In fact it is discontinuous on the subset $\mathbb{D} := \{(2r + 1)/2^k : r, k \in \mathbb{Z}\}$ of dyadic rationals: basically because these admit two distinct binary expansions like, e.g. $(0.1000 \dots)_2 = \frac{1}{2} = (0.01111 \dots)_2$. For non-dyadic reals, on the other hand, such a conversion is computably possible [52, Theorem 4.1.13.1]. Now consider the problem of computing only the first n bits of the binary expansion of x . While 2^n -fold advice (namely the n bits) is trivially sufficient, one can do much better:

Example 2. $\text{Adic}_2^{(n)} : [0, 1] \rightrightarrows \{0, 1\}^n$ is (ρ, ν) -computable with 2-fold advice; namely when providing, in addition to a ρ -name of x , also the n -th bit of (some of) its binary expansion.

Note however that, implicitly, n is given here.

Proof. Suppose that $[0, 1] \ni x = \sum_{i=1}^{\infty} b_i 2^{-i}$ with $b_i \in \{0, 1\}$ and $b_n = 0$. (The other case $b_n = 1$ proceeds analogously.) Then, corresponding to the 2^{n-1} possible choices of $(b_1, \dots, b_{n-1}, b_n)$ with $b_n = 0$, it holds $x \in [0, 2^{-n}] \cup [2 \cdot 2^{-n}, 3 \cdot 2^{-n}] \cup \dots \cup [(2^n - 2) \cdot 2^{-n}, (2^n - 1) \cdot 2^{-n}]$:

$$x \in \left(\left(2k - \frac{1}{2} \right) \cdot 2^{-n}, \left(2k + \frac{3}{2} \right) \cdot 2^{-n} \right) \tag{1}$$

for some (unique) $k \in \{0, 1, \dots, 2^{n-1}\}$. Conversely, Eq. (1) implies (since $b_n = 0$) $x \in [(2k) \cdot 2^{-n}, (2k + 1) \cdot 2^{-n}]$ and $(b_1, \dots, b_{n-1}) = \text{bin}(k)$. Since *strict* real inequalities are semi-decidable (formally: ρ -r.e. open in the sense of [52, Definition 3.1.3.2]), dovetailing can search for k to satisfy Eq. (1). \square

Many problems in analysis involving compact (hence bounded) sets are discontinuous unless provided with some integer bound; compare e.g. [52, Section 5.2] or the next

Example 3. Differentiation of smooth functions, that is the mapping

$$\partial : C^\infty[0, 1] \rightarrow C^\infty[0, 1], \quad f \mapsto f'$$

is discontinuous w.r.t. the uniform norm $\| \cdot \|$. However when given, in addition to uniform approximations to f , some upper bound on $\|f''\|$, ∂ does become computable.

Proof. Cf. [52, Exercise 6.4.9] and the proof of [52, Corollary 6.4.8.2]. \square

For a more involved illustration from computational linear algebra, we report from [57, Section 3.5] the following

Example 4. Given a real symmetric $d \times d$ matrix A (in form of approximations $A_n \in \mathbb{Q}^{d \times d}$ with $|A - A_n| \leq 2^{-n}$), it is generally impossible, for lack of continuity and even in the multivalued sense, to compute (approximations to) any eigenvector of A . However when providing, in addition to A itself, the number $\text{Card } \sigma(A)$ of distinct eigenvalues (i.e. *not* counting multiplicities in the spectrum $\sigma(A)$) of A , finding the entire spectral resolution (i.e. an orthogonal basis of eigenvectors) becomes computable.

Another case study on the benefit of additional discrete advice turning nonuniform into uniform computability is taken from [43, Lemma 2.8]:

Example 5. A closed subset $A \subseteq \mathbb{R}^d$ is called ψ^d -computable if one can, given $\vec{x} \in \mathbb{R}^d$, approximate the distance

$$d_A(\vec{x}) = \min \{ \|\vec{x} - \vec{a}\|_2 : \vec{a} \in A \} \tag{2}$$

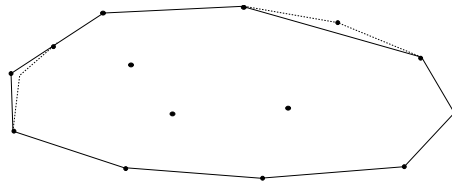


Fig. 1. The convex hull of some points in 2D. Infinitesimal perturbation can heavily affect the (number and) subset of extreme points.

from below; more formally: upon input of a sequence $\vec{q}_n \in \mathbb{Q}^d$ with $\|\vec{x} - \vec{q}_n\| \leq 2^{-n}$, output a sequence $p_m \in \mathbb{Q}$ with $\sup_m p_m = d_A(\vec{x})$; compare [52, Section 5.1]. Similarly, $\psi_{<}^d$ -computability of A means approximation of d_A from above.

- (a) A finite set $A = \{\vec{v}_1, \dots, \vec{v}_N\} \subseteq \mathbb{R}^d$ is $\psi_{<}^d$ -computable iff it is ψ^d -computable iff each element \vec{v}_i is computable.
- (b) Neither of the three nonuniform equivalences in (a) holds uniformly.
- (c) However if the cardinality of A is given as additional information, $\psi_{<}^d$ -computability becomes uniformly equivalent to computability of A 's members.
- (d) Whereas $\psi_{>}^d$ -computability still remains uniformly strictly weaker than the other two.

Our next example treats a standard problem from computational geometry [3, Section 1.1]:

Example 6. For a set $S \subseteq \mathbb{R}^d$, its *convex hull* is the least convex set containing S :

$$\text{chull}(S) := \bigcap \{C : S \subseteq C \subseteq \mathbb{R}^d, C \text{ convex}\}.$$

A *polytope* is the convex hull of finitely many points, $\text{chull}(\{\vec{p}_1, \dots, \vec{p}_N\})$. For a convex set C , point $\vec{p} \in C$ is called *extreme* (written " $\vec{p} \in \text{ext}(C)$ ") if it does not lie on the interior of any line segment contained in C :

$$\vec{p} = \lambda \cdot \vec{x} + (1 - \lambda) \cdot \vec{y} \wedge \vec{x}, \vec{y} \in C \wedge 0 < \lambda < 1 \Rightarrow \vec{x} = \vec{y}.$$

For a set X , let $\binom{X}{k} := \{\{x_1, \dots, x_k\} : x_i \in X \text{ pairwise distinct}\}$. The problem

$$\text{extchull}_N : \binom{\mathbb{R}^d}{N} \ni \{\vec{x}_1, \dots, \vec{x}_N\} \mapsto \{\vec{y} \text{ extreme point of } \text{chull}(\vec{x}_1, \dots, \vec{x}_N)\} \tag{3}$$

of identifying the extreme points of the polytope C spanned by given pairwise distinct $\vec{x}_1, \dots, \vec{x}_N$, is discontinuous (and hence uncomputable) already in dimension $d = 2$ and for $N = 3$ with respect to output encoding $\psi_{>}$, cf. Fig. 1:

Let $\vec{x}_1 := (0, 0)$, $\vec{x}_2 := (1, 0)$, and $\vec{x}_3 := (\frac{1}{2}, \epsilon)$: For $\epsilon = 0$, these points get mapped to $\{(0, 0), (1, 0)\}$; whereas for $\epsilon \neq 0$, the set of extreme points is $\{(0, 0), (1, 0), (\frac{1}{2}, \epsilon)\}$.

Trivially, extchull_N does become computable when giving, in addition to approximations to the points $\vec{x}_1, \dots, \vec{x}_N$, one bit $b_i \in \{0, 1\}$ for each $i = 1, \dots, N$ (that is, totally and in binary an integer between 0 and $2^N - 1$) indicating whether $\vec{x}_i \in \text{ext chull}(\vec{x}_1, \dots, \vec{x}_N)$. However in Proposition 24 below we shall show that, in order to compute extchull_N , it suffices to know merely the number $M \in \{2, \dots, N\}$ of extreme points of $\text{chull}(\vec{x}_1, \dots, \vec{x}_N)$ —and that $(N - 1)$ -fold discrete advice is in fact necessary.

Instead of providing discrete advice, the convex hull problem can of course also be made computable by redefining the problem [15].

We finally demonstrate how a little discrete advice can, even for an already (continuous and) computable real function, have a tremendous impact on its computational complexity:

Example 7. Let $A \subseteq \mathbb{N}$ be arbitrary. Fix some 'canonical pulse' $\varphi : \mathbb{R} \rightarrow [0, 1]$ similar to [42, p.52], that is an infinitely often differentiable (i.e. C^∞) polynomial-time computable function with $\varphi(1/2) = 1$ and $\varphi^{(k)}(x) = 0 = \varphi^{(k)}(x)$ for every $k \in \mathbb{N}$ and every $x \notin [\frac{1}{4}, \frac{3}{4}]$.

- (a) Deciding A is (up to polynomial overhead) as hard as evaluating the C^∞ -function

$$h_A : [0, 1] \rightarrow [0, 1], \quad x \mapsto \sum_{n \in A} 2^{-n^2} \cdot \varphi(x \cdot 2^n - 1)$$

in the sense of [29, Section 2.4].

In particular for A decidable but not in $\mathcal{P}/\text{PSPACE}/\text{EXP}$, h_A is computable but not within polynomial time/polynomial space/exponential time.

(b) When providing, in addition to some $x \in [2^{-n}, 2^{-n+1}]$, also the two-fold advice of whether $n \in A$ holds or not, this renders h_A polynomial-time computable.

Proof. Note that h_A is composed from disjoint pulses. In particular, according to the chain rule of differentiation, its k -th derivative $h_A^{(k)}(x)$ equals $2^{nk} \cdot 2^{-n^2} \varphi^{(k)}(x \cdot 2^n - 1)$ for $x \in [2^{-n}, 2^{-n+1}]$; which is bounded independently of n , thus showing that h_A is C^∞ .

Next, $h_A(2^{-n} \cdot 3/2) = 2^{-n^2}$ if $n \in A$ and $h_A(2^{-n} \cdot 3/2) = 0$ otherwise: Evaluating h_A up to error 2^{-n^2} reveals whether $n \in A$ holds or not.

Conversely, given $x \in (2^{-n}, 2^{-n+1})$, evaluating $h_A(x)$ amounts to calculating $\varphi(x \cdot 2^n - 1)$ in polynomial time in case $n \in A$; and $h_A(x) = 0$ in case $n \notin A$. Also, $h_A(x) = 0$ for $x \in (\frac{7}{4} \cdot 2^{-n}, \frac{10}{4} \cdot 2^{-n})$. So in order to approximate $h_A(x)$ up to error 2^{-m^2} , it suffices to detect whether x belongs to one of the intervals $(\frac{7}{4} \cdot 2^{-n}, \frac{10}{4} \cdot 2^{-n})$ or $(2^{-n}, 2^{-n+1})$, $n \leq m$; and, in the latter case, to know whether $n \in A$ holds. Since the above intervals overlap by at least $\frac{1}{4} \cdot 2^{-m}$, the former can be achieved in time polynomial in m . \square

It seems conceivable that other (and perhaps more natural) examples from analysis, too, exhibit trade-offs between their computational complexity and the amount of advice provided. This perspective may also be helpful in devising upper bounds: starting off with a non-uniform algorithm and trying to gradually reduce the discrete advice employed, hopefully eventually arriving at a uniform one.

1.2. Complexity measure of nonuniform computability

We are primarily interested in problems over real Euclidean spaces \mathbb{R}^d , $d \in \mathbb{N} = \{1, 2, \dots\}$. Yet for reasons of general applicability to arbitrary spaces U of continuum cardinality, let us recall from Weihrauch’s TTE framework [52, Section 3] the concept of a so-called *representation* $\alpha : \subseteq \Sigma^\omega \rightarrow U$, that is an encoding of all elements $u \in U$ as infinite binary strings. For a countable space U , on the other hand, a *notation* $\alpha : \subseteq \Sigma^* \rightarrow U$ encodes all elements $u \in U$ as finite binary strings. Now a (α, β) -realizer of a function $f : U \rightarrow V$ maps all α -encodings of $u \in U$ to β -encodings of $f(u) \in V$; and f is called (α, β) -computable if some Turing machine with one-way (i.e. non-rewritable) output tape can compute an (α, β) -realizer of f .

Providing *discrete advice* to f amounts to presenting to the Turing machine, in addition to an infinite binary string encoding $u \in U$, some integer (or ‘color’) i ; and doing so for each u , means to colorize U . Now it is natural to wonder about the least advice (i.e. the minimum number of colors) needed:

Definition 8. (a) For a function $f : \subseteq A \rightarrow B$ between topological spaces A and B and covering Δ of $\text{dom}(f) = \bigcup_{D \in \Delta} D$, f is Δ -continuous if $f|_D$ is continuous for each $D \in \Delta$.

Call $\mathfrak{C}_t(f) := \min\{\text{Card}(\Delta) : f \text{ is } \Delta\text{-continuous}\}$ the *cardinal of discontinuity* of f .

(b) A function $f : \subseteq A \rightarrow B$ between represented spaces (A, α) and (B, β) is k -wise (α, β) -continuous if there exists a partition Δ of $\text{dom}(f)$ of $\text{Card}(\Delta) = k$ such that $f|_D$ is (α, β) -continuous on each $D \in \Delta$.

(c) Call f *nonuniformly* (α, β) -computable if, for every α -computable $a \in \text{dom}(f)$, $f(a)$ is β -computable.

(d) We say that f is (α, β) -computable with k -fold advice (or simply k -computable whenever α, β are clear from the context) if there exists a partition Δ of $\text{Card}(\Delta) = k$ and a notation δ of Δ such that the mapping $f_\Delta : (a, D) \mapsto f(a)$ is (α, δ, β) -computable on $\text{dom}(f_\Delta) := \{(a, D) : a \in D \in \Delta\}$.

Call $\mathfrak{C}_c(f) = \mathfrak{C}_c(f, \alpha, \beta) := \min\{k : f \text{ is } (\alpha, \beta)\text{-computable with } k\text{-fold advice}\}$ the *complexity of nonuniform* (α, β) -computability of f .

A function computable with finite advice is obviously nonuniformly computable, because a δ -name of $D \in \Delta$ with $a \in D$ is finite and can thus be incorporated into a machine computing $f(a)$. This justifies *complexity of nonuniform computability* as a notion and quantitative refinement of nonuniform computability.

Note that in (a) one can always proceed from a covering to a partition of $\text{dom}(f)$ by breaking ties arbitrarily. Now continuous functions are exactly the 1-continuous ones; and computability is equivalent to computability with 1-fold advice. We record, as an extension of the Main Theorem of Recursive Analysis, the following immediate

Observation 9. If α, β are admissible representations in the sense of [52, Definition 3.2.7], then every k -wise (α, β) -computable function is k -continuous (but not vice versa); that is $\mathfrak{C}_t(f) \leq \mathfrak{C}_c(f)$ holds.

More precisely, every k -wise (α, β) -computable function $f : \subseteq A \rightarrow B$ has a k -continuous (α, β) -realizer in the sense of [52, Definition 3.1.3.4], hence is k -wise (α, β) -continuous.

¹ Although the Continuum Hypothesis is *not* needed in order to make this minimum well-defined, in view of Item (d) we shall only be concerned with countable (and usually even finite) advice anyway.

The above examples illustrate some interesting discontinuous functions to be computable with k -fold advice for certain $k \in \mathbb{N}$. Specifically [Example 4](#) shows that diagonalizing real symmetric $d \times d$ -matrices is d -computable; and [Theorem 47](#) below will reveal this value d to be optimal. In fact the present work determines for some natural computational problems in linear algebra explicitly both their cardinal of continuity and their complexity of nonuniform computability.

Remark 10. We advertise Computability with Finite Advice as a generalization of classical Recursive Analysis:

- (a) It captures the concept of a hybrid approach to discrete&continuous computation.
- (b) It complements Type-2 oracle computation:
In the discrete realm, every function $f : \mathbb{N} \rightarrow \mathbb{N}$ becomes computable when employing an appropriate oracle; whereas in the Type-2 case, exactly the *continuous* functions $f : \mathbb{R} \rightarrow \mathbb{R}$ are computable relative to some oracle [[58](#), Corollary 6]. On the other hand, 2-fold advice can make a continuous function computable which, without advice, may have arbitrarily high degree of uncomputability; see [Proposition 11\(d\)](#).
- (c) Discrete advice avoids a common major point of criticism against Recursive Analysis, namely that it denounces even simplest discontinuous functions as uncomputable;
- (d) and such kind of advice is very practical: In applications additional discrete information about the input is often actually available and should be used. For instance a given real matrix may be known to be non-degenerate (as is often exploited in numerics) or, slightly more generally, to have k eigenvalues coincide for some known $k \in \mathbb{N}$.

Interestingly, this idea has also been expressed in the review [[33](#)] of a book on aspects of discrete recursion theory.

1.3. Related work, in particular Kolmogorov complexity

[Definition 8](#) goes back to [[51](#), Definition 3.3]; see also [[41](#), Definition 5.8] where our quantity $\mathfrak{C}_t(f)$ it is called “basesize”. Providing discrete advice can also be considered as yet another instance of enrichment in mathematics [[31](#), p.238/239].

Various other approaches have been pursued in the literature in order to make discontinuous functions accessible to nontrivial computability investigations.

Exact Geometric Computation considers the arguments \bar{x} as exact rational numbers [[35](#)].

Special encodings of discontinuous functions motivated by spaces in Functional Analysis, are treated e.g. in [[62](#)]; however these do not admit function evaluation.

Weakened notions of computability may refer to stronger models of computation [[12](#)]; provide more information on (e.g. the binary encoding of, rather than rational approximations with error bounds to) the argument x [[38,40](#)]; or expect less information on (e.g. no error bounds for approximations to) the value $f(x)$ [[55](#)].

A taxonomy of discontinuous functions, namely their *degrees* of Borel measurability, is investigated in [[10,59,60](#)]:

Specifically, a function $f : \subseteq A \rightarrow B$ is continuous ($=\Sigma_1$ -measurable) iff, for every closed $T \subseteq B$, its preimage $f^{-1}[T]$ is closed in $\text{dom}(f) \subseteq A$; and f is computable iff this mapping $T \mapsto f^{-1}[T]$ on closed sets is $(\psi_{\leq}^d, \psi_{\leq}^d)$ -computable.

A degree relaxation, f is called Σ_2 -measurable iff, for every closed $T \subseteq B$, $f^{-1}[T]$ is an F_δ -set.

Weihrauch degrees compare functions f, g by considering f ‘at most as discontinuous/uncomputable’ as g if there exist continuous/computable functions ϕ, ψ such that $f = \phi \circ (\text{id}, g \circ \psi)$ [[5](#)]; cmp. also [[52](#), Section 8.2]. [[41](#), Theorem 5.9] shows that Weihrauch degrees are a (considerable) refinement of least discrete advice; cmp. [Lemmas 15 and 39](#).

Levels of discontinuity are studied in [[20,22,23](#)]:

Take the set $\text{LEV}'(f, 1) \subseteq \text{dom}(f) =: \text{LEV}'(f, 0)$ of points of discontinuity of $f = f|_{\text{LEV}'(f, 0)}$; then the set $\text{LEV}'(f, 2) \subseteq \text{LEV}'(f, 1)$ of points of discontinuity of $f|_{\text{LEV}'(f, 1)}$ and so on: the least index k for which $\text{LEV}'(f, k) = \emptyset$ holds is f 's level of discontinuity $\text{Lev}'(f)$.

A variant, $\text{Lev}(f)$, considers $\text{LEV}(f, 1)$ the closure of $\text{LEV}'(f, 1)$ in $\text{dom}(f)$, then $\text{LEV}(f, 2)$ the closure of points of discontinuity of $f|_{\text{LEV}(f, 1)}$ and so on until $\text{LEV}(f, k) = \emptyset$.

Our approach superficially resembles the fourth and last ones above. A minor difference, they correspond to *ordinal* measures whereas the size of the partition considered in [Definition 8](#) is a *cardinal*. As a major difference we now establish these measures as logically largely independent.

Proposition 11. (a) *There exists a 2-computable function $f : [0, 1] \rightarrow \{0, 1\}$ which is not measurable nor on any level of discontinuity.*

(b) *There exists a Δ_2 -measurable function $f : [0, 1] \rightarrow [0, 1]$ which is not k -continuous for any finite k .*

(c) *If f is on the k -th level of discontinuity, it is k -continuous; in formula: $\mathfrak{C}_t(f) \leq \text{Lev}'(f) \leq \text{Lev}(f)$.*

(d) *To any oracle $\mathcal{O} \subseteq \Sigma^*$ there exists a continuous, 2-computable function $f : \subseteq [0, 1] \rightarrow [0, 1]$ which is not computable even relative to \mathcal{O} .*

- (e) There are nonuniformly computable functions not k -computable for any $k \in \mathbb{N}$.
- (f) There even exists a nonuniformly computable $f : \mathbb{R} \rightarrow \mathbb{R}$ with $\mathfrak{C}_t(f) = c$, the cardinality of the continuum.

Any real function is trivially c -continuous by partitioning its domain into singletons. Item (f) is due to ANDREJ BAUER, personal communication. Item (c) appears also in [41, Theorem 5.10]. The last paragraph of [41, Section 5.1] includes our Item (e) and partly extends Item (a) by exhibiting, to any ordinal λ and cardinal $\beta \leq \lambda$, a function $f : \subseteq \mathbb{N}^\lambda \rightarrow \beta$ with $\mathfrak{C}_t(f) = \beta$ and $\text{Lev}(f) = \lambda$. Complementing Item (e), conditions where nonuniform computability does imply (even) 1-computability have been devised in [8].

Proof of Proposition 11. (a) Consider a non Borel-measurable subset $S \subseteq [0, 1]$; e.g. exceeding the Borel hierarchy [24,39] by being complete for Δ_1^1 . (Using the Axiom of Choice, S can even be chosen as non Lebesgue-measurable.) Then its characteristic function $\mathbf{1}_S$ is not measurable and totally discontinuous, hence $[0, 1] = \text{LEV}'(\mathbf{1}_S, 1) = \text{LEV}'(\mathbf{1}_S, 2) = \dots$; whereas $(S, [0, 1] \setminus S)$ gives a 2-decomposition of $\text{dom}(\mathbf{1}_S)$ with $\mathbf{1}_{S|_S} \equiv 1$ and $\mathbf{1}_{S|[0,1]\setminus S} \equiv 0$.

- (b) See Example 23(b).
- (c) By definition, f is continuous on $\text{LEV}'(f, 0) \setminus \text{LEV}'(f, 1)$, on $\text{LEV}'(f, 1) \setminus \text{LEV}'(f, 2)$, and so on—until $\text{LEV}'(f, k-1)$ on which f is continuous because $\text{LEV}'(f, k) = \emptyset$. Therefore $\Delta = (\text{LEV}'(f, 0) \setminus \text{LEV}'(f, 1), \text{LEV}'(f, 1) \setminus \text{LEV}'(f, 2), \dots, \text{LEV}'(f, k-1))$ constitutes a partition with the desired properties.
- (d) Fix any uncomputable $t \in [0, 1]$ and consider

$$f : \subseteq [0, 1] \rightarrow [0, 1], \quad f(x) := 0 \text{ for } x < t, \quad f(x) := 1 \text{ for } x > t, \quad f(t) := \perp$$

which is obviously continuous (because the ‘jump’ $x = t$ is not part of $\text{dom}(f)$) and 2-computable (namely on $[0, t)$ and $(t, 1]$). Since t is uncomputable, $t \notin \mathbb{Q}$. So if f were computable, we could evaluate it at any $x \in \mathbb{Q}$ to conclude whether $x < t$ or $x > t$; and apply bisection to compute t itself: contradiction. In fact we may choose t uncomputable relative to any prescribed oracle [56,1].

- (e) Let $f|_D$ be computable on each $D \in \Delta$. Then $f(x)$ is computable for each computable $x \in D$; hence also for each computable $x \in \text{dom}(f) = \bigcup \Delta$.

Example 23(b) has range $\{0\} \cup \{1/k : k \in \mathbb{N}\}$ consisting of computable (even rational) numbers only.

- (f) Consider a Sierpiński–Zygmund Function [26, Theorem 5.2] $f : \mathbb{R} \rightarrow \mathbb{R}$, i.e. such that $f|_D$ is discontinuous for any $D \subseteq \text{dom}(f)$ of $\text{Card}(D) = c$. Observe that this property is not affected by arbitrary modifications of f on any subset $X \subseteq \text{dom}(f)$ of $\text{Card}(X) < c$: If the restriction $f|_{\text{dom}(f) \setminus X}$ is continuous on $D \setminus X$ for some $D \subseteq \text{dom}(f)$ of $\text{Card}(D) = c$, then so is f on $D \setminus X$ —contradicting $\text{Card}(D \setminus X) = c$.

We may therefore modify the original function to be, say, identically 0 on the countable subset $X := \mathbb{R}_c$ of recursive reals, thus rendering it nonuniformly computable. Now suppose Δ is any partition of \mathbb{R} of $\text{Card}(\Delta) < c$. Then, e.g. by Cori and Lascar [13, Exercise 7.13],

$$c = \text{Card}(\mathbb{R}) = \sum_{D \in \Delta} \text{Card}(D) = \max(\text{Card}(\Delta), \sup_{D \in \Delta} \text{Card}(D))$$

requires $\text{Card}(D) = c$ for some $D \in \Delta$; but $f|_D$ is discontinuous, hence $\mathfrak{C}_t(f) \geq c$. \square

Further related research includes

Computational Complexity of real functions; see e.g. [29] and [52, Section 7]. Note, however, that Definition 8 refers to a purely information-theoretic notion of complexity of a function and is therefore more in the spirit of

Information-based Complexity in the sense of [48]. There, on the other hand, inputs are considered as real number entities given exactly; whereas we consider approximations to real inputs enhanced with discrete advice.

Finite Continuity is being studied for *Darboux Functions* in [36,37]. It amounts to d -continuity for some $d \in \mathbb{N}$ according to Definition 8.

Promise Problems in discrete complexity theory consider relaxations of classical decision problems (i.e. languages $L \subseteq \mathbb{N}$) where input instances do not range over entire \mathbb{N} , say, but are ‘promised’ to belong to some fixed subset $D \subseteq \mathbb{N}$.

This can be regarded as a kind of advice, cmp. [17, Section 7.1]; and, conversely, k -fold advice corresponds to k promises $D_1, \dots, D_k \subseteq \mathbb{N}$ that render the problem total in covering the entire input space: $\bigcup_j D_j = \mathbb{N}$.

Kolmogorov Complexity has been investigated for finite strings and, asymptotically, for infinite ones; cf. e.g. [32, Section 2.5] and [47]. Also a kind of advice is part of that theory in form of *conditional* complexity [32, Definition 2.1.2].

We quote from [32, EXERCISE 2.3.4ace] the following

Fact 12. An infinite string $\bar{\sigma} = (\sigma_n)_{n \in \omega} \in \Sigma^\omega$ is computable (e.g. printed onto a one-way output tape by some so-called Type-2 or monotone machine; cf. [52,44])

- (a) iff its initial segments $\bar{\sigma}_{1:n} := (\sigma_1, \dots, \sigma_n)$ have Kolmogorov complexity $\leq \mathcal{O}(1)$ conditionally to n , i.e., iff $C(\bar{\sigma}_{1:n}|n)$ is bounded by some $c = c(\bar{\sigma}) \in \mathbb{N}$ independent of n .

(b) Equivalently: the uniform complexity $C_u(\bar{\sigma}_{1:n}) := C(\bar{\sigma}_{1:n}; n)$ in the sense of [32, Exercise 2.3.3] is bounded by some c for infinitely many n .

Recall that $C(\bar{\sigma}_{1:n}; n)$ is defined as the least size of a program computing any (not necessarily proper) extension of the function $\{1, \dots, n\} \ni i \mapsto \sigma_i$ [32, Exercise 2.1.12]; i.e. in contrast to $C(\bar{\sigma}_{1:n}|n)$, only lower bounds i to n are provided.

Proof (Claim b). If $\bar{\sigma}$ is computable by some machine M , then obviously a minor (and constant size) modification M' of it will, given $n \in \mathbb{N}$, print $\bar{\sigma}_{1:n}$. Hence $C_u(\bar{\sigma}_{1:n}) \leq |M'|$.

Concerning the converse implication, observe that there are only $\mathcal{O}(1)^c$ machines of size $\leq c$. And for each of the infinitely many n , at least one of them prints all initial segments of length up to n . Hence by pigeonhole principle, a single one of them does so for infinitely many n . Which implies it does so even for all n . \square

Definition 13. (a) For $\bar{\sigma} \in \Sigma^\omega$, write $C(\bar{\sigma}) := \sup_n C(\bar{\sigma}_{1:n}|n)$ and $C(\bar{\sigma}|\bar{\tau}) := \sup_n C(\bar{\sigma}_{1:n}|n, \bar{\tau})$, where the Kolmogorov complexity conditional to an infinite string is defined literally as for a finite one [32, Definition 2.1.1].

(b) Similarly, let $C_u(\bar{\sigma}_{1:n}|\bar{\tau})$ denote the uniform complexity relative to the infinite string $\bar{\tau}$ and abbreviate $C_u(\bar{\sigma}|\bar{\tau}) := \sup_n C_u(\bar{\sigma}_{1:n}|\bar{\tau})$.

(c) For a represented space (A, α) and $a \in A$, write $C^\alpha(a) := \inf\{C(\bar{\sigma}) : \alpha(\bar{\sigma}) = a\}$; similarly for C_u^α .

Note that we purposely do not consider some normalized form of Kolmogorov Complexity for infinite strings like $C(\bar{\sigma}_{1:n}|n)/n$ in order to establish the following

Proposition 14. A function $F : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ is computable with finite advice iff the Kolmogorov complexity $C_u(F(\bar{\sigma})|\bar{\sigma})$ is bounded by some c independent of $\bar{\sigma} \in \text{dom}(F)$.

It seems that (at least the proof in [34] of) Fact 12(a) is ‘too nonuniform’ for Proposition 14 to hold with C_u replaced by C , even for compact $\text{dom}(F)$.

Proof. Suppose $\bar{\sigma} \mapsto F(\bar{\sigma})$ is computable for $\bar{\sigma} \in D_i$ by Turing machine M_i . Then obviously $C_u(F(\bar{\sigma})|\bar{\sigma}) \leq |M_i| + |\text{bin}(i)|$ is bounded independent of $i \leq d$.

Conversely consider, as in the proof of Fact 12(b), the $d \leq \mathcal{O}(1)^c$ machines M_i of size $\leq c$; and remember that, for each $\bar{\sigma} \in \text{dom}(F)$ and given $\bar{\sigma}$, some M_i outputs the entire (as opposed to just some initial segments of the) infinite string $F(\bar{\sigma})$. Let $D_i \subseteq \text{dom}(F)$ denote the set of those $\bar{\sigma}$ for which M_i does so. Then M_i computes $F|_{D_i}$ and $\text{dom}(F) = \bigcup_{i=1}^d D_i$: F is computable with d -fold advice. \square

1.4. Overview

Section 2 introduces and explores witnesses of k -discontinuity as a tool for proving lower bounds on the complexity of nonuniform computability. Section 3 extends this concept from (single-valued) functions to multivalued functions aka relations. Section 4 then applies it to determine the complexity of nonuniform computability for four typical tasks in high-school and scientific mathematics. We close in Section 5 with two generalizations.

2. Properties of the complexity of nonuniform computability

Lemma 15. (a) Let $f : A \rightarrow B$ be d -continuous (computable) and $A' \subseteq A$. Then the restriction $f|_{A'}$ is again d -continuous (computable).

(b) Let $f : A \rightarrow B$ be d -continuous (computable) and $g : B \rightarrow C$ be k -continuous (computable). Then $g \circ f : A \rightarrow C$ is $d \cdot k$ -continuous (computable).

(c) If $f : A \rightarrow B$ is (α, β) -computable with d -fold advice and $\alpha' \leq \alpha$ and $\beta \leq \beta'$, then f is also (α', β') -computable with d -fold advice.

(d) Let I denote some finite set with discrete topology and corresponding notation. Then $f : I \times A \rightarrow B$ is d -continuous (computable) iff each $f(i, \cdot) : A \rightarrow B$ is d -continuous (computable).

Proof. (a) Obviously, any partition Δ of A induces one $\Delta' := \{D \cap A' : D \in \Delta\}$ of A' of at most the same cardinality.

(b) If f is continuous (computable) on $A_i \subseteq A$ and g is continuous (computable) on $B_j \subseteq B$, then $g \circ f$ is continuous (computable) on $A_i \cap f^{-1}[B_j]$: f is on any subset of A_i ; and so is g on any subset of B_j , particularly on the image of $A_i \cap f^{-1}[B_j] \subseteq B_j$ under f .

(c) obvious.

(d) If f is d -continuous (computable), so is by (a) each restriction $f(i, \cdot)$.

Conversely, let $\Delta_i = (D_{i,j})_{j=1, \dots, d}$ be partitions of A such that $f(i, \cdot)|_{D_{i,j}}$ is continuous (computable) for each $i \in I$ and each $j = 1, \dots, d$. Since I bears the discrete topology (and is finite), f is continuous (computable) on each $D_j := \bigcup_{i \in I} \{i\} \times D_{i,j} \subseteq I \times A$ ($j = 1, \dots, d$). Now $\Delta := (D_j)_{j=1, \dots, d}$ constitutes a d -element partition of $I \times A$. \square

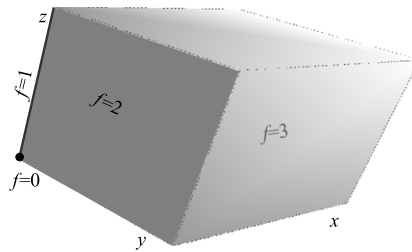


Fig. 2. Sketch of the function from Example 18(d).

A minimum size partition Δ of $\text{dom}(f)$ to make f computable on each $D \in \Delta$ need not be unique: Alternative to Example 1, we

Remark 16. Given a ρ -name of $x \in \mathbb{R}$ and indicating whether $\lfloor x \rfloor \in \mathbb{Z}$ is even or odd suffices to compute $\lfloor x \rfloor$: Suppose $\lfloor x \rfloor = 2k \in 2\mathbb{Z}$ (the odd case proceeds analogously). Then $x \in [2k, 2k + 1)$. Conversely, $x \in [2k - 1, 2k + 2)$, together with the promise $\lfloor x \rfloor \in 2\mathbb{Z}$, implies $\lfloor x \rfloor = 2k$. Hence, given $(q_n) \in \mathbb{Q}$ with $|x - q_n| \leq 2^{-n}$, $k := 2 \cdot \lfloor q_1/2 + \frac{1}{4} \rfloor$ (calculated in exact rational arithmetic) will yield the answer. \square

2.1. Witness of k -discontinuity

Recall that the partition Δ in Definition 8 need not satisfy any (e.g. topological regularity) conditions. The following notion turns out as useful in lower bounding the cardinality of such a partition:

- Definition 17.** (a) A d -flag \mathcal{F} in a topological space X is a $(d + 1)$ -tuple F_0, F_1, \dots, F_d of subsets of X such that F_i is contained in the closure $\overline{F_{i+1}}$ of F_{i+1} for each $i = 0, \dots, d - 1$.
 (b) Let $\mathcal{F} = (F_0, \dots, F_d)$ and $\mathcal{G} = (G_0, \dots, G_d)$ be d -flags with $G_i \subseteq F_i$ for $i = 0, \dots, d$. Then \mathcal{G} is called a subflag of \mathcal{F} . \mathcal{F} is trivial if $F_0 = \emptyset$.
 (c) For a partition Δ of X , a flag \mathcal{F} in X is Δ -monochromatic if, to every $i = 0, \dots, d$, there exists some $D \in \Delta$ with $F_i \subseteq D$.
 (d) For X, Y metric spaces and $f : X \rightarrow Y$, a witness of d -discontinuity of f is a nontrivial d -flag \mathcal{F} in $\text{dom}(f)$ such that, for each $0 \leq k < \ell \leq d$ and each $x \in F_k$ and each sequence $(x_n) \subseteq F_\ell$ with $x = \lim_n x_n$, $f(x_n)$ does not converge to $f(x)$.

Example 18. Let X, Y be metric spaces.

- (a) A function $f : X \rightarrow Y$ is discontinuous iff it admits a witness of 1-discontinuity:
 Indeed, suppose \mathcal{F} is a witness of 1-discontinuity. Then there exists $x \in F_0$, and, since $F_0 \subseteq \overline{F_1}$, $x = \lim_n x_n$ for some $(x_n) \subseteq F_1$. Now $f(x_n) \not\rightarrow f(x)$ shows that f is discontinuous at x . Conversely if f is discontinuous, then $f(x_n) \not\rightarrow f(x)$ for some $x, x_n \in \text{dom}(f)$ with $x_n \rightarrow x$. Thus, $F_0 := \{x\}$ and $F_1 := \{x_n : n \in \mathbb{N}\}$ constitutes a witness of 1-discontinuity.

(b) Let

$$x, (x_n)_n, (x_{n,m})_{n,m}, (x_{n,m,\ell})_{n,m,\ell}, \dots, (x_{n_1, \dots, n_d})_{n_1, \dots, n_d}$$

denote a family of (multi)sequences in X such that, for each $0 \leq k < d$ and each $\bar{n} \in \mathbb{N}^k$, it holds $\lim_m x_{\bar{n}, m} = x_{\bar{n}}$. Then (F_0, F_1, \dots, F_d) with $F_0 := \{x\}, F_1 := \{x_n : n \in \mathbb{N}\}, F_k := \{x_{\bar{n}} : \bar{n} \in \mathbb{N}^k\}$ constitutes a nontrivial d -flag in X .

- (c) Let $\mathcal{F} = (F_0, \dots, F_d)$ be a nontrivial d -flag in X and $f : X \rightarrow Z$ a function mapping to some discrete space Z with $f[F_k] \cap f[F_\ell] = \emptyset$ for each $0 \leq k < \ell \leq d$. Then \mathcal{F} is a witness of d -discontinuity.
 (d) Consider the function $f : \subseteq \mathbb{R}^3 \rightarrow \{-\infty, 1, 2, 3\}$ with $\text{dom}(f) = [0, \infty)^3$ defined by $f : (x_1, x_2, x_3) \mapsto \max\{i : x_i > 0\}$; cmp. Fig. 2. Then $(\{(0, 0, 0)\}, \{(x, 0, 0) : x > 0\}, \{(x, y, 0) : y > 0\}, \{(x, y, z) : z > 0\})$ is a witness of 3-discontinuity of f .

Lemma 19. (a) Let X, Y be metric spaces and $f : X \rightarrow Y$ a function. A nontrivial subflag of a witness of d -discontinuity of f is again a witness of d -discontinuity of f .

- (b) Let Δ be a finite partition of X and \mathcal{F} a nontrivial d -flag. Then there exists a nontrivial Δ -monochromatic subflag \mathcal{G} of \mathcal{F} .
 (c) If \mathcal{F} is a Δ -monochromatic witness of d -discontinuity of Δ -continuous $f : X \rightarrow Y$, then $\text{Card}(\Delta) > d$.
 (d) If f admits a witness of d -discontinuity, then f is not d -wise continuous.

Item (d) gives only a sufficient condition for d -wise discontinuity: Observation 31 shows that its converse in general fails for $d \geq 2$.

Proof. (a) obvious.

(b) In case $d = 0, \emptyset \neq F_0 = F_0 \cap \bigcup \Delta = \bigcup_{D \in \Delta} (F_0 \cap D)$ implies $G_0 := F_0 \cap D \neq \emptyset$ for some $D \in \Delta$. Now proceed inductively from d to $d + 1$, supposing that (F_0, \dots, F_d) already is monochromatic. For $D \in \Delta$ consider $G_{d+1,D} := F_{d+1} \cap D$ and $G_{k,D} := F_k \cap \overline{G_{k+1,D}}, k = 0, \dots, d$. Then, for each $D \in \Delta, \mathcal{G}_D := (G_{0,D}, G_{1,D}, \dots, G_{d,D}, G_{d+1,D})$ constitutes a monochromatic subflag of \mathcal{F} . It remains to find \mathcal{G}_D nontrivial for some $D \in \Delta$. To this end observe $\bigcup_{D \in \Delta} G_{d+1,D} = F_{d+1}$ by distributivity and, inductively,

$$\bigcup_{D \in \Delta} G_{k,D} = F_k \cap \bigcup_{D \in \Delta} \overline{G_{k+1,D}} = F_k \cap \overline{\bigcup_{D \in \Delta} G_{k+1,D}} = F_k \cap \overline{F_{k+1}} = F_k$$

because topological closure commutes with finite unions. So $F_0 \neq \emptyset$ implies $G_{0,D} \neq \emptyset$ for some $D \in \Delta$.

(c) Since \mathcal{F} is Δ -monochromatic, there exists a mapping $\delta : \{0, 1, \dots, d\} \rightarrow \Delta$ with $F_k \subseteq \delta(k)$ for each $k = 0, \dots, d$. We now show that δ must be injective. \mathcal{F} is nontrivial, thus $\emptyset \neq F_0, F_k$ for each k . Therefore there exists, to each $0 \leq k < \ell \leq d$, some $x \in F_k \subseteq \delta(k)$ and $(x_n) \subseteq F_\ell \subseteq \delta(\ell)$ with $x_n \rightarrow x$ and $f(x_n) \not\rightarrow f(x)$. Since $f|_{\delta(\ell)}$ is continuous by hypothesis, this discontinuity of f at $x \in \delta(k)$ requires $\delta(k) \neq \delta(\ell)$.

(d) Suppose that f is Δ -continuous for some $\text{Card}(\Delta) \leq k$. By $a + b$, one can proceed to a Δ -monochromatic witness of d -discontinuity of f . Now (c) raises a contradiction. \square

2.2. More examples

We begin with a real variant of (the d -fold iteration of) the *limited principle of omniscience* from constructive mathematics, already shown d -wise discontinuous in [51, Theorem 3.5]:

Example 20. The mapping

$$\text{LPO}_d : \mathbb{R}^d \ni (x_1, \dots, x_d) \mapsto \text{Card}\{i : 1 \leq i \leq d, x_i \neq 0\} \in \{0, 1, 2, \dots, d\}$$

is $(d + 1)$ -continuous: namely even constant on each $D_k := \{(x_1, \dots, x_d) : \text{Card}\{i : x_i \neq 0\} = k\}, k = 0, \dots, d$. It is not d -continuous, because Example 18(c) (and then Lemma 19(d)) applies to the following d -flag: $F_0 := \{0^d\}, F_1 := \{(1/n, 0^{d-1}) : n \in \mathbb{N}\}, F_2 := \{(1/n, 1/n+1/m, 0^{d-2}) : n, m \in \mathbb{N}\}, F_k := \{(1/n_1, 1/n_1+1/n_2, \dots, 1/n_1+\dots+1/n_k, 0^{d-k}) : n_1, \dots, n_k \in \mathbb{N}\}$.

Example 21. The flag (F_0, \dots, F_{d-1}) above also shows that the mapping

$$\text{Card}_d : \mathbb{R}^d \ni (x_1, \dots, x_d) \mapsto \text{Card}\{x_1, \dots, x_d\} \in \{1, 2, \dots, d\} =: [d]$$

is not $(d - 1)$ -continuous (but trivially d -continuous).

In particular, Example 5(c) is best possible in the following sense: For $d \in \mathbb{N}$ and given $X = \{x_1, \dots, x_d\} \subseteq \mathbb{R}$, knowing $k := \text{Card}(X) \in \{1, \dots, d\}$ (i.e. d -fold advice according to Example 21) is obviously necessary to even state the ρ^k -computability of some k -tuple $(x_{i_1}, \dots, x_{i_k})$ with $k = \text{Card}\{x_{i_1}, \dots, x_{i_k}\}$, that is for enumerating X 's members *without* repetition; whereas X 's members *with* repetition can be enumerated without any advice according to [52, Lemma 5.1.10].

Example 22. The mapping

$$\text{Class}_{1+d} : \mathbb{R}^{1+d} \ni (x, y_1, \dots, y_d) \mapsto \{1 \leq i \leq d : y_i = x\} \in 2^{[d]}$$

is (with respect to representation ρ^{1+d} on its domain and the discrete topology on its co-domain) $(d + 1)$ -computable, but not d -continuous.

Proof. Suppose Class_{d+1} is d -continuous. Then $\text{LPO}_d(x_1, \dots, x_d) = [d] \setminus \text{Class}_{d+1}(0, x_1, \dots, x_d)$ would imply d -continuity of LPO_d by Lemma 15(b): contradicting Example 20.

Conversely, observe $\text{Class}_{1+d}(x, y_1, \dots, y_d) = [d] \setminus \text{LPO}_d(x - y_1, \dots, x - y_d)$; hence $(d + 1)$ -computability of LPO_d yields the same for Class_{1+d} . \square

Example 23. Fix some bijection $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}, (x, y) \mapsto \langle x, y \rangle$; e.g. $\langle x, y \rangle := 2^{x-1} \cdot (2y - 1)$.

(a) For $\bar{n} \in \mathbb{N}^*$, let $\langle \bar{n} \rangle := \sum_i 2^{-(i, n_i)}$; and map the empty tuple to 0. This mapping $\langle \cdot \rangle : \mathbb{N}^* \rightarrow [0, 1] \cap \mathbb{D}$ is well-defined and injective. For each k , the range $\langle \mathbb{N}^k \rangle$ belongs to Borel class Δ_2 .

(b) Consider $f : [0, 1] \rightarrow [0, 1]$ well-defined by $f(x) := 1/(k + 1)$ for $x = \langle \bar{n} \rangle$ with $\bar{n} \in \mathbb{N}^k; f(x) := 0$ for $x \notin \langle \mathbb{N}^* \rangle$. Then f is Δ_2 -measurable but not d -continuous for any $d \in \mathbb{N}$.

Proof of Example 23. (a) Let $\langle n \rangle_i := 2^{-(i, n_i)}$ and, for $1 \leq i_1 < \dots < i_k, \langle n_1, \dots, n_k \rangle_{\{i_1, \dots, i_k\}} := \sum_{\ell=1}^k \langle n_\ell \rangle_{i_\ell}$. Hence $\langle n_1, \dots, n_k \rangle = \langle \bar{n} \rangle_{\{1, 2, \dots, k\}}$. Now observe that forming the topological closure of $\langle \mathbb{N}^1 \rangle_i$ means including $\{0\} = \langle \mathbb{N}^0 \rangle$.

Similarly, for $i_1 \neq i_2$,

$$\overline{\langle\langle \mathbb{N}^2 \rangle\rangle_{\{i_1, i_2\}}} = \langle\langle \mathbb{N}^2 \rangle\rangle_{\{i_1, i_2\}} \uplus \langle\langle \mathbb{N}^1 \rangle\rangle_{i_1} \uplus \langle\langle \mathbb{N}^1 \rangle\rangle_{i_2} \uplus \langle\langle \mathbb{N}^0 \rangle\rangle = \langle\langle \mathbb{N}^2 \rangle\rangle_{\{i_1, i_2\}} \uplus (\overline{\langle\langle \mathbb{N}^1 \rangle\rangle_{i_1}} \cup \overline{\langle\langle \mathbb{N}^1 \rangle\rangle_{i_2}})$$

where “ \uplus ” denotes disjoint union. More generally, for $\bar{i} := \{i_1, \dots, i_k\}$, it holds

$$\overline{\langle\langle \mathbb{N}^k \rangle\rangle_{\bar{i}}} \stackrel{(*)}{=} \biguplus_{\bar{j} \subseteq \bar{i}} \langle\langle \mathbb{N}^{\text{Card } \bar{j}} \rangle\rangle_{\bar{j}} \stackrel{(**)}{=} \langle\langle \mathbb{N}^k \rangle\rangle_{\bar{i}} \uplus \bigcup_{\bar{j} \subsetneq \bar{i}} \overline{\langle\langle \mathbb{N}^{\text{Card } \bar{j}} \rangle\rangle_{\bar{j}}}$$

where “ $\bar{j} \subseteq \bar{i}$ ” means index running over all (finitely many) subsets \bar{j} of \bar{i} . In particular, $\langle\langle \mathbb{N}^k \rangle\rangle_{\bar{i}}$ is the difference of the two closed sets $\overline{\langle\langle \mathbb{N}^k \rangle\rangle_{\bar{i}}}$ and $\bigcup_{\bar{j} \subsetneq \bar{i}} \overline{\langle\langle \mathbb{N}^{\text{Card } \bar{j}} \rangle\rangle_{\bar{j}}}$, hence in Δ_2 .

- (b) Well-definition of f follows from a). Moreover, $f^{-1}(1/(k+1)) = \langle\langle \mathbb{N}^k \rangle\rangle$ is in Δ_2 . Since $\text{range}(f) = \{1/(k+1) : k \in \mathbb{N}\} \cup \{0\}$, the preimage $f^{-1}[V]$ of any open set $V \ni 0$ is a union of finitely many $f^{-1}(1/(k+1))$ and therefore in Δ_2 , too; whereas the preimage of open $V \ni 0$ misses finitely many $f^{-1}(1/(k+1))$ and thus also belongs to Δ_2 .

Let $F_0 := \{0\}$, $F_1 := \{2^{-(1,n)} : n \in \mathbb{N}\}$, $F_2 := \{2^{-(1,n)} + 2^{-(2,m)} : n, m \in \mathbb{N}\}$, ..., $F_d := \{\sum_{i=1}^d 2^{-(i,n_i)} : n_1, \dots, n_d \in \mathbb{N}\}$. This constitutes a nontrivial d -flag; and $f|_{F_k} \equiv 1/(k+1)$ shows the restriction $f : F_0 \cup F_1 \cup \dots \cup F_d \rightarrow \{1, \dots, 1/(d+1)\}$ to be d -discontinuous. \square

Finally recall Example 6 of computing (or rather identifying) from a given N -tuple $(\vec{x}_1, \dots, \vec{x}_N)$ of distinct points in \mathbb{R}^d those extremal to (i.e. minimal and spanning) the convex hull $\text{chull}(\vec{x}_1, \dots, \vec{x}_N)$. In the 1D case, this problem $(x_1, \dots, x_N) \mapsto \text{extchull}_N(x_1, \dots, x_N)$ is computable: simply return the two (distinct!) numbers $\min\{x_1, \dots, x_N\}$ and $\max\{x_1, \dots, x_N\}$. We have already seen that in 2D it generally lacks ψ^2 -computability because of discontinuity.

Proposition 24. Let $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^d$ be pairwise distinct and $C := \text{chull}(\vec{x}_1, \dots, \vec{x}_N)$.

- (a) Let $\vec{y} \in \text{ext}(C)$. Then there exists a closed halfspace

$$H_{\vec{u}, t}^+ = \{\vec{z} \in \mathbb{R}^d : \sum_i z_i u_i \geq t\} \subseteq \mathbb{R}^d$$

with rational normal (although not necessarily unit) vector $\vec{u} \in \mathbb{Q}^d \setminus \{0\}$ and real $\mathbb{R} \ni t > 0$ such that $H_{\vec{u}, t}^+ \cap \{\vec{x}_1, \dots, \vec{x}_N\} = \{\vec{y}\} = \{\vec{x}_j\}$ for some $1 \leq j \leq N$.

- (b) Conversely $H_{\vec{u}, t}^+ \cap \{\vec{x}_1, \dots, \vec{x}_N\} = \{\vec{x}_j\}$ with $\vec{u} \neq 0$ implies $\vec{x}_j \in \text{ext}(C)$.
- (c) Given $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^d$ as above and for $1 \leq j \leq N$, “ $\vec{x}_j \in \text{ext}(C)$ ” is semi-decidable. More formally, the following set is $(\rho^{d \times N}, \nu)$ -r.e. open:

$$\{(\vec{x}_1, \dots, \vec{x}_N, j) : \vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^d, \mathbb{N} \ni j \leq N, \vec{x}_j \in \text{extchull}_N(\vec{x}_1, \dots, \vec{x}_N)\}.$$

- (d) The mapping extchull_N from Eq. (3) is $(\rho^{d \times N}, \psi^d)$ -computable.
- (e) For $N \geq 2$ and given the number $M := \text{Card } \text{ext}(C) \in \{2, \dots, N\}$ of extreme points, the set of their indices, i.e.

$$\{i_1, \dots, i_M\} \subseteq \{1, \dots, N\} \quad \text{s.t.} \quad \text{ext}(C) = \{\vec{x}_{i_1}, \dots, \vec{x}_{i_M}\}$$

becomes $(\rho^{d \times N}, \nu)$ -computable.

In particular, extchull_N is $(\rho^{d \times N}, \psi^d)$ -computable with $(N - 1)$ -fold advice.

- (f) It is however $(N - 2)$ -wise $(\rho^{d \times N}, \psi^d)$ -discontinuous in dimension $d = 2$.

Proof. (a) and (b): It is well-known [19] that extreme points \vec{y} of a polytope C (although not necessarily of a general convex body) are precisely its exposed points, i.e. satisfy $\{\vec{y}\} = C \cap H_{\vec{u}, t}^+$ for some $t > 0$ and $\vec{u} \in \mathbb{R} \setminus \{0\}$. Equivalently: $\langle \vec{u}, \vec{x}_j \rangle > \langle \vec{u}, \vec{x}_i \rangle$ for all $i \neq j$ —obviously a condition open in \vec{u} , which therefore may be chosen from the dense subset $\mathbb{Q}^d \subseteq \mathbb{R}^d$.

- (c) Follows from (a + b) by dovetailed search for some $\vec{u} \in \mathbb{Q}^d \setminus \{0\}$ with $\langle \vec{u}, \vec{x}_j \rangle :=: t > \langle \vec{u}, \vec{x}_i \rangle$ for all $i \neq j$, where $\langle \vec{u}, \vec{x} \rangle := u_1 x_1 + \dots + u_d x_d$.
- (d) Follows from (c) by trying all $j = 1, \dots, N$. Indeed, a ψ^d -name (but not a ψ^d -name) permits to ‘increase’ at any time the set to be output.
- (e) similarly to (d), now trying all M -tuples $(i_1 < i_2 < \dots < i_M)$ in $\{1, \dots, N\}$. Note that indeed $\text{Card } \text{ext}(C) \geq 2$ because the \vec{x}_i are pairwise distinct.
- (f) We might construct a witness of $(N - 2)$ -discontinuity, but take the more elegant approach of a reduction by virtue of Lemma 15(b). To this end observe that semi-decidability of inequality makes $\text{Card}_n : \mathbb{R}^n \ni (x_1, \dots, x_n) \rightarrow \text{Card}\{x_1, \dots, x_n\}$ $(\rho^n, \rho^<)$ -computable, i.e. upper semi-continuous; hence by Example 21, Card_n must be $(n - 1)$ -wise lower semi-discontinuous.

Now let $x_1, \dots, x_n \in \mathbb{R}$ be given. According to [52, Exercise 4.3.15] suppose w.l.o.g. $x_1 \geq x_2 \geq \dots \geq x_{n-1} \geq x_n = 0$. Then proceed to the following collection X of $N := n + 1$ points in 2D: $(0, 0), (1, x_1), (2, x_1 + x_2), \dots, (n, x_1 + \dots + x_n)$;

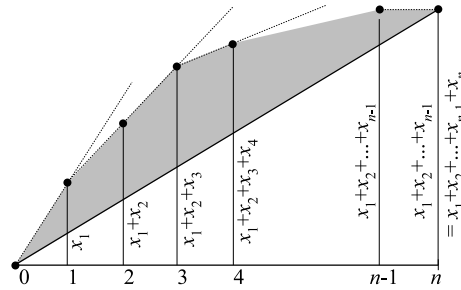


Fig. 3. Knowing in 2D which points are/not extreme to their convex hull can be used to conclude which real numbers are in-/equal.

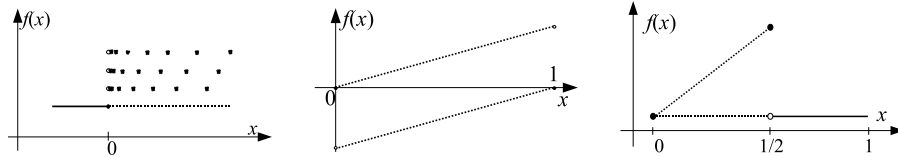


Fig. 4. (a) The cardinal of discontinuity cannot be lower bounded by the number of limit points. b) A 2-continuous function which, after identifying arguments $x = 0$ and $x = 1$, exhibits mere 3-continuity yet admits no witness of 2-discontinuity; see Observation 31. (c) A function where a greedy meta-algorithm may fail to produce a smallest decomposition into continuous restrictions.

cf. Fig. 3. Let $f_n : \mathbb{R}^n \rightarrow \mathbb{R}^{2(n+1)}$ denote this computable mapping $(x_1, \dots, x_n) \mapsto X$. Observe that the sequence of slopes from points $\#i$ to $\#i + 1$ is non-increasing because $x_i \geq x_{i+1}$; and two successive slopes $(\#i - 1 \rightarrow \#i)$ and $(\#i \rightarrow \#i + 1)$ coincide iff $x_i = x_{i+1}$; which in turn is equivalent to point $\#i$ being not extreme to $\text{chull}(X)$, $1 \leq i < n$. In fact from a ψ^2 -name of $\text{extchull}(X)$ one can semi-decide $(i, x_1 + \dots + x_i) \notin \text{extchull}(X)$ by verifying $d_{\text{extchull}(X)}(i, x_1 + \dots + x_i) > 0$ [52, Lemma 5.1.7]. Subtracting $(0, 0)$ which is always extreme to $\text{chull}(X)$, this yields a $(\psi^2, \rho_>)$ -computable mapping $h_n : \text{extchull}_{n+1}(X) \mapsto \text{Card } \text{extchull}_{n+1}(X) - 1 = \text{Card}\{x_1, \dots, x_n\}$ defined on the image of $\text{extchull}_{n+1} \circ f_n$. Now since $h_n \circ \text{extchull}_{n+1} \circ f_n = \text{Card} : \mathbb{R}^n \rightarrow \{1, \dots, n\}$ is $(n - 1)$ -wise lower semi-discontinuous by the above considerations, Lemma 15(b) requires that extchull_{n+1} be $(n - 1)$ -wise (ρ^{n+1}, ψ^2) -discontinuous where we chose $N = n + 1$. \square

Question 25. Fix $N \geq d \geq 2$. Let $\vec{x}_1, \dots, \vec{x}_N \in \mathbb{R}^d$ be affinely independent. Restricted to such inputs, what is the least advice that renders extchull continuous/computable?

2.3. Further remarks

For some time the author had felt that when $\text{dom}(f)$ is sufficiently ‘nice’ and for $x \in \text{dom}(f)$, the cardinal of discontinuity of f could be lower bounded in terms of the number of distinct limits of f at x , that is the cardinality of

$$\text{Lim}(f, x) := \{ \lim_{n \rightarrow \infty} f(x_n) : \text{dom}(f) \ni x_n \rightarrow x \}.$$

However the following example (cf. also Fig. 4(a)) shows that this is not the case:

$$f : [-1, 1] \rightarrow [0, 1], \quad 2^{-n} \cdot 3^{-m} \mapsto 3^{-m} \quad (n, m \in \mathbb{N}), \quad f(x) \equiv 0 \text{ otherwise.}$$

Here $\text{Lim}(f, 0)$ is infinite but f is continuous on $D_1 := \{2^{-n} \cdot 3^{-m} : n, m \in \mathbb{N}\}$ (because the latter set contains no accumulation point) and $f \equiv 0$ on $D_2 := [-1, 1] \setminus D_1$; hence $\mathcal{C}_t(f) = 2$.

Finally we remark that in the case of finite advice, the notation δ in Definition 8(d) usually arises straightforwardly and naturally; although an artificially bad choice is possible even for 2-wise computable functions:

Example 26. The characteristic function $\chi_H : \mathbb{N} \rightarrow \{0, 1\}$ of the Halting problem $H \subseteq \mathbb{N}$ is obviously 2-wise (ν, ν) -computable by virtue of $\Delta = \{H, \mathbb{N} \setminus H\}$, namely for $\delta : \subseteq \Sigma^* \rightarrow \Delta$ with $1 \mapsto H$ and $0 \mapsto \mathbb{N} \setminus H$. Whereas with respect to the following notation $\tilde{\delta}$, χ_H is equally obviously not $(\nu, \tilde{\delta}, \nu)$ -computable:

$$\tilde{\delta} : \Sigma^* \rightarrow \Delta, \quad \bar{x} \mapsto H \text{ for } \text{bin}(\bar{x}) \in H, \quad \bar{x} \mapsto \mathbb{N} \setminus H \text{ for } \text{bin}(\bar{x}) \notin H.$$

Definition 8 raises the question of how to determine for an arbitrary given function f (the cardinality of) a least partition Δ of $\text{dom}(f)$ such that $f|_D$ is continuous/computable for each $D \in \Delta$. Such a question of course arises generically for any complexity theory. For the complexity measure of ‘levels of discontinuity’ (recall Section 1.3) such a partition can (at least in principle) be determined by a straightforward meta-algorithm: Let D_i be $\text{dom}(f)$ with all points of discontinuity removed, then repeat with $i + 1$ and f restricted to said set of points of discontinuity. Of course this algorithm in general does not yield a least partition in the (more general) sense of Definition 8(a+d); recall Proposition 11(a).

Instead the following meta-algorithm may seem promising:

- Take some (w.r.t. set-inclusion) maximal subset D_i of $\text{dom}(f)$ such that $f|_{D_i}$ is continuous.
- Then repeat with the restriction of f to $\text{dom}(f) \setminus D_i$
- until arriving at the empty function.

Remark 27. (a) In computer science, algorithms of this type are called *greedy*.

(b) For our meta-algorithm, each iteration amounts to one invocation of Zorn’s Lemma.

(c) The number of iterations is (like the *level*) in general an ordinal rather than a cardinal.

(d) Even in the case of finite cardinal (=ordinal) of continuity, a maximal subset D_1 as above need not be unique

(e) and an unfavorable choice can lead to a suboptimal partition.

To illustrate the latter, consider the function

$$f : [0, 1] \rightarrow \mathbb{R}, \quad \mathbb{Q} \cap [0, 1/2] \ni x \mapsto x, \quad (1/2, 1] \ni x \mapsto 0, \quad [0, 1] \setminus \mathbb{Q} \ni x \mapsto 0$$

depicted in Fig. 4(c). Then f is continuous on both $D_1 := ([0, 1] \setminus \mathbb{Q}) \cup (1/2, 1]$ and $D_2 := [0, 1/2] \cap \mathbb{Q}$; hence they form a 2-element partition $\Delta = \{D_1, D_2\}$. On the other hand, f is also continuous on $D'_1 := ([0, 1/2) \cap \mathbb{Q}) \cup (1/2, 1]$ but not on any proper superset of D'_1 . But once the above greedy meta-algorithm has chosen this D'_1 , the restriction $f|_{[0,1] \setminus D'_1}$ remains discontinuous at $1/2$ and hence finally results in a 3-element partition Δ' of $[0, 1]$.

2.4. Weak k -fold advice

Recalling Observation 9, k -wise (α, β) -computability of $f : \subseteq A \rightarrow B$ implies k -wise (α, β) -continuity from which in turn follows weak k -wise (α, β) -continuity in the following sense:

Definition 28. Consider a function $f : A \rightarrow B$ between represented spaces (A, α) and (B, β) .

- (a) Call f weakly k -wise (α, β) -continuous if there exists a k -continuous (α, β) -realizer $F : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ of f in the sense of [52, Definition 3.1.3].
- (b) Call f weakly k -wise (α, β) -computable if it admits a k -computable (α, β) -realizer.

However conversely, and as opposed to the classical [52, Theorem 3.2.11] (i.e. the case $k = 1$), weak 2-wise (α, β) -continuity in generally does not imply 2-wise (α, β) -continuity. Basically the reason is that a partition of $\text{dom}(f)$ yields a partition of $\text{dom}(F)$; whereas a partition Δ of $\text{dom}(F)$ need not be compatible with the representation in that different α names for the same argument a may belong to different elements of Δ :

Example 29. Consider the $f : [0, 1] \rightarrow [-1, +1]$ depicted in Fig. 4(b):

$$[0, 1) \cap \mathbb{Q} \ni x \mapsto x =: g(x), \quad \mathbb{R} \setminus \mathbb{Q} \ni x \mapsto x - 1 =: h(x), \quad f(1) := 0.$$

This function is continuous on both $\mathbb{Q} \cap [0, 1)$ and on $\{1\} \cup \mathbb{R} \setminus \mathbb{Q}$; hence 2-continuous, and admits a 2-continuous (ρ, ρ) -realizer.

Now proceed from $[0, 1]$ to \mathcal{S}^1 , i.e. identify $x = 0$ with $x = 1$; formally, consider the representation $\tilde{\rho} := \iota \circ \rho : \subseteq \Sigma^\omega \rightarrow \mathcal{S}^1$ where $\iota : \mathbb{R} \rightarrow [0, 1), x \mapsto x \bmod 1$. Since $f(0) = 0 = f(1)$, this induces a well-defined function $\tilde{f} : \mathcal{S}^1 \rightarrow [-1, +1]$; which admits a 2-continuous $(\tilde{\rho}, \rho)$ -realizer: namely the 2-continuous (ρ, ρ) -realizer of f . But \tilde{f} itself is not 2-continuous: Suppose $\mathcal{S}^1 = D_1 \uplus D_2$ where $\tilde{f}|_{D_1}$ and $\tilde{f}|_{D_2}$ are both continuous. W.l.o.g. $0 \in D_1$. Observe that $\tilde{f}(0) = 0 = g(0) \neq h(0) = -1$. Hence, as \mathbb{Q} is dense and because continuous h is different from continuous g , continuity of $\tilde{f}|_{D_1}$ requires it to coincide with g : first just locally at $x = 0$, but then also globally—which implies $\lim_{x \nearrow 1} \tilde{f}|_{D_1}(x) = g(1) = 1$, contradicting $\tilde{f}|_{D_1}(1) = 0$. \square

As already mentioned, Example 29 illustrates that the implication from k -wise (α, β) -continuity to weak k -wise (α, β) -continuity cannot be reversed in general—even for admissible representations. Indeed, $\tilde{\rho}$ can be shown equivalent to the standard representation $\delta_{\mathcal{S}^1}$ of \mathcal{S}^1 as an effective topological space [52, Definition 3.2.2].

Applying Lemma 15 to realizers yields the following counterpart for weak advice:

Remark 30. Fix represented spaces (A, α) , (B, β) , and (C, γ) .

- (a) Let $f : A \rightarrow B$ be weakly d -wise (α, β) -continuous/computable and $A' \subseteq A$. Then the restriction $f|_{A'}$ is again weakly d -wise (α, β) -continuous/computable.
- (b) Let $f : A \rightarrow B$ be weakly d -wise (α, β) -continuous/computable and $g : B \rightarrow C$ be weakly k -wise (β, γ) -continuous/computable. Then $g \circ f : A \rightarrow C$ is weakly $d \cdot k$ -wise (α, γ) -continuous/computable.
- (c) If $f : A \rightarrow B$ is weakly d -wise (α, β) -continuous (computable) and $\alpha' \preceq_t \alpha$ ($\alpha' \preceq \alpha$) and $\beta \preceq_t \beta'$ ($\beta \preceq \beta'$), then f is also weakly d -wise (α', β') -continuous (computable).

Notice that property (b) does not carry over to *multi-representations* in the sense of [53]; cf. the discussion preceding Lemma 39 below.

We also observe that Lemma 19 does not admit a converse, even for total functions between compact spaces:

Observation 31. *The function $\tilde{f} : \mathcal{S}^1 \rightarrow [-1, +1]$ from Example 29 is not 2-continuous yet has no witness of 2-discontinuity.*

Proof. Suppose $(\{x\}, F_1, F_2)$ is a witness of 2-discontinuity of \tilde{f} . Consider the

- case $x \in (0, 1) \cap \mathbb{Q}$. Since $F_1 \ni x_n \rightarrow x$ has $\tilde{f}(x_n) \not\rightarrow \tilde{f}(x) = x$, w.l.o.g. $0 < x_n < 1$ and $x_n \notin \mathbb{Q}$: otherwise proceed to an appropriate subsequence. Now take $F_2 \ni x_{n,m} \rightarrow x_n$, convergence w.l.o.g. holding uniformly in n ; in particular $x_{m,m} \rightarrow x$. Then $\tilde{f}(x_{n,m}) \not\rightarrow \tilde{f}(x_n) = x_n - 1$ requires, by definition of \tilde{f} , $\tilde{f}(x_{n,m}) = x_{n,m}$ for almost all m and n : contradicting that a witness of discontinuity is required to satisfy $\tilde{f}(x_{m,m}) \not\rightarrow \tilde{f}(x)$.
- Case $x \in (0, 1) \setminus \mathbb{Q}$: similarly.
- Case $x = 0 \equiv 1$: As $F_1 \ni x_n \rightarrow x$ and since $\tilde{f}(x_n) \not\rightarrow \tilde{f}(x) = 0$, we may consider two subcases:
- Subcase $x_n \in (1/2, 1) \cap \mathbb{Q}$ for almost all n :
Now take $F_2 \ni x_{n,m} \rightarrow x_n$ uniformly. Then $\tilde{f}(x_{n,m}) \not\rightarrow \tilde{f}(x_n) = x_n$ requires, by definition of \tilde{f} , $\tilde{f}(x_{n,m}) = x_{n,m} - 1$ for almost all m and n : contradicting $\lim_m x_{m,m} = x$ and $\tilde{f}(x_{m,m}) \not\rightarrow \tilde{f}(x) = 0$.
- Subcase $x_n \in (0, 1/2) \setminus \mathbb{Q}$ for almost all n : similarly. \square

Note that $\text{dom}(\tilde{f}) = \mathcal{S}^1 \cong \{(a, b) : a^2 + b^2 = 1\} \subseteq [-1, +1]^2$ in Observation 31 is not simply connected. On the other hand $\tilde{f} : [-1, +1]^2 \rightarrow [-1, 2]$, defined by

$$\tilde{f}(\vec{x}) := \tilde{f}(\vec{x}) \in [-1, +1] \quad \text{for } \vec{x} \in \mathcal{S}^1, \quad \tilde{f}(\vec{x}) := 2 \quad \text{for } \vec{x} \notin \mathcal{S}^1$$

has convex domain, is not 3-continuous, yet has no witness of 3-discontinuity.

Question 32. *Suppose f has simply connected (or even convex) domain and is not 2-continuous. Does it then admit a witness of 2-discontinuity?*

3. Multivalued functions, i.e. relations

Many applications involve functions which are ‘non-deterministic’ (or non-extensional) in the sense that, for a given input argument x , several values y are acceptable as output; recall e.g. Items (i) and (ii) in Section 1. In linear algebra for instance, given a singular matrix A , we want to find *some* (say normed) vector \vec{v} such that $A \cdot \vec{v} = 0$. This is reflected by relaxing the mapping $f : x \rightarrow y$ to be not a function but a relation (also called multivalued function). We write $f : X \rightrightarrows Y$ and $x \rightrightarrows f(x)$ instead of $f : X \rightarrow 2^Y \setminus \{\emptyset\}$ and $x \mapsto f(x)$ to indicate that, on input $x \in X$, *any single* $y \in f(x)$ is permitted as output. Many practical problems have been shown computable as multivalued functions but admit no computable single-valued selection; cf. e.g. [52, Exercise 5.1.13], [57, Lemma 12 or Proposition 17], and the left of Fig. 5 below. On the other hand, even relations often lack computability merely for reasons of continuity—and appropriate additional discrete advice renders them computable, recall Example 4 above.

Now Definition 8 of the complexity of nonuniform computability straight-forwardly extends from single-valued to multivalued functions; and so does Observation 9 for (single-valued) realizers which can then be treated using Lemma 19. However a direct generalization of Lemma 19 to multivalued mappings turns out to be more convenient. This approach requires a notion of (dis-)continuity for relations rather than for functions.

3.1. Topological power of multivaluedness

It is well-known that multivaluedness has both practical relevance and theoretically strictly enhances the computing capabilities over real numbers: Giving a Type-2 Machine a choice renders problems computable (and thus continuous) which are discontinuous for any fixed single-valued selection: cmp. e.g. the left of Fig. 5 or [52, Section 6.3]. It is thus to be expected that also the degree of discontinuity increases when proceeding from a multivalued (i.e. ‘nondeterministic’) mapping to some single-valued (‘deterministic’) choice. We illustrate this with a natural example exhibiting an exponential jump in complexity: Recall (Example 22) that the mapping $\text{Class}_n : \mathbb{R}^n \ni (x_0; x_1, \dots, x_{n-1}) \mapsto \{1 \leq j < n : x_0 = x_j\}$ is n -computable but not $(n - 1)$ -continuous.

Theorem 33. *Let $\text{Class}_{n,\ell}(x_1, \dots, x_n) := \text{Class}_{n+1}(x_\ell; x_1, \dots, x_\ell, \dots, x_n) = \{1 \leq j \leq n : x_j = x_\ell\} \subseteq [n]$. Consider the multivalued mapping*

$$\text{SomeClass}_n : \mathbb{R}^n \ni (x_1, \dots, x_n) \rightrightarrows \{\text{Class}_{n,\ell}(x_1, \dots, x_n) : \ell = 1, \dots, n\}$$

yielding, for some ℓ , all indices i with $x_i = x_\ell$. This map is $\lfloor 1 + \log_2 n \rfloor$ -continuous.

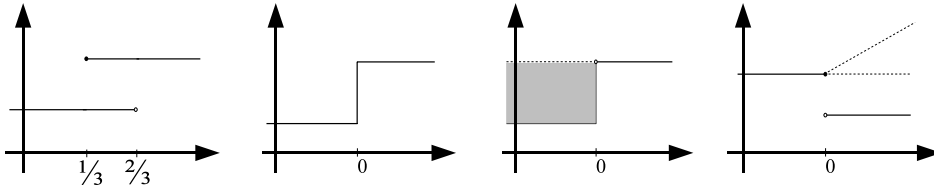


Fig. 5. (a) (ρ, ρ) -computable relation which is not hemicontinuous nor admits a continuous selection. (b) Quantification over all $y \in f(x)$ is generally necessary to capture discontinuity of a multivalued function. (c) Example of a (nontrivially) discontinuous relation. (d) A continuous relation with no continuous realizer.

More precisely, given (a ρ^n -name of \vec{x} and)

$$d := \lfloor \log_2 m \rfloor \quad \text{with } m := \min_{1 \leq \ell \leq n} \text{Card Class}_{n,\ell}(\vec{x}),$$

one can compute $\text{Class}_{n,\ell}(\vec{x})$ for some ℓ with $m = \text{Card Class}_{n,\ell}(\vec{x})$.

Our proof proceeds by applying $k := 2^d$ to Item (d) of the following combinatorial

Lemma 34. Let $x_1, \dots, x_n \in \mathbb{R}$ and $m = \min_{1 \leq \ell \leq n} \text{Card Class}_{n,\ell}(\vec{x})$ as above.

- (a) For each k, ℓ , it either holds $\text{Class}_{n,\ell}(\vec{x}) = \text{Class}_{n,k}(\vec{x})$ or $\text{Class}_{n,\ell}(\vec{x}) \cap \text{Class}_{n,k}(\vec{x}) = \emptyset$. Also, $\bigcup_{\ell} \text{Class}_{n,\ell}(\vec{x}) = [n]$.
- (b) Consider $I \subseteq [n]$ with $1 \leq \text{Card}(I) < 2m$ such that

$$x_i \neq x_j \quad \text{for all } i \in I \quad \text{and all } j \in [n] \setminus I. \tag{4}$$

Then $I = \text{Class}_{n,\ell}(\vec{x})$ for some ℓ .

- (c) Suppose $k \in \mathbb{N}$ is such that $k \leq m < 2k$. Then there exists ℓ with $k \leq \text{Card}(\text{Class}_{n,\ell}(\vec{x})) < 2k$ satisfying (4). Conversely every $I \subseteq [n]$ with $k \leq \text{Card}(I) < 2k$ satisfying (4) has $I = \text{Class}_{n,\ell}(\vec{x})$ for some ℓ .
- (d) Given a ρ^n -name of (x_1, \dots, x_n) and given $k \in \mathbb{N}$ with $k \leq m < 2k$, one can computably find some $\text{Class}_{n,\ell}(\vec{x})$.

Proof. (a) Obvious.

- (b) Take $i \in I$ with $i \in \text{Class}_{n,\ell}(\vec{x})$ for some ℓ . Then $I \supseteq \text{Class}_{n,\ell}(\vec{x})$ follows, because $j \in \text{Class}_{n,\ell}(\vec{x}) \setminus I$ would imply $x_i = x_j$: contradicting Eq. (4).

It remains to show $I \subseteq \text{Class}_{n,\ell}(\vec{x})$. Suppose that $x_i \neq x_{i'}$ for some $i' \in I$. Then $i' \in \text{Class}_{n,\ell'}(\vec{x})$ for some $\ell' \neq \ell$, i.e. with $\text{Class}_{n,\ell}(\vec{x}) \cap \text{Class}_{n,\ell'}(\vec{x}) = \emptyset$. Thus condition “ $x_i \neq x_j$ ” fails for all $j \in \text{Class}_{n,\ell}(\vec{x})$; and “ $x_{i'} \neq x_j$ ” fails for all $j \in \text{Class}_{n,\ell'}(\vec{x})$: i.e. for a total of $\text{Card}(\text{Class}_{n,\ell}(\vec{x})) + \text{Card}(\text{Class}_{n,\ell'}(\vec{x})) \geq 2m$ choices of $j \in [n]$, whereas by Eq. (4) it is supposed to hold for all $j \in [n] \setminus I$: a total of $> n - 2m$ choices—contradiction.

- (c) For the first claim, simply choose ℓ with $\text{Card}(\text{Class}_{n,\ell}(\vec{x})) = m$. Concerning the second claim, observe that $k \leq m$ and $\text{Card}(I) < 2k$ imply $\text{Card}(I) < 2m$; hence Item (b) applies.
- (d) Recall that inequality of real numbers is ‘semi-decidable’: formally: $\{(x, y) : x \neq y\} \subseteq \mathbb{R}^2$ is ρ^2 -r.e. open in \mathbb{R}^2 in the sense of [52, Definition 3.1.3.2]. Hence we may simultaneously try every $I \subseteq [n]$ with $k \leq \text{Card}(I) < 2k$ and semi-decide Condition (4): according to Item (c) this will succeed precisely for $I = \text{Class}_{n,\ell}(\vec{x})$. \square

3.2. Continuity for relations

Like single-valued computable functions (recall the Main Theorem), also computable relations satisfy certain topological conditions. However for such multivalued mappings, the literature knows a variety of easily confusable notions like [28, §7], [7], or [45]. *Hemicontinuity* [14, Definitions 1.1+2.1] for instance is not necessary for real computability; cf. Example 36(a) below. It may be tempting to regard *computing* a multivalued mapping f as the task of calculating, given x , the set-value $f(x)$ [46]. In our example applications, however, one wants to capture that a machine is permitted, given x , to ‘nondeterministically’ choose and output *some* value $y \in f(x)$. Note that this coincides with [52, Definition 3.1.3]. In particular we do not insist that, upon input x , *all* $y \in f(x)$ occur as output for *some* nondeterministic choice—as required in [9, Section 7].

Instead, let us generalize Definition 17 as in Item (c) of the following.

Definition 35. Fix some possibly multivalued mapping $f : \subseteq X \Rightarrow Y$ and write $\text{dom}(f) := \{x \in X : f(x) \neq \emptyset\}$.

- (a) Upper hemicontinuity of f at $x \in \text{dom}(f)$ means that to every open $V \supseteq f(x)$ there exists a neighborhood U of x such that $f(z) \subseteq V$ for all $z \in U$ (equivalently: for all $z \in U \cap \text{dom}(f)$).
- (b) Lower hemicontinuity of f at $x \in \text{dom}(f)$ means that to every open V with $V \cap f(x) \neq \emptyset$ there exists a neighborhood U of x such that $f(z) \cap V \neq \emptyset$ for all $z \in U \cap \text{dom}(f)$.
- (c) Call f continuous at $x \in \text{dom}(f)$ if there is some $y \in f(x)$ such that for every open neighborhood V of y there exists a neighborhood U of x such that $f(z) \cap V \neq \emptyset$ for all $z \in U \cap \text{dom}(f)$.

- (d) Based on (c), extend the notions of d -wise continuity, d -wise (α, β) -continuity, and d -wise (α, β) -computability (Definition 8(a) literally from functions to relations; similarly for weak d -wise (α, β) -continuity and weak d -wise (α, β) -computability (Definition 28).

For an ordinary (i.e. single-valued) function f , $\text{dom}(f)$ amounts to the usual notion; and such f is obviously (upper/lower hemi-)continuous (at x) iff it is continuous (at x) in the original sense. What we call *continuous* is denoted as *weak continuity* in [7]; who require for (*strong*) *continuity* our Definition 35(c) to hold for *all* $y \in f(x)$ —which however is not necessary for computability (consider $x = 1/3$ at Example 36(a) and thus unsuitable for our purpose of proving *lower* bounds on the complexity of nonuniform computability. (Weak) continuity on the other hand is necessary (Lemma 38(b)) but, as opposed to the single-valued case [52, Theorem 3.2.11], not sufficient for admitting a continuous realizer: consider Example 36(d) taken from [7, Proposition 2.3(3)].

Example 36. (a) Consider in Fig. 5(a) the multivalued $f : [0, 1] \rightrightarrows [0, 1]$,

$$1/3 > x \rightrightarrows \{0\}, \quad [1/3, 2/3) \ni x \rightrightarrows \{0, 1\}, \quad 2/3 \leq x \rightrightarrows \{1\}.$$

Then f is neither lower nor upper hemicontinuous—yet (ρ, ρ) -continuous, even computable: Given $(q_n) \subseteq \mathbb{Q}$ with $|x - q_n| \leq 2^{-n}$, test q_3 : if $q_3 \leq 1/2$ output 0, otherwise output 1. Indeed, $|x - q_3| \leq 1/8$ implies $x \leq 5/8 < 2/3$ for $q_3 \leq 1/2$, hence $0 \in f(x)$; whereas $q_3 > 1/2$ implies $x \geq 3/8 > 1/3$, hence $1 \in f(x)$.

- (b) Referring to Fig. 5(b), the multivalued function

$$g : [-1, 1] \rightrightarrows [0, 1], \quad [-1, 0) \ni x \rightrightarrows \{0\}, \quad 0 \rightrightarrows [0, 1], \quad (0, 1] \ni x \rightrightarrows \{1\}$$

is not continuous at 0 w.r.t. any $y \in g(0) = [0, 1]$ although $g(0)$ itself does intersect $g(z)$ for all z .

- (c) Consider the multivalued function

$$h : [-1, +1] \rightrightarrows [0, 1], \quad 0 \geq x \rightrightarrows [0, 1], \quad 0 < x \rightrightarrows \{1\}$$

sketched in Fig. 5(c). It is discontinuous at $x := 0$: To any $y \in h(x) = [0, 1]$, the open neighborhood $V := (-1, \frac{1+y}{2})$ of y does not intersect $h(z)$ for all $z > 0$.

- (d) Consider the multivalued function $s : \mathbb{R} \rightrightarrows \mathbb{R}$ with graph

$$\{(x, 0) : x \leq 0\} \cup \{(x, -1) : x > 0\} \cup \{(x, 0) : 0 < x \in \mathbb{Q}\} \cup \{(x, x) : 0 < x \in \mathbb{R} \setminus \mathbb{Q}\}$$

cf. Fig. 5(d): it is continuous at every x but not computable nor admits a continuous (ρ, ρ) -realizer.

The following multivalued variant of Example 20 had been established in [51, Theorem 5.2.2]:

Example 37. For $d \in \mathbb{N}$, let

$$\text{MLPO}_d : \{(x_1, \dots, x_d) : x_i \in \mathbb{R}, \exists j : x_j = 0\} \ni (x_1, \dots, x_d) \rightrightarrows \{j : 1 \leq j \leq d, x_j = 0\}.$$

Then MLPO_d is (trivially d -continuous but) not $(d - 1)$ -continuous.

For a proof, refer to Section 3.4.

Lemma 38. (a) Let X, Y be metric spaces, $x \in X$, and $f : \subseteq X \rightrightarrows Y$ some relation. Then f is continuous at x in the sense of Definition 35(c) iff the following holds:

$$\exists y \in f(x) \forall \epsilon > 0 \exists \delta > 0 \forall z \in B(x, \delta) \cap \text{dom}(f) : y \in B(f(z), \epsilon). \tag{5}$$

- (b) Let (A, α) and (B, β) be effective metric spaces² with corresponding Cauchy representations and $f : \subseteq A \rightrightarrows B$ a possibly multivalued mapping. If f is d -wise (α, β) -continuous, then it is d -wise continuous.

- (c) Let X_1, \dots, X_k be closed subsets of \mathbb{R}^n , X_1 compact, and $\delta > 0$. Then there exists $\epsilon > 0$ such that $\bigcap_{i=1}^k B(X_i, \epsilon) \subseteq B(\bigcap_{i=1}^k X_i, \delta)$.

Proof. (a) Suppose $\emptyset \neq f(z) \cap B(y, \epsilon)$ for all $z \in B(x, \delta) \cap \text{dom}(f)$. That is, to $z \in B(x, \delta) \cap \text{dom}(f)$, there exists some $w \in f(z) \cap B(y, \epsilon)$. Then $y \in B(w, \epsilon) \subseteq B(f(z), \epsilon)$.

Conversely suppose $y \in B(f(z), \epsilon)$, that is, $y \in B(w, \epsilon)$ for some $w \in f(z)$. Then $w \in B(y, \epsilon) \cap f(z) \neq \emptyset$.

- (b) It suffices to consider the case $d = 1$. But this has already been established in [7, Proposition 4.5+2.3].

² Cf. [52, Section 8.1] for a formal definition and imagine Euclidean spaces \mathbb{R}^k as major examples and focus of interest for our purpose.

- (c) By induction, it suffices to consider the case $k = 2$. Write $X := X_1$ and $Y := X_2$. First consider the disjoint case $X \cap Y = \emptyset$. Then the distance function d_Y from Eq. (2) is positive on X . Moreover d_Y is continuous and therefore, on compact X , bounded from below by some $2\epsilon > 0$. Hence $B(X, \epsilon) \cap B(Y, \epsilon) = \emptyset$. In the general case, $Z := X \cap Y$ is not necessarily empty but closed. Now consider $X' := X \setminus B(Z, \delta/2)$ and $Y' := Y \setminus B(Z, \delta/2)$: X' is compact and disjoint from closed Y' ; hence $B(X', \epsilon) \cap B(Y', \epsilon) = \emptyset$ for some $0 < \epsilon \leq \delta/2$ according to the first case. Since $X \subseteq X' \cup B(Z, \delta/2)$,

$$\begin{aligned} B(X, \epsilon) \cap B(Y, \epsilon) &\subseteq \underbrace{B(X' \cup B(Z, \delta/2), \epsilon)}_{\subseteq B(X', \epsilon) \cup B(Z, \delta/2, \epsilon)} \cap \left(B(Y', \epsilon) \cup \underbrace{B(B(Z, \delta/2), \epsilon)}_{=B(Z, \delta/2+\epsilon) \subseteq B(Z, \delta)} \right) \\ &\subseteq \underbrace{(B(X', \epsilon) \cap B(Y', \epsilon))}_{=\emptyset} \cup (B(X', \epsilon) \cap B(Z, \delta)) \cup (B(Z, \delta) \cap B(Y', \epsilon)) \cup B(Z, \delta) \end{aligned}$$

is contained in $B(Z, \delta)$. \square

Lemma 15(a) literally applies also to multivalued mappings $f : A \rightrightarrows B$. Similarly generalizing **Lemma 15(b)** is quite cumbersome: For $B = \bigcup_i B_i$, the preimages $f^{-1}[B_i]$,

- if defined as $\{a \in A : f(a) \subseteq B_i\}$, need not cover A
- if defined as $\{a \in A : f(a) \cap B_i \neq \emptyset\}$, need not be mapped to within B_i by f .

On the other hand, already the following partial generalization of **Lemma 15(b)** turns out as useful:

- Lemma 39.** (a) Let $f : A \rightarrow B$ be single-valued and $g : B \rightrightarrows C$ multivalued. If f is d -continuous (computable) and g is k -continuous (computable), then $g \circ f : A \rightrightarrows C$ is $(d \cdot k)$ -continuous (computable).
 (b) Let $f : A \rightrightarrows B$ and $g : B \rightrightarrows C$ be multivalued. If f is d -continuous (computable) and g is continuous (computable), then $g \circ f : A \rightrightarrows C$ is again d -continuous (computable).
 (c) Let I denote some finite set with discrete topology. Then $f : I \times A \rightrightarrows B$ is d -continuous (computable) iff each $f(i, \cdot) : A \rightrightarrows B$ is d -continuous (computable).

- Proof.** (a) Since f is single-valued, the set $A_i \cap f^{-1}[B_j]$ is unambiguous and mapped by f to a subset of B_j ; that is the proof of **Lemma 15(b)** carries over.
 (b) If f is continuous (computable) on each A_i , then so is $g \circ f$.
 (c) Similar to **Lemma 15(d)**. \square

3.3. Witness of multivalued discontinuity and hypergraph coloring

Lemma 39 provides a means for proving d -discontinuity of some function by reduction to a function already known d -discontinuous. In fact, [51, Theorem 3.7] has established that every d -discontinuous (even multivalued) function $f : \mathbb{R}^* \rightarrow \mathbb{N}$ can be proven d -discontinuous by reduction from LPO_d . However we are particularly interested in multivalued functions with continuous codomain; and therefore now generalize **Definition 17**:

- Definition 40.** (a) Let $T = (V, E)$ denote a tree with edges directed from the root. Call vertex $v \in V$ a direct successor of $u \in V$ (and u a direct ancestor of v) if $(u, v) \in E$. A successor (ancestor) of u is u itself or any direct successor (direct ancestor) of a successor (ancestor) of u . If u is neither a successor nor ancestor of v , they are unrelated. The degree of $u \in V$, $\text{deg}(u)$, is the number of direct successors of u .
 (b) For a tree T , a T -flag \mathcal{F} in a topological space X is a family of sets $F_v \subseteq X$, $v \in V$, satisfying $F_u \subseteq \overline{F_v}$ for each edge $(u, v) \in E$.
 (c) Another T -flag \mathcal{G} is a subflag of \mathcal{F} if it holds $G_v \subseteq F_v$ for all $v \in V$. \mathcal{F} is trivial if the root r of T has $F_r = \emptyset$.
 (d) For a partition Δ of X , a T -flag \mathcal{F} in X is Δ -monochromatic if, to every $v \in V$, there exists some $D \in \Delta$ with $F_v \subseteq D$.
 (e) For $\ell \in \{1, 2, \dots\}$, call tree $S_\ell := (\{0, 1, \dots, \ell\}, \{(0, 1), \dots, (0, \ell)\})$ an ℓ -star. Fix metric spaces X, Y and $f : \subseteq X \rightrightarrows Y$. An ℓ -witness of discontinuity of f is a nontrivial S_ℓ -flag $\mathcal{F} = (F_0, F_1, \dots, F_\ell)$ in $\text{dom}(f)$ such that, for every $x \in F_0$ and every choice of sequences $(x_{n,j})_n \subseteq F_j$ with $\lim_n x_{n,j} = x$ ($j = 1, \dots, \ell$), there exists some $\epsilon > 0$ such that, for almost all $n \in \mathbb{N}$, it holds

$$f(x) \cap \bigcap_{j=1}^{\ell} B(f(x_{n,j}), \epsilon) = \emptyset. \tag{6}$$

- (f) Let $T = (V, E)$ denote a tree. A hyperedge in T is a set $\{v, w_1, \dots, w_\ell\} \subseteq V$ with $w_1, \dots, w_\ell \in V$ ($\ell \in \mathbb{N}$) pairwise unrelated successors of v . A hypergraph on the tree (V, E) is a triple $\mathcal{T} = (V, E, \mathcal{W})$ where (V, E) constitutes a tree and \mathcal{W} is a set of hyperedges in (V, E) .

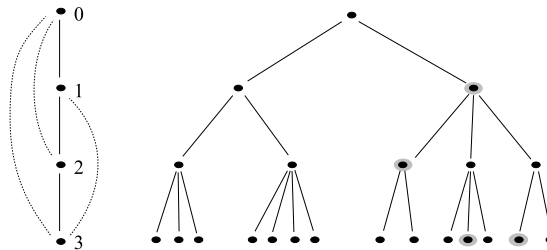


Fig. 6. Left: a linear chain L_3 as in Example 41(d). Right: tree of uniform depth 3 as in Lemma 42(a); a canonical sample hyperedge in this tree is indicated with gray vertices.

- (g) A proper c -coloring ($c \in \mathbb{N}$) of a set \mathcal{W} of hyperedges in (V, E) is a mapping $C : V \rightarrow \{1, \dots, c\}$ such that no hyperedge becomes monochromatic. In other words:
For each $\{v, w_1, \dots, w_\ell\} \in \mathcal{W}$, there must exist $1 \leq i, j \leq \ell$ with $C(v) \neq C(w_i)$ or $C(w_i) \neq C(w_j)$.
- (h) For a hypergraph $\mathcal{T} = (V, E, \mathcal{W})$ on a tree (V, E) , a witness of \mathcal{T} -discontinuity of f is a (V, E) -flag $\mathcal{F} = (F_u)_{u \in V}$ in $\text{dom}(f)$ such that, for every hyperedge $\{v, w_1, \dots, w_\ell\} \in \mathcal{W}$, the flag $(F_v, F_{w_1}, \dots, F_{w_\ell})$ is an ℓ -witness of discontinuity of f .

Hypergraph coloring (which also Lemma 34abc can be considered a case of) is of course a standard topic in combinatorics as well as in theoretical computer science. Proposition 43(e) below connects witnesses of discontinuity to colorings of hypergraphs on trees. A characterization of Weihrauch degrees (for relations with discrete range) in terms of colored forests (i.e. collections of trees, rather than hypergraphs on trees) had been established in [21, Theorem 3.8]. Connecting (and maybe even simplifying) both concepts is certainly desirable but beyond the purpose of the present work.

- Example 41.** (a) For a (single-valued) function $f : X \rightarrow Y$, “1-witness of discontinuity” in the sense of Definition 40(e) is synonymous with “witness of 1-discontinuity” in the sense of Definition 17(d): recall Example 18(a) and note that $\lim_n x_n = x$ and $f(x) \in B(f(x_n), \epsilon)$ for every $\epsilon > 0$ and infinitely many n implies $f(x_{n_m}) \rightarrow f(x)$ for some subsequence $(x_{n_m})_m$ of $(x_n)_n$.
- (b) An ℓ -witness of discontinuity (Definition 40(e)) is a witness of S_ℓ -discontinuity (Definition 40h) where we identify the tree S_ℓ with the hypergraph over S_ℓ having all vertices of S_ℓ as single hyperedge.
 - (c) Let $T = (V, E)$ denote a tree with root r and some T -flag $\mathcal{F} = (F_v)_{v \in V}$ in X . For $v \in V$ consider the unique path $(r, v_1), (v_1, v_2), \dots, (v_{d-1}, v)$ from r to v in T . Then $(F_r, F_{v_1}, \dots, F_{v_{d-1}}, F_v)$ is a d -flag.
 - (d) Consider vertices $V_d := \{0, 1, \dots, d\}$, edges $E_d := \{\{j - 1, j\} : j = 1, \dots, d\}$, and hyperedges $\mathcal{W}_d := \{\{i, j\} : i < j\}$; cf. Fig. 6. Then
 - (i) $L_d := (V_d, E_d)$ is a tree
 - (ii) and “ L_d -flag” is synonymous with “ d -flag” in the sense of Definition 17.
 - (iii) For (single-valued) $f : X \rightarrow Y$, “witness of d -wise discontinuity” is synonymous with “witness of $(V_d, E_d, \mathcal{W}_d)$ -discontinuity”.
 - (iv) The hypergraph \mathcal{W}_d admits a proper $(d + 1)$ -coloring but no proper d -coloring.
- (e) A (V, E) -flag may well have $F_u \cap F_v \neq \emptyset$ for some $(u, v) \in E$. The definition of an ℓ -witness $(F_0, F_1, \dots, F_\ell)$ of discontinuity, however, implicitly requires $\bigcap_{i=0}^\ell F_i = \emptyset$.

We now generalize Example 41d iv) and Example 18(c):

Lemma 42. Let T denote a tree and consider the canonical hypergraph \mathcal{T} on T , defined to have precisely the sets $\{v, w_1, w_2, \dots, w_{\text{deg}(v)}\}$ as hyperedges, where $v \in V$, $u_1, \dots, u_{\text{deg}(v)}$ are the direct successors of v , and w_i run through all successors of u_i .

- (a) Suppose T has uniform depth $d \in \mathbb{N}$; that is, all leaves have the same distance d to the root. (Recall that a single vertex has depth 0, and S_ℓ has depth 1.) Then \mathcal{T} does not admit a proper d -coloring.
- (b) Let $f : X \rightrightarrows Y$ denote a multivalued mapping between metric spaces and \mathcal{F} a nontrivial T -flag in X , where T denotes a tree. Suppose that the following hold:
 - (i) f is compact in the sense that $f(x) \subseteq Y$ is compact for each $x \in X$.
 - (ii) f is \mathcal{F} -hereditary in the sense that, whenever v is a successor of u in T , it holds $\bigcap_{x \in F_u} f(x) \supseteq \bigcup_{y \in F_v} f(y)$.
 - (iii) f is \mathcal{F} -constant: $f(x) = f(y)$ for all $x, y \in F_v$ for each $v \in T$.
 - (iv) For each $v \in T$ and $u_1, \dots, u_{\text{deg}(v)}$ its direct successors, it holds $f[F_v] \cap f[F_{u_1}] \cap \dots \cap f[F_{u_{\text{deg}(v)}}] = \emptyset$.
 Then \mathcal{F} is a witness of \mathcal{T} -discontinuity of f .

Proof. (a) By induction on d , the case $d = 1$ being obvious. Now T consists of a root v with $\ell := \text{deg}(v)$ direct successors u_1, \dots, u_ℓ , each u_i in turn root of a subtree T_i of uniform depth d . Any proper coloring C of T is also one on T_i ; and

thus requires by induction hypothesis at least $d + 1$ colors. In particular, each of these $d + 1$ colors occurs in each T_i , $i = 1, \dots, \ell$. Therefore there exist vertices $w_i \in T_i$ with $C(v) = C(w_i)$. But, w_i being a successor of u_i , \mathcal{W} by definition contains the hyperedge $\{v, w_1, \dots, w_\ell\}$ which just turned out to be monochromatic: contradiction.

- (b) Write $\ell := \text{deg}(v)$. Since $f[F_v], f[F_{u_i}]$ are compact, their (finite) intersection being empty means $B(f[F_v], \epsilon) \cap B(f[F_{u_1}], \epsilon) \cap \dots \cap B(f[F_{u_\ell}], \epsilon) = \emptyset$ for some sufficiently small $\epsilon > 0$ according to Lemma 38(c). Let w_i denote some successor of u_i , $i = 1, \dots, \ell$; then heredity implies $B(f[F_v], \epsilon) \cap B(f[F_{w_1}], \epsilon) \cap \dots \cap B(f[F_{w_\ell}], \epsilon) = \emptyset$, that is Eq. (6) holds. \square

Proposition 43. Let $\mathcal{T} = (V, E, \mathcal{W})$ denote a hypergraph on tree $T = (V, E)$, X, Y metric spaces, $f : X \rightrightarrows Y$ a (possibly multivalued) function, and Δ a partition of X .

- (a) A nontrivial subflag of a witness of \mathcal{T} -discontinuity of f is again a witness of \mathcal{T} -discontinuity of f .
- (b) Let $U \subseteq V$ be a nonempty subset of T 's vertices such that any two $v, w \in U$ have some common ancestor $u \in U$. Then

$$T[U] := (U, E') \quad \text{with } E' := \{(u, v) : u \text{ closest proper ancestor in } U \text{ of } v\}$$

is again a tree. If $\mathcal{F} = (F_v)_{v \in V}$ is a (nontrivial) T -flag, its restriction $\mathcal{F}|_U := (F_u)_{u \in U}$ is a (nontrivial) $T[U]$ -flag.

- (c) Let Δ be a finite partition of X and \mathcal{F} a nontrivial T -flag. Then there exists a nontrivial Δ -monochromatic T -subflag \mathcal{G} of \mathcal{F} .
- (d) Suppose that f is Δ -continuous and \mathcal{F} a Δ -monochromatic ℓ -witness of discontinuity of f ; $F_j \subseteq \delta(j) \in \Delta$ for all $j = 0, 1, \dots, \ell$. Then $C(j) := \delta(j)$ defines a non-monochromatic coloring of S_ℓ . In particular, it holds $\text{Card}(\Delta) > 1$.
- (e) Suppose that f is Δ -continuous and let \mathcal{F} be a Δ -monochromatic witness of \mathcal{T} -discontinuity of f . Then \mathcal{W} admits a proper $\text{Card}(\Delta)$ -coloring.
- (f) Let T denote a tree of uniform depth d and \mathcal{W} its canonical hyperedges. Suppose that f admits a witness of (T, \mathcal{W}) -discontinuity. Then f is not d -wise continuous.

Proof. Note that the definition of a T -flag $\mathcal{F} = (F_v)_v$ implies $F_u \subseteq \overline{F_v}$ and $F_v \subseteq \overline{F_w}$ for edges $(u, v), (v, w)$. In particular $F_u \subseteq \overline{F_w}$ whenever w is a successor of u ; and if the root r of T has $F_r \neq \emptyset$ (i.e. if \mathcal{F} is nontrivial), then $F_v \neq \emptyset$ for every vertex v of T .

- (a) Let $(F'_v)_v$ be a nontrivial subflag of the witness $(F_v)_v$ of \mathcal{T} -discontinuity of f . Then the above note shows $\emptyset \neq F'_v$ for every hyperedge $\{v, w_1, \dots, w_\ell\}$ of \mathcal{T} ; i.e. the subflag $(F'_v, F'_{w_1}, \dots, F'_{w_\ell})$ of $(F_v, F_{w_1}, \dots, F_{w_\ell})$ is nontrivial. And since Eq. (6) was required to hold for $(F_v, F_{w_1}, \dots, F_{w_\ell})$ for every $x_0 \in F_v$ and every choice of sequences $(x_{n,j})_n$ in F_{w_j} , it holds a fortiori for $(F'_v \subseteq F_v, F'_{w_1} \subseteq F_{w_1}, \dots, F'_{w_\ell} \subseteq F_{w_\ell})$.
- (b) $T[U]$ is weakly connected by hypothesis; and devoid of cycles because T was. Moreover for every $(u, v) \in E'$, v is an ancestor of u of T ; hence $F_u \subseteq F_v$ holds by the above note. Thus, $\mathcal{F}|_U$ is a $T[U]$ -flag; and nontrivial if \mathcal{F} was.
- (c) Let r denote the root of T and u a leaf of T . The path $[r, u] := \{v \in V : v \text{ ancestor of } u\}$ satisfies the hypothesis of Item (b); hence $\mathcal{F}|_{[r,u]}$ is a nontrivial $[r, u]$ -flag. In fact a d -flag according to Example 41(d), where d denotes the depth of u in T . Hence we may apply Lemma 19(b) to obtain a nontrivial Δ -monochromatic $[r, u]$ -subflag $(F'_v)_{v \in [r,u]}$ of $\mathcal{F}|_{[r,u]}$. Next observe that this subflag extends back to a nontrivial T -subflag \mathcal{F}' of \mathcal{F} via $F'_v := F_v$ for $v \in V \setminus [r, u]$: the condition " $F'_v \subseteq \overline{F'_w}$ " for a successor w of v easily follows from " $F_v \subseteq \overline{F_w}$ ", taking into account that $w \in [r, u]$ (hence $F'_w \subseteq F_w$) implies $v \in [r, u]$ and $w \notin [r, u]$ means $F'_w = F_w$. By its very construction, the restriction $\mathcal{F}'|_{[r,u]}$ of this subflag \mathcal{F}' of \mathcal{F} is Δ -monochromatic.
Now consider some enumeration u_1, \dots, u_N of all leaves of T . By iterating the above argument, we obtain a sequence $\mathcal{F} =: \mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_N$ of T -flags such that \mathcal{F}_{n+1} is a nontrivial subflag of \mathcal{F}_n and $\mathcal{F}_n|_{[r,u_n]}$ is Δ -monochromatic. Since subflags of Δ -monochromatic flags are Δ -monochromatic, it follows that the restrictions $\mathcal{F}_n|_{[r,u_n]}$ are Δ -monochromatic for every $n \leq N$. Now the paths from the root to all leaves cover the entire tree; formally: $\bigcup_n [r, u_n] = V$. This shows that \mathcal{F}_N itself must be Δ -monochromatic.
- (d) Suppose $\delta(j) = \delta(i) \in D$ for all $0 \leq i, j \leq \ell$. By hypothesis there exists $x \in F_0 \subseteq D$ and $x_{n,j} \in F_j \subseteq D$ with $x = \lim_n x_{n,j}$ satisfying Eq. (6) in Definition 40(e). Now since $f|_D$ is continuous at x by presumption, it satisfies Eq. (5) in Lemma 38(a): contradicting Eq. (6).
- (e) Similarly to (d), take $\delta : \mathcal{F} \rightarrow \Delta$ with $F_v \subseteq \delta(F_v)$ for each $F_v \in \mathcal{F}$; and define $C(v) := \delta(F_v)$ for each $v \in T$. We claim that this constitutes a proper coloring of \mathcal{W} . Indeed, according to Definition 40(h), for every hyperedge $\{v, w_1, \dots, w_\ell\} \in \mathcal{W}$, $(F_v, F_{w_1}, \dots, F_{w_\ell})$ constitutes a ℓ -witness of discontinuity of f —hence (Item d) is colored non-monochromatically.
- (f) Suppose f is Δ -continuous for some partition Δ of $\text{Card}(\Delta) \leq d$. By (c), we may proceed to a Δ -monochromatic witness of (T, \mathcal{W}) -discontinuity. By (e), it follows that (T, \mathcal{W}) admits a proper d -coloring: contradicting Lemma 42(a). \square

3.4. Example: alternative proof that MLPO_d is not $(d - 1)$ -continuous

Fix $d \in \mathbb{N}$. Consider the tree T_d with vertices V_d consisting of all ordered k -tuples without repetition over $[d] := \{1, 2, \dots, d\}$ for $k = 0, 1, \dots, d$; where tuple \bar{w} becomes a successor of \bar{v} iff \bar{v} is an initial segment of \bar{w} ; cmp. Fig. 7.

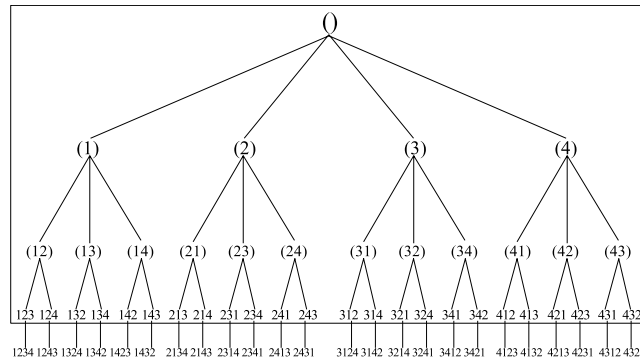


Fig. 7. The tree T_4 from Section 3.4. The part in the rectangle is T'_4 according to Lemma 44.

More formally, define $V_d := \bigcup_{X \subseteq [d]} \mathcal{S}_X$, where \mathcal{S}_X denotes the set of all bijections from $[X] := \{1, 2, \dots, |X|\}$ to X . In particular $X = \text{range}(\bar{v})$ for $\bar{v} \in \mathcal{S}_X$. Now consider $\bar{w} \in \mathcal{S}_Y$ a successor of $\bar{v} \in \mathcal{S}_X$ iff $|X| \leq |Y|$ and $\bar{v} = \bar{w}|_{[X]} : [X] \rightarrow Y$ (and in particular $X = \text{range}(\bar{v}) \subseteq \text{range}(\bar{w}) = Y$) holds. T_d is a tree of uniform depth d : To $\bar{v} \in \mathcal{S}_X$ with $|X| < d$, there exists $i \in [d] \setminus X$ and extension \bar{w} (i.e. successor) of \bar{v} ,

$$\bar{w} \in \mathcal{S}_Y, \quad Y := X \uplus \{i\}, \quad \bar{w}|_{[X]} := \bar{v}, \quad \bar{w}(|X| + 1) := i.$$

Lemma 44. Let T'_d denote the subtree of T_d with vertices $V'_d := \bigcup_{X \subseteq [d]} \mathcal{S}_X = V_d \setminus \mathcal{S}_{[d]}$ (i.e. cut off the last row in Fig. 7). Then T'_d has uniform depth $d - 1$. Write \mathcal{T}'_d for the canonical hypergraph over T'_d according to Lemma 42. For $X \subsetneq [d]$ and $\bar{v} \in \mathcal{S}_X$, let

$$F_{\bar{v}} := \prod_{i=1}^d \begin{cases} (0, 1] & : i \in X \\ \{0\} & : i \notin X \end{cases} \subseteq \mathbb{R}^d.$$

Then $\mathcal{F} := (F_{\bar{v}})_{\bar{v} \in \mathcal{S}_X, X \subseteq [d]}$ is a witness of \mathcal{T}'_d -discontinuity of MLPO_d .

In view of Proposition 43(f), Example 37 thus follows.

Proof. Note that $F_{\bar{v}} \subseteq \text{dom}(\text{MLPO}_d)$ depends only on $\text{range}(\bar{v})$ and has $\text{MLPO}_d(\bar{x}) = [d] \setminus \text{range}(\bar{v})$ for all $\bar{x} \in F_{\bar{v}}$: MLPO_d is \mathcal{F} -hereditary and \mathcal{F} -constant with compact (in fact finite) co-domain. Moreover, since $\{0\} \subseteq (0, 1]$, $\text{range}(\bar{v}) \subseteq \text{range}(\bar{w})$ implies $F_{\bar{v}} \subseteq F_{\bar{w}}$. So \mathcal{F} is a nontrivial T'_d -flag.

Let $\bar{v} \in T'_d$ with direct successors $\bar{u}_1, \dots, \bar{u}_\ell \in T'_d$, $\ell := \text{deg}(\bar{v})$. Write $X := \text{range}(\bar{v})$; then $\ell = d - |X|$, and $\text{range}(\bar{u}_i) = X \uplus \{j_i\}$, where $\{j_1, \dots, j_\ell\} = [d] \setminus X$. In particular, $\bigcup_{i=1}^\ell \text{range}(\bar{u}_i) = \bigcup_{i=1}^\ell X \uplus \{j_i\} = [d]$. According to Lemma 42(b), \mathcal{F} is thus a witness of \mathcal{T}'_d -discontinuity of MLPO_d . \square

4. Applications

Based on Lemma 19(b), we now determine the complexity of nonuniform computability for several concrete (multivalued) functions including the examples from Section 1.

Example 45. Consider the space $\mathbb{R}^{N \times M}$ of rectangular matrices and the mapping $\text{rank}_{N,M} : \mathbb{R}^{N \times M} \rightarrow \{0, 1, \dots, d\}$, $d := \min(N, M)$. This rank function is trivially $(d + 1)$ -computable; but not d -continuous: in fact its restriction to diagonal matrices is not d -continuous. To this end observe that the following function $f_{N,M} : \mathbb{R}^d \rightarrow \mathbb{R}^{N \times M}$ is $(\rho^d, \rho^{N \times M})$ -computable:

$$\mathbb{R}^d \ni (x_1, \dots, x_d) \mapsto \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & x_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & x_3 & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & x_d & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{N \times M}$$

and has $\text{rank} f(x_1, \dots, x_d) = \text{LPO}_d(x_1, \dots, x_d)$: now combine Lemma 39 with Example 20. \square

4.1. Linear equation solving

Consider the problem of solving a system of linear equations; more precisely of finding a nonzero vector in the kernel of a given singular matrix:

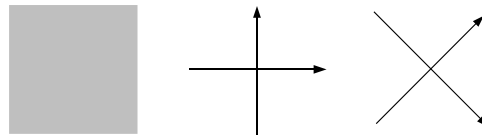


Fig. 8. Breaking 2-fold degeneracy (left) of an eigenspace to $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in two ways $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} \epsilon & \epsilon \\ \epsilon & \epsilon \end{pmatrix}$ (middle and right) admitting no common eigenvectors: $\text{EVecBase}'_2(A_0) = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}$, $\text{EVecBase}'_2(A_1) = \{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \}$, and $\text{EVecBase}'_2(A) = \mathcal{O}(\mathbb{R}^2)$, the orthogonal group.

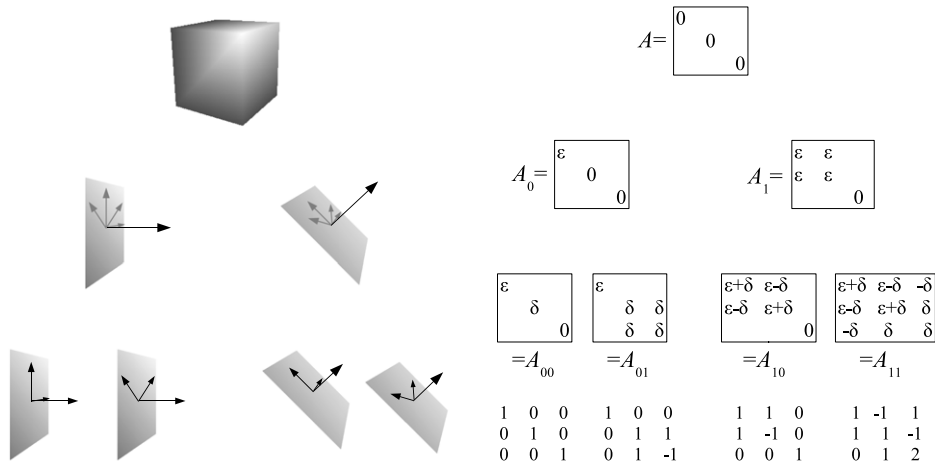


Fig. 9. Construction similar to Fig. 8, now iterated in 3D. The last row right lists eigenvectors of each A_{ij} .

4.2. Symmetric matrix diagonalization

According to Example 4, the number $\text{Card } \sigma(A) \in \{1, \dots, d\}$ is sufficient advice to computably find a basis of eigenvectors for a given symmetric $A \in \mathbb{R}^{d \times d}$. This happens to be optimal:

Theorem 47. Fix $d \in \mathbb{N}$ and consider the space $\mathbb{R}^{\binom{d}{2}}$ of real symmetric $d \times d$ matrices. Then the multivalued mapping

$$\text{EVecBase}_d : \mathbb{R}^{\binom{d}{2}} \ni A \mapsto \{ (\vec{w}_1, \dots, \vec{w}_d) \text{ basis of } \mathbb{R}^d \text{ of eigenvectors to } A \}$$

has complexity $\mathfrak{C}_t(\text{EVecBase}_d) = \mathfrak{C}_c(\text{EVecBase}_d, \rho^{\binom{d}{2}}, \rho^{d \times d}) = d$.

The lack of continuity of the mapping EVecBase is closely related to inputs with degenerate eigenvalues [57, Example 18]. In fact our below proof yields a witness of $(d - 1)$ -discontinuity by constructing an iterated sequence of symmetry breakings in the sense of Mathematical Physics; cf. Fig. 9. On the other hand even in the non-degenerate case, EVecBase is inherently multivalued since, e.g., any permutation of a basis constitutes again a basis.

Remember that a non-zero linear combination of eigenvectors to the same eigenvalue is again an eigenvector; whereas eigenvectors to different eigenvalues are orthogonal and w.l.o.g. normalized. In view of Lemma 39, it suffices to prove $(d - 1)$ -discontinuity for the mapping $\text{EVecBase}'_d := \text{GramSchmidt}_d \circ \text{EVecBase}_d : \mathbb{R}^{\binom{d}{2}} \rightarrow \mathcal{O}(\mathbb{R}^d)$ where

$$\text{GramSchmidt}_d : \{ (\vec{x}_1, \dots, \vec{x}_d) \text{ linearly independent} \} \subseteq \mathbb{R}^{d \times d} \rightarrow \mathcal{O}(\mathbb{R}^d)$$

denotes (some single-valued choice of) Gram--Schmidt Orthonormalization which is well-known computable and yields an orthogonal matrix as value. Note that the co-domain of $\text{EVecBase}'_d$ is compact but not finite nor almost-discrete as required in [21]. We also remark that $\text{EVecBase}'_d(A)$ is a closed set for each $A \in \mathbb{R}^{\binom{d}{2}}$: $\mathcal{O}(\mathbb{R}^d) \ni U_n \rightarrow U$ implies $U_n^\dagger \rightarrow U^\dagger$ and $A = U_n \cdot \text{diag}(x_1, \dots, x_d) \cdot U_n^\dagger \rightarrow U \cdot \text{diag}(x_1, \dots, x_d) \cdot U^\dagger$. Theorem 47 thus follows from Proposition 43(f) in conjunction with Item (d) of

Lemma 48. Let $\mathcal{O}(\mathbb{A}^d)$ denote the space of orthogonal $d \times d$ -matrices with algebraic reals as entries.

(a) Let $O_2 := (\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}) \in \mathcal{O}(\mathbb{A}^2)$ and, for $x, y \in \mathbb{R}$, $A := \text{diag}(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Then A and $A' := O_2 \cdot A \cdot O_2^\dagger$ have no eigenvectors in common in case $x \neq y$, whereas in case $x = y$ it holds $A = A'$.

- (b) For $d \geq 2$, consider the direct matrix sum $\mathcal{O}(\mathbb{A}^d) \ni \mathcal{O}_d := \mathcal{O}_2 \oplus \text{id}_{d-2} : \mathbb{R}^2 \times \mathbb{R}^{d-2} \ni (\vec{x}, \vec{y}) \mapsto (\mathcal{O}_2 \cdot \vec{x}, \vec{y})$. Moreover let $A_0 = \text{diag}(x_1, x_2, \dots, x_d) \in \mathbb{R}^{\binom{d}{2}}$ where $x_1, \dots, x_d \in \mathbb{R}$. Then A_0 and $A_1 := \mathcal{O}_d \cdot A_0 \cdot \mathcal{O}_d^\dagger$
- (i) have no eigenvector to eigenvalue x_1 in common in case $x_1 \notin \{x_2, \dots, x_d\}$
 - (ii) whereas in case $x_1 = x_2 = \dots = x_d$, it holds $A_0 = A_1$.
- (c) There exist a family $(U_{i_1, \dots, i_k})_{0 \leq k < d, i_j \in \{0, 1\}}$ in $\mathcal{O}(\mathbb{A}^d)$ satisfying $U_{i_1, \dots, i_k, 0} = U_{i_1, \dots, i_k}$ and $U = U_0 = \text{id}$ such that, for $x_1, \dots, x_d \in \mathbb{R}$ and

$$A_{\vec{i}} := U_{\vec{i}} \cdot A \cdot U_{\vec{i}}^\dagger \in \mathbb{R}^{\binom{d}{2}} \quad \text{with } A := \text{diag}(x_1, \dots, x_d),$$

the following dichotomy holds for $k \geq 1$:

- (i) In case x_1, \dots, x_k are pairwise distinct and $\notin \{x_{k+1}, \dots, x_d\}$, $A_{i_1, \dots, i_{k-1}, 0}$ and $A_{i_1, \dots, i_{k-1}, 1}$ have no eigenvectors to eigenvalue x_k in common.
 - (ii) In case $x_k = x_{k+1} = \dots = x_d$, it holds $A_{i_1, \dots, i_{k-1}, 0} = A_{i_1, \dots, i_{k-1}, 1}$.
- (d) Consider the complete binary tree T of depth $d - 1$ on vertex set

$$V := \{(i_1, \dots, i_k) : k \in \{0, 1, \dots, d - 1\}, i_j \in \{0, 1\}\}$$

where $\vec{i} = (i_1, \dots, i_k)$ is considered a successor of $\vec{j} = (j_1, \dots, j_\ell)$ iff $k \geq \ell$ and $(j_1, \dots, j_\ell) = (i_1, \dots, i_\ell)$. Next define $F_k \subseteq \mathbb{R}^d$ for $0 \leq k < d$ as

$$\{(x_1, \dots, x_k, x_{k+1}, \dots, x_d) : x_1, \dots, x_{k+1} \in \mathbb{R} \text{ pairwise distinct, } x_{k+1} = \dots = x_d\}$$

and $F_{i_1, \dots, i_k} := \{U_{\vec{i}} \cdot \text{diag}(\vec{x}) \cdot U_{\vec{i}}^\dagger : \vec{x} \in F_k\}$. Then $\mathcal{F} := (F_{\vec{i}})_{\vec{i} \in V}$ is a witness of \mathcal{T} -discontinuity of $\text{EVecBase}'_d$, where \mathcal{T} denotes the canonical hypergraph over T according to Lemma 42.

Note that in Item (c) $U_{\vec{i}} (\vec{i} \in \{0, 1\}^k)$ is defined also for $k = 0$ (namely as $U = \text{id}$) although Claims (i+ii) refer only to $k \geq 1$. \mathcal{F} in Item (d), however, is (and has to be) defined also for the case $k = 0$.

Proof. (a) straight forward.

- (b) Note that the eigenspace to x_1 is one-dimensional in case (i).
- (c) by induction on k , the starting case $k = 1$ already covered by (b). For the induction step $k \mapsto k + 1$, take U_{i_1, \dots, i_k} according to the induction hypothesis and consider $\mathcal{O}_{d-k} \in \mathcal{O}(\mathbb{A}^{d-k})$ according to (b). That is, $A'_0 := \text{diag}(x_{k+1}, \dots, x_d)$ has no eigenvector to eigenvalue x_{k+1} in common with $A'_1 := \mathcal{O}_{d-k} \cdot A'_0 \cdot \mathcal{O}_{d-k}^\dagger$ in case $x_{k+1} \notin \{x_{k+2}, \dots, x_d\}$; whereas in case $x_{k+1} = x_{k+2} = \dots = x_d$, $A'_0 = A'_1$ holds. Now let $U_{i_1, \dots, i_k, 0} := U_{i_1, \dots, i_k}$ and $U_{i_1, \dots, i_k, 1} := U_{i_1, \dots, i_k} \cdot (\text{id}_k \oplus \mathcal{O}_{d-k}) \in \mathcal{O}(\mathbb{A}^d)$. In case x_1, \dots, x_k, x_{k+1} are pairwise distinct and different from all x_{k+2}, \dots, x_d , it holds in particular $x_{k+1} \notin \{x_{k+2}, \dots, x_d\}$; from which we have already concluded that A'_0 has no eigenvector to eigenvalue x_{k+1} in common with $A'_1 = \mathcal{O}_{d-k} \cdot A'_0 \cdot \mathcal{O}_{d-k}^\dagger$; hence neither do A and $(\text{id}_k \oplus \mathcal{O}_{d-k}) \cdot A \cdot (\text{id}_k \oplus \mathcal{O}_{d-k})^\dagger$; and this property is furthermore maintained under the joint orthogonal transform $U_{i_1, \dots, i_k} : A_{i_1, \dots, i_k, j} = U_{i_1, \dots, i_k, j} \cdot A \cdot U_{i_1, \dots, i_k, j}^\dagger (j = 0, 1)$ have no eigenvector to eigenvalue x_{k+1} in common.
- In case $x_{k+1} = x_{k+2} = \dots = x_d$, on the other hand, $A'_1 = \mathcal{O}_{d-k} \cdot A'_0 \cdot \mathcal{O}_{d-k}^\dagger$ implies $A = (\text{id}_k \oplus \mathcal{O}_{d-k}) \cdot A \cdot (\text{id}_k \oplus \mathcal{O}_{d-k})^\dagger$ and $A_{i_1, \dots, i_k, 1} = A_{i_1, \dots, i_k} = A_{i_1, \dots, i_k, 0}$.

- (d) Note that, obviously, $F_k \subseteq F_{k+1}$; hence (F_0, \dots, F_{d-1}) is a nontrivial d -flag; and $\mathcal{F}' := (F_{|\vec{i}|})_{\vec{i} \in V}$ a T -flag. From continuity of orthogonal mappings

$$\lim_n \vec{x}_n = \vec{x} \Rightarrow \lim_n (U_{\vec{i}} \cdot \text{diag}(\vec{x}_n) \cdot U_{\vec{i}}^\dagger) = U_{\vec{i}} \cdot \text{diag}(\vec{x}) \cdot U_{\vec{i}}^\dagger$$

it follows with c(ii) that \mathcal{F} is a T -flag as well. Moreover, each $B_{\vec{i}} \in F_{\vec{i}}$ has the form $B_{i_1, \dots, i_k} = A_{i_1, \dots, i_k}(x_1, \dots, x_d)$ as in (c) with x_1, \dots, x_k, x_{k+1} pairwise distinct and $x_{k+1} = \dots = x_d$.

Note that $A_{i_1, \dots, i_k}(x_1, \dots, x_d)$ has as eigenvectors to (non-degenerate) eigenvalues x_1, \dots, x_k precisely the first k columns of U_{i_1, \dots, i_k} ; and to $(d-k)$ -fold degenerate eigenvalue x_{k+1} as eigenvectors any linear combination of the last $d-k$ columns of U_{i_1, \dots, i_k} . In particular, $\text{EVecBase}'_d(A_{i_1, \dots, i_k}(\vec{x}))$ depends only on (i_1, \dots, i_k) but not on $\vec{x} \in F_k$; that is, $\text{EVecBase}'_d$ is constant on F_{i_1, \dots, i_k} with values closed in compact $\mathcal{O}(\mathbb{R}^d)$.

According to c(i), $B_{i_1, \dots, i_{k-1}, 0}$ and $B_{i_1, \dots, i_{k-1}, 1}$ have no eigenvector to eigenvalue x_k in common. Now recall that any element of $\text{EVecBase}_d(B_{i_1, \dots, i_{k-1}, j})$ is by definition a basis of \mathbb{R}^d of eigenvectors to $B_{i_1, \dots, i_{k-1}, j}$ and must in particular include some eigenvector to eigenvalue x_k . Therefore $\text{EVecBase}_d(B_{i_1, \dots, i_{k-1}, 0})$ is disjoint from $\text{EVecBase}_d(B_{i_1, \dots, i_{k-1}, 1})$: $\text{EVecBase}'_d(B_{i_1, \dots, i_{k-1}, 0}) \cap \text{EVecBase}'_d(B_{i_1, \dots, i_{k-1}, 1}) = \emptyset$.

We finally show that $\text{EVecBase}'_d$ is \mathcal{F} -hereditary in the sense of Lemma 42(b): indeed, recall that the first k columns of $U_{i_1, \dots, i_{k-1}, j}$ are unique eigenvectors of $A_{i_1, \dots, i_{k-1}, j}(x_1, \dots, x_k, x_{k+1}, \dots, x_{k+1}) = B_{i_1, \dots, i_{k-1}, j}$; and, since $U_{\vec{i}}$ in (c) does not depend on \vec{x} , its columns remain eigenvectors (although not necessarily unique) of $A_{i_1, \dots, i_{k-1}, j}(\vec{x})$ also for $\vec{x} \in F_{k-1}$, that is, for $B_{i_1, \dots, i_{k-1}}$. Whereas eigenvectors of $B_{i_1, \dots, i_{k-1}, j}$ to eigenvalue x_{k+1} are linear combinations of the last $d-k$ columns of $U_{i_1, \dots, i_{k-1}, j}$ —again independent of \vec{x} and in particular also for $\vec{x} \in F_{k-1}$.

Now the claim follows from Lemma 42(b). \square

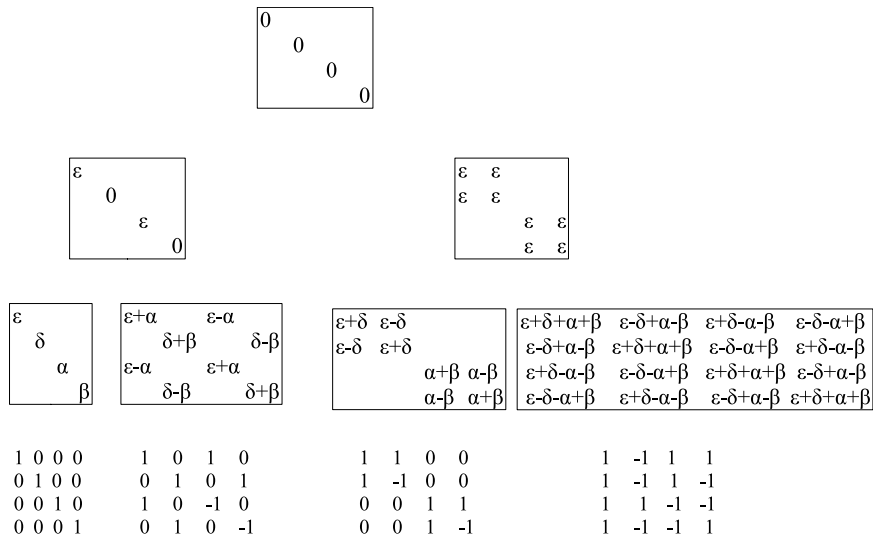


Fig. 10. Variant of Fig. 9 showing that finding some eigenvector in 4D is 2-discontinuous.

4.3. Finding some eigenvector

Instead of computing an entire basis of eigenvectors, we now turn to the problem of determining just one arbitrary eigenvector to a given real symmetric matrix. This turns out to be considerably less ‘complex’:

Theorem 49. (a) For a real symmetric $n \times n$ -matrix A , consider the multiplicity $m(A)$ of some least-degenerate eigenvalue,

$$m(A) := \min \{ \dim \text{kernel}(A - \lambda \text{id}) : \lambda \in \sigma(A) \} \in \{1, \dots, n\}.$$

Given $d := \lfloor \log_2 m \rfloor$ and a $\rho^{(d)}$ -name of A , one can ρ^n -compute (i.e. effectively find) some eigenvector of A .

(b) The multivalued mapping

$$\text{SomeEVec}_n : \mathbb{R}^{(d)} \ni A \mapsto \{ \vec{w} \text{ eigenvector of } A \} \subseteq \mathbb{R}^n \setminus \{0\}$$

is $(\rho^{(d)}, \rho^n)$ -computable with $\lfloor 1 + \log_2 n \rfloor$ -fold advice

(c) but not $\lfloor \log_2 n \rfloor$ -continuous.

Proof. (a) Compute some $(\rho^n$ -name of an) n -tuple of eigenvalues $(\lambda_1, \dots, \lambda_n)$ of A , repeated according to their multiplicities; cmp. e.g. [57, Proposition 17]. Now according to [57, Theorem 11], (an entire basis of, and in particular some eigenvector in) the eigenspace $\text{kernel}(A - \lambda_i \text{id})$ can be computably found when knowing $d_i := \text{rank}(A - \lambda_i \text{id})$ (recall Theorem 46), that is when knowing the multiplicity of λ_i in the multi-set $(\lambda_1, \dots, \lambda_n)$. But the multiplicity and index of some least-degenerate eigenvalue is computable from advice d by virtue of Theorem 33.

(b) follows from (a).

(c) The normalization function $\text{norm} : \mathbb{R}^n \setminus \{0\} \rightarrow \{ \vec{x} \in \mathbb{R}^n : \|\vec{x}\|_2 = 1 \} = \mathcal{S}^{n-1}, \vec{x} \mapsto \vec{x} / \sum_i |x_i|^2$ is computable. In view of Lemma 39 it thus suffices to prove $\lfloor \log_2 n \rfloor$ -discontinuity of $\text{SomeEVec}'_n := \text{norm} \circ \text{SomeEVec}_n : \mathbb{R}^{(d)} \rightarrow \mathcal{S}^{n-1}$. Moreover, the complexity of $\text{SomeEVec}'_{n+1}$ is obviously at least as large as that of $\text{SomeEVec}'_n$; hence without loss of generality we shall restrict to the case of $n = 2^d$ being a power of two. This case follows from Proposition 43(f) by constructing a witness of d -discontinuity in Lemma 50(c) below. \square

Note that, although A_{00} and A_{01} in Fig. 9 admit no simultaneous diagonalization, they do have $(1, 0, 0)$ as eigenvector in common; and A_0, A_1 share the eigenvector $(0, 0, 1)$. So in order to get $\text{SomeEVec}'(A_{i_1, \dots, i_k, 0}) \cap \text{SomeEVec}'(A_{i_1, \dots, i_k, 1}) = \emptyset$, we establish the following variant of Lemma 48; cmp. also Fig. 10:

Lemma 50. (a) Let \mathbb{F} denote a field and $A, B \in \mathbb{F}^{n \times n}$ matrices.

(i) If A, B have no eigenvalues in common, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathbb{F}^{2n \times 2n}$ and $\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix} / 2$ have no eigenvector in common.

(ii) If $A = B$, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix} / 2$.

(b) There exist a family $(U_{i_1, \dots, i_k})_{0 \leq k \leq d, i_j \in \{0, 1\}}$ in $\mathcal{O}(\mathbb{A}^{2^d})$ satisfying $U_{i_1, \dots, i_k, 0} = U_{i_1, \dots, i_k}$ and $U = U_0 = \text{id}$ such that, for $x_1, \dots, x_{2^d} \in \mathbb{R}$ and

$$A_i := U_i \cdot A \cdot U_i^\dagger \in \mathbb{R}^{(2^d)} \quad \text{with } A := \text{diag}(x_1, \dots, x_{2^d}),$$

the following dichotomy holds for $k \geq 1$:

(i) In case

$$(x_1, \dots, x_{2^d}) = \underbrace{(y_1, \dots, y_{2^k})}_{1st} \underbrace{(y_1, \dots, y_{2^k}, \dots, y_1, \dots, y_{2^k})}_{2nd} \dots \underbrace{(y_1, \dots, y_{2^k})}_{2^{d-k}\text{-th}}$$

with y_1, \dots, y_{2^k} pairwise distinct, $A_{i_1, \dots, i_{k-1}, 0}$ and $A_{i_1, \dots, i_{k-1}, 1}$ have no eigenvector in common.

(ii) In case $(x_1, \dots, x_{2^d}) = (y_1, \dots, y_{2^{k-1}}, y_1, \dots, y_{2^{k-1}}, y_1, \dots, y_{2^{k-1}},$

$y_1, \dots, y_{2^{k-1}}, \dots, y_1, \dots, y_{2^{k-1}}, y_1, \dots, y_{2^{k-1}})$ (2^{d-k+1} -times), it holds $A_{i_1, \dots, i_{k-1}, 0} = A_{i_1, \dots, i_{k-1}, 1}$.

(c) Consider the complete binary tree T of depth d on vertex set

$$V := \{(i_1, \dots, i_k) : k \in \{0, 1, \dots, d\}, i_j \in \{0, 1\}\}.$$

Define $F_k \subseteq \mathbb{R}^{2^d}$ for $0 \leq k \leq d$ as

$$\{(y_1, \dots, y_{2^k}, y_1, \dots, y_{2^k}, \dots, y_1, \dots, y_{2^k}) : y_1, \dots, y_{2^k} \in \mathbb{R} \text{ pairwise distinct}\}$$

and $F_i := \{U_i \cdot \text{diag}(\vec{x}) \cdot U_i^T : \vec{x} \in F_k\}$. Then $\mathcal{F} := (F_i)_{i \in V}$ is a witness of \mathcal{T} -discontinuity of $\text{SomeEVec}'_{2^d}$, where \mathcal{T} denotes the canonical hypergraph over T according to Lemma 42.

Proof. a(i) Suppose $\begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \in \mathbb{R}^{2^d}$ is an eigenvector of $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to eigenvalue λ . Then $A \cdot \vec{u} = \lambda \vec{u}$ and $B \cdot \vec{v} = \lambda \vec{v}$ require $\vec{u} = 0$ or $\vec{v} = 0$: otherwise λ would be a common eigenvalue. Now $\begin{pmatrix} A+B & A-B \\ A-B & A+B \end{pmatrix} / 2 \cdot \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} = \lambda \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}$ implies $B \cdot \vec{u} = A \cdot \vec{u} = \lambda \vec{u}$: contradiction.

(b) Notice that the case $d > k$ can be obtained as a 2^{d-k} -fold direct sum of the case $d = k$: if $A_0, A_1 \in \mathbb{R}^{n \times n}$ have no common eigenvector, neither do $A_0 \oplus A_0, A_1 \oplus A_1 \in \mathbb{R}^{2n \times 2n}$; whereas $A_0 = A_1$ implies $A_0 \oplus A_0 = A_1 \oplus A_1$. Hence it suffices to proceed by induction on $k = d$.

The starting case $k = 1 = d$ is Lemma 48(a). In case $k = d > 1$, take $U_{i_1, \dots, i_{d-1}} \in \mathcal{O}(\mathbb{A}^{2^{d-1}})$ according to the induction hypothesis; then define $U_{i_1, \dots, i_{k-1}, 0} := U_{i_1, \dots, i_{k-1}} \oplus U_{i_1, \dots, i_{k-1}} \in \mathcal{O}(\mathbb{A}^{2^d})$ and

$$U_{i_1, \dots, i_{k-1}, 1} := 2^{-k/2} \begin{pmatrix} + \text{id}_{2^{d-1}} & + \text{id}_{2^{d-1}} \\ + \text{id}_{2^{d-1}} & - \text{id}_{2^{d-1}} \end{pmatrix} \cdot U_{i_1, \dots, i_{k-1}, 0} \in \mathcal{O}(\mathbb{A}^{2^d}).$$

The claimed dichotomy then holds according to (a).

(c) $F_{k-1} \subseteq F_k$ is clear. Similarly to Lemma 48(d), conclude that (F_{i_1, \dots, i_k}) is a T -flag; $\text{SomeEVec}'(\vec{x})$ depends only on i_1, \dots, i_k and not on the choice of $\vec{x} \in F_{i_1, \dots, i_k}$; $\text{SomeEVec}'(\vec{x})$ is a closed subset of compact \mathcal{S}^{2^d-1} ; $\text{SomeEVec}'(\vec{x}) \subseteq \text{SomeEVec}'(\vec{y})$ for $\vec{x} \in F_{i_1, \dots, i_k}$ and $\vec{y} \in F_{i_1, \dots, i_{k-1}}$; and $\text{SomeEVec}'(\vec{x}) \cap \text{SomeEVec}'(\vec{y}) = \emptyset$ for $\vec{x} \in F_{i_1, \dots, i_{k-1}, 0}$ and $\vec{y} \in F_{i_1, \dots, i_{k-1}, 1}$: thus Lemma 42(b) applies. \square

4.4. Root finding

We now address the effective Intermediate Value Theorem [52, Theorem 6.3.8.1]. Closely related is the problem of selecting from a given closed non-empty interval some point, recall Example 5(d). Both are treated quantitatively within our complexity-theoretic framework.

Specifically concerning Example 5(d), observe that any non-degenerate interval $[a, b]$ contains a rational (and thus computable) point x ; and providing an integer numerator and denominator of x makes the problem of computably selecting some x from given $[a, b]$ trivial. On the other hand, rational numbers may require arbitrarily large descriptions; even more, there are intervals containing rationals only of such large Kolmogorov Complexity; cf. Claim (e) of the following

Remark 51. For $q \in \mathbb{Q}$, let $C_{\mathbb{Q}}(q) := \min\{C(r, s) : r, s \in \mathbb{Z}, q = r/s\}$.

- (a) There exists $c \in \mathbb{N}$ such that, for all $r, s \in \mathbb{Z}$ coprime, it holds $C(r, s) \leq C_{\mathbb{Q}}(r/s) + c$ and, in the sense of Definition 13(c), $C_{\mathbb{Q}}^{\rho}(r/s) \leq C_{\mathbb{Q}}(r/s) + c$.
- (b) For $a, b \in \mathbb{Q}$, $C_{\mathbb{Q}}(a+b), C_{\mathbb{Q}}(a-b), C_{\mathbb{Q}}(a \cdot b), C_{\mathbb{Q}}(a/b) \leq C_{\mathbb{Q}}(a) + C_{\mathbb{Q}}(b) + \mathcal{O}(1)$.
And for $a, b \in \mathbb{R}$, $C_{\mathbb{Q}}^{\rho}(a+b), C_{\mathbb{Q}}^{\rho}(a-b), C_{\mathbb{Q}}^{\rho}(a \cdot b), C_{\mathbb{Q}}^{\rho}(a/b) \leq C_{\mathbb{Q}}^{\rho}(a) + C_{\mathbb{Q}}^{\rho}(b) + \mathcal{O}(1)$.
- (c) There exists an unbounded function $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that the Kolmogorov Complexity $C(m)$ of any integer $m \geq n$ is at least $\varphi(n)$.
- (d) Let $x \in \mathbb{R}$ be algebraic of degree 2 (e.g. $x = \sqrt{p} + q$ for some prime number $p \in \mathbb{P}$ and $q \in \mathbb{Q}$). Then there exists $\varepsilon > 0$ such that for all $r, s \in \mathbb{Z}$ with $s > 0, |x - r/s| > \varepsilon/s^2$.
- (e) Given $N \in \mathbb{N}$, there exist $a, b \in \mathbb{Q} \cap [0, 1]$ such that all $q \in \mathbb{Q} \cap [a, b]$ have $C_{\mathbb{Q}}(q) \geq N$.

Item (a) says that the minimum in the definition of $C_{\mathbb{Q}}(r/s)$ is ‘almost’ attained by (unique) coprime r, s .

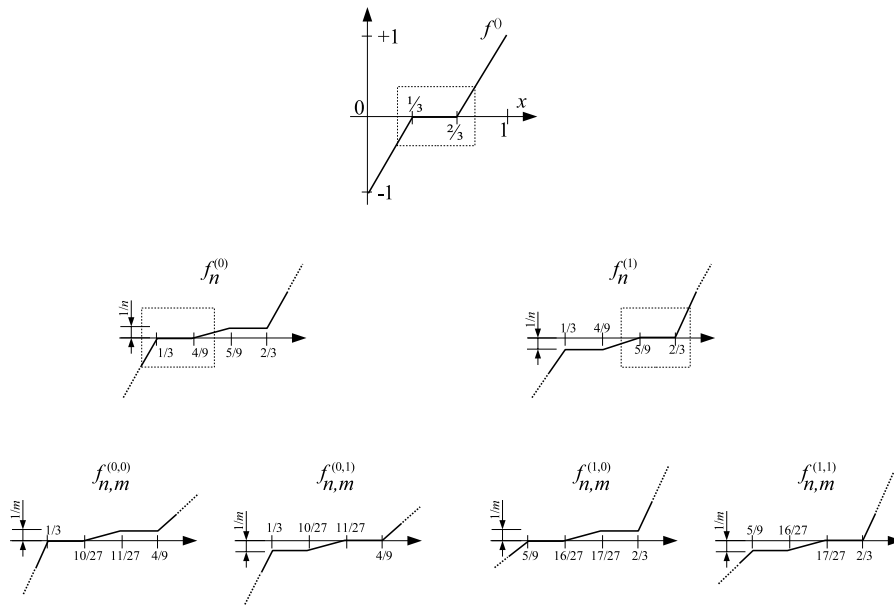


Fig. 11. Witness of \mathcal{T}_2 -discontinuity for the effective Intermediate Value Theorem.

- Proof.** (a) A constant-size program can easily convert (r', s') attaining the minimum to coprime $r/s = r'/s'$. Similarly, there is a fixed program converting (r, s) into the constant rational sequence $(r/s, r/s, \dots)$ which is a ρ -name of r/s .
- (b) It is easy, and uses only constant size overhead, to combine Turing machines computing a and b into ones computing $a + b, a - b, a \times b$, and a/b , respectively.
- (c) is from [32, Theorem 2.3.1(i)].
- (d) is Liouville’s Theorem on Diophantine approximation.
- (e) Take $x \in (1/3, 2/3)$ algebraic of degree 2, $\varepsilon > 0$ according to (d). Choose $0 < \delta < 1/3$ such that $\varphi(\lceil \sqrt{\varepsilon/\delta} \rceil) > N$. Then $\mathbb{Q} \ni r/s \in [x - \delta, x + \delta]$ with r, s coprime requires $\varepsilon/s^2 < \delta$, hence $s > \sqrt{\varepsilon/\delta}$ and $C_{\mathbb{Q}}(r, s) \geq C(s) \geq N$. Finally apply (a). \square

Note that Remark 51(e) applies only to rational numbers; that is $[a, b]$ might still contain, say, algebraic reals x with ρ -names of low Kolmogorov complexity. We now extend the claim to computable elements: referring to Proposition 14, Corollary 53 below shows that, even with the help of negative information about (i.e. a $\psi_{>}$ -name of) a given interval $[a, b]$, unbounded discrete advice is in general necessary to find (a ρ -name of) some $x \in [a, b]$.

Theorem 52. Finding a zero of a given continuous function $f : [0, 1] \rightarrow [-1, +1]$ with $f(0) = -1$ and $f(1) = +1$, that is the multivalued mapping $\text{Intermed} : \subseteq C[0, 1] \rightrightarrows [0, 1], f \rightrightarrows f^{-1}[0]$ on

$$\text{dom}(\text{Intermed}) := \{f : [0, 1] \rightarrow [-1, +1] \text{ continuous}, f(1) = 1 = -f(0)\},$$

has $\mathcal{C}_c(\text{Intermed}, [\rho \rightarrow \rho], \rho) = \mathcal{C}_t(\text{Intermed}) = \omega$.

Discontinuity of Intermed is well-known due to, and to occur for, arguments f which ‘hover’ [52, Theorem 6.3.2]. We iterate this property to obtain a witness of d -discontinuity for arbitrary $d \in \mathbb{N}$ in the following

Proof of Theorem 52. We first argue that countable advice suffices: as has been frequently exploited before [52, Section 6.3], $f \in \text{dom}(\text{Intermed})$ has an entire interval of zeros or has some isolated root or both. In the former case, that interval contains a rational one—which can be provided explicitly by its numerator and denominator as (unbounded but) countable discrete advice. Otherwise, [52, Theorem 6.3.7] applies and permits to find a root.

In order to show that bounded discrete advice does not suffice, we consider the complete binary tree T_d on $V_d = \{(i_1, \dots, i_k) : 0 \leq k \leq d, i_j = 0, 1\}$ of uniform depth d and construct a nontrivial T_d -flag $F_{i_1, \dots, i_k} := \{f_{n_1, \dots, n_k}^{(i_1, \dots, i_k)} : n_1, \dots, n_k \in \mathbb{N}\}$ in $\text{dom}(\text{Intermed})$ and prove it to be a witness of \mathcal{T}_d -discontinuity of Intermed , where \mathcal{T}_d denotes the canonical hypergraph over T_d : recall Lemma 42, noting that $C[0, 1]$ is metric with respect to the norm $\|f\| = \sup_{0 \leq x \leq 1} |f(x)|$ (written $\text{ulim}_n f_n = f$ for $\|f - f_n\| \rightarrow 0$) and the co-domain of Intermed is compact. Moreover Intermed will turn out as (iii) constant on each F_{i_1, \dots, i_k} , $\text{Intermed}(f_{n_1, \dots, n_k}^{(i_1, \dots, i_k)}) = I^{(i_1, \dots, i_k)}$ a (i) compact interval in $[0, 1]$ satisfying (ii) heredity $I^{(i_1, \dots, i_k)} \subseteq I^{(i_1, \dots, i_k, j)}$ and (iv) $I^{(i_1, \dots, i_k, 0)} \cap I^{(i_1, \dots, i_k, 1)} = \emptyset$.

First consider the piecewise linear, continuous function $f^0 := f : [0, 1] \rightarrow [-1, +1]$,

$$f(x) := 3x - 1 \text{ for } x \in [0, 1/3], \quad f(x) := 0 \text{ on } [1/3, 2/3], \quad f(x) := 3x - 2 \text{ for } x \in [2/3, 1].$$

Then $I^{(0)} := [1/3, 4/9]$ and $I^{(1)} := [5/9, 2/3]$ are closed, disjoint, and lie in $f^{-1}[0] = [1/3, 2/3] = I$. Moreover to $n > 0$ there are piecewise linear continuous functions $f_n^{(0)}, f_n^{(1)} \in \text{dom}(\text{Intermed})$ with $(f_n^{(0)})^{-1}[0] = I^{(0)}$ and $(f_n^{(1)})^{-1}[0] = I^{(1)}$ and $\|f - f_n^{(0)}\|, \|f - f_n^{(1)}\| < 1/n$; cf. Fig. 11. This settles the case $d = 1$.

Iterating the above construction as sketched in Fig. 11 we obtain, to every $d \in \mathbb{N}$ and $(i_1, \dots, i_d) \in \{0, 1\}^d$, closed intervals $I^{(i_1, \dots, i_d)}$

- of length 3^{-d-1}
- with $I^{(i_1, \dots, i_{d-1}, i_d)} \subseteq I^{(i_1, \dots, i_{d-1})}$
- and $I^{(i_1, \dots, i_{d-1}, 0)} \cap I^{(i_1, \dots, i_{d-1}, 1)} = \emptyset$

and multi-sequences of functions $f_{n_1, \dots, n_d}^{(i_1, \dots, i_d)} \in \text{dom}(\text{Intermed})$ with

- $f_{n_1, \dots, n_d}^{(i_1, \dots, i_d)} = \text{ulim}_m f_{n_1, \dots, n_d, m}^{(i_1, \dots, i_d, j)}$
- and $(f_{n_1, \dots, n_d}^{(i_1, \dots, i_d)})^{-1}[0] = \text{Intermed}(f_{n_1, \dots, n_d}^{(i_1, \dots, i_d)}) = I^{(i_1, \dots, i_d)}$. \square

Corollary 53. *The problem*

$$\text{Select} : [a, b] \Rightarrow [a, b], \quad \text{dom}(\text{Select}) := \{[a, b] : 0 \leq a < b \leq 1\},$$

of selecting some point from a given closed nondegenerate interval has $\mathcal{C}_c(\text{Select}, \psi_-, \rho) = \omega$.

Proof. [6, Section 6] has shown that the Weihrauch degrees of Select and Intermed coincide. Now apply [41, Theorem 5.9]. \square

5. Conclusion, extensions, and perspectives

We argue that a major source of criticism against Recursive Analysis misses the point: although computable functions *are* necessarily continuous when given approximations to the argument *x only*, many practical *f*'s do become computable when providing in addition some discrete information about *x*. Such ‘advice’ often consists of some very natural and mathematically explicit integer value from a bounded range (e.g. the rank of the matrix under consideration) and may be readily available in practical applications.

We have then turned this observation into a complexity theory, investigating the minimum size (=cardinal) of the range this discrete information comes from. We have devised tools for proving lower bounds on this quantity. And we have determined it explicitly for several common computational problems from linear algebra: calculating the rank of a given matrix, solving a system of linear equalities, diagonalizing a symmetric matrix, and finding some eigenvector to a given symmetric matrix. The latter three are inherently multivalued. And they exhibit a considerable variety in complexity: for input matrices *A* of format $n \times n$, usually discrete advice of order $\Theta(n)$ is necessary and sufficient; whereas some single eigenvector can be found using only $\Theta(\log n)$ -fold advice: specifically, the quantity $\lfloor \log_2 \min \{ \dim \text{kernel}(A - \lambda \text{id}) : \lambda \in \sigma(A) \} \rfloor$. The algorithm exploits this data based on some combinatorial considerations—which nicely complement the heavily analytical and topological arguments usually dominant in Recursive Analysis.

Our lower bound proofs assert *d*-discontinuity of the function under consideration. They can be extended (yet become even more tedious when trying to do so formally) to *weak d*-discontinuity. Also the major tool for such proofs, namely that of witnesses of *d*-discontinuity, would deserve generalizing from effective metric to computable topological spaces.

Question 54. (a) *In view of Example 2: Does bounded advice suffice for converting a (ρ -name of) given $x \in \mathbb{D}$ to some entire binary expansion of x ?*

(b) *In view of Example 3: Is differentiation $\partial : C^1[0, 1] \rightarrow C[0, 1]$ computable/continuous with countable advice?*

5.1. Non-integral advice

Theorem 46 shows that *d*-fold advice does not suffice for effectively finding a nontrivial solution \vec{x} to a homogeneous equation $A \cdot \vec{x} = 0$; whereas $(d + 1)$ -fold advice, namely providing $\text{rank}(A) \in \{0, \dots, d\}$, does suffice.

- Since the rank can be effectively approximated from below (i.e. is $\rho_{<}$ -computable) [57, Theorem 7], it in fact suffices to provide complementing upper approximations (i.e. a $\rho_{>}$ -name) to $\text{rank}(A)$. One may say that this constitutes strictly less than $(d + 1)$ -fold information.

- Similarly concerning diagonalization of a real symmetric $n \times n$ -matrix A , since the number $\text{Card } \sigma(A)$ of distinct eigenvalues can be effectively approximated from below, it suffices to provide only complementing upper approximations—cmp. [57, Theorem 19]—which may be regarded as strictly less than n -fold advice.
- Similarly, with respect to the problem of finding some eigenvector of A , again strictly less than $\lfloor 1 + \log_2 n \rfloor$ -fold advice suffices: namely lower approximations to $\lfloor \log_2 m(A) \rfloor$ (with $m(A)$ from Theorem 49) based on the following

Observation 55. The mapping $\mathbb{R}^{\binom{n}{2}} \ni A \mapsto \lfloor \log_2 m(A) \rfloor$ is $(\rho^{n \cdot (n-1)/2}, \rho_{>})$ -computable.

Proof. Given λ , $\dim \text{kernel}(A - \lambda \text{id}) = n - \text{rank}(A - \lambda \text{id})$ is $\rho_{>}$ -computable by [57, Theorem 7(i)]; hence so is its minimum $m(A)$ over all $\lambda \in \sigma(A)$, cmp. [57, Proposition 17] and [52, Exercise 4.2.11]. Moreover, $\log_2 : (0, \infty) \ni x \mapsto \ln(x) / \ln(2) \in \mathbb{R}$ is (ρ, ρ) -computable; and by monotonicity also $(\rho_{>}, \rho_{>})$ -computable. Finally, $x \mapsto \lfloor x \rfloor$ is $(\rho_{>}, \rho_{>})$ -computable on \mathbb{R} (although we need that only on \mathbb{N}). \square

The above examples suggest refining k -fold advice to non-integral values of k :

Definition 56. Let $f : X \rightarrow Y$ be a function and Z a topological T_0 space.

- (a) Call f continuous with Z -advice if there exists a function $g : X \rightarrow Z$ such that the function $f|_g$, defined as follows, is continuous:

$$\text{dom}(f|_g) := \{(x, z) : x \in X, g(x) = z\} \subseteq X \times Z, \quad (x, z) \mapsto f(x). \tag{7}$$

- (b) Let Z be finite and fix some injective notation $v_Z : \subseteq \Sigma^* \rightarrow \tau$ of the (finitely many) open subsets of Z . Then the representation $\delta_Z = \delta_{Z, v_Z} : \subseteq \Sigma^\omega \rightarrow Z$ is defined as follows:
 $\bar{\sigma} \in \Sigma^\omega$ is a $\delta_{Z, v}$ -name of z iff it is a v -enumeration (with arbitrary repetition) of all open sets containing Z .
- (c) For effective metric spaces (X, α) and (Y, β) , call f (α, β) -computable with Z -advice if there exists some $g : X \rightarrow Z$ such that the function $f|_g : \subseteq X \times Z \rightarrow Y$ from (a) is $(\alpha \times \delta_Z, \beta)$ -computable.
- (d) For finite T_0 spaces Z, W , write “ $Z \leq W$ ” if, for every choice of effective metric spaces (X, α) and (Y, β) , each function $f : X \rightarrow Y$ (α, β) -computable with Z -advice is also (α, β) -computable with W -advice.

The domain of $f|_g$ is an instance of a fibered product or pullback.
 Restricting Z to discrete spaces, one recovers Definition 8(a):

Lemma 57. For $d \in \mathbb{N}$ let Z_d denote the set $\{0, 1, \dots, d - 1\}$ equipped with the discrete topology.

- (a) A function $f : X \rightarrow Y$ is d -continuous iff it is continuous with Z_d -advice.
- (b) The representation δ_{Z_d} of Z_d is computably equivalent to $v_{Z_d} : \delta_{Z_d} \equiv v_{Z_d}$. Whereas in general, v_Z is only computably reducible to (but not from) $\delta_Z : v_Z \preceq \delta_Z$.
- (c) A function $f : X \rightarrow Y$ is (α, β) -computable with d -wise advice iff it is (α, β) -computable with Z_d advice.
- (d) (Z, δ_Z) is admissible. In particular if (X, α) and (Y, β) are admissible and if function $f : X \rightarrow Y$ is (α, β) -computable with Z -advice, then f is continuous with Z -advice.
- (e) Let $h : \subseteq W \rightarrow Z$ be surjective and continuous ((δ_W, δ_Z) -computable). Then it holds $Z \leq W$.

Proof. (a) Let $\Delta = \{D_0, D_1, \dots, D_{d-1}\}$ denote an d -element partition of X such that $f|_{D_z}$ is continuous for each $z = 0, 1, \dots, d - 1$. Define $g : X \ni x \mapsto$ the unique $z \in Z_d$ with $x \in D_z$. Then, for open $V \subseteq Y$,

$$(f|_g)^{-1}[V] = \bigoplus_{z \in Z} (f^{-1}[V] \cap \underbrace{g^{-1}[\{z\}]}_{=D_z}) \times \{z\} = \bigoplus_{z \in Z} \underbrace{(f|_{D_z})^{-1}[V]}_{\text{open in } D_z} \times \{z\} \tag{8}$$

is relatively open in $\bigoplus_{z \in Z} D_z \times \{z\} = \text{dom}(f|_g)$, i.e. $f|_g$ continuous.

Conversely let $f|_g$ be continuous for $g : X \rightarrow Z_d$. Define $\Delta := \{D_0, D_1, \dots, D_{d-1}\}$ where $D_z := g^{-1}[\{z\}]$ for $z \in Z_d$. Then Eq. (8) requires that $\bigcup_z (f^{-1}[V] \cap D_z) \times \{z\}$ be open in $\bigcup_z D_z \times \{z\}$. Now $D_z \times \{z\}$ is open by definition of the product topology and because $z \in Z_d$ is discrete, this implies that also the intersection $(f|_g)^{-1}[V] \cap (D_z \times \{z\}) = (f^{-1}[V] \cap D_z) \times \{z\}$ be open in $D_z \times \{z\}$, i.e. that $f|_{D_z}^{-1}[V]$ is open in D_z ; hence $f|_{D_z}$ is continuous for each $z \in Z_k$.

- (b) Since Z is finite and $v_Z : \subseteq \Sigma^* \rightarrow Z$ is injective, everything is bounded a priori. For instance, given a v_Z -name of $z \in Z$, one can easily produce a pre-stored list of all (finitely many) open sets containing this z : thus showing $v_Z \leq \delta_Z$.
 For the converse, exploit that Z_d bears the discrete topology and therefore is effectively T_1 [54]: Given an enumeration of all (finite) open sets U_i containing $z \in Z_d$, their intersection $\bigcap_i U_i$ becomes a singleton after finite time, thereby identifying z .
 In the Sierpiński space \mathbb{S} from Example 58(a) below, (some $\delta_{\mathbb{S}}$ -names of) $1 = \perp$ cannot continuously be distinguished from (a $\delta_{\mathbb{S}}$ -name of) $0 = \top$.

- (c) Suppose that $f|_{D_z}$ be (α, β) -computable for each $z \in Z_d$. Since Z_d is finite, it follows that $f|^\mathbb{S} : (x, z) \mapsto f|_{D_z}(x)$ is $(\alpha \times \nu_{Z_d}, \beta)$ -computable and hence $(\alpha \times \delta_{Z_d}, \beta)$ -computable by (b). Conversely let $f|^\mathbb{S}$ be $(\alpha \times \delta_{Z_d}, \beta)$ -computable. Then, similarly to (a) and since each $z \in Z_d$ is δ_{Z_d} -computable, it follows that also the all restrictions $f|_{D_z}$ be (α, β) -computable for $z \in Z_d$ where $D_z = g^{-1}[\{z\}]$.
- (d) Observe that δ_Z coincides with the *standard representation* of the (finite, hence effective) T_0 -space Z ; compare [52, Section 3.2]. Concerning the second claim, [52, Corollary 3.2.12] reveals that $f|^\mathbb{S}$ is continuous.
- (e) Let $f : X \rightarrow Y$ be continuous (computable) with Z -advice $g : X \rightarrow Z$ and let $\tilde{h} : Z \rightarrow W$ denote any left-inverse to h , i.e. s.t. $\tilde{h} \circ h = \text{id}_Z$. We show that f is also computable with W -advice $\tilde{h} \circ g : X \rightarrow W$: Given x and $w := \tilde{h} \circ g(x)$, one can by prerequisite compute $h(w) =: z$ which has $w = g(x)$; similarly for continuity. \square

In this sense, **Example 1** turns out to suffice with even strictly less than 2-fold advice:

- Example 58.** (a) Consider as Z the Sierpiński space \mathbb{S} , i.e. the set $\{0, 1\}$ equipped with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$ as open sets. Then the characteristic function of the *complement* of the Halting problem $\mathbf{1}_{\mathbb{N} \setminus H} : \mathbb{N} \rightarrow \mathbb{S}$ is $(\nu, \delta_{\mathbb{S}})$ -computable, but $\mathbf{1}_H$ itself is not.
- (b) The Gauß Staircase function $f := \lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ is (ρ, ρ) -computable with \mathbb{S} -advice.
 - (c) Generalizing $\mathbb{S} =: \mathbb{S}_1$, denote by \mathbb{S}_d the set $\{0, 1, \dots, d\}$ equipped with the following topology: $\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \dots, \{0, 1, 2, \dots, d\}\}$. Then $\text{rank} : \mathbb{R}^{d \times d} \rightarrow \{0, 1, \dots, d\}$ is $(\rho^{d \times d}, \rho)$ -computable with \mathbb{S}_d -advice.
 - (d) It holds $\mathbb{S}_{d-1} \preceq \mathbb{S}_d \preceq Z_{d+1} \preceq Z_{d+2}$.
 - (e) On the other hand, the Dirichlet Function $\mathbf{1}_{\mathbb{Q}} : [0, 1] \subseteq \mathbb{R} \rightarrow \{0, 1\}$ is computable with Z_2 -advice but not continuous with \mathbb{S}_d -advice for any $d \in \mathbb{N}$: $\mathbb{S}_d \not\preceq Z_2$.

In view of [57, Theorem 11], **Example 58(c)** shows that \mathbb{S}_d -advice renders also $\text{LinEq}_{n,m}$ computable for $d := \min(n, m - 1)$.

Proof of Example 58. (a) Simulate the given Turing machine M , and for each step append “ $\{0, 1\}$ ” to the $\delta_{\mathbb{S}}$ -name of $1 = \mathbf{1}_{\mathbb{N} \setminus H}(\langle M \rangle)$ to output in case that M does not terminate; whereas if and when M does turn out to terminate, start appending “ $\{0\}$ ” to the output, thus indeed producing a $\delta_{\mathbb{S}}$ -name of 0. Since any $\delta_{\mathbb{S}}$ -name of 0 must include the set $\{0\}$ in its enumeration, one can distinguish it in finite time from a $\delta_{\mathbb{S}}$ -name of 1. $\delta_{\mathbb{S}}$ -computing $\mathbf{1}_H(\langle M \rangle) = 0$ would thus amount to detecting the non-termination of M , contradicting that H is not co-r.e.

- (b) Intuitively, “ $x \notin \mathbb{Z}$ ” is semi-decidable; hence suffices to provide only half-sided advice for the case “ $x \in \mathbb{Z}$ ”. Formally, define $g : \mathbb{R} \rightarrow \mathbb{S}$ as the characteristic function of $\mathbb{R} \setminus \mathbb{Z}$, i.e., $g : \mathbb{Z} \ni x \mapsto 0$ and $g : \mathbb{R} \setminus \mathbb{Z} \ni x \mapsto 1$. Observe $\text{dom}(f|^\mathbb{S}) = (\mathbb{Z} \times \{0\}) \uplus (\mathbb{R} \setminus \mathbb{Z} \times \{1\})$. Hence, for $y \in \mathbb{N}$, it is $(f|^\mathbb{S})^{-1}[\{y\}] = ((y, y + 1) \times \{1\}) \uplus (\{y\} \times \{0\})$ with $(y, y + 1) \times \{1\} = ((y, y + 1) \times \mathbb{S}) \cap \text{dom}(f|^\mathbb{S})$ and $\{y\} \times \{0\} = (\mathbb{R} \times \{0\}) \cap \text{dom}(f|^\mathbb{S})$ both open in $\text{dom}(f|^\mathbb{S})$.
- (c) Let $g := \text{rank} : \mathbb{R}^{k \times k} \rightarrow \mathbb{S}_k$. Then it holds $\text{rank} \upharpoonright^{\text{rank}}(A, i) = \text{rank}(A) = i$ on $\text{dom}(\text{rank} \upharpoonright^{\text{rank}}) = \{(A, i) : A \in \mathbb{R}^{k \times k}, i \in \mathbb{N}, \text{rank}(A) = i\}$. In particular $(\text{rank} \upharpoonright^{\text{rank}})^{-1}[\{j\}] = (\{A : \text{rank}(A) \geq j\} \times \{0, 1, \dots, j\}) \cap \text{dom}(\text{rank} \upharpoonright^{\text{rank}})$ is relatively open in $\text{dom}(\text{rank} \upharpoonright^{\text{rank}})$, because $\{A : \text{rank}(A) \geq j\} \subseteq \mathbb{R}^{k \times k}$ is open by [57, Theorem 7] and $\{0, 1, \dots, j\} \subseteq \mathbb{S}_k$ is open by definition.
- (d) The identity mapping $\text{id} : Z_{k+1} \setminus \{0, 1, \dots, k\} \rightarrow \{0, 1, \dots, k\} = \mathbb{S}_k$ is surjective and, in view of the discrete topology on Z_{k+1} , trivially continuous/computable: hence $\mathbb{S}_k \preceq Z_{k+1}$ by **Lemma 57(e)**. Similarly, $Z_{k+1} \preceq Z_{k+2}$ follows from the surjection $Z_{k+2} = \{0, 1, \dots, k + 1\} \ni i \mapsto \max\{i, k\} \in Z_{k+1}$. And $\mathbb{S}_{k-1} \preceq \mathbb{S}_k$ is established by the surjection $h_k : \mathbb{S}_k \rightarrow \mathbb{S}_{k-1}$ defined as $0 < i \mapsto i - 1$ and $0 \mapsto 0$, whose continuity can be seen from $h_k^{-1}[\{0, 1, \dots, i\}] = \{0, 1, \dots, i, i + 1\}$ for $0 \leq i < k$.
- (e) Suppose $\mathbf{1}_{\mathbb{Q}}|^\mathbb{S} : \subseteq \mathbb{R} \times \mathbb{S}_d \rightarrow \{0, 1\}$ is continuous for $g : \mathbb{R} \rightarrow \mathbb{S}_d$. Then the restriction $\mathbf{1}_{\mathbb{Q}}|_{D_z}$ is continuous (i.e. constant) on each $D_z := g^{-1}[\{z\}]$; that is, for each $z = 0, 1, \dots, d$, it either holds $D_z \subseteq \mathbb{Q}$ or $D_z \subseteq \mathbb{R} \setminus \mathbb{Q}$. First observe that there exist k, ℓ and two sequences $x_n \in D_k$ of rationals and $y_n \in D_\ell$ of irrationals with $|x_n - y_n| \rightarrow 0$. Indeed, $y_n := y$ arbitrary irrational belongs to D_ℓ for $\ell := g(y)$; and, since \mathbb{Q} is dense, there exists $(x_n) \subseteq \mathbb{Q}$ with $x_n \rightarrow y$; where, by pigeon-hole, $x_n \in D_k$ for some k and infinitely many (by proceeding to a subsequence w.l.o.g. all) n . We treat the case $k < \ell$; $k > \ell$ works similarly. By construction, it holds $\mathbf{1}_{\mathbb{Q}}(y_n) = 0$ and $g(y_n) = \ell$; hence $(y_n, \ell) \in (\mathbf{1}_{\mathbb{Q}}|^\mathbb{S})^{-1}[\{-\frac{1}{2}, \frac{1}{2}\}] =: V$ for all n . Since V is open in $\text{dom}(\mathbf{1}_{\mathbb{Q}}|^\mathbb{S}) \subseteq \mathbb{R} \times \mathbb{S}_d$, it follows $(x_n, k) \in V$ for all sufficiently large n ; recall that the topology on \mathbb{S}_d has $k \in U$ for open $U \subseteq \mathbb{S}_d$ and $k < \ell \in U$. But $\mathbf{1}_{\mathbb{Q}}(x_n) = 1$ contradicts $(x_n, k) \in V$. \square

Since \mathbb{S} -advice is strictly less than 2-fold advice (**Example 58d+e**) strictly richer than 1-fold (i.e. no) advice (**Example 58b**), it is consistent to quantify \mathbb{S} -advice as $(1 + \frac{1}{2})$ -fold. In fact, Z_2 and \mathbb{S} are (up to homeomorphism) the only 2-element T_0 spaces; but according to **Example 58(e)**, \mathbb{S}_d -advice is *not* more (nor less, for $d \geq 2$) than Z_2 -advice and hence cannot justly be called $(d + \frac{1}{2})$ -fold.

In fact the **Definition 8** of integral and cardinal k -continuity has an important structural advantage: the complexities of two functions are always comparable—either $\mathfrak{C}_t(f) < \mathfrak{C}_t(g)$, or $\mathfrak{C}_t(f) > \mathfrak{C}_t(g)$, or $\mathfrak{C}_t(f) = \mathfrak{C}_t(g)$; Whereas when refining

beyond integral advice, non-comparability emerges. As a matter of fact [Definition 56](#) may be related to Weihrauch Degrees with their complicated structure [51,41,5]. In this way, these degrees resembles classical structural Turing-complexity in terms of (say, polynomial-time) reducibility; whereas the cardinal of discontinuity corresponds to resource-bounded Turing complexity classes like $\text{DTIME}(f(n))$.

Question 59. *By assigning weights to the advice values $z \in Z$ and to the measurable subsets of $\text{dom}(X)$, can one obtain a notion of average advice in the spirit of SHANNON's entropy?*

5.2. Topologically-restricted advice

[Definition 8](#) asks for the number of color classes needed to make f continuous/computable on each such class—unconditional to the topological complexity of the classes themselves: in principle, they may be arbitrarily high on the Borel Hierarchy or even non-measurable (subject to the axiom of choice).

From our point of view, determining the discrete advice to (i.e. the color c of) some input x to f is a non-computational process preceding the evaluation of f . For instance in the Finite Element Method approach to solving a partial differential equation on some surface S , its discretization via triangulation gives rise to a matrix A known a-priori to have 3-band form: its band-width need not be 'computed', nor does one have to explicitly represent the subset of all 3-band matrices within the collection of all matrices. In fact, since the optimal color classes themselves (rather than the number of colors) are usually far from unique, this freedom may be exploited to choose them not too wild.

On the other hand, [Definition 8](#) can easily be adapted to take into account topological restrictions:

Definition 60. Let $f : A \rightarrow B$ denote a function between topological spaces A, B (represented spaces (A, α) and (B, β)); and let $\mathcal{A} \subseteq 2^A$ denote a class of subsets of $A = \text{dom}(f)$.

- (a) $\mathfrak{C}_t(f; \mathcal{A}) := \min \{ \text{Card}(\Delta) : \Delta \subseteq \mathcal{A} \text{ partition of } A, f|_D \text{ is continuous } \forall D \in \Delta \}$
- (b) $\mathfrak{C}_c(f, \alpha, \beta; \mathcal{A}) := \min \{ \text{Card}(\Delta) : \Delta \subseteq \mathcal{A} \text{ partition of } A, f|_D \text{ is } (\alpha, \beta)\text{-computable } \forall D \in \Delta \}$

Hence for $\mathcal{A} := 2^{\text{dom}(f)}$ the powerset of $\text{dom}(f)$ one recovers the previous, unrestricted [Definition 8](#): $\mathfrak{C}_t(f; 2^{\text{dom}(f)}) = \mathfrak{C}_t(f)$ and $\mathfrak{C}_c(f, \alpha, \beta, 2^{\text{dom}(f)}) = \mathfrak{C}_c(f, \alpha, \beta)$; whereas restricting the topology of the color classes may increase, but not decrease, the number of colors needed: $\mathfrak{C}_t(f; \mathcal{A}) \leq \mathfrak{C}_t(f; \mathcal{B})$ for $\mathcal{B} \subseteq \mathcal{A}$; similarly for \mathfrak{C}_c . Also notice that, unless \mathcal{A} is closed under finite unions and intersections, it may now well matter whether Δ is a partition or a covering of $\text{dom}(f)$.

Concerning the applications considered in [Section 4](#), [Corollary 62](#) below shows that the optimal advice (namely the matrix rank and the number of distinct eigenvalues) gives rise to topologically very tame color classes. In order to formalize this claim, recall that for a metrizable space X , each level of the Borel Hierarchy $\Sigma_t(X), \Pi_t(X) \subseteq \Sigma_t(X) \cup \Pi_t(X) \subseteq \Sigma_{t+1}(X) \cap \Pi_{t+1}(X)$ of open/closed ($t = 1$) set, F_σ/G_δ ($t = 2$) sets and so on, is strictly refined by the Hausdorff difference hierarchy; whose second level $2-\Sigma_t(X) = 2-\Pi_t(X)$ consists of all sets of the form $U \setminus V$ with $U, V \in \Sigma_t(X)$ (equivalently: of the form $A \setminus B$ with $A, B \in \Pi_t(X)$); cf. e.g. [25, Section 22.E]. We can now strengthen [Proposition 11a+c](#):

Lemma 61. *Extend HERTLING's notions*

$$\text{LEV}'(f, i + 1) = \{x \in \text{LEV}'(f, i) : f|_{\text{LEV}'(f,i)} \text{ discontinuous at } x\}$$

$$\text{LEV}(f, i + 1) = \overline{\{x \in \text{LEV}(f, i) : f|_{\text{LEV}(f,i)} \text{ discontinuous at } x\}}$$

literally from functions to relations in the sense of [Definition 35\(c\)](#).

- (a) Let X be a metric space and $f : X \rightrightarrows Y$. Then, in addition to the inequalities $\mathfrak{C}_t(f; 2-\Sigma_2) \leq \mathfrak{C}_t(f; 2-\Sigma_1)$ and $\text{Lev}'(f) \leq \text{Lev}(f)$, it also holds $\mathfrak{C}_t(f; 2-\Sigma_1) \leq \text{Lev}(f)$.
Moreover in case f is single-valued, it holds $\mathfrak{C}_t(f; 2-\Sigma_2) \leq \text{Lev}'(f)$.
- (b) The Dirichlet Function, i.e. the characteristic function $\mathbf{1}_Q : [0, 1] \subseteq \mathbb{R} \rightarrow \{0, 1\}$, has $\mathfrak{C}_c(\mathbf{1}_Q, \rho, \rho; 2-\Sigma_2) = \mathfrak{C}_t(\mathbf{1}_Q; 2-\Sigma_2) = 2$ but $\text{Lev}'(\mathbf{1}_Q) = \text{Lev}(\mathbf{1}_Q) = \infty$.
- (c) Let $f : X \rightrightarrows Y$ be such that $f|_U$ is continuous on open $U \subseteq X$. Then it holds $\text{LEV}(f, 1) \subseteq X \setminus U$; and the prerequisite that U be open is essential.
- (d) More generally, if $U_i \subseteq X \setminus (U_1 \cup \dots \cup U_{i-1})$ is relatively open and $f|_{U_i}$ continuous thereon for all $i \leq k$, then $\text{LEV}(f, k) \subseteq X \setminus (U_1 \cup \dots \cup U_k)$.

Proof. We record that [23, Lemma 2.5] remains valid for relations instead of functions.

- (a) By definition, f is continuous on $\text{LEV}(f, i) \setminus \text{LEV}(f, i + 1)$: the difference of two closed sets and thus in $2-\Pi_1$. Ranging $i = 0, \dots, \text{Lev}(f) - 1$ thus yields a $\text{Lev}(f)$ -element partition of $\text{dom}(f)$ as required.

For the case of $\text{Lev}'(f)$, recall that the set $\text{LEV}'(f, i + 1)$ of discontinuities of (single-valued!) $f|_{\text{LEV}'(f,i)}$ is always F_σ , i.e. in $\Sigma_2(\text{LEV}'(f, i))$; and by induction in $\Sigma_2(\text{dom}(f))$ since F_σ sets are closed under finite intersection. Now proceed as above.

- (b) Observe that, since $\mathbb{Q} \in F_\sigma$, both $\mathbb{Q} \cap [0, 1]$ and $[0, 1] \setminus \mathbb{Q}$ belong to $2-\Sigma_2$, thus showing $\mathfrak{C}_c(\mathbf{1}_\mathbb{Q}; 2-\Sigma_2) = 2$. However the subset $\text{LEV}'(f, 1)$ of discontinuities of $\mathbf{1}_\mathbb{Q}$ coincides with $[0, 1] = \text{LEV}'(f, 0)$; therefore it holds $[0, 1] = \text{LEV}'(f, k) = \text{LEV}(f, k) \neq \emptyset$ for all (even transfinite) k .
- (c) From [23, Lemma 2.5.3], it follows that U is disjoint from $\text{LEV}'(f, 1)$, i.e. $\text{LEV}'(f, 1) \subseteq X \setminus U$ a closed set; therefore $\text{LEV}(f, 1) = \overline{\text{LEV}'(f, 1)}$, the least closed set containing $\text{LEV}'(f, 1)$, is a subset of $X \setminus U$. Recall from (b) the example of $\mathbf{1}_\mathbb{Q} : [0, 1] \rightarrow \{0, 1\}$ continuous on \mathbb{Q} , yet \mathbb{Q} is certainly not disjoint from $\text{LEV}(\mathbf{1}_\mathbb{Q}, 1) = \text{LEV}'(\mathbf{1}_\mathbb{Q}, 1) = [0, 1]$.
- (d) proceeds by induction on k , the case $k = 1$ been handled in (c). First observe that $\text{LEV}(f, k + 1) = \text{LEV}(f|_{\text{LEV}(f,k)}, 1)$ since the topological closure implicit on the left hand side coincides with the closure relative to (closed) $\text{LEV}(f, k)$ on the right hand side. Moreover, the induction hypothesis $\text{LEV}(f, k) \subseteq X \setminus (U_1 \cup \dots \cup U_k)$ implies $\text{LEV}(f|_{\text{LEV}(f,k)}, 1) \subseteq \text{LEV}(f|_{X \setminus (U_1 \cup \dots \cup U_k)}, 1)$ by [23, Lemma 2.5.4], which is in turn contained in $(X \setminus (U_1 \cup \dots \cup U_k)) \setminus U_{k+1}$ according to (c). \square

Lemma 61(a + b) indicates that the greedy meta-algorithm underlying the definitions of $\text{Lev}(f)$ and $\text{Lev}'(f)$ yields topologically mild color classes on the one hand, but on the other hand not necessarily the least number of such classes. For the problems in linear algebra considered above, however, greedy is optimal:

Corollary 62. (a) Fix $n, m \in \mathbb{N}$ and recall from **Theorem 46** the problem $\text{LinEq}_{n,m}$ of finding to a given $A \in \mathbb{R}^{n \times m}$ $\text{rank}(A) \leq d := \min(n, m - 1)$ some non-zero $\vec{x} \in \mathbb{R}^m$ such that $A \cdot \vec{x} \neq 0$. It holds

$$\text{Lev}'(\text{LinEq}_{n,m}) = \text{Lev}(\text{LinEq}_{n,m}) = \mathfrak{C}_t(\text{LinEq}_{n,m}) = \mathfrak{C}_c(\text{LinEq}_{n,m}, \rho^{n \times m}, \rho^m; 2-\Sigma_1) = d + 1.$$

(b) Fix $d \in \mathbb{N}$ and recall from **Theorem 47** the problem EVecBase_d of finding to a given real symmetric $d \times d$ -matrix A a basis of eigenvectors. It holds

$$\text{Lev}'(\text{EVecBase}_d) = \text{Lev}(\text{EVecBase}_d) = \mathfrak{C}_t(\text{EVecBase}_d) = \mathfrak{C}_c(\text{EVecBase}_d, \rho^{\binom{d}{2}}, \rho^{d \times d}; 2-\Sigma_1) = d.$$

(c) Fix $n \in \mathbb{N}$ and recall from **Theorem 49** the problem SomeEVec_n of finding, to a given real symmetric $d \times d$ -matrix A , some eigenvector. It holds

$$\text{Lev}'(\text{SomeEVec}_n) = \text{Lev}(\text{SomeEVec}_n) = \mathfrak{C}_t(\text{SomeEVec}_n) = \mathfrak{C}_c(\text{SomeEVec}_n, \rho^{\binom{n}{2}}, \rho^n; 2-\Sigma_1) = \lfloor 1 + \log_2 n \rfloor.$$

More precisely, the class $2-\Sigma_1$ of pairwise differences of open sets above may be replaced by the class $2-\Sigma_1$ of pairwise differences of r.e. open sets, i.e. by the second Hausdorff level on the ground level Σ_1 of the effective Borel Hierarchy.

Proof. (a) **Theorem 46** refers to arbitrary color classes and shows that, there, d -fold advice is insufficient to continuity: $\mathfrak{C}_t(\text{LinEq}_{n,m}) > d$. In view of **Lemma 61(a)** it thus suffices to show $\text{Lev}(\text{LinEq}_{n,m}) \leq d + 1$. Indeed, the set $\text{rank}^{-1}(\geq k)$ of matrices of rank at least k is effectively open a subset of $X := \mathbb{R}^{n \times m}$ because $A \mapsto \text{rank}(A)$ is lower-computable [57, Theorem 7(i)]. In particular, $U_{d-k+1} := V_k := \text{rank}^{-1}(k) = \text{rank}^{-1}(\geq k) \cap \text{rank}^{-1}(\leq k)$ is effectively open in $\text{rank}^{-1}(\leq k) = \text{dom}(\text{LinEq}_{n,m}) \setminus (V_d \cup \dots \cup V_{k+1})$; and $\text{LinEq}_{n,m}$ is computable and continuous thereon by [57, Theorem 11]. Now apply **Lemma 61(d)** to conclude

$$\text{LEV}(\text{LinEq}_{n,m}, d + 1) \subseteq \text{dom}(\text{LinEq}_{n,m}) \setminus (V_d \cup \dots \cup V_0) = \emptyset.$$

(b) Similarly to (a) and in view of **Theorem 47** it suffices to show $\text{Lev}(\text{EVecBase}_d) \leq d$. Now, again, the set $V_k := \{A : \text{Card } \sigma(A) = k\} = \{A : \text{Card } \sigma(A) \geq k\} \cap \{A : \text{Card } \sigma(A) \leq k\}$ of symmetric real $d \times d$ -matrices A with exactly k distinct eigenvalues is effectively open in $\{A : \text{Card } \sigma(A) \leq k\} = \text{dom}(\text{EVecBase}_d) \setminus (V_d \cup \dots \cup V_{k+1})$; because $A \mapsto \text{Card } \sigma(A)$ is lower-computable and lower-continuous [57, Proposition 17]. And EVecBase_d is computable and continuous on V_k by [57, Theorem 19], so **Lemma 61(d)** yields the claim.

(c) Again, in order to show $\text{Lev}(\text{SomeEVec}_n) \leq \lfloor 1 + \log_2 n \rfloor$, consider the sets $U_k := \{A \in \mathbb{R}^{n \times (n-1)/2} : \lfloor \log_2 m(A) \rfloor = k\}$, $k = 0, \dots, \lfloor \log_2 n \rfloor$, on which SomeEVec_n is computable by **Theorem 49**. This time $\{A : \lfloor \log_2 m(A) \rfloor \leq k\}$ (rather than “ $\geq k$ ”) are, according **Observation 55**, effectively open subsets of $\text{dom}(\text{SomeEVec}_n)$. Hence U_k is relatively open in $\{A : \lfloor \log_2 m(A) \rfloor \geq k\} = \mathbb{R}^{n \times (n-1)/2} \setminus (U_0 \cup \dots \cup U_{k-1})$; now apply **Lemma 61(d)**. \square

In the discrete realm, the Church–Turing Hypothesis is generally accepted and bridges the gap between computational practice and formal recursion theory:

every function which would naturally be regarded as computable is computable under his definition, i.e. by one of his (i.e. Turing’s) machines [27, p.376]

In the real number setting, the Type-2 Machine has not attained such universal acceptance—mostly due to its inability to compute any discontinuous function. Hence we propose as a real counterpart to the discrete Church–Turing Hypothesis something along the following lines:

The class of real functions f which would naturally be regarded as computable coincides with those functions computable by a Type-2 Machine with finite discrete advice of color classes in $2-\Sigma_1(\text{dom } f)$.

- Question 63.** (a) In view of *Lemma 61(a)*: What is the Borel complexity of the set of points of discontinuity (which is F_σ for single-valued functions) in the multivalued case?
- (b) Is the rank the (up to permutation) unique least advice rendering LinEq computable/continuous?
- (c) Is the number of distinct eigenvalues the (up to permutation) unique least advice rendering EVecBase computable/continuous?
- (d) More generally, what are sufficient conditions for the sets $\text{LEV}(f, i)$ ($i = 1, \dots, \text{Lev}(f)$) to be the unique least-size partition of $\text{dom}(f)$ into subsets where f is continuous?

Recall that in the proof of *Corollary 62*, we have repeatedly employed *Lemma 61(d)* giving a sufficient condition for the sets $\text{LEV}(f, i)$ to constitute a least-size partition of $\text{dom}(f)$ into subsets where f is continuous.

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Additional extensions and generalizations can be found in [11,2].

Question 63(a) has been answered in [18].

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