Separably closed fields with Hasse derivations

Martin Ziegler

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Abstract

In [6] Messmer and Wood proved quantifier elimination for separably closed fields of finite Ershov invariant e equipped with a (certain) Hasse derivation. We propose a variant of their theory, using a sequence of e commuting Hasse derivations. In contrast to [6] our Hasse derivations are iterative.

1 Introduction

Definition. Let R be a commutative ring. A Hasse derivation is a family $D = (D_0, D_1, \dots)$ of additive maps $D_n : R \to R$ such that[1]

$$D_0(x) = x \tag{1.1}$$

$$D_n(xy) = \sum_{a+b=n} D_a(x) D_b(y) \tag{1.2}$$

$$D_a D_b = \binom{a+b}{a} D_{a+b} \quad . \tag{1.3}$$

Two Hasse derivations D and E commute if $D_m E_n = E_n D_m$ for all m, n.

We fix for the rest of the paper a natural number e and a prime p.

The following notion was introduced by Okugawa in [7]: A *$\mathcal{D}$–field* is a pair $(K, \mathbf{D})$, where K is a field of characteristic p and $\mathbf{D} = (\mathbf{D}_1, \dots, \mathbf{D}_e)$ is a sequence of commuting Hasse derivations on K. The *field of constants*[2] C consists of those elements of K on which all derivations $\mathbf{D}_{i,1}$ $(i = 1, \dots, e)$ vanish. Clearly C contains K^p. $(K, \mathbf{D})$ is a *strict* $\mathcal{D}$–field if $C = K^p$.

Definition. Let L_e be the natural language for $\mathcal{D}$–fields, which contains symbols $\{0, 1, +, -, \cdot\}$ for the field operations and unary function symbols $D_{i,n}$ $(i \in$

[1] Equation (1.3) means that we consider only *iterative* Hasse derivations.

[2] The definition used here differs from the definition given in [7], where the constants are killed by all $\mathbf{D}_{i,j}$ $(j > 0)$

$\{1, \ldots, e\}, n \in \mathbb{N}$). We denote by $\mathrm{SCH}_{p,e}$ the L_e–theory of all separably closed, strict $\mathcal{D}$–fields which have degree of imperfection e.[3]

The aim of this article is to prove the following theorem:

Theorem 1.1.

1. $\mathrm{SCH}_{p,e}$ *is complete and has quantifier elimination.*
2. *Every $\mathcal{D}$–field can be extended to a model of* $\mathrm{SCH}_{p,e}$.
3. *Every separably closed field of degree of imperfection e can be expanded to a model of* $\mathrm{SCH}_{p,e}$.

Our theory is a variant of the theory given by M. Messmer and C. Wood in [6], where a single, non–iterative Hasse derivation was used. For $e = 1$ our two approaches coincide and Theorem 1.1 was proved (slightly differently) in [6].[4]

We will prove the theorem in Section 3. The main algebraic ingredient is the amalgamation property of the class of $\mathcal{D}$–fields, which we prove in Section 2, Proposition 2.6. In Section 4 we give an alternative proof for quantifier elimination.

I thank Anand Pillay for various helpful discussions.

2 Amalgamation

We will prove in this section that the class of $\mathcal{D}$–fields has the *amalgamation property*: Any two extensions of a $\mathcal{D}$–field K can be jointly embedded in a third extension of K.

Lemma 2.1. *For any $\mathcal{D}$–field K the index of its field C of constants is bounded by p^e. Let K' be an extension of K with constant field C'. Then C' and K are linearly disjoint over C.*

Proof. We write d_i for the derivation $\mathbf{D}_{i,1}$ and C_i for its field of constants in K. By reordering $\mathbf{D}$ we may assume that C is the irredundant intersection of the first f of the C_i. So the $B_i = C_1 \cap \cdots \cap C_i$ form a properly descending sequence

$$K = B_0 \supset B_1 \supset \cdots \supset B_f = C.$$

The formula (1.3) implies that $d_i^p = 0$ for all i. Since the d_i commute, each d_i maps B_{i-1} into itself. By Theorem 27.3 of [3] we find elements $x_i \in B_{i-1}$ with $d_i(x_i) = 1$ and for any such choice $1, x_i, \ldots, x_i^{p-1}$ is a basis of B_{i-1} over B_i.

[3] I.e. $[K : K^p] = p^e$

[4] M. Messmer and C. Wood have asked me to point out, that, for $e > 1$, there is a gap in the proof of the main theorem [6], as well as a false claim about the product rule in the non-iterative case.

Whence the $x = x_1^{e_1} \cdots x_f^{e_f}$, $(e_i < p)$ form a basis of K over $B_f = C$. For the same reason these elements form a basis of K' over $B'_f \supset C'$. Thus the x are independent over C'.

□

An alternative proof uses the Wronskian matrix: Let $\theta_1, \ldots, \theta_{p^e}$ be an enumeration of all operators of the form $\mathbf{D}_{1,n_1}\mathbf{D}_{2,n_2}\cdots\mathbf{D}_{e,n_e}$, $(n_i < p)$ (or, equivalently, $d_1^{n_1} d_2^{n_2} \cdots d_e^{n_e}$). It can easily be proved by a standard argument that a sequence $x_1, \ldots, x_N$ is linearly independent over C iff the matrix $(\theta_\alpha(x_\beta))$ has rank N (see [7, Proposition 5.1]). The Lemma follows immediately from this.

Corollary 2.2. *Let K be a strict $\mathcal{D}$–field and F a $\mathcal{D}$–field which extends K. Then F is a separable extension of K. If $[K : K^p] = p^e$, F is strict iff K and F have a common p–basis.*

(See [1] for the definition of p–basis and its basic properties.)

Proof. By the Lemma K and the field C constants of F are linearly disjoint over K^p. This implies that K and F^p are linearly disjoint over K^p. Thus F is separable over K. We have $[K : K^p] \leq [F : C] \leq p^e$. So, if $[K : K^p] = p^e$, we have $[F : C] = p^e$. Therefore $C = F^p$ iff $[F : F^p] = p^e$, which proves the second part. □

Lemma 2.3. *Let $(K, \mathbf{D})$ be a $\mathcal{D}$–field and F a field extension of K. Assume that K and F have a common p–basis. Then $\mathbf{D}$ extends uniquely to a sequence $\mathbf{E}$ of commuting Hasse derivations on F. Furthermore, if $(F', \mathbf{E}')$ is an extension of $(K, \mathbf{D})$ which contains F, the functions in $\mathbf{E}'$ map F into itself, so that $\mathbf{E} = \mathbf{E}' \upharpoonright F$.*

In the special case that F is a separably algebraic extension of K the Lemma is due to F.K. Schmidt ([2]) for $e = 1$ and to Okugawa ([7, Proposition 2.8]) for arbitrary e. We will deduce the general case from a theorem of Matsumara.

Proof. For a single Hasse derivation the lemma follows from the fact that field extensions with a common p–basis are *0–étale* , see [3, 26.7 and 27.2]. So, if $\mathbf{E}$ is a sequence of Hasse derivations of F which extends $\mathbf{D}$, it remains to show that the $\mathbf{E}_i$ commute. Let us prove that $\mathbf{E}_1$ and $\mathbf{E}_2$ commute, i.e. that $\mathbf{E}_{1,i}$ and $\mathbf{E}_{2,j}$ commute for all i, j, by induction on $i + j$. Fix m and n and assume that $\mathbf{E}_{1,i}$ and $\mathbf{E}_{2,j}$ commute for all $i + j < m + n$. It is easy to check (use (1.2)) that then $\mathbf{E}_{1,m}\mathbf{E}_{2,n} - \mathbf{E}_{2,n}\mathbf{E}_{1,m}$ is a derivation. Since $\mathbf{D}_1$ and $\mathbf{D}_2$ commute, this derivation vanishes on K and therefore also on F. □

The uniqueness stated in the Lemma follows also from the following recursive formula, which shows that $\mathbf{D}$ can be computed from its values on a basis[5] of

[5] Even the values on a p–basis would suffice.

F/F^p: Let D be any Hasse derivation. Then (2.2) below implies for $r < p$ that

$$D_{pn+r}(x^p b) = \sum_{m \leq n} D_m(x)^p D_{p(n-m)+r}(b). \tag{2.1}$$

Lemma 2.4. *Any $\mathcal{D}$–field K has a smallest strict extension K^{strict}, which is a purely inseparable extension of K.*

Proof. Consider an arbitrary Hasse derivation D. We note first that (1.2) implies

$$D_n(x^p) = \begin{cases} D_{\frac{n}{p}}(x)^p & \text{if } p|n \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

Also, by (1.3), if $D_1(x) = 0$, we have $D_m(x) = 0$ for all m which are not divisible by p. It follows that

$$D' = (D_0, D_p, D_{2p}, \dots)$$

is a Hasse derivation on the constant field of D_1.[6]

Let C be the constant field of $(K, \mathbf{D})$. Since the $\mathbf{D}_i$ commute, all $\mathbf{D}_{i,n}$ map C into itself. By the last remark $\mathbf{D}' = (\mathbf{D}'_1, \dots, \mathbf{D}'_e)$ is a sequence of commuting Hasse derivations on C. We transport $\mathbf{D}'$ from C to $K^* = C^{\frac{1}{p}}$ via the Frobenius map:

$$\mathbf{D}^*_{i,n}(x) = \mathbf{D}_{pn,i}(x^p)^{\frac{1}{p}}.$$

$\mathbf{D}^*$ extends $\mathbf{D}$ by (2.2). We repeat this process and get an infinite sequence of purely inseparable extensions. The union of this sequence is K^{strict}. □

Note that K^{strict} is separably closed if K is separably closed.

Lemma 2.5. *Let F and L be $\mathcal{D}$–fields which both extend the $\mathcal{D}$–field K. Assume that, in a common field extension, F and L are linearly disjoint over K. Then FL has a unique $\mathcal{D}$–structure which extends the $\mathcal{D}$–structures of F and L.*

Proof. A *$\mathcal{D}$–module* over K is a K–vector space V with a family $\mathbf{D}_{i,n}$ $(i \in \{1, \dots, e\}, n \in \mathbb{N})$ of commuting additive maps $V \to V$ such that for all $D = \mathbf{D}_i$, $x \in K$ and $v \in V$.

$$D_0(v) = v \tag{2.3}$$

$$D_n(xv) = \sum_{a+b=n} D_a(x) D_b(v) \tag{2.4}$$

$$D_a D_b(v) = \binom{a+b}{a} D_{a+b}(v) \quad . \tag{2.5}$$

A commutative K–algebra R is a *$\mathcal{D}$–algebra* if it is a $\mathcal{D}$–module and the $\mathbf{D}_i$ are Hasse derivations on R.

The following statements are easy to check (cf. [4]):

[6] Note that $\binom{n}{i} \equiv \binom{pn}{pi} \mod p$.

- If V and W are $\mathcal{D}$–modules over K, the tensor product $V \otimes_K W$ becomes a $\mathcal{D}$–module by the definition

$$\mathbf{D}_{i,n}(v \otimes w) = \sum_{a+b=n} \mathbf{D}_{i,a}(v) \otimes \mathbf{D}_{i,b}(w).$$

- If R and S are $\mathcal{D}$–algebras over K, their tensor product is also a $\mathcal{D}$–algebra.

If R and S have unit–elements, R and S are subrings of $R \otimes_K S$. It is clear that the $\mathcal{D}$–structure of $R \otimes_K S$ is the only common extension of the $\mathcal{D}$–structures of R and S[7].

If F and L are linearly disjoint over D, FL is the quotient field of $F \otimes_K L$. By [2] and [7, Proposition 2.3] a sequence of commuting Hasse derivations on a domain extends uniquely to the quotient field. This proves the Lemma. □

Proposition 2.6. *The class of $\mathcal{D}$–fields has the amalgamation property.*

Proof. Let F and L be $\mathcal{D}$–fields which both extend the $\mathcal{D}$–field K. If we apply Lemma 2.3 and Lemma 2.4 to the separable algebraic closures of F and L, we see that we may assume that F and L are separably closed and strict. Then $(K^{\mathrm{sep}})^{\mathrm{strict}}$ is a $\mathcal{D}$–subfield of F and L (again by Lemmas 2.3 and 2.4), so we may assume that K is separably closed and strict. We may also assume that F and L are situated in a common extension field and are algebraically independent over K. By Corollary 2.2 F is a separable extension of K and therefore a regular extension, since K is separably closed. This implies that F and L are linearly independent over K and that we can extend the $\mathcal{D}$–structure of F and L to FL. □

3 Proof of the Theorem

1. Quantifier elimination and completeness

To prove that $\mathrm{SCH}_{p,e}$ has quantifier elimination, we have to show that the following is true: If F and L are models of $\mathrm{SCH}_{p,e}$ with a common substructure R, we can embed F over R in an elementary extension of L. Let K be the quotient field of R in F and K' the copy of K in L.

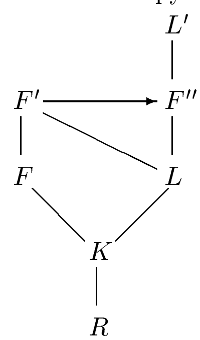
For all Hasse derivations D we have the recursion formula

$$D_n\left(\frac{r}{s}\right) = \frac{D_n(r) - \sum_{a<n} D_a(\frac{r}{s}) D_{n-a}(s)}{s}$$

This shows that K and K' are $\mathcal{D}$–subfields of F and L and are, over R, isomorphic as $\mathcal{D}$–fields. So we can assume that $R = K$.

[7] Note that $D_n(1) = 0$ for any Hasse derivation D and $n > 0$.

By amalgamation we find a $\mathcal{D}$–field F' which extends F and L. We may assume that F' is strict. By Corollary 2.2 F' is a separable extension of L. Since L is separably closed, we can embed F' over L in an elementary extension L' of L (see [1, Claim 2.2]). Let F'' be the copy of F' in L'.



It remains to show, that F'' is a $\mathcal{D}$–subfield of L' which, over L, is isomorphic to F'. But this follows immediately from Lemma 2.3, since F' and L have a common p–basis by Corollary 2.2. Note that the assumption that F is a model of $\mathrm{SCH}_{p,e}$ was not used.

$\mathrm{SCH}_{p,e}$ is complete since it is consistent by part 3 below and since all models contain the trivial $\mathcal{D}$–field $\mathbb{F}_p$.

2. Every $\mathcal{D}$–field is contained in a model

Let F be a $\mathcal{D}$–field and L be any model of $\mathrm{SCH}_{p,e}$. (We will see below that $\mathrm{SCH}_{p,e}$ is consistent.). By the proof of quantifier elimination we can embed F (over $\mathbb{F}_p$) in an elementary extension of L.

3. Every separably closed field with degree of imperfection e can be expanded to model

Let F be a separably closed field of imperfection degree e. Let $b_1, \ldots, b_e$ be a p–basis of F. Then the b_i are algebraically independent over $\mathbb{F}_p$ and form a p–basis of $K = \mathbb{F}_p(b_1, \ldots, b_e)$. Define a sequence of commuting Hasse derivations on K by

$$f(b_1, \ldots, b_i + t, \ldots, b_e) = \sum_{n=0}^{\infty} \mathbf{D}_{i,n}(f(b_1, \ldots, b_e))\, t^n. \tag{3.1}$$

or, equivalently, by

$$\mathbf{D}_{i,n}(b_1^{k_1} \cdots b_e^{k_e}) = \binom{k_i}{n} b_1^{k_1} \cdots b_i^{k_i - n} \cdots b_e^{k_e} \tag{3.2}$$

It is easy to check, and well–known, that this definition turns K into a strict $\mathcal{D}$–field (see [7, Section I.1]). By Lemma 2.3 we can extend $\mathbf{D}$ to F. F is strict since $[F : F^p] = p^e$ (Corollary 2.2). So $(F, \mathbf{D})$ is a model of $\mathrm{SCH}_{p,e}$.

4 Remarks

Stability and elimination of imaginaries

Using the methods of [1] and [6] it is easy to prove that $\mathrm{SCH}_{p,e}$ is stable and has elimination of imaginaries. The stability of $\mathrm{SCH}_{p,e}$ can also be derived directly from the stability of separably closed fields ([8]) as follows: Let F be a separably closed field with p–basis $b_1,\ldots,b_e$ and D a Hasse derivation of F. The formula (2.1) shows that all D_n are definable in the field F using the parameters $D_n(b_1^{k_1}\cdots b_e^{k_e})$.[8] This implies that F together with any sequence of Hasse derivations is stable.

Let me also indicate why $\mathrm{SCH}_{p,e}$ has elimination of imaginaries, following [5]. One notes first, that, working in fields, it suffices to show that $\mathrm{SCH}_{p,e}$ has weak elimination of imaginaries (see [5, Fact 5.5]). By a theorem of Evans, Pillay and Poizat (see [5, Proposition 5.8]) it is enough to show that every type $q(x_1,\ldots,x_m)$ over a model $(F,\mathbf{D})$ has a canonical base. Let $\theta_1,\theta_2,\ldots$ be an enumeration of all operators of the form $\mathbf{D}_{1,n_1}\mathbf{D}_{2,n_2}\cdots\mathbf{D}_{e,n_e}$, $(n_i=0,1,\ldots)$ and let I_q be the ideal of all polynomials $f\in F[X_{\alpha,j}]_{\alpha=1,2,\ldots;\,j=1,\ldots,m}$ such that the formula $f(\theta_\alpha(x_j))\doteq 0$ belongs to q. By quantifier elimination q is determined by I_q. Thus the field of definition of I_q serves as a canonical base of q.

Canonical p-bases

Let $(F,\mathbf{D})$ be a $\mathcal{D}$–field with degree of imperfection e. A p–basis $b_1,\ldots,b_e$ is *canonical* if for all $n>0$

$$\mathbf{D}_{i,n}(b_j)=\begin{cases}1 & \text{if } n=1 \text{ and } i=j\\ 0 & \text{otherwise}\end{cases}. \tag{4.1}$$

Lemma 4.1. *Let F be a field with with degree of imperfection e. Every p–basis of F is a canonical p–basis of a uniquely determined sequence of commuting Hasse derivations.*

Proof. A canonical p–basis $b_1,\ldots,b_e$ determines $\mathbf{D}$ uniquely: $\mathbf{D}_{i,n}(b_1^{k_1}\cdots b_e^{k_e})$ is given by (3.2). To compute $\mathbf{D}_{i,n}(x)$ for arbitrary x, write

$$x=\sum_{0\le k_1,\ldots,k_e<p^m} x_{k_1\ldots k_e}^{p^m} b_1^{k_1}\cdots b_e^{k_e}$$

for some m with $n<p^m$. Then

$$\mathbf{D}_{i,n}(x)=\sum_{0\le k_1,\ldots,k_e<p^m} x_{k_1\ldots k_e}^{p^m}\mathbf{D}_{i,n}(b_1^{k_1}\cdots b_e^{k_e}). \tag{4.2}$$

Now let $b_1,\ldots,b_e$ be any p–basis. The construction at the end of the last section shows that (3.2) and (4.2) define a sequence $\mathbf{D}$ of commuting Hasse derivations with canonical p–basis $b_1,\ldots,b_e$. □

[8] Actually the parameters b_i and $D_{p^m}(b_i)$ suffice.

The $\mathbf{D}$ constructed in in the last part of the proof is strict. So we conclude, that only a strict sequence $\mathbf{D}$ can have a canonical p–basis. The converse is true if $(F, \mathbf{D})$ is ω-saturated:

Remark. *Every ω–saturated strict $\mathcal{D}$–field has a canonical p–basis.*

I will give the proof only in the following special case, which will be used later.

Corollary 4.2. *Every ω–saturated model of* $\mathrm{SCH}_{p,e}$ *has a canonical p–basis.*

Proof. To have a canonical p–basis means that a certain countable set $\Sigma(x_1, \ldots, x_e)$ of formulas is realized. Since $\mathrm{SCH}_{p,e}$ is complete, it is enough to show that *some* model of $\mathrm{SCH}_{p,e}$ has a canonical p–basis. For this take a separably closed field F of imperfection degree e and fix a p–basis $\bar{b}$. Let $\mathbf{D}$ be the unique sequence which has $\bar{b}$ as a canonical p–basis. $(F, \mathbf{D})$ is a model of $\mathrm{SCH}_{p,e}$. □

Lemma 4.1 and the last remark allow us to determine all strict sequences of commuting Hasse derivations of an ω–saturated field F. We note first that, if $b_1, \ldots, b_e$ is a canonical p–basis for $\mathbf{D}$, then $b'_1, \ldots, b'_e$ is a canonical p–basis for $\mathbf{D}$ iff the differences $b_i - b'_i$ belong to

$$F^{p^\infty} = \bigcap_{k=1}^{\infty} F^{p^k} = \big\{a \in F \mid \mathbf{D}_{i,n}(x) = 0 \;\; (i = 1, \ldots, e\,;\; n = 1, 2, \ldots)\big\}.$$

This gives

Remark. *Let F be an ω–saturated field with degree of imperfection e. There is a natural 1–1–correspondence between the set of all strict sequences of commuting Hasse derivations and the set of all p–bases modulo F^{p^∞}.*

Lambda functions

Let $b_1, \ldots, b_e$ be a p–basis of F. The functions $\lambda^m_{k_1 \ldots k_e}$ are defined by

$$x = \sum_{0 \le k_1, \ldots, k_e < p^m} \lambda^m_{k_1 \ldots k_e}(x)^{p^m}\, b_1^{k_1} \cdots b_e^{k_e}$$

Fix a natural number m. For a multi-index $\kappa = (k_1, \ldots, k_e) \in \{0, \ldots, p^m - 1\}^e$ and a sequence $\mathbf{D}$ of Hasse derivations let us use the notations

$$b^\kappa = b_1^{k_1} \cdots b_e^{k_e} \qquad \text{and} \qquad \mathbf{D}_\kappa = \mathbf{D}_{1,k_1} \mathbf{D}_{2,k_2} \cdots \mathbf{D}_{e,k_e}.$$

If we apply $\mathbf{D}_\kappa$ to the equation

$$x = \sum_\mu \lambda^m_\mu(x)^{p^m}\, b^\mu,$$

we obtain

$$\mathbf{D}_\kappa(x) = \sum_\mu \lambda_\mu^m(x)^{p^m}\,\mathbf{D}_\kappa(b^\mu).$$

If $\mathbf{D}$ is strict, the Wronski matrix $\big(\mathbf{D}_\kappa(b^\mu)\big)$ is always regular. If $b_1,\ldots,b_e$ is canonical for $\mathbf{D}$, its entries are, up to factors from $\mathbb{F}_p$, monomials in the b_i. It is also easy to see that the determinant is 1. This yields

Lemma 4.3. *Let $(F,\mathbf{D})$ be a $\mathcal{D}$–field with canonical p–basis $b_1,\ldots,b_e$. Then the functions $(\lambda_\mu^m(x))^{p^m}$ are polynomials in $b_1,\ldots,b_e$ and the $\mathbf{D}_\kappa(x)$.* □

Quantifier elimination

Let $\mathrm{T}_{p,e}$ denote the theory of separably closed fields F of characteristic p with a named p–basis $b_1,\ldots,b_e$. It is shown in [1] that $\mathrm{T}_{p,e}$ is complete and has quantifier elimination if one adds function symbols for the λ_μ^m to the language.[9]

This fact can be used to give an alternative proof for the quantifier elimination of $\mathrm{SCH}_{p,e}$: Let $\phi(\bar{x})$ be an L_e–formula and $(F,\mathbf{D})$ a saturated model of $\mathrm{SCH}_{p,e}$. By Corollary 4.2 we can find a canonical p–basis $b_1,\ldots,b_e$. Since we can define the $\mathbf{D}_{i,n}$ in $(F,b_1,\ldots,b_e)$, $\phi(\bar{x})$ is equivalent to a Boolean combination of polynomial equations between $b_1,\ldots,b_e$ and terms of the form $\lambda_\mu^m(x_i)$, for sufficiently large m. By taking p^m-th powers we can replace the $\lambda_\mu^m(x_i)$ by $\lambda_\mu^m(x_i)^{p^m}$. By the last lemma we obtain an equivalent Boolean combination of equations of the form

$$\sum_\kappa q_\kappa(\bar{x})\,b^\kappa \doteq 0 \qquad (4.3)$$

where the $q_\kappa(\bar{x})$ are terms in the $\mathbf{D}_\kappa(x_i)$. The equivalence holds for any choice of the canonical p–basis $b_1,\ldots,b_e$. Since F^{p^∞} is infinite, we can find the $b_1,\ldots,b_e$ algebraically independent over any given tuple $\bar{x}$. This shows that we can replace each equation (4.3) by $\bigwedge_\kappa q_\kappa(\bar{x}) \doteq 0$. We observe finally that the resulting quantifier free L_e–formula does not depend on the choice of F.

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Mathematisches Institut
Eckerstrasse 1
79104 Freiburg, Germany

ziegler@uni-freiburg.de