Raising to powers in algebraically closed fields

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20 November 2003

Abstract

We study structures on the fields of characteristic zero obtained by introducing (multivalued) operations of raising to power. Using Hrushovski-Fraisse construction we single out among the structures exponentially-algebraically closed once and prove, under certain Diophantine conjecture, that the first order theory of such structures is model complete and every its completion is superstable.

Keywords: Superstable, fields, Schanuel conjecture

2000 Mathematics Subject Classification: 03C45, 03C98

1 Introduction

This paper deals with some of the issues discussed in [Z1] and is part of the program of applying ideas around Hrushovski's construction of 'new strongly minimal structures' for understanding classical analytic structures.

We consider here the class of two-sorted structures of the form (D, ex, R) where D (the domain of ex) is an infinite-dimensional vector space over a fixed field K of characteristic zero in the usual language of vector spaces, R (the range) a field of characteristic zero and ex is a homomorphism of the additive group of D onto the multiplicative group $R^\times$ of the field.

In these structures, for $a \in K$, one can consider the relation

$$\exists z(x = \mathrm{ex}(z) \& y = \mathrm{ex}(a \cdot z))$$

which in the case of $D = R = \mathbb{C}$ and ex = exp is represented locally by a transcendental analytic function $y = x^a$. Also, in the structures where the kernel ker of ex is an infinite cyclic group one may consider definable finitely generated groups of the form $a_1 \cdot \ker + \ldots + a_n \cdot \ker$ for $a_1, \ldots, a_n \in K$. At the same time $\mathrm{ex}(a_1 \cdot \ker + \ldots + a_n \cdot \ker)$ is a finitely generated multiplicative subgroup in the field R. Thus the structures carry some interesting Diophantine geometry.

We introduce a predimension δ for finite subsets $X \subseteq D$:

$$\delta(X) = \mathrm{l.d.}_{\mathrm{K}}(X) + \mathrm{tr.d.}(\mathrm{ex}(X)) - \mathrm{l.d.}_{\mathrm{Q}}(X)$$

where $\mathrm{l.d.}_{\mathrm{K}}(X)$ is the dimension of the vector space over K generated by X, $\mathrm{l.d.}_{\mathrm{Q}}(X)$ is the dimension of the vector space over $\mathbb{Q}$ generated by X, tr.d. the transcendence degree.

Given a non-negative integer d, we consider the subclass $\mathcal{E}_d$ of the class defined by the condition

$$\delta(X) \geq -d \ \text{ for any finite } X \subseteq D. \tag{1}$$

The class is always non-empty. The condition (1) is satisfied for the complex numbers (as D and R) and ex = exp if K is a subfield of $\mathbb{C}$ of a finite transcendence degree d and the Schanuel Conjecture holds.

This class proves to have a very nice model theory provided a number-theoretical conjecture on intersections of varieties with tori holds. In the terminology of [Z2] a **basic torus** is an algebraic subgroup of the multiplicative group $(R^\times)^n$ (virtually given by a set of equations of the form $y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} = 1$ with integer powers) and a **torus** is a coset of a basic torus. Notice also that $(R^\times)^n$ is a torus itself, so we say that a torus $T \subseteq (R^\times)^n$ is a proper subtorus, if $T \neq (R^\times)^n$.

The conjecture **CIT** states:

Let $W \subseteq R^n$ be a $\mathbb{Q}$-definable algebraic variety irreducible over $\mathbb{Q}$. Then there is a finite family $\tau(W)$ of proper subtori of $(R^\times)^n$ such that for any basic torus $T \subseteq (R^\times)^n$ and any irreducible component S of the intersection $W \cap T$ satisfying

$$\dim S > \dim W + \dim T - n$$

there is $T_i \in \tau(W)$ *with* $S \subseteq T_i$.

The **Schanuel conjecture** states that for any additively independent complex numbers $x_1, \ldots, x_n$

$$\text{tr.d.}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq n.$$

In [Z2] we formulate and discuss connections between a stronger **Uniform Schanuel conjecture** and CIT.

Under the assumption that CIT holds we prove that
(i) the class $\mathcal{E}_d$ is axiomatizable;
(ii) the subclass $\mathcal{EC}_d$ of $\mathcal{E}_d$-existentially closed structures is the model completion of $\mathcal{E}_d$ in the existential expansion of the language, its theory allows elimination of quantifiers in the expanded language and any completion of the theory is superstable.

This allows us to study the classical structure, the field of the complex numbers with raising to real powers. It corresponds to the case $D = R = \mathbb{C}$, ex = exp and $K = \mathbb{R}$. Using [Z2] (which is based on works of D.Bernstein, A.Kushnirenko, A.Khovanski and B.Kazarnovski) we give a complete set of axioms for the structure and prove that it is superstable and allows elimination of quantifiers to the level of existential formulas, provided the Schanuel Conjecture along with CIT hold. In fact the Uniform Schanuel conjecture is sufficient.

The author is thankful to the referee of the paper for suggesting many useful improvements.

2 Definitions and notation

This section along with definitions and notations discusses basic ingredients of Hrushovski's construction which is standard enough, so the reader can guess the proofs if they seem too short or are absent.

We use here some of the terminology of [Z2], slightly improved, where we discussed K-linear and affine spaces, tori and their intersections with algebraic varieties.

For technical reasons we find it more convenient to represent the two-sorted structures (D, R) in the equivalent way as one sorted structures in the language $\mathcal{L}_K$ which is the extension of the language of vector spaces over $\mathbb{Q}$ by:

an equivalence relation E,

n-ary predicates $L(x_1, \dots, x_n)$ for linear subspaces $L \subseteq D^n$ given by a set of K-linear equations in $x_1, \dots, x_n$,

n-ary predicates EW for algebraic varieties $W \subseteq R^n$ definable and irreducible over $\mathbb{Q}$.

The interpretation can be explained in the above mentioned terms as follows:

$E(x, y) \equiv [\mathrm{ex}(x) = \mathrm{ex}(y)]$,

$L(x_1, \dots, x_n) \equiv [\langle x_1, \dots, x_n \rangle \in L]$,

$EW(x_1, \dots, x_n) \equiv [\langle \mathrm{ex}(x_1), \dots, \mathrm{ex}(x_n) \rangle \in W]$.

Definition $\mathcal{E}(K)$ is the class of structures D in language $\mathcal{L}_K$ with axioms saying that D is an infinite-dimensional vector space over K, E is an equivalence relation on D which is congruent with respect to the relations $EW(x_1, \dots, x_n)$, $R^\times = D/E$ can be identified with the multiplicative group of a field of characteristic zero and the predicates EW define its algebraic varieties over $\mathbb{Q}$. The canonical mapping

$$\mathrm{ex} : D \to R^\times$$

is a homomorphism of the additive group of D into the multiplicative group $R^\times$ of the field.
The set of axioms above we denote $\mathrm{PF}(K)$ (K-powered field of characteristic zero).

Notation For finite $X, X' \subseteq D$, $Y, Y' \subseteq R$

$\mathrm{l.d.}_{\mathrm{K}}(X)$ the dimension of the vector space $sp_K(X)$ generated by X over K;

$\mathrm{l.d.}_{\mathrm{Q}}(X)$ the dimension of the vector space $sp_{\mathbb{Q}}(X)$ generated by X over $\mathbb{Q}$;

$\mathrm{tr.d.}(Y)$ the transcendence degree of Y;

$\delta(X)$ **the predimension** of finite $X \subseteq D$:

$$\delta(X) = \mathrm{l.d.}_{\mathrm{K}}(X) + \mathrm{tr.d.}(\mathrm{ex}(X)) - \mathrm{l.d.}_{\mathrm{Q}}(X);$$

$\delta(X/X') = \delta(X \cup X') - \delta(X')$;

For infinite $Z \subseteq A$ and $k \in \mathbb{Z}$ $\delta(X/Z) \geq k$ by definition means that for any $Y \subseteq_{fin} Z$ there is $Y \subseteq_{fin} Y' \subseteq Z$ such that $\delta(X/Y') \geq k$, and $\delta(X/Z) = k$ means $\delta(X/Z) \geq k$ and not $\delta(X/Z) \geq k+1$.

We let also
$\text{l.d.}_{\text{K}}(X/X') = \text{l.d.}_{\text{K}}(X \cup X') - \text{l.d.}_{\text{K}}(X')$;
$\text{tr.d.}(Y/Y') = \text{tr.d.}(Y \cup Y') - \text{tr.d.}(Y')$;
$\text{l.d.}_{\text{Q}}(X/X') = \text{l.d.}_{\text{Q}}(X \cup X') - \text{l.d.}_{\text{Q}}(X')$;
ker is the name of a unary predicate of type EW : $x \in \ker \equiv \text{ex}(x) = 1$. We write $\ker_{|A}$ for the realisation of this predicate in A.

Given $d \in \mathbb{Z}$ denote $\mathcal{E}_d(K)$ the subclass of $\mathcal{E}(K)$ consisting of all D satisfying the condition:

$$\delta(X) \geq -d \text{ for all finite } X \subseteq D.$$

Below we fix K and write simply $\mathcal{E}_d$ instead of $\mathcal{E}_d(K)$.

Denote $\mathcal{SE}$ the class of the substructures of the structures of $\mathcal{E}$ in the language $\mathcal{L}_K$.

Given an integer d denote $\mathcal{SE}_d$ the subclass of $\mathcal{SE}$ consisting of A which satisfy $\delta(X) \geq -d$ for any finite $X \subseteq A$.

Remark For any structure A in $\mathcal{E}$ and any $X \subseteq \ker_{|A}$ in the structure

$$\delta(X) \leq 0$$

and thus $\mathcal{E}_d$ is empty for $d < 0$.

Notation Denote $\mathcal{E}^0$ (correspondingly $\mathcal{SE}^0$) the subclass of $\mathcal{E}$ ($\mathcal{SE}$) consisting of the structures A such that

$$\delta(X) = 0$$

holds for any $X \subseteq \ker_{|A}$.

Remark Evidently $\mathcal{E}_0 \subseteq \mathcal{E}^0$.

Notation Denote $\mathcal{E}^0_d = \mathcal{E}^0 \cap \mathcal{E}_d$, $\mathcal{SE}^0_d = \mathcal{SE}_d \cap \mathcal{SE}^0$.

Definition A subspace $L \subseteq D^n$ is said to be K-**linear** if there are $k_{i,j} \in K \quad (i \leq r,\ j \leq n)$ such that

$$L = \{\langle x_1, \ldots, x_n \rangle \in D^n : k_{i,1}x_1 + \ldots + k_{i,n}x_n = 0\}.$$

Define $\dim L$ to be the corank of the matrix $(k_{i,j})$, equivalently, the Morley rank of the definable subset L of the vector space D.
Let $L \subseteq D^{n+l}$ be a K-linear subspace, $\bar{a} = \langle a_1, \ldots, a_l \rangle$. Denote

$$L(\bar{a}) = \{\langle x_1, \ldots, x_n \rangle \in D^n : \langle x_1, \ldots, x_n, a_1, \ldots, a_l \rangle \in L\}.$$

An affine subspace $V \subseteq D^n$ is said to be K-**affine defined over the set** $C \subseteq D$ if $V = L(\bar{c})$ for some K-linear subspace $L \subseteq D^{n+l}$ and $\bar{c} \in C^l$.
The same terminology is applied for $\mathbb{Q}$ instead of K.

For $L \subseteq D^n \quad K$-linear denote $\bar{L}$ the minimal $\mathbb{Q}$-linear subspace of D^n containing L.
For $W \subseteq R^{n+l}$ an algebraic variety, $\bar{b} = \langle b_1, \ldots, b_l \rangle$ denote

$$W(\bar{b}) = \{\langle x_1, \ldots, x_n \rangle \in R^n : \langle x_1, \ldots, x_n, b_1, \ldots, b_l \rangle \in W\}.$$

Remark A K-affine subspace is defined over a set C iff
$V = L_0 + \bar{a}$ for some $\bar{a} \in D^n \cap sp_K(C)$ and $L_0 \quad K$-linear.

Lemma 2.1 *If* $X = \{x_1, \ldots x_{n+l}\} \subseteq D, \quad X' = \{x_{n+1}, \ldots x_{n+l}\}$, $\bar{x} = \langle x_1, \ldots x_{n+l} \rangle, \quad \bar{x}' = \langle x_{n+1}, \ldots x_{n+l} \rangle$ *then:*
$\mathrm{l.d.}_{\mathrm{K}}(X) = \dim L$, *for* $L \subseteq D^{n+l}$ *the minimal* K*-linear subspace containing* $\bar{x}$;
$\mathrm{l.d.}_{\mathrm{Q}}(X) = \dim \bar{L}$;
$\mathrm{tr.d.}(\mathrm{ex}(X))$ *is the dimension of the minimal variety over* $\mathbb{Q}$ *containing* $\mathrm{ex}(\bar{x})$;
$\delta(X/X') = \mathrm{l.d.}_{\mathrm{K}}(X/X') + \mathrm{tr.d.}(\mathrm{ex}(X)/\mathrm{ex}(X')) - \mathrm{l.d.}_{\mathrm{Q}}(X/X')$;
$\mathrm{l.d.}_{\mathrm{K}}(X/X') = \dim L(0^l)$, *where* 0^l *is a string of* l *zeroes;*
$\mathrm{l.d.}_{\mathrm{Q}}(X/X') = \dim \bar{L}(0^l)$.

Proof Immediate from the definitions.□

Notation For $A, B \in \mathcal{SE}$ denote by $A \leq B$ the fact that $A \subseteq B$ as structures and $\delta(X/A) \geq 0$ for all finite $X \subseteq B$.

Lemma 2.2 *For any structure A of the class $\mathcal{SE}$ and finite $X, Y, Z \subseteq A$:*

(i) If $\mathrm{sp}_Q(X') = \mathrm{sp}_Q(X)$ *then* $\delta(X') = \delta(X)$.

(ii) If $\mathrm{sp}_Q(X'Y) = \mathrm{sp}_Q(XY)$ *then* $\delta(X/Y) = \delta(X'/Y)$.

(ii) If $\mathrm{sp}_Q(Y) = \mathrm{sp}_Q(Y')$ *then* $\delta(X/Y) = \delta(X/Y')$.

(iv) $\delta(XY/Z) = \delta(X/YZ) + \delta(Y/Z)$.

Lemma 2.3 *For $A, B, C \in \mathcal{SE}$*

(i) if $A \leq B$ and $B \leq C$, then $A \leq C$;

(ii) if $A \leq B$, $Y \subseteq B$, $\delta(Y/A) = 0$, then $AY \leq B$.

Proof Immediate from the definitions.□

Notation Let $A \in \mathcal{SE}_d$ and $X \subseteq A$ finite. Denote

$$\partial_A(X) = \min\{\delta(X') : X \subseteq X' \subseteq A\}.$$

Lemma 2.4 *Let $A \in \mathcal{SE}_d$ and $X \subseteq A$ finite. Choose $X' \subseteq A$ finite such that*

$$\delta(X') = \partial_A(X).$$

Then $X' \leq A$.

Proof Immediate from the definitions.□

Lemma 2.5 *Let $A, B \in \mathcal{SE}_d$, $A \leq B$ and X a finite subset of A. Then*

$$\partial_A(X) = \partial_B(X).$$

Proof Immediate from the definitions.□

Lemma 2.6 *Suppose $A \in \mathcal{SE}_d$, $A' \in \mathcal{SE}$, $A' = \mathrm{sp}_Q(AX)$, and $\delta(X'/A) \geq 0$ for all finite $X' \subseteq \mathrm{sp}_Q X$. Then $A' \in \mathcal{SE}_d$ and $A \leq A'$.*

Proof We may assume that X is $\mathbb{Q}$-linearly independent over A. Let $Z \subseteq A'$, $Z = \{z_1, \ldots z_n\}$, and $z_i = x_i + y_i$ for some $x_i \in \mathrm{sp}_Q(X)$,

$y_i \in A$. Let $\{x_1, \ldots x_k\}$ be a $\mathbb{Q}$-linear base of $\{x_1, \ldots x_n\}$. Then, using Lemma 2.2,

$$\delta(Z) = \delta(x_1 + y_1, \ldots x_k + y_k, y'_{k+1}, \ldots, y'_n)$$

for $y'_{k+1}, \ldots, y'_n$ appropriate $\mathbb{Q}$-linear combinations of $y_1, \ldots y_n$.

Rewrite

$$\delta(Z) = \delta(\{x_1 + y_1, \ldots x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) + \delta(y'_{k+1}, \ldots, y'_n).$$

By the assumtions $\delta(y'_{k+1}, \ldots, y'_n) \geq -d$. On the other hand

$\delta(\{x_1 + y_1, \ldots x_k + y_k\}/\{y'_{k+1}, \ldots, y'_n\}) \geq \delta(\{x_1, \ldots x_k\}/A) \geq 0$

since

$\mathrm{l.d.}_{\mathrm{K}}(\{x_1+y_1, \ldots x_k+y_k\}/\{y'_{k+1}, \ldots, y'_n\}) \geq \mathrm{l.d.}_{\mathrm{K}}(\{x_1+y_1, \ldots x_k+y_k\}/A) \geq \mathrm{l.d.}_{\mathrm{K}}(\{x_1, \ldots x_k\}/A)$,

$\mathrm{tr.d.}(\mathrm{ex}\{x_1 + y_1, \ldots x_k + y_k\}/\mathrm{ex}\{y'_{k+1}, \ldots, y'_n\}) \geq \mathrm{tr.d.}(\mathrm{ex}\{x_1 + y_1, \ldots x_k + y_k\}/\mathrm{ex}A) \geq \mathrm{tr.d.}(\mathrm{ex}\{x_1, \ldots x_k\}/\mathrm{ex}A)$

and

$\mathrm{l.d.}_{\mathrm{Q}}(\{x_1+y_1, \ldots x_k+y_k\}/\{y'_{k+1}, \ldots, y'_n\}) = k = \mathrm{l.d.}_{\mathrm{Q}}(\{x_1, \ldots x_k\}/A)$.

Thus

$$\delta(Z) \geq -d.$$

The same argument shows that

$$\delta(Z/A) \geq 0.$$

□

Lemma 2.7 *There is an $A \in \mathcal{SE}_d$.*

Proof Take an additive subgroup $A = \omega \cdot \mathbb{Q} \subseteq D$ for ω a non-zero element in D. Define $H = A/\omega\mathbb{Z}$. Then H, considered as a multiplicative group, is characterized by the property that it is a torsion group such that any equations of the form $x^n = h$ has for any h exactly n solutions. In other words H is isomorphic to the torsion subgroup of an algebraically closed field R of characteristic 0. Define ex as the canonical homomorphism $A \to R^\times$ corresponding to this isomorphism. Obviously, $\delta(X) = 0$ for any finite $X \subseteq A$.□

Lemma 2.8 *Suppose $A \in \mathcal{SE}_d$ and* $\mathrm{ex}(A)$ *contains the torsion subgroup of the field. Then there is $D \in \mathcal{E}_d$ and an embedding of A into D such that $A \leq D$ and* $\ker_{|D} = \ker_{|A}$.

Proof Choose algebraically closed fields D and R of characteristic zero such that $A \subseteq D$, $A/E \subseteq R^\times$ and $\mathrm{l.d.}_{\mathrm{K}}(D/A) = \mathrm{tr.d.}(R/\mathrm{ex}A) \geq \aleph_0$. We want to define $\mathrm{ex} : D \to R^\times$ extending ex_A so that $\mathbf{D} \in \mathcal{E}_d$.

Denote $A_0 = A$, $\mathrm{ex}_0 = \mathrm{ex}_A$ and $H_0 = \mathrm{ex}_0(A_0)$.

Proceed by induction defining A_α, H_α and an endomorphism

$$\mathrm{ex}_\alpha : A_\alpha \to H_\alpha$$

by choosing:

On the even steps: the first element $a \in D \setminus A_\alpha$ and define $\mathrm{ex}_{\alpha+1}(a)$ to be any element in $R^\times \setminus acl(H_\alpha)$. Put $A_{\alpha+1} = A_\alpha + \mathbb{Q} \cdot a$ and extend $\mathrm{ex}_{\alpha+1}$ to $A_{\alpha+1}$ as a group homomorphism. Put $H_{\alpha+1} = \mathrm{ex}_{\alpha+1}(A_{\alpha+1})$.

On the odd steps: the first element $h \in R^\times \setminus H_\alpha$ and define a to be any element in $D \setminus \mathrm{sp}_K(A_\alpha)$ and $\mathrm{ex}_{\alpha+1}(a) = h$. Put $A_{\alpha+1} = A_\alpha + \mathbb{Q} \cdot a$ and extend $\mathrm{ex}_{\alpha+1}$ to $A_{\alpha+1}$ as a group homomorphism. Put $H_{\alpha+1} = \mathrm{ex}_{\alpha+1}(A_{\alpha+1})$.

On both even and odd steps it follows from Lemma 2.6 that $A_{\alpha+1} \in \mathcal{SE}_d$ and $A_\alpha \leq A_{\alpha+1}$.

Also,

$$\ker_{|A_{\alpha+1}} = \ker_{|A_\alpha}$$

since if $\mathrm{ex}(qa + a') = 1$ for some rational $q = \frac{m}{n}$ and $a' \in A_\alpha$ then $h^m = g^n$ for $h = \mathrm{ex}(a)$, $g = \mathrm{ex}(a') \in H_\alpha$. Since by assumptions H_α contains a root of degree m of g^n, and $h \notin H_\alpha$, only $q = 0$ is possible. $\square$

3 Exponentially-algebraically closed structures

Definition A structure $\mathbf{D}$ in $\mathcal{E}_d^0$ is said to be **$\mathcal{E}_d^0$-exponentially-algebraically closed** (e.a.c.) if for any $\mathbf{D}' \in \mathcal{E}_d^0$, such that $\mathbf{D} \leq \mathbf{D}'$, any finite quantifier-free type over $\mathbf{D}$ which is realized in $\mathbf{D}'$ has a realization in $\mathbf{D}$.

Denote $\mathcal{EC}_d^0$ the class of $\mathcal{E}_d^0$-exponentially-algebraically closed structures, or, in the shorter form, $\mathcal{EC}$.

It follows from the transitivity of $\leq$-embedding and the inductiveness of the class $\mathcal{E}_d$ in the standard way

Proposition 1 *For any* $\mathbf{D}$ *in* $\mathcal{E}_d^0$ *there exists* $\mathcal{E}_d^0$*-e.a.c. structure containing* $\mathbf{D}$.

Below $\mathbf{D}$ is always an $\mathcal{E}_d^0$-exponentially-algebraically closed structure.

By Lemma 2.5 we may omit D when writing ∂_D.

Notation $\mathrm{cl}(A) = \{b : \partial(Ab) = \partial(A)\}$

Lemma 3.1 *The operator* $A \mapsto \mathrm{cl}$ *in* $\mathbf{D}$ *is a closure operator, i.e. it satisfies*

(i) $A \subseteq B$ *implies* $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$;

(ii) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;

(ii) For any $b, c \in D: \quad b \in \mathrm{cl}(A, c) \setminus \mathrm{cl}(A) \to c \in \mathrm{cl}(A, b)$.

Proof Standard.□

We want to find out now what are the systems of equations and inequalities that have solutions in any e.a.c.-structure.

Definition For $C \subseteq D$, an K-affine variety $V \subseteq D^n$ and an algebraic variety $W \subseteq R^n$ it is said that the **pair** (V, W) **is definable over** C if V is definable over $sp_Q(C)$ and the variety W is definable over the field $\mathbb{Q}(\mathrm{ex}(sp_Q(C)))$ (we often say 'defined over $\mathrm{ex}(sp_Q(C))$).

If W is irreducible over the corresponding set, then the pair is said to be **irreducible over** C.

V is said to be **free of additive dependencies over** C if there is no proper $\mathbb{Q}$-affine subspace of D^n containing V.

W is said to be **free of multiplicative dependencies over** C if no connected component of W lies in a proper subtorus of $(R^\times)^n$.

A pair (V, W) is said to be **a free pair** if both V is free of additive dependencies and W is free of multiplicative dependencies.

Let $W \subseteq R^n$ be an algebraic variety defined and irreducible over some $\mathrm{ex}(C)$ for some $C = sp_Q(C) \subseteq D$. A pair (V, W) is said to be a **normal pair over** C if in some extension of $\mathbf{D}$ there are $\langle a_1, \ldots, a_n \rangle \in V$ and $\langle b_1, \ldots, b_n \rangle \in W$ such that for any $k \leq n$ independent integer vectors $m_i = \langle m_{i,1}, \ldots m_{i,n} \rangle$, $i = 1, \ldots, k$, and

$$a'_i = m_{i,1}a_1 + \ldots + m_{i,n}a_n, \quad b'_i = b_1^{m_{i,1}} \cdot \ldots \cdot b_n^{m_{i,n}}$$

we have

$$\mathrm{l.d.}_{\mathrm{K}}(\langle a'_1, \ldots, a'_k \rangle / C) + \mathrm{tr.d.}(\langle b'_1, \ldots, b'_k \rangle / \mathrm{ex}(C)) \geq k.$$

Lemma 3.2 *Let $C, A \in \mathcal{SE}$ finite, $C \leq A$, $\bar{c}$ be the string of all elements of C and $\bar{a}$ be the string of elements of A. Let L be the minimal K-linear space containing $\bar{a}\bar{c}$ and W the minimal algebraic variety over $\mathbb{Q}$ containing $\mathrm{ex}(\bar{a}\bar{c})$. Then the pair $(L(\bar{c}), W(\mathrm{ex}(\bar{c})))$ is normal.*

Proof Take $\bar{a}$ for $\langle a_1, \ldots, a_n \rangle$ and $\mathrm{ex}(\bar{a})$ for $\langle b_1, \ldots, b_n \rangle$ in the definition of normality. Then $C \leq A$ implies the inequalities required in the definition. $\square$

To formulate an equivalent definition of normality we introduce the following:

Notation Let $V \subseteq D^n$ be an affine K-space defined with parameters $\bar{c}$. Choose a generic n-tuple $\bar{a}$ in the space. Given a matrix $\bar{m}$ of integer vectors $m_i = \langle m_{i,1}, \ldots m_{i,n} \rangle$, $i = 1, \ldots, k$, consider $a'_i = m_{i,1}a_1 + \ldots + m_{i,n}a_n$ and denote $\bar{m}V$ the minimal K-affine subspace over $\bar{c}$ containing $\langle a'_1, \ldots, a'_k \rangle$, (the K-locus over $\bar{c}$).

Similarly, for an algebraic variety $W \subseteq R^n$ defined over $\bar{d}$ and the same $\bar{m}$ choose a generic n-tuple $\bar{b}$ in W, consider $b'_i = b_1^{m_{i,1}} \cdot \ldots \cdot b_n^{m_{i,n}}$ and denote $W^{\bar{m}}$ the algebraic locus over $\bar{d}$ of $\langle b'_1, \ldots, b'_k \rangle$.

Evidently, the definitions do not depend on the choice of the generic tuples.

Lemma 3.3 *The pair (V, W) is normal if and only if for any independent integer vectors $m_i = \langle m_{i,1}, \ldots m_{i,n} \rangle$, $i = 1, \ldots, k$*

$$\dim(\bar{m}V) + \dim(W^{\bar{m}}) \geq k.$$

Proof Immediate from the definitions.□

Lemma 3.4 *Let $C \subseteq D$ and (V,W) a normal free irreducible pair over C. Let $V' \subseteq V$ be a finite union of proper K-affine subspaces definable over C and $W' \subseteq W$ a proper algebraic $\mathrm{ex}(C)$-definable subvariety. Then there is $\bar{a}$ in $\mathbf{D}$ such that $\bar{a} \in V \setminus V'$ and $\mathrm{ex}(\bar{a}) \in W \setminus W'$. Moreover in some extension $\mathbf{D}' \geq \mathbf{D}$ $\bar{a}$ can be chosen generic in V over C and $\mathrm{ex}(\bar{a})$ generic in W over $\mathrm{ex}(C)$.*

Proof Take $\bar{a}$ in some extension of D to be generic in V over D and $\bar{b}$ in some extension of R generic in W over R. Choose a sequence $\{\bar{b}^{\frac{1}{l}} : l \in \mathbb{N}\}$ associated with $\bar{b}$ in the following sense:

$\bar{b}^1 = \bar{b}$ and $(\bar{b}^{\frac{1}{ml}})^m = \bar{b}^{\frac{1}{l}}$ for any $m, l \in \mathbb{N}$, and $(\bar{x})^m$ is understood coordinatewise.

Define ex on $A = D + \mathrm{sp}_Q(a_1, \dots a_n)$ as:

$$\mathrm{ex}(\sum_i \frac{m_i}{l} a_i + d) = \prod_i (b_i^{\frac{1}{l}})^{m_i} \cdot \mathrm{ex}(d)$$

for any integers m_i, $l \neq 0$ and element $d \in D$. The definition is consistent since V is free of additive dependencies. Evidently the formula defines a homomorphism. The kernel of the homomorphism coincides with that of ex on D, since W has no multiplicative dependencies. Thus $A \in \mathcal{SE}$. Notice that by the normality for any k independent integer vectors $m_i = \langle m_{i,1}, \dots m_{i,n} \rangle$, $i = 1, \dots, k$, it holds $\delta(\{m_1\bar{a}, \dots, m_k\bar{a}\}/D) \geq 0$.

Thus $D \subseteq A$ satisfy the assumptions of Lemma 2.6 and hence $A \in \mathcal{SE}_d$, $D \leq A$. By the choice $\bar{a} \in V \setminus V'$ and $\mathrm{ex}(\bar{a}) \in W \setminus W'$. Since $\mathbf{D} \in \mathcal{EC}$ there is a realization of the type in $\mathbf{D}$. □

Proposition 2 *A structure $\mathbf{D}$ in $\mathcal{E}_d^0$ is e.a.c. iff given any $C \subseteq D$, a normal free pair (V,W) over C and a finite union $V' \subseteq V$ of proper K-affine subspaces definable over C, there is $\bar{a} \in V \cap \ln W$ such that $\bar{a} \notin V'$.*

4 Definability of normality and freeness conditions

This technical section heavily relies on [Z2] where in particular a theorem of J.Ax is used. Later, while preliminary versions of the present paper were put on my web-page, B.Poizat [P] and K.Holland [H] used a preliminary version of [Z2] and the theorem of Ax to prove technical results very similar to the main result in this section, for their own purposes (however, linked with a Hrushovski style construction). So it would be difficult to resolve the priority question if such one happens to arise.

Let $V(a) \subseteq D^n$ be a K-affine subspace defined over some finite tuple a from D, $V'(a)$ a finite union of K-affine subspaces defined over a, $W(b)$ an algebraic variety defined over b, a tuple from R. In fact, we may assume $a \in D^n$ is a vector such that $V(a) = L + a$ and L is K-linear. Thus $\dim V(a) = \dim L$ does not depend on a. Also, $V(a)$ is free of additive dependencies iff L is. It is evident that the set of a for which $V'(a)$ is a proper subset of $V(a)$ is quantifier-free definable in the K-vector space language. Also, by basic algebraic geometry the set of b satisfying for a given l the statement:

$$W(b) \text{ is irreducible and } \dim W(b) \geq l$$

is quantifier-free definable in the language of fields.

Our further arguments use the following statement (Corollary 3 of [Z2]).

Fact 1 *Given an algebraic variety $W(\bar{a}) \subseteq \mathbb{C}^n$ there is a finite collection $\mu(W)$ of non-zero integer vectors such that for any torus $T \subseteq \mathbb{C}^{*n}$ and an infinite atypical component $S \subseteq W(\bar{a}) \cap T$ of the intersection there is $\bar{m} \in \mu(W)$ and a constant c (depending on a and T) such that all $(a_1, \ldots, a_n)$ in the component satisfy $a_1^{m_1} \cdot \ldots \cdot a_n^{m_n} = c$.*

Lemma 4.1 *The set of b such that $W(b)$ is of dimension l, irreducible and free of multiplicative dependencies is quantifier-free definable in the language of fields.*

Proof Given $W(b)$ which is not free of multiplicative dependencies, $W(b) \subseteq T$ for some proper torus. This implies that $W(b)$ is an atypical component of $W(b) \cap T$. By Fact 1 this is equivalent to the statement

$$\bigvee_{\bar{m} \in \mu(W)} \forall \bar{y} \in W(b) \;\; \bar{y}^{\bar{m}} = \text{const.}$$

□

Notation Given a basic torus $T \subseteq (R^{\times})^n$ there is a uniquely determined algebraic (group) variety $(R^{\times})/T$ and the corresponding regular homomorphism

$$[T] : (R^{\times})^n \to (R^{\times})^n/T.$$

We write W/T for the image of W under the homomorphism instead of $[T](W)$. Also, since T is uniquely determined by any of it cosets, we use the notation also when T is a non-basic torus.

Let $T \subseteq P$ be tori, $W \subseteq P$. We say that W/T is an **atypical image with respect to** P if

$$\dim W/T < \min\{\dim P/T, \dim W\}.$$

Easy dimension calculations show for irreducible $W \subseteq P$, W/T atypical image, that for any generic $w \in W$ it holds

$$\dim W \cap Tw > 0 \tag{2}$$

and

$$\dim W \cap Tw > \dim W - \dim P/T. \tag{3}$$

The following Fact has been proved in [Z2]: The statement of the Proposition 1 of [Z2] (or rather its reformulation in the proof) is stronger than the Fact. The proof assumes CIT but the Corollary 3 of [Z2] (Fact 1 above, the function field case of CIT) in an obvious way replaces CIT in this proof to yield:

Fact 2 *Let $W \subseteq R^N$ be an algebraic variety, $a \in R^r$, some $r < N$, $P \subseteq R^N$ a torus and*

$$\{y \in P : y^\frown a \in W\} = W(a) \subseteq P.$$

Then there is a finite collection $\pi_P(W)$ of basic subtori of P depending on W only, such that given a torus $T \subseteq P$, for any connected infinite atypical component X of $W(a) \cap T$, there exists $Q \in \pi_P(W)$ and $c \in P$ such that $X \subseteq Q \cdot c$ and X is typical in $W(a) \cap T$ with respect to $Q \cdot c$.

Proposition 3 [1] *Given $W(a) \subseteq P = (F^*)^n$ an irreducible algebraic variety, for any basic torus $T \subseteq P$ with atypical image $W(a)/T$ with respect to P, there is $Q \in \pi_P(W)$ such that*

$$\dim W(a)/Q = \dim W(a)/T - \dim Q/(Q \cap T)$$

and

$$\dim W(a)/T = \dim W(a)/(Q \cap T).$$

Proof Let $w \in W(a)$ be generic and $X \subseteq W(a) \cap T \cdot w$ be a component of the intersection of maximal dimension. Then by the additive formula

$$\dim W(a)/T = \dim W(a) - \dim X \tag{4}$$

and $\dim X = \dim W(a) \cap T \cdot w > 0$. We may assume $w \in X$. By Fact 2 there is $Q \in \pi_P(W)$ such that (i) $X \subseteq Q \cdot w$ and (ii) X is a typical component of the intersection $(W(a) \cap Qw) \cap Tw$ with respect to Qw. By (i) and the maximality of $\dim X$, we have $\dim W(a)/T = \dim W(a)/(Q \cap T)$. And (ii) means that, given a connected component $Y \supseteq X$ of the variety $W(a) \cap Qw$, we have

$$\dim X = \dim Y + \dim Q \cap T - \dim Q. \tag{5}$$

But Y is a component of a generic fiber of the mapping $W(a) \to W(a)/Q$, and by the classical theorem on dimension of fibers ([S], Chapter 1, s.6, Thm 7)

$$\dim Y = \dim W(a) \cap Qw = \dim W(a) - \dim W(a)/Q. \tag{6}$$

Combining (4), (5) and (6) we get the requred equality on $\dim W(a)/Q$.□

In case $P = (R^\times)^n$ we write $\pi(W)$ instead of $\pi_P(W)$.

Lemma 4.2 *If a pair $(V, W(a))$ in n-spaces is not normal then either $\dim V + \dim W(a) < n$, or there is $Q \in \pi(W)$ defined by a matrix q on $l = \text{codim } Q$ independent integer n-rows as $Q = \{y \in (F^\times)^n : y^q = 1\}$ such that*

$$\dim qV + \dim W(a)^q < l.$$

[1] I am grateful to Kitty Holland for detecting a serious error in the formulation of the Proposition in the previous version of the paper. The present version is quite similar to an unpublished result of her's, and her proof is based on the similar results from Section 5 of [Z2]

Proof Suppose $\dim V + \dim W(a) \geq n$, and the pair is not normal, which is witnessed by $\bar{m}$, a set of $k < n$ independent integer n-vectors, as

$$\dim \bar{m}V + \dim W(a)^{\bar{m}} < k.$$

By definitions the mapping $x \to \bar{m}x$ is is a linear surjective mapping $D^n \to D^k$ and $y \to y^{\bar{m}}$ is a surjective homomorphism $(R^\times)^n \to (R^\times)^k$. Denote the kernel of the second one T, thus the latter mapping in notations above is $P \to P/T$, and $W(a)^{\bar{m}} = W(a)/T$.

Claim. $W(a)/T$ is an atypical image.

Suppose not. Then, in case $\dim P/T \leq \dim W(a)$, we get $\dim W(a)/T = \dim P/T = k$, a contradiction. In case $\dim W(a) < \dim P/T$ we get $\dim W(a)/T = \dim W(a)$. It follows $\dim \bar{m}V + \dim W(a)^m = \dim mV + \dim W(a) \geq \dim V - \dim T + \dim W(a) \geq n - \dim T = \dim P/T$, which contradicts the assumptions again. Claim proved.

By Proposition 3 there is $Q \in \pi(W)$ with $\dim W(a)/Q = \dim W(a)/T - \dim Q/(Q \cap T)$ and $\dim W(a)/(Q \cap T) = \dim W(a)/T$.

Claim 2. W.l.o.g. we may assume that $Q \supseteq T$.

Indeed, the basic torus $Q \cap T$ is given by a sytem of $k' = \text{codim } Q \cap T \geq k$ independent equations $y^{m'} = 1$.

By definition m' defines a linear surjective mapping $m' : D^n \to D^{k'}$, with $\ker m' \subseteq \ker m$, so m can be obtained as the composition of m' with another linear mapping with fibers of dimension $k' - k$. Thus,

$$\dim m'V \leq \dim mV + k' - k.$$

On the other hand

$$\dim W(a)^{m'} = \dim W(a)/(Q \cap T) = \dim W(a)/T = \dim W(a)^m.$$

Thus

$$\dim m'V + \dim W(a)^{m'} \leq \dim mV + \dim W(a)^m + k' - k < k'.$$

In other words, we can replace T by $Q \cap T$, and so m by m', and still witness the failure of normality. Claim proved.

Let now the above basic torus $Q \supseteq T$ be given by $l = \text{codim } Q \leq k$ equations of the form $y^q = 1$, and the matrix q induce the surjective mappings

$$D^n \to D^l \text{ and } (R^*)^n \to (R^*)^l.$$

Since $Q \supseteq T$ we have $\dim qV \leq \dim mV$, while on R we have $\dim W(a)^q = \dim W(a)^m - (k-l)$, by the definition of Q.

The two last formulas yield

$$\dim qV + \dim W(a)^q \leq \dim mV + \dim W(a)^m + l - k.$$

It follows

$$\dim qV + \dim W(a)^q < l.$$

□

Corollary 1 *Given $V(a), V'(a), W(\mathrm{ex}(a))$, as defined in the beginning of the section, the statement about parameters a :*

$(V(a), W(\mathrm{ex}(a)))$ is a free normal pair and $V'(a)$ is a proper subset of $V(a)$

is quantifier-free definable in $\mathcal{L}_K$.

Denote the formula from the corollary $\mathrm{NF}_{V,V',W}(a)$. Denote EC the set of axioms of the form

$$\forall x[\mathrm{NF}_{V,V',W}(x) \rightarrow \exists y((y \in V(x))\&(y \notin V'(x))\&(\mathrm{ex}(y) \in W(\mathrm{ex}(x))))]$$

It follows from Proposition 2

Corollary 2 *For any $\mathbf{D} \in \mathcal{E}_d^0$*

$$\mathbf{D} \models EC \text{ iff } \mathbf{D} \text{ is exponentially-algebraically closed.}$$

5 Axiomatizing $\mathcal{E}_d$

Notation Denote $\mathcal{E}_{d/\ker}$ the subclass of $\mathcal{E}$ for which $\delta(X/\ker) \geq -d$ holds for any finite $X \subseteq D$.
Denote $\mathcal{E}^0_{d/\ker} = \mathcal{E}^0 \cap \mathcal{E}_{d/\ker}$.

Lemma 5.1 *For $\{x_1, \ldots, x_n\} \subseteq \ker$*

(i) $\delta(x_1, \ldots, x_n) = 0$ iff $\mathrm{l.d.}_{\mathrm{K}}(x_1, \ldots, x_n) = \mathrm{l.d.}_{\mathrm{Q}}(x_1, \ldots, x_n)$;

(ii) the condition $\delta(X) = 0$ for all $X \subseteq \ker$ is equivalent to: $x_n = k_1x_1 + \ldots + k_{n-1}x_{n-1}$ with $k_1, \ldots, k_{n-1} \in K$ and $0 \neq x_i \in \ker$ for any $i \leq n$ implies $k_1, \ldots, k_{n-1} \in \mathbb{Q}$.

Proof (i) is immediate from the definitions. To see (ii) assume first that the condition on ker holds and $x_n = k_1x_1 + \ldots + k_{n-1}x_{n-1}$ is a minimal counterexample. By (i) then one gets $x_n = q_1x_1 + \ldots + q_{n-1}x_{n-1}$ for some $q_1, \ldots, q_{n-1} \in \mathbb{Q}$. Combining the two linear combinations one comes to $k_i = q_i$ for all $i < n$. The converse is obvious.$\Box$

Lemma 5.2 *Let $A \in \mathcal{SE}^0$, $B \in \mathcal{SE}$ and $A \leq B$. Then $B \in \mathcal{SE}^0$.*

Proof Let $X \subseteq B$ and $X \subseteq \ker$. Then, since tr.d.$(X/A) = 0$,

$$0 \leq \delta(X/A) = \mathrm{l.d.}_{\mathrm{K}}(X/A) - \mathrm{l.d.}_{\mathrm{Q}}(X/A)$$

and hence $\mathrm{l.d.}_{\mathrm{K}}(X/A) = \mathrm{l.d.}_{\mathrm{Q}}(X/A)$.

We want to prove that if $x_n = k_1x_1 + \ldots + k_{n-1}x_{n-1}$ for $x_1, \ldots, x_n \in X$ then all $k_i \in \mathbb{Q}$.
Suppose $x_1, \ldots, x_n$ is a counterexample with $k_{n-1} \in K \setminus \mathbb{Q}$, $x_1, \ldots, x_l \in A$ and $x_{l+1}, \ldots, x_n \in B \setminus A$ with $n - l$ minimal possible.

Notice that then $\mathrm{l.d.}_{\mathrm{K}}(x_{l+1}, \ldots, x_n/A) = n - l - 1$ and hence $\mathrm{l.d.}_{\mathrm{Q}}(x_{l+1}, \ldots, x_n/A) = n - l - 1$. Thus there are non-trivial integer coefficients $m_{l+1}, \ldots, m_n$ such that $m_{l+1}x_{l+1} + \ldots + m_nx_n = y \in A$. It follows $y \in \ker$. Combining this with the initial combination one contradicts minimality.$\Box$

Lemma 5.3 $\mathcal{E}^0_{d/\ker} = \mathcal{E}^0_d$

Proof We assume $d \geq 0$. Let $D \in \mathcal{E}^0_{d/\ker}$, $X \subseteq D$. Then $\delta(X/\ker) \geq -d$ which means that $\delta(X \cup Y) \geq -d$ for appropriate finite $Y \subseteq \ker$.
Claim $\delta(X \cup Y) \leq \delta(X)$.
It is enough to prove that $\delta(Xy) \leq \delta(X)$ for any $y \in \ker$. If $y \in sp_Q(X)$ then the equality holds. If $y \notin sp_Q(X)$ then $\mathrm{l.d.}_{\mathrm{Q}}Xy = \mathrm{l.d.}_{\mathrm{Q}}X + 1$, $\mathrm{l.d.}_{\mathrm{K}}Xy \leq \mathrm{l.d.}_{\mathrm{K}}X + 1$, $\mathrm{tr.d.ex}(Xy) = \mathrm{tr.d.ex}(X)$.
Claim proved.
It follows from the Claim that $\mathbf{D} \in \mathcal{E}^0_d$ and thus $\mathcal{E}^0_{d/\ker} \subseteq \mathcal{E}^0_d$.
Assume now $\mathbf{D} \in \mathcal{E}^0_d$. Then for any finite $X \subseteq D$ any $Y \subseteq \ker$ one has $\delta(X \cup Y) \geq -d$. It follows $\delta(X/\ker) \geq -d$. Thus $\mathbf{D} \in \mathcal{E}^0_{d/\ker}$. $\Box$

Notation Let AK be the following set of axioms:
for any $k_1, \dots, k_n \in K$ such that $1, k_1, \dots, k_n$ are $\mathbb{Q}$-linearly independent there is an axiom stating:

$$\forall \bar{x}(x_1, \dots, x_n \in \ker \setminus \{0\} \to k_1 \cdot x_1 + \ldots + k_n \cdot x_n \notin \ker).$$

Remark If ker is a cyclic subgroup then AK holds in the structure.

Lemma 5.4 *The subclass of* $\mathcal{E}$ *axiomatized by* AK *is exactly* $\mathcal{E}^0$.

Proof By Lemma 5.1 AK holds for any $\mathbf{D} \in \mathcal{E}^0$.
To prove the converse, by the same Lemma, we need to prove that for any $k_1, \dots, k_n \in K$ and $a_1, \dots, a_n, b \in \ker$ if $k_1 \cdot a_1 + \ldots + k_n \cdot a_n = b$ then there are $q_1, \dots, q_n \in \mathbb{Q}$ such that $q_1 \cdot a_1 + \ldots + q_n \cdot a_n = b$.

Suppose w.l.o.g. that $a_1, \dots, a_n$ are $\mathbb{Q}$-linearly independent and $k_1 \cdot a_1 + \ldots + k_n \cdot a_n = b \in \ker$. By the axioms there are integers $m_1, \dots, m_{n+1}$ such that $k_1 \cdot m_1 + \ldots + k_n \cdot m_n + m_{n+1} = 0$ with, say, $m_1 \neq 0$. It follows that

$$m_1 \cdot b = k_2 \cdot (m_1 \cdot a_2 - m_2 \cdot a_1) + \ldots + k_n \cdot (m_1 \cdot a_n - m_n \cdot a_1) - m_{n+1} \cdot a_1.$$

Since $m_1 \cdot a_i - m_i \cdot a_1 \in \ker$ for $i = 2, \dots, n$ and $m_1 \cdot b + m_{n+1} \cdot a_1 \in \ker$, by induction hypothesis $m_1 \cdot b + m_{n+1} \cdot a_1$ is a $\mathbb{Q}$-linear combination of $m_1 \cdot a_i - m_i \cdot a_1$ for $i = 2, \dots, n$ and thus b is a $\mathbb{Q}$-linear combination of $a_1, \dots, a_n$. $\square$

From now on we have to use the conjecture CIT formulated in the introduction and discussed in [Z2]. Recall that $\tau(W)$ is the finite set of basic tori stipulated in the conjecture.

Notation Let, for a definable K-linear $L \subseteq D^{n+l}$, a natural number l and an algebraic variety $W \subseteq R^n$ defined and irreducible over $\mathbb{Q}$, the formula

$$\mathrm{A}_{L,W}(\bar{x}) := \quad (\forall \bar{z} \in \ker^l)[(\bar{x}\bar{z} \in L)\&(\mathrm{ex}(\bar{x}) \in W) \to \bigvee_{T \in \tau(W)} \mathrm{ex}(\bar{x}) \in T].$$

Definition The pair (L, W) as above is said to be m-**special** if the minimal torus $\mathrm{T}(W)$ containing W is $\mathrm{ex}(\bar{L}(0^l))$ and

$$\dim L(0^l) + \dim W < m + \dim \bar{L}(0^l).$$

Equivalently, for an $\bar{a}$ generic in L and a $\bar{b}$ generic in W,

$$\mathrm{l.d.}_{\mathrm{K}}(\bar{a}/\ker) + \mathrm{tr.d.}(\bar{b}) - \mathrm{l.d.}_{\mathrm{Q}}(\bar{a}/\ker) < m.$$

Lemma 5.5 *For a structure* $\mathbf{D}$ *in* $\mathcal{E}$ *and* $\bar{c} \in D^n$*, given* $m \in \mathbb{Z}$

$$\delta(\bar{c}/\ker) \geq m \qquad \textit{iff} \qquad \bigwedge_{(L,W)\ \textit{is } m\textit{-special}} \mathrm{A}_{L,W}(\bar{c})$$

Proof Suppose, given m and $\bar{c}$, all the formulae hold in $\mathbf{D}$.

Let l, $\bar{a} \in \ker^l$ and L be chosen so that $\bar{c}\bar{a} \in L$ and $\dim L(\bar{a}) = \mathrm{l.d.}_{\mathrm{K}}(\bar{c}/\ker)$, $\dim \bar{L}(\bar{a}) = \mathrm{l.d.}_{\mathrm{Q}}(\bar{c}/\ker)$ (see Lemma 2.1.) Remember that $\dim L(\bar{a}) = \dim L(0^l)$ and $\dim \bar{L}(\bar{a}) = \dim \bar{L}(0^l)$. Let W be the minimal algebraic variety over $\mathbb{Q}$ containing $\mathrm{ex}(\bar{c})$. We claim that

$$\delta(\bar{c}/\ker) = \dim L(0^l) + \dim W - \dim \bar{L}(0^l) \geq m.$$

Suppose the opposite is true. Then (L, W) is m-special and

$$\mathbf{D} \models \mathrm{A}_{L,W}(\bar{c}).$$

Hence, by the choice of L, $\bar{a}$ and W, necessarily $\mathrm{ex}(\bar{c}) \in T$ for a proper torus $T \in \tau(W)$. This contradicts the minimality of W. The right-to-left implication in the statement is proved.

To prove the converse suppose that

$$\delta(\bar{c}/\ker) \geq m \text{ and for some } m\text{-special } (L, W) \ \ \mathbf{D} \models \neg A_{L,W}(\bar{c}).$$

Then for some $\bar{a} \in \ker^l$

$$\bar{c}\bar{a} \in L \ \&\ \mathrm{ex}(\bar{c}) \in W \setminus \bigcup_{T \in \tau(W)} T.$$

Let $\bar{b} \in \ker^r$ and N be a $\mathbb{Q}$-linear subspace of D^{n+r} such that $\bar{c}\bar{b} \in N$ and

$$\dim N(\bar{b}) = \dim N(0^r) = \mathrm{l.d.}_{\mathrm{Q}}(\bar{c}/\ker).$$

Notice that $\mathrm{ex}(N(0^l))$ is then equal to the minimal torus T_c containing $\mathrm{ex}(\bar{c})$ and $\dim T_c = \dim N(0^l)$. Also notice that

$$\mathrm{l.d.}_{\mathrm{K}}(\bar{c}/\ker) \leq \dim L(0^l),$$

$$\mathrm{tr.d.}(\bar{c}/\mathrm{ex}(\ker)) = \mathrm{tr.d.}(\bar{c}) \leq \dim W \cap T_c.$$

Since $\delta(\bar{c}/\ker) \geq m$,

$$\dim L(0^l) + \dim W \cap T_c - \dim T_c \geq m.$$

By our assumptions T_c is not a subtorus of any $T \in \tau(W)$, thus by CIT

$$\dim W \cap T_c = \dim W + \dim T_c - \dim \mathrm{T}(W).$$

Notice that by assumptions $\mathrm{T}(W) = \mathrm{ex}(\bar{L}(0^l))$ and $\dim \mathrm{T}(W) = \dim \bar{L}(0^l)$. Hence

$$\dim L(0^l) + \dim W - \dim \bar{L}(0^l) \geq m,$$

which contradicts the fact that (L, W) is m-special. □

Corollary 3 *The subclass $\mathcal{E}_{d/\ker}$ of $\mathcal{E}$ is axiomatized by the set of axioms:*

AS_d $\qquad (\forall \bar{x})\mathrm{A}_{L,W}(\bar{x}) \qquad (L, W)$ *is* $(-d)$*-special.*

Corollary 3 and Lemma 5.4 immediately imply

Theorem 1 *The subclass of $\mathcal{E}$ axiomatized by* AS_d *and* AK *is exactly $\mathcal{E}^0_d$. The class of structures axiomatized by* $\mathrm{PF}(K)$, AS_d, AK *and* EC *is exactly $\mathcal{EC}^0_d$.*

Notation Denote $\mathrm{PCF}_d(K)$ the theory of $\mathcal{EC}^0_d(K)$. In what follows we omit K.

6 The theory of algebraically closed K-powered fields of characteristic zero

Definition The extension of the initial language $\mathcal{L}_K$ by predicates

$$E_P(\bar{x}) \equiv \exists \bar{y} P(\bar{x}, \bar{y}),$$

where P is a quantifier-free formula, is denoted $\mathcal{L}^E_K$, and these predicates are called E-predicates.
Notice that negations of $\mathrm{A}_{L,W}(\bar{x})$ are equivalent to E-predicates.

Lemma 6.1 *For $M, N \in \mathcal{E}^0$, if $M \subseteq N$ in the language $\mathcal{L}_K^E$, then $M \leq N$.*

Proof Given a finite $X \subseteq N$, its $\mathcal{L}_K$-quantifier-free type obviously tells the value of $\delta(X)$.

Also, the statement '$\partial_M(X) = \delta(X)$' follows from the $\mathcal{L}_K^E$-quantifier-free type of X. Indeed, using the Claim from the proof of Lemma 5.3, one easily sees that, if $m = \delta(X)$, the statement is equivalent to

$$\delta(X) = m \ \& \ \forall Z \ \delta(XZ/\ker) \geq m.$$

By Lemma 5.5 the second part of the expression is given by negations of E-predicates.

It follows from general properties of $\leq$ that for any $Y \subseteq N$, given $X \subseteq M$ such that $\partial_M(X) = \delta(X)$, one has $\delta(Y/X) \geq 0$. Thus $M \leq N$.□

Lemma 6.2 *Assume $M_1, M_2 \in \mathcal{E}_d^0$ and both satisfy* EC.

(i) Suppose $A \subseteq M_1$, $A \subseteq M_2$ and $\bar{b}_i \in M_i^n$ are such that $A\bar{b}_i \leq M_i$ for $i = 1, 2$ and the $\mathcal{L}_K$-quantifier-free types of $\bar{b}_1$ and $\bar{b}_2$ over A coincide. Then the $\mathcal{L}_K^E$-quantifier-free types of $\bar{b}_1$ and $\bar{b}_2$ over A coincide.

(ii) Suppose $M_2 \leq M_1$. Then $M_2 \subseteq M_1$ in the language $\mathcal{L}_K^E$.

Proof (i) Let $\exists \bar{x} P_{b_1}(\bar{x})$ be an E-predicate with parameters in $A\bar{b}_1$ which holds in M_1, with P_{b_1} quantifier-free. Let $\bar{d}$ be a string in M_1 for which $P_{b_1}(\bar{d})$ holds. Then $P_{b_1}(\bar{x})$ is a consequence of a formula $P_{b_1}^0(\bar{x})$ of the form

$$\bar{x}\bar{a}\bar{b}_1 \in L \setminus (L^0 \cup \ldots \cup L^k) \ \& \ (\mathrm{ex}(\bar{x}\bar{a}\bar{b}_1) \in W \setminus W^0),$$

where $\bar{a}$ is the string of all elements of A, L is the minimal K-linear space containing $\bar{d}\bar{a}\bar{b}_1$ and W is the minimal algebraic variety over $\mathbb{Q}$ containing $\mathrm{ex}(\bar{d}\bar{a}\bar{b}_1)$, $L^i \subseteq L$ are K-linear subspaces and $W^0 \subseteq W$ is an algebraic subvariety over $\mathbb{Q}$. By Lemma 3.2 it follows that (L, W) is normal over $\bar{a}\bar{b}_1$. Moreover, since normality is expressible quantifier-freely in $\mathcal{L}_K$, the pair is normal over $\bar{a}\bar{b}_2$. It follows from axioms EC that $\exists \bar{x} \bar{P}_{b_2}^0(\bar{x})$ holds in M_2 and hence $\exists \bar{x} P_{b_2}(\bar{x})$ holds. Thus $\bar{b}_1$ and $\bar{b}_2$ satisfy the same E-predicates over A.

(ii) Let $A = M_2 \leq M_1$ and $b_1, b_2 \in A$ be of the same $\mathcal{L}_K$-quantifier-free type. Then we have the assumptions of (i) satisfied, and the argument above proves that every E-predicate with parameters in M_2 which holds in M_1 must also hold in M_2. The converse is obvious, thus $M_2 \subseteq M_1$ as an $\mathcal{L}_K^E$-substructure. $\square$

Corollary 4 *For* $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{EC}_d^0$

$$\mathbf{D}_1 \subseteq \mathbf{D}_2 \textit{ as } \mathcal{L}_K^E\textit{-structures} \quad \textit{iff} \quad \mathbf{D}_1 \leq \mathbf{D}_2.$$

Notation Define ID to be the set of axioms of the form

$$\exists x_1, \ldots, x_m \forall y_1, \ldots, y_n \mathrm{A}_{L,W}(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

for positive integers m and (L, W) ranging over all the m-special pairs.

Remark If for any m there is $X \subseteq D$ such that $\partial(X) \geq m$ then $\mathbf{D} \models$ ID. Thus any e.a.c. infinite dimensional structure satisfies ID. It is probable that any $\mathbf{D} \in \mathcal{EC}_d^0$ satisfies ID, so this set of axioms is redundant.

Proposition 4 *Suppose* $\mathbf{D} \models \mathrm{PCF}_d$+ID. *Then any finite* $\mathcal{L}_K^E$*-quantifier-free type which is realized in an* $\mathcal{E}_d^0$*-extension of* $\mathbf{D}$ *is realized in* $\mathbf{D}$ *itself.*

Proof Consider an elementary extension $\mathbf{D}^*$ of $\mathbf{D}$. We prove that for any finite $A \subseteq D^*$ and $\bar{c}$ in an extension $\mathbf{D}'$, with $D^* \leq D'$, the $\mathcal{L}_K^E$-quantifier-free type q of $\bar{c}$ over A is realized in $\mathbf{D}^*$. W.l.o.g. we assume that $A \leq D^*$.

Consider first the case $\partial(\bar{c}/A) = 0$. Choose $\bar{c}'$ in D' extending $\bar{c}$ such that $\partial(\bar{c}/A) = \delta(\bar{c}'/A) = 0$. Since $A \leq D'$ by Lemma 3.2 and axioms EC the $\mathcal{L}_k$-quantifier-free type of $\bar{c}'$ over A is realized in $\mathbf{D}^*$. Let $\bar{b}'$ be the realization. Since $\partial(\bar{c}'/A) = \delta(\bar{c}'/A)$ we have $A\bar{c}' \leq D'$. Since $\delta(\bar{b}'/A) = 0$ and $A \leq D^*$ we have $A\bar{b}' \leq D^*$. Hence by Lemma 6.2 the $\mathcal{L}_K^E$-quantifier-free types of $\bar{b}'$ and $\bar{c}'$ over A coincide. Hence the corresponding substring of $\bar{b}'$ is a realization we sought for.

Let now $\partial(\bar{c}/A) = k > 0$ and $\{c_1, \ldots, c_k\}$ be a ∂-base of $\bar{c}$ over A. It follows from ID and the saturatedness that there are $\{b_1, \ldots, b_k\}$ in $\mathbf{D}^*$ which are ∂-independent over A. Evidently the $\mathcal{L}_K$-quantifier-free types of $\langle b_1, \ldots, b_k \rangle$ and $\langle c_1, \ldots, c_k \rangle$ over A coincide. Also $A \cup$

$\{b_1, \ldots, b_k\} \leq D^*$ and $A \cup \{c_1, \ldots, c_k\} \leq D'$. Hence the $\mathcal{L}_K^E$-quantifier-free types of the strings coincide too. Thus we may identify the strings and since $\partial(\bar{c}/A \cup \{c_1, \ldots, c_k\}) = 0$ by the case considered above the $\mathcal{L}_K$-quantifier-free type of $\bar{c}$ over A is realized in $\mathbf{D}^*$. □

We say that a (partial) map $\varphi : \mathbf{D}_1 \to \mathbf{D}_2$ is an $\mathcal{L}_K^E$-monomorphism, if it is injective and for any k-ary E-predicate S and any k-tuple a from the domain of φ

$$\mathbf{D}_1 \models S(a) \text{ iff } \mathbf{D}_2 \models S(\varphi(a)).$$

Lemma 6.3 *Let* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *satisfy* $\mathrm{PCF}_d + \mathrm{ID}$, *and* $A_1 \leq D_1$, $A_2 \leq D_2$ *such that there is an* $\mathcal{L}_K^E$*-monomorphism*

$$\varphi : A_1 \to A_2.$$

or $A_1 = A_2 = \emptyset$.

Let $\mathbf{D}_1^A$ *and* $\mathbf{D}_2^A$ *be the expansions of* $\mathbf{D}_1$, $\mathbf{D}_2$ *by the set of constants naming elements of* A_1 *and* A_2 *in correspondence with* φ. *Then*

$$\mathbf{D}_1^A \equiv \mathbf{D}_2^A.$$

Proof We prove that given ω-saturated elementary extensions $\mathbf{D}_1^*$ of $\mathbf{D}_1$ and $\mathbf{D}_2^*$ of $\mathbf{D}_2$, given finite $B \subseteq D_1^*$, $c \in D_1^*$ and a $\mathcal{L}_K^E$-monomorphism φ of $A_1 \cup B$ into $\mathbf{D}_2^*$ one can extend the monomorphism to c. By symmetry, this yields a winning strategy for the Ehrenfeucht-Fraisse game, and we are done.

We may assume that φ is the identity and $A_1 \cup B = A = \varphi(A)$. It is enough to show that under the assumption for any $c \in D_1^*$ we can extend φ to some $A' \supseteq Ac$ as an $\mathcal{L}_K^E$-monomorphism and $A' \leq D_1^*$, $\varphi(A') \leq D_2^*$.

If $\partial(c/A) = 1$ then define $A' = Ac$ and $\varphi(c)$ to be any element from D_2^* which is not in the ∂-closure of A in $\mathbf{D}_2^*$. Then A' and $\varphi(A')$ are as required.

If $\partial(c/A) = 0$ then extend c to a finite string $\bar{c}$ from D_1^* so that $\delta(\bar{c}/A) = 0$. Again, as in the proof of Proposition 4, there is $\bar{b}$ in D_2^* which realizes the $\mathcal{L}_K$-quatifier-free type of $\bar{c}$ over A. Since $\delta(\bar{b}/A) = 0$, $A\bar{b} \leq D_2^*$ holds. Thus by Lemma 6.2 the $\mathcal{L}_K^E$-quatifier-free types of $\bar{b}$ and $\bar{c}$ over A coincide. Put $\varphi(\bar{c}) = \bar{b}$, $A' = A\bar{c}$.□

To apply the Lemma we need A_1, A_2 satisfying the assumption. In particular, we can not start with $A_1 = A_2 = \emptyset$ if $d > 0$. The next lemma solves the problem for $\mathbf{D}_1$ and $\mathbf{D}_2$ satisfying the same $\mathcal{L}_K$-existential sentences.

Lemma 6.4 *For any finite subset A_1 of a structure $\mathbf{D}_1$, model of* $\mathrm{PCF}_d + \mathrm{ID}$, *there is a finite subset $\tilde{A}_1$ such that*

(i) $\tilde{A}_1 \leq D_1$;

(ii) if A_2 is a subset of an ω-saturated model $\mathbf{D}_2$ of $\mathrm{PCF}_d + \mathrm{ID}$ *and there is a $\mathcal{L}_K^E$-monomorphism $\varphi : A_1 \to A_2$, then φ can be extended to $\tilde{A}_1$ and*

$$\varphi(\tilde{A}_1) = \tilde{A}_2 \leq D_2.$$

Proof Let $\bar{a}_1$ be the string of all elements of A_1 and $\bar{c}$ in $\mathbf{D}_1$ such that $\delta(\bar{a}_1\bar{c}) = \partial(\bar{a}_1)$. It follows $A_1\bar{c} \leq D_1$. Let $m = \partial(\bar{a}_1)$.

Let $q^0(\bar{x}\bar{y})$ be the $\mathcal{L}_K$-quantifier-free type of $\bar{a}_1\bar{c}$. Let $\bar{a}_2$ be a string in D_2 which is $\mathcal{L}_K^E$-monomorphic to $\bar{a}_1$. Then the E-predicates guarantee that $q^0(\bar{a}_2\bar{y})$ is consistent and thus $\partial(\bar{a}_2) \leq m = \partial(\bar{a}_1)$. By symmetry $\partial(\bar{a}_2) = m = \partial(\bar{a}_1)$. Let $\bar{y} = \bar{d}$ be the realization of $q^0(\bar{a}_2\bar{y})$ in $\mathbf{D}_2$. Since $\delta(\bar{a}_2\bar{d}) = \partial(\bar{a}_2)$, we have $\bar{a}_2\bar{d} \leq D_2$. Now Lemma 6.2 says that $\bar{a}_2\bar{d}$ is of the same $\mathcal{L}_K^E$-quantifier-free type as $\bar{a}_1\bar{c}$. $\square$

Theorem 2 *The theory* $\mathrm{PCF}_d + \mathrm{ID}$ *is a model completion of* $\mathrm{PF} + \mathrm{AK} + \mathrm{AS_d} + \mathrm{ID}$ *in the language $\mathcal{L}_K^E$. The theory has quantifier elimination in this language.*

Proof It follows from Lemmas 6.3 and 6.4 that the theory is submodel complete. Thus (see e.g. Theorem 13.1 of [S]) it has elimination of quantifiers. $\square$

Remark In fact, given a model $\mathbf{D}$ of $\mathrm{PCF}_d + \mathrm{ID}$ we may assume that there is a finite $A \subseteq D$ with $\delta(A) = -d$ (otherwise $\mathbf{D}$ is a model of $\mathrm{PCF}_{d'}$ for some $d' < d$) and thus $A \leq \mathbf{D}$. The fact that there exists $X \cong A$ in the basic language can be expressed by the formula $\exists X \mathrm{A}_{L,W}(X)$ for some pair (L, W) witnessing the fact that

$\delta(A) = \delta(A/ker) = -d$ (see Lemma 5.5). Then, by Lemmas 6.3 and 6.2,

$$\mathrm{PCF}_d + \mathrm{ID} + \exists X \mathrm{A}_{L,W}(X)$$

is a complete theory.

Theorem 3 *Any completion of* $\mathrm{PCF}_d + \mathrm{ID}$ *is superstable.*

Proof Let $\mathbf{D} \in \mathcal{EC}_d^0$ satisfy ID and card $D = \lambda$. We want to establish the cardinality of the set $S(D)$ of complete 1-types over D. Let $\mathbf{D}^*$ be an elementary extension of $\mathbf{D}$ which realizes all n-types over $\mathbf{D}$ for all natural n. Let $S^{\#}(D)$ the set of all complete n-types over $\mathbf{D}$ which are realized in $\mathbf{D}^*$ by n-tuples $\bar{b} = \langle b_1, \ldots, b_n \rangle$ such that $\delta(\bar{b}/D) = \partial(b_1/D)$. It follows that card $S(D) \leq$ card $S^{\#}(D)$. From general properties of $\leq$ we get $D\bar{b} \leq D^*$, and by Lemma 6.2 the $\mathcal{L}_K^E$-quantifier-free type of $\bar{b}$ over D is determined by the $\mathcal{L}_K$-quantifier-free type of that. By quantifier elemination the complete type of $\bar{b}$ over $\mathbf{D}$ is determined by the $\mathcal{L}_K$-quantifier-free subtype. Thus card $S(D)$ is less or equal to the cardinality of $QS(D)$, the set of all $\mathcal{L}_K$-quantifier-free complete types over $\mathbf{D}$, which is of power $\lambda + 2^\omega$, since each such type is uniquely determined by $(V, W, \{W^{\frac{1}{l}} : l \in \mathbb{N}\})$ for some K-affine space V, an algebraic variety W and an associated sequence of varieties $\{W^{\frac{1}{l}} : l \in \mathbb{N}\}$. □

7 Raising to real powers in the complex field

Let $K \subseteq \mathbb{C}$ be of finite transcendence degree d. Notice that

$$\mathrm{l.d.}_{\mathrm{K}}(X) \geq \mathrm{tr.d.}(X/K) \geq \mathrm{tr.d.}(X) - \mathrm{tr.d.}(K)$$

in this case. Thus

$$\mathrm{l.d.}_{\mathrm{K}}(X) + \mathrm{tr.d.}(\exp(X)) - \mathrm{l.d.}_{\mathrm{Q}}(X) \geq [\mathrm{tr.d.}(X) + \mathrm{tr.d.}(\exp(X)) - \mathrm{l.d.}_{\mathrm{Q}}(X)] - d.$$

Assuming the Schanuel conjecture, the expression in the brackets is non-negative. Thus one gets

$$\delta(X) \geq -d$$

for

$$\delta(X) = \text{l.d.}_{\text{K}}(X) + \text{tr.d.}(\exp(X)) - \text{l.d.}_{\text{Q}}(X).$$

Assume now K is a subfield of the reals $\mathbb{R}$ and has transcendence degree d, $D = R = \mathbb{C}$, ex = exp, and let $\mathbb{C}^{(K)} = (\mathbb{C}, \exp, \mathbb{C})$ be the corresponding two-sorted structure on the complex numbers in the language $\mathcal{L}_K$.

Lemma 7.1 *(i) Assume* SchC*. Then* $\mathbb{C}^{(K)}$ *satisfies* PF+AK*;*
(ii) Assume also CIT*. Then* $\mathbb{C}^{(K)}$ *satisfies* AS_d*.*

Proof (i) Follows from the remarks above.
(ii) Again, Schanuel's conjecture implies $\mathbb{C}^{(K)} \in \mathcal{E}_{d/\ker}$, so the statement follows from Corollary 3.□

Theorem 4 *Assuming* SchC+CIT*, for any field* $K \subseteq \mathbb{R}$ *of finite transcendence degree* d*, the structure* $\mathbb{C}^{(K)}$ *satisfies* PCF_d+ ID*. Thus the theory of the structure allows quantifier elimination in the language* $\mathcal{L}_K^E$ *and is superstable.*

Proof The main result of [Z2], Theorem 5 followed by a Remark, state under the assumtions of the theorem under the proof.

Fact 3 *Let* $L \subseteq \mathbb{C}^n$ *be an* $\mathbb{R}$*-linear subspace and* W *a family of algebraic varieties such that* $(L, W(a))$ *is normal and free for any* a *in a definable set of parameters* $C(W)$*. Then there is a positive real constant* $R(L, W)$ *such that, given a ball* $B \subseteq \text{Re}(L)$ *of radius* $R(L, W)$*, there is a point*

$$x \in (\text{Re}(L) + \imath B) \cap \ln W(a) \text{ (notice that } (\text{Re}(L) + \imath B) \subseteq L). \quad (7)$$

Moreover, for any number l *we can choose a real constant* $R(L, W, l)$ *such that, given any* $\mathbb{R}$*-affine hyperplanes* $H_i \subseteq \mathbb{C}^n$*,* $(i = 1, \ldots, l)$ *and a ball* $B \subseteq \text{Re}(L)$*, there is an* x *satisfying (7) with*

$$x \notin \bigcup_{i=1}^{l} H_i.$$

The Fact yields condition EC. Thus $\mathbb{C}^{(K)}$ satisfies $\mathrm{PCF}_d(K)$, so it is a structure from $\mathcal{EC}^0_d(K)$.

Claim. Given countable $A \leq \mathbb{C}$ there are countably many 0-dimensional analytic subsets S_i of $\mathbb{C}^n$, for all n, such that any $\bar{b} \in \mathbb{C}^n$ satisfying $\delta(\bar{b}/A) = 0$ belongs to one of the $S_i's$.

Proof. We may assume that A is closed under taking K-spans and under the operation $\ln(\mathrm{acl}(\exp(A)))$. We also assume that $\bar{b}$ is $\mathbb{Q}$-independent over A. Let $V \subseteq \mathbb{C}^n$ be the minimal K-affine space over A containing $\bar{b}$, and $W \subseteq \mathbb{C}^n$ be the minimal algebraic variety definable over $\exp(A)$ containing $\exp(\bar{b})$. Since $\delta(\bar{b}/A) = 0$, we have

$$\dim V + \dim W = n.$$

If the dimension of the analytic set $V \cap \ln W$ is 0, then we take S_i to be this set. Otherwise, by Corollary 2 of [Z2] (under SchC+CIT), there are finitely many tori (of the form $\exp(M_i + c_i)$ for M_i a $\mathbb{Q}$-linear subspace, $c_i \in \mathbb{C}^n$, $i = 1, \ldots, l$) such that any infinite analytic component of $\exp(V) \cap W$ belongs to one of the tori. Moreover, Lemma 3.1 of [Z2] proves that any such torus intersects W atypically. It follows immediately that $\exp(c_i)$ can be chosen in $\mathrm{acl}(\exp(A))$, thus $c_i \in A^n$. Then $\exp(\bar{b}) \notin \exp(M_i + c_i)$, by our assumptions. It follows that $\bar{b}$ belongs to

$$V \cap \ln W \setminus \bigcup_{i=1}^{l} M_i + c_i + 2\pi i \mathbb{Z}^n$$

which is a countable analytic subset of $\mathbb{C}^n$. Claim proved.

It follows immediately from the claim that for countable $A \leq \mathbb{C}$ the ∂-closure of A is countable. Hence the ∂-basis of $\mathbb{C}$ is uncountable. In particular, ID holds. □

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